Meshless Local Petrov-Galerkin (MLPG) Method for Convection-Diffusion Problems

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Abstract: Due to the very general nature of the Meshless Local Petrov-Galerkin (MLPG) method, it is very easy and natural to introduce the upwinding concept (even in multi-dimensional cases) in the MLPG method, in order to deal with convection-dominated flows. In this paper, several upwinding schemes are proposed, and applied to solve steady convection-diffusion problems, in one and two dimensions. Even for very high Peclet number flows, the MLPG method, with upwinding, gives very good results. It shows that the MLPG method is very promising to solve the convection-dominated flow problems, and fluid mechanics problems.

keyword: MLPG, MLS, convection-dominated flow, upwinding.

1 Introduction

Although the finite element method (FEM) and the closely related finite volume method (FVM) are well-established numerical techniques for computer modeling in engineering and sciences, they are not without shortcomings. First of all, their reliance on a mesh leads to complications for certain classes of problems. The generation of good quality meshes presents significant difficulties in the analysis of engineering systems (especially in 3D). These difficulties can be overcome by the so-called meshless methods, which have attracted considerable interest over the past decade. A number of meshless methods have been developed by different authors, such as Smooth Particle Hydrodynamics (SPH) [Lucy (1977)], Diffuse Element Method (DEM) [Naroles, Touzot, and Villon (1992)], Element Free Galerkin method (EFG) [Belytcshko, Lu, and Gu (1994)], Reproducing Kernel Particle Method (RKPM) [Liu, Jun, and Zhang (1995)], hp-clouds method [Duarte and Oden (1996)], Finite Point Method (FPM) [Oñate, Idelsohn, Zienkiewicz, and Taylor (1996)], Partition of Unity Method (PUM) [Babuška and Melenk (1997)], Local Boundary Integral Equation method (LBIE) [Zhu, Zhang, and Atluri (1998a,b)], Meshless Local Petrov-Galerkin method (MLPG) [Atluri and Zhu (1998a,b); Atluri (1999); Atluri and Zhu (2000)]. Most of these methods, in reality, are not really meshless method, since they use a background mesh for the numerical integration of the weak form. To be a truly meshless method, both interpolation and integration should be performed without a mesh. The finite point method (FPM) [Oñate, Idelsohn, Zienkiewicz, and Taylor (1996)] is a truly meshless method. A non-element interpolation scheme - weighted least squares (WLS) is used and there is no integration required. However, this method is based on point collocation, and is very sensitive to the choice of collocation points. As discussed in Atluri and Zhu (1998a,b), and in Zhu, Zhang, and Atluri (1998a,b), MLPG and LBIE are truly meshless methods, because, a traditional non-overlapping, continguous mesh is not required, either for interpolation purpose or for integration purpose. As pointed out in Atluri, Kim, and Cho (1999b), the LBIE approach can be treated simply as a special case of the MLPG scheme. The MLPG method is based on a weak form computed over a local sub-domain, which can be any simple geometry like a sphere, cube or ellipsoid in 3D. The trial and test function spaces can be different or the same. It offers a lot of flexibility to deal with different boundary value problems. A wide range of problems has been solved by Atluri and his coauthors. The objective of this paper is to extend the MLPG method to solve steady convection-diffusion problems.

In fluid mechanics, the existence of the convection term makes the problem non-self-ajoint. A special treatment is needed to stabilize the numerical approximation for these kinds of problems. Schemes related to upwinding are the most general techniques to stabilize Finite Difference Method (FDM), FEM and FVM. The same concept is needed in the meshless methods, so as to obtain a good accuracy for convection-dominated flows.

Only very few works were reported by using the so-called meshless methods, to solve convection-dominated flows. In Oñate, Idelsohn, Zienkiewicz, and Taylor (1996), the FPM method was applied, with upwinding for the first derivative or with characteristic approximation. However, there is a significant drawback of the FPM as discussed above, and, in multidimensional cases, it is not easy to define the critical distance, which is important to stabilize the method and obtain good accuracy. In Liu, Jun, Sihling, Chen, and Hao (1997), the RKPM method, combined with the Streamline Upwind/Petrov-Galerkin (SUPG) form of variational formulation, was used. As pointed out before, the RKPM method is not a truly meshless method, since a background mesh is used to integrate the weak form. In fact, a truly meshless method, such as the MLPG method, is much easier and more flexible for introducing the upwinding concept, with a very clear physical meaning. This will be illustrated in this paper, and numeri-

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cal examples for the steady convection-diffusion problems will be used for verification purposes.

2 Local Weak Form

Let Ω be a bounded region in $\mathbf{R}^{n_{sd}}$, where n_{sd} is the number of space dimensions, and assume that Ω has a piecewise smooth boundary Γ. Then, the governing equation for steady convection-diffusion problems in Ω can be written as:

$$v_j \frac{\partial \phi}{\partial x_j} = \frac{\partial}{\partial x_j}(K\frac{\partial \phi}{\partial x_j}) + f \tag{1}$$

where, v_j is the flow velocity, K is the diffusivity coefficient, f is a source term, and repeated indices are to be summed. The boundary conditions are assumed to be:

- Essential Boundary Conditions:

$$\phi = \bar{\phi} \quad on \quad \Gamma_\phi \tag{2}$$

- Natural Boundary Conditions:

$$K\frac{\partial \phi}{\partial x_j} n_j = \bar{t} \quad on \quad \Gamma_t \tag{3}$$

where, $\bar{\phi}$ and $\bar{t}$ are given, n_j is the outward unit normal vector to Γ, Γ_ϕ and Γ_t are subsets of Γ satisfying $\Gamma_\phi \cap \Gamma_t = \emptyset$ (the empty set) and $\Gamma_\phi \cup \Gamma_t = \Gamma$.

Different from other meshless methods, the MLPG method is based on a local weak form over a local sub-domain Ω_s, which is located entirely inside the global domain Ω. It is noted that the local sub-domain can be of an arbitrary shape containing the node position $\mathbf{x}$ in question.

To satisfy Eq. (1) in a local sub-domain Ω_s with a piecewise smooth boundary Γ_s, Eq. (1) is weighted by a test function w and integrated over the local sub-domain such that the local weighted residual equation can be written as:

$$\int_{\Omega_s} [v_j \frac{\partial \phi}{\partial x_j} - \frac{\partial}{\partial x_j}(K\frac{\partial \phi}{\partial x_j}) - f] w \, d\Omega = 0 \tag{4}$$

for all w. By using the integration by parts, Eq. (4) is recast into a local weak form as:

$$\int_{\Omega_s} [v_j \frac{\partial \phi}{\partial x_j} + K\frac{\partial \phi}{\partial x_j}\frac{\partial w}{\partial x_j} - fw] \, d\Omega - \int_{\Gamma_s} K\frac{\partial \phi}{\partial x_j} n_j w \, d\Gamma = 0 \tag{5}$$

for all continuous trial functions ϕ and continuous test functions w. In general, the boundary Γ_s of the local sub-domain Ω_s may intersect with the boundary of the global domain Ω. Therefore,

$$\Gamma_s = \Gamma_{sI} \cup \Gamma_{s\phi} \cup \Gamma_{st} \tag{6}$$

where, Γ_{sI} is the part of Γ_s which is inside the global domain, $\Gamma_{s\phi} = \Gamma_s \cap \Gamma_\phi$, and $\Gamma_{st} = \Gamma_s \cap \Gamma_t$.

The local weak form provides a very clear concept for a local non-element integration, which does not need any background integration cells which are continguous over the entire domain. Also the MLPG method leads to a natural way to construct the global stiffness matrix through the integration over a local sub-domain. To solve this local weak form, some kind of meshless interpolation schemes is needed. This will be discussed in the next section.

3 The MLS approximation scheme

In order to preserve the local character of the numerical implementation, a meshless method uses a local interpolation or approximation to represent the trial/test functions with the values (or the fictitious values) of the unknown variable at some randomly located nodes. There are a lot of local interpolation schemes, such as MLS, PUM, RKPM, hp-clouds, Shepard function, etc., available to achieve this aim.

The Moving Least Square (MLS) method is generally considered as one of the schemes to interpolate data with a reasonable accuracy. Therefore, the MLS scheme is chosen in this paper.

Consider the approximation of a function $u(\mathbf{x})$ in a domain Ω with a number of scattered nodes $\{\mathbf{x}_i\}$, $i = 1, 2, \ldots, n$, the moving least-square approximant $u^h(\mathbf{x})$ of $u(\mathbf{x})$, $\forall \mathbf{x} \in \Omega$, can be defined by

$$u^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \tag{7}$$

where, $\mathbf{p}^T(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_m(\mathbf{x})]$ is a complete monomial basis of order m, which, for example, can be chosen as linear:

$$\mathbf{p}^T(\mathbf{x}) = [1, x, y], \quad m = 3; \tag{8}$$

or quadratic:

$$\mathbf{p}^T(\mathbf{x}) = [1, x, y, x^2, xy, y^2], \quad m = 6 \tag{9}$$

for 2D problems, and $\mathbf{a}(\mathbf{x})$ is a vector containing coefficients $a_j(\mathbf{x})$, $j = 1, 2, \ldots, m$ which are functions of the space coordinates $\mathbf{x}$, and determined by minimizing a weighted discrete L_2 norm, defined as:

$$\begin{aligned} \mathbf{J}(\mathbf{x}) &= \sum_{i=1}^{n} w_i(\mathbf{x})[\mathbf{p}^T(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - \hat{u}_i]^2 \\ &= [\mathbf{P}\cdot \mathbf{a}(\mathbf{x}) - \hat{\mathbf{u}}]^T \cdot \mathbf{W} \cdot [\mathbf{P}\cdot \mathbf{a}(\mathbf{x}) - \hat{\mathbf{u}}] \end{aligned} \tag{10}$$

where $w_i(\mathbf{x})$ is the weight function associated with the node i, with $w_i(\mathbf{x}) > 0$ for all $\mathbf{x}$ in the support of $w_i(\mathbf{x})$, $\mathbf{x}_i$ denotes the value of $\mathbf{x}$ at node i, n is the number of nodes in Ω for which the weight functions $w_i(\mathbf{x}) > 0$, the matrices $\mathbf{P}$ and $\mathbf{W}$ are defined as

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}^T(\mathbf{x}_1) \\ \mathbf{p}^T(\mathbf{x}_2) \\ \cdots \\ \mathbf{p}^T(\mathbf{x}_n) \end{bmatrix}_{n \times m} \tag{11}$$

$$\mathbf{W} = \begin{bmatrix} w_1(\mathbf{x}) & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & w_n(\mathbf{x}) \end{bmatrix} \tag{12}$$

and

$$\hat{\mathbf{u}}^T = [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n] \tag{13}$$

where, $\hat{u}_i, i = 1, 2, \ldots, n$ are the fictitious nodal values and not the nodal values of the unknown trial function $u^h(\mathbf{x})$ in general.

The stationarity of $\mathbf{J}$ in Eq. (10) with respect to $\mathbf{a}(\mathbf{x})$ leads to the following relation between $\mathbf{a}(\mathbf{x})$ and $\hat{\mathbf{u}}$:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\hat{\mathbf{u}} \tag{14}$$

where the matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are defined by

$$\mathbf{A}(\mathbf{x}) = \mathbf{P}^T\mathbf{W}\mathbf{P} = \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i) \tag{15}$$

$$\mathbf{B}(\mathbf{x}) = \mathbf{P}^T\mathbf{W} = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \ldots, w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)] \tag{16}$$

Solving this for $\mathbf{a}(\mathbf{x})$ and substituting it into Eq. (7), we get the MLS approximation as

$$u^h(\mathbf{x}) = \Phi^T(\mathbf{x}) \cdot \hat{\mathbf{u}} = \sum_{i=1}^{n} \phi_i(\mathbf{x})\hat{u}_i \quad \forall \mathbf{x} \in \Omega \tag{17}$$

where, the nodal shape function corresponding to nodal point $\mathbf{x}_i$ is given by

$$\Phi^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \tag{18}$$

It should be noted that the MLS approximation is well defined only when the matrix $\mathbf{A}$ in Eq. (14) is non-singular. It can be seen that this is the case if and only if the rank of $\mathbf{P}$ equals m. A necessary condition for a well-defined MLS approximation is that at least m weight functions are non-zero (i.e. $n \geq m$) for each sample point $\mathbf{x} \in \Omega$ and that the nodes in Ω will not be arranged in a special pattern such as on a straight line.

The partial derivatives of $\phi_i(\mathbf{x})$ can be obtained as

$$\phi_{i,k} = \sum_{j=1}^{m} [p_{j,k}(\mathbf{A}^{-1}\mathbf{B})_{ji} + p_j(\mathbf{A}^{-1}\mathbf{B}_{,k} + \mathbf{A}^{-1}_{,k}\mathbf{B})_{ji}] \tag{19}$$

in which $\mathbf{A}^{-1}_{,k}$ is given by

$$\mathbf{A}^{-1}_{,k} = -\mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1} \tag{20}$$

and the index following a comma indicates a spatial derivative.

It is known that the smoothness of the shape functions $\phi_i(\mathbf{x})$ is determined by that of the basis functions and of the weight functions. Let $C^k(\Omega)$ be the space of k-th continuously differentiable functions. If $w_i(\mathbf{x}) \in C^k(\Omega), i = 1, 2, \ldots, n$ and $p_j(\mathbf{x}) \in C^l(\Omega), j = 1, 2, \ldots, m$, then $\phi_i(\mathbf{x}) \in C^r(\Omega)$ with $r = min(k, l)$. A number of choices are available for the basis functions and the weight functions. In this paper, the linear basis is chosen and a spline weight function as in Atluri and Zhu (1998a) is used:

$$w_i(\mathbf{x}) = \begin{cases} 1 - 6(\frac{d_i}{r_i})^2 + 8(\frac{d_i}{r_i})^3 - 3(\frac{d_i}{r_i})^4 & 0 \leq d_i \leq r_i \\ 0 & d_i \geq r_i \end{cases} \tag{21}$$

where $d_i = |\mathbf{x} - \mathbf{x}_i|$ is the distance from node $\mathbf{x}_i$ to point $\mathbf{x}$, and r_i is the size of the support for the weight function w_i. It can be easily seen that the spline weight function is C^1 continuous over the entire domain.

From the above discussion, it shows that the MLS shape functions don't have the Kronecker delta property. This causes the difficulty to impose the essential boundary conditions. Several methods have been proposed, e.g., see Belytcshko, Krongauz, Organ, Fleming, and Krysl (1996) and Zhu and Atluri (1998).

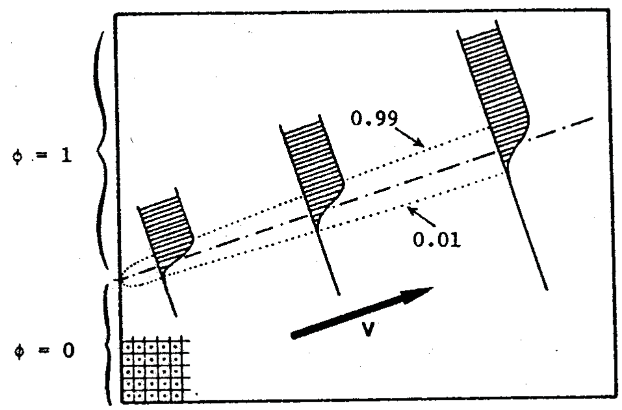
In this paper, the penalty method used in Zhu and Atluri (1998) and the method recently proposed by Atluri, Kim, and Cho (1999b) are used to deal with the essential boundary conditions.

4 Upwinding Schemes

4.1 Overview

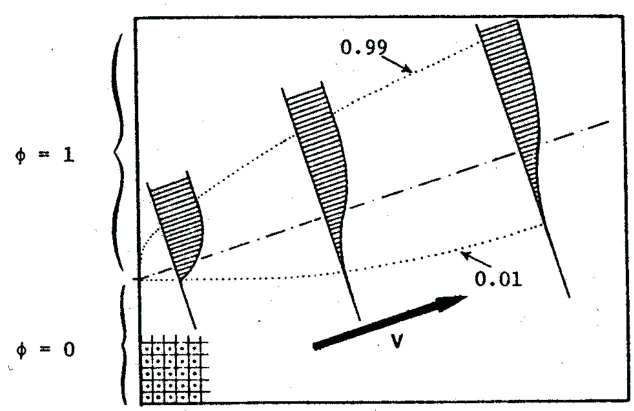
It is well known that convection-dominated flows are some of the most difficult problems to solve numerically. The presence of the convection term causes serious numerical difficulties, appearing in the form of "wiggles" (oscillatory solutions), when the convection term is dominant. This problem could be solved somewhat heuristicaly, by using upwinding. A number of upwinding schemes has been developed, for FDM, FEM and FVM. In the one dimensional cases, optimal upwinding schemes may be designed so as to result in exact nodal solutions. However, in the multidimesional cases, generalizations of traditional upwinding schemes were unsuccessful due to the crosswind diffusion problem (see Fig.1-Fig.2).

It was apparent that the upwind effect, arrived at by whatever means, was needed only in the direction of flow. However, It is not easy to design such methods for multidimensional cases. Hughes and Brooks (1979) introduced the 'Streamline Upwind (SU) method', where the artificial diffusion operator is constructed to act only in the flow direction, a priori eliminating the possibility of any crosswind diffusion. A similar idea was described in Kelly, Nakazawa, Zienkiewicz, and Heinrich (1980) as 'anisotropic balancing dissipation'. As shown in Hughes and Brooks (1979), one could obtain much better solutions for the crosswind problem by using the streamline upwind method, but several deficiencies remained. The upwinded convection term was not consistent with the centrally weighted source and transient terms, resulting in excessively diffuse solutions when these terms were present (see Fig.3-Fig.4).



(a)

Figure 1 : Illustration of the crosswind diffusion problem with $Pe = 100$ (from Leonard (1979)): (a) Exact.



(b)

Figure 2 : Illustration of the crosswind diffusion problem with $Pe = 100$ (from Leonard (1979)): (b) Optimal Upwinding.

Clearly, upwind weighting of all terms in the equation was needed, i.e., some kind of Petrov-Galerkin method is needed for consistency purposes. Therfore, in order to solve the crosswind problem and be consistent with all terms, more refined versions of the upwinding concept should be based on Petrov-Galerkin methods, which utilize upwinding only in the streamline direction. As known, most popular among such methods is the Streamline Upwind Petrov-Galerkin method (SUPG)[Brooks and Hughes (1982)], which consistently introduces an additional stability term in the upwind direction. This method has better stability and accuracy properties than the standard Galerkin formulation for convection-dominated flows. However, some drawbacks still remain. The choice of the related parameters is still not so straightfoward (another definition of the related parameters can be seen in Franca, Frey,

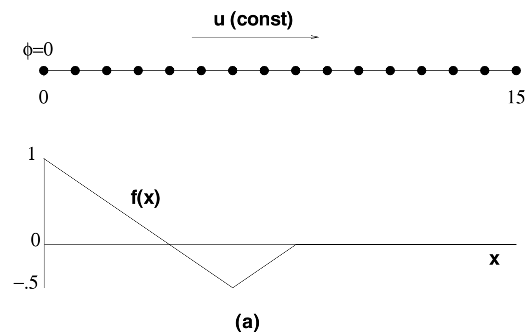


Figure 3 : Pure convection with a source term: (a) Problem Statement.

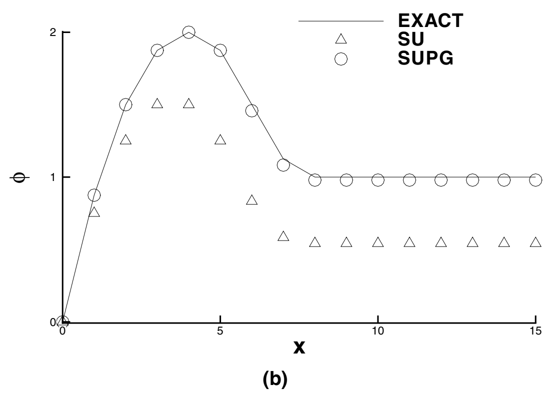


Figure 4 : Pure convection with a source term: (b) Results for SU and SUPG.

and Hughes (1992)). In addition, for non-regular solutions, spurious oscillations remain in the neighborhoods containing sharp layers. In order to prevent those 'wiggles', many methods have been proposed by adding some kind of discontinuity-capturing perturbation term (see e.g. Hughes, Mallet, and Mizukami (1986), Almeida and Silva (1997) and references therein).

For meshless methods, the same kind of consideration should be taken to deal with convection-dominated flows. As pointed out in the introduction, the very general nature of truly meshless methods, such as the MLPG method, makes it easier to introduce the upwind concept more clearly and effectively. In this paper, two upwinding schemes for the MLPG method are proposed in the following.

4.2 MLPG Upwinding Scheme I

The MLPG method is based on the Petrov-Galerkin weighted-residual procedures. Different spaces for the test and trial functions can be used, as shown in Fig.5.

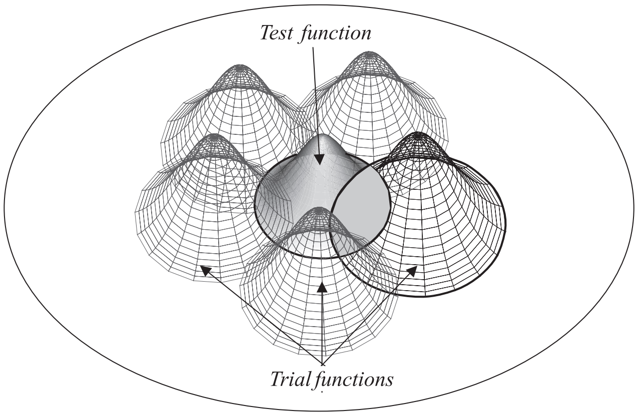


Figure 5 : The MLPG method without Upwinding.

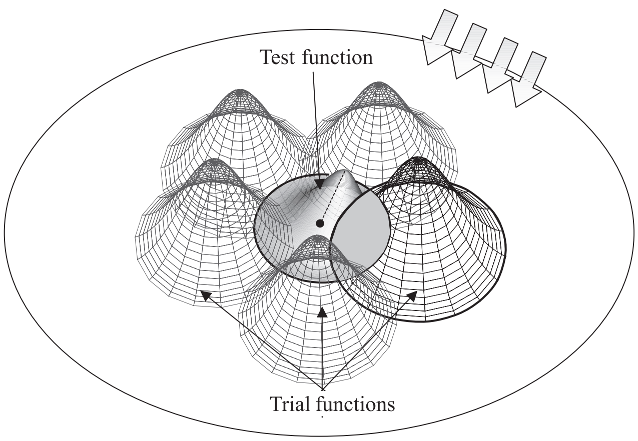


Figure 6 : MLPG Upwinding Scheme I (US-I).

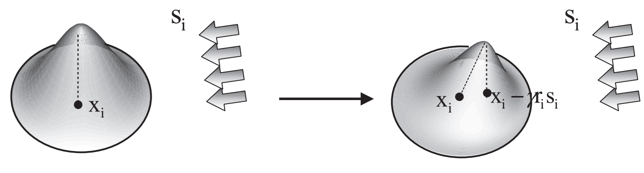


Figure 7 : MLPG Upwinding Scheme I (US-I): Specification.

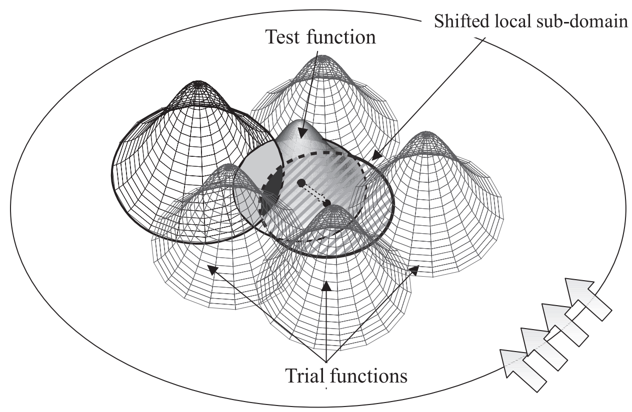


Figure 8 : MLPG Upwinding Scheme II (US-II).

Therefore one of the very natural ways to construct upwinding schemes is to choose different trial and test functions. This can be done by a lot of ways. For example, in order to apply upwinding in the streamline direction, we can skew the test function opposite to the streamline direction as shown in Fig.6. For convenience, we denote this as Upwinding Scheme I (US-I).

As an illustration, we choose a skewed weight function as the test function. The skewed weight function is given as follows: using the same form of weight function as in Eq. (21), we shift the position of the maximum of $w_i(\mathbf{x})$ from $\mathbf{x}_i$ to $\mathbf{x}_i - \gamma r_i \mathbf{s}_i$, as shown in Fig.7,

where, s_i is the unit vector of the streamline direction at $\mathbf{x}_i$, r_i is the size of the support for the test functions at $\mathbf{x}_i$, and γ is given by

$$\gamma = \frac{1}{2}\coth(\frac{Pe}{2}) - \frac{1}{Pe} \tag{22}$$

in which Pe is a local Peclet number defined as:

$$Pe = \frac{ur_i}{K} \tag{23}$$

The size of the support for the trial functions also equal to r_i at $\mathbf{x}_i$, and the local sub-domain at $\mathbf{x}_i$ is coincided with the support for the test functions at $\mathbf{x}_i$.

4.3 MLPG Upwinding Scheme II

In fact, because the MLPG method is based on a local weak form over a local sub-domain, there is another very convenient way to design upwinding schemes. We can use the Bubnov-Galerkin procedure to discretize the local weak form, as done in Atluri, Kim, and Cho (1999b), and shift the local sub-domain opposite to the streamline direction, as shown in Fig.8. We denote this as Upwinding Scheme II (US-II).

Here, the same spaces for the trial and test functions are used, that is, the same support and the same interpolation scheme (MLS) for the trial functions and the test functions are employed. In this case, the local sub-domain at $\mathbf{x}_i$ is no longer coincident with the support for the test functions at $\mathbf{x}_i$, but the size is the same. (It should be noted that in the usual MLPG method, we usually choose the test functions such that the integration term along the boundary Γ_{sI} equals to zero, but, in general, this is not true for the MLPG method with US-II. Therefore, in the local weak form, the integration term along the boundary Γ_{sI} should be retained.)

In particular, the shifting distance of the local sub-domain can

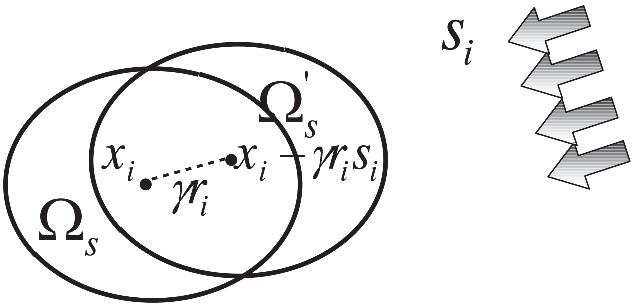


Figure 9 : MLPG Upwinding Scheme II (US-II): Specification.

be specified as γr_i, where, r_i is the size of the support for the test functions, which is equal to the size of the local domain, at $\mathbf{x}_i$, and γ is given by

$$\gamma = \coth(\frac{Pe}{2}) - \frac{2}{Pe} \tag{24}$$

in which Pe is a local Peclet number, defined as:

$$Pe = \frac{2ur_i}{K} \tag{25}$$

The direction of the shifting is opposite to the streamline direction $\mathbf{s}_i$ at $\mathbf{x}_i$, as shown in Fig.9.

5 Numerical examples

To assess the effectiveness of the methods described herein, a series of numerical calculations is performed. For convenience, we note the MLPG method without upwinding as MLPG, the MLPG method with US-I as MLPG1 and the MLPG method with US-II as MLPG2. Here, the MLPG method without upwinding is based on the Galerkin procedure to approximate the local weak form and the local sub-domain coincides with the support of the test functions. Examples for 1D and 2D problems are given in the following.

5.1 One-dimensional problems

For 1D problems, Eq. (1) can be written as:

$$u\frac{d\phi}{dx} - K\frac{d^2\phi}{dx^2} - f = 0 \tag{26}$$

First, reconsider the problem as shown in Fig.3. Here, $K = 0$, $u = 1$ and $f = \begin{cases} 1-\frac{1}{4}x, & (0 < x \le 6) \\ \frac{1}{4}x-2, & (6 < x \le 8) \\ 0, & (8 < x \le 15) \end{cases}$ The results for MLPG1 and MLPG2 are shown in Fig.10. It shows that both MLPG1 and MLPG2 give very good solutions.

Then we consider another problem, with the domain $0 \le x \le 1$. Two different cases are considered:

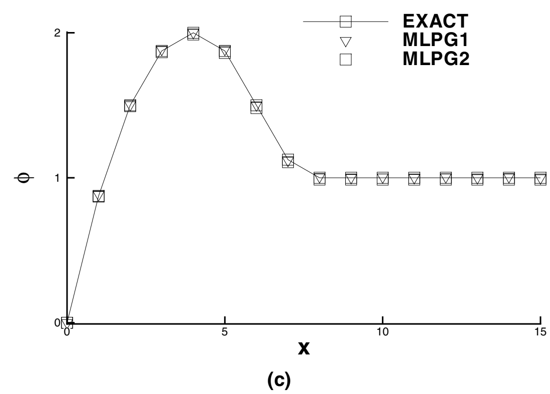


Figure 10 : Pure convection with a source term: (c) Results for MLPG1 and MLPG2.

- Case 1:

$$\begin{aligned} &K = 1, \quad f = 0, \\ &\phi = 0 \quad \text{at} \quad x = 0, \\ &\phi = 1 \quad \text{at} \quad x = 1. \end{aligned} \tag{27}$$

- Case 2:

$$\begin{aligned} &K = 1, \quad f = 100, \\ &\phi = 0 \quad \text{at} \quad x = 0, \\ &\phi = 0 \quad \text{at} \quad x = 1. \end{aligned} \tag{28}$$

u may have different values, leading to different Peclet numbers Pe, which is defined as:

$$Pe = \frac{uL}{K} \tag{29}$$

For the above cases, $L = 1$ and $K = 1$, thus, $Pe = u$.

11 points and the value of 2.3 times nodal distance ($2.3\Delta x$) for radius of support are used to solve these problems. Penalty method is chosen to deal with essential boundary conditions due to its simplicity. Results for the value of ϕ are shown in Fig.11-Fig.12.

These results show that both MLPG1 and MLPG2 produce very good solutions when Peclet number is very high, however, the MLPG method without upwinding gives oscillationary solutions when Peclet number becomes large. For low Peclet number, all methods get good results. It also shows that, in general, MLPG2 gives better solutions than MLPG1. It can be said that the choice for the test function in MLPG1 is not the best one and more care should be taken. So, for 2D problems tested in the following, we will only use the MLPG2, i.e., Upwinding Scheme II.

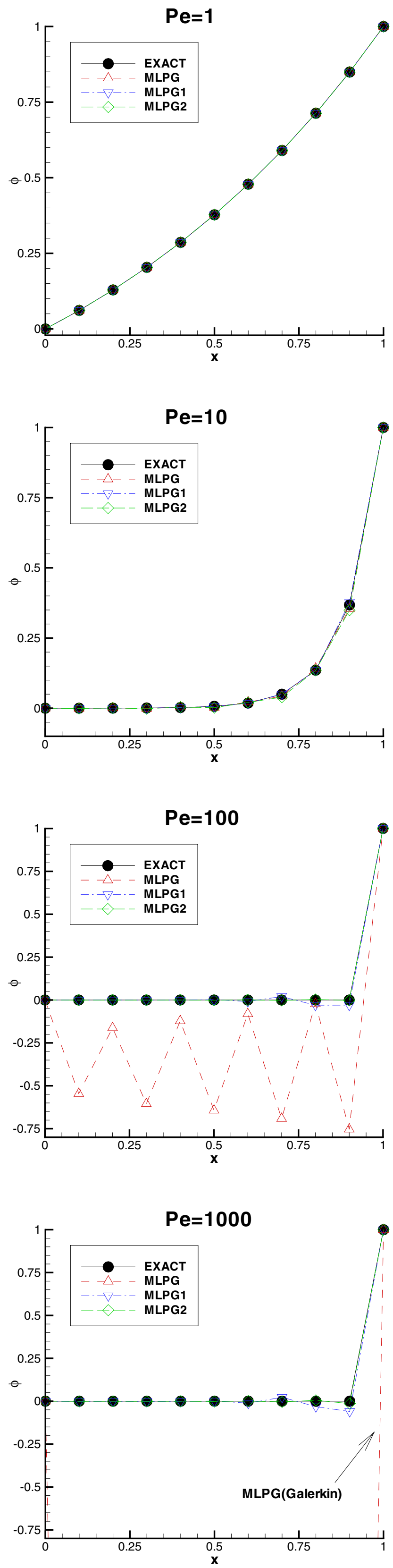


Figure 11 : Case 1 for 1D problems.

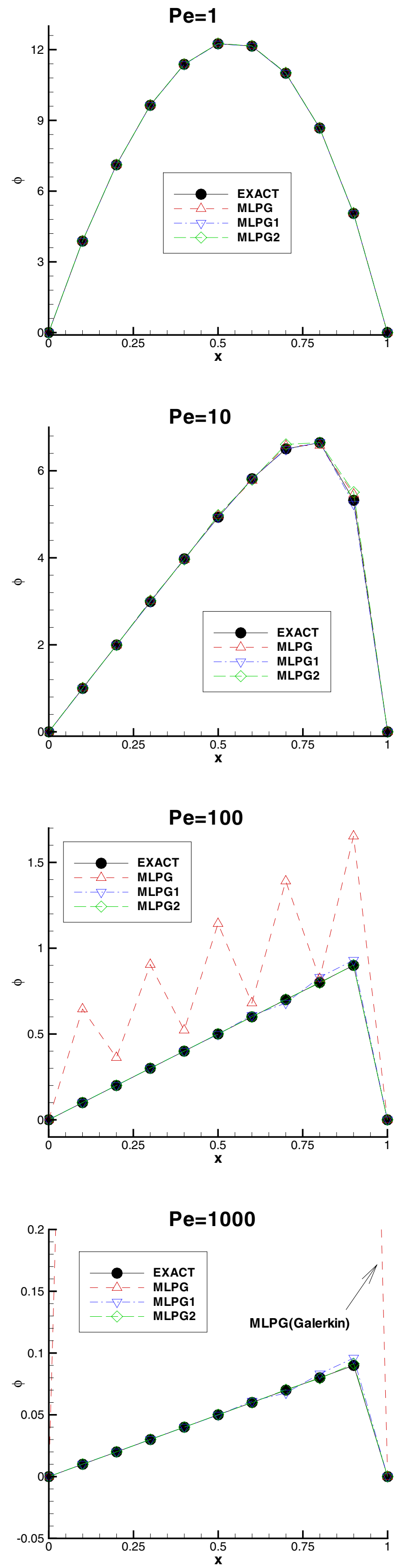


Figure 12 : Case 2 for 1D problems.

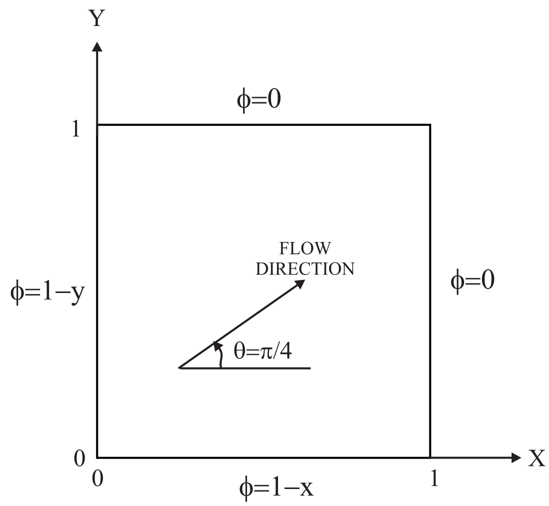


Figure 13 : Convection skew to the mesh-I: problem statement.

5.2 *Two-dimensional problems*

For 2D problems, Eq. (1) can be written as:

$$u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}-K\frac{\partial^2\phi}{\partial x^2}-K\frac{\partial^2\phi}{\partial y^2}-f=0 \tag{30}$$

In this section, several cases are considered in a unit square domain given by $0 \leq x, y \leq 1$. These problems have been widly studied in literature, e.g., see Hughes, Mallet, and Mizukami (1986) and Almeida and Silva (1997), as tests for the accuracy of various numerical schemes. The first two cases are used to assess the performance of the MLPG and MLPG2 methods for different Peclet numbers. The last three cases involve very high Peclet numbers, and are used to assess solutions which are essentially of pure convection flows. For the purpose of comparison, the standard Galerkin finite element method (GFEM) and the streamline-upwind Petrov-Galerkin method (SUPG) have been included in the results. Linear finite elements are used for GFEM and SUPG. The parameters for SUPG are chosen as those in Brooks and Hughes (1982). Except for the last case, a regular mesh (SUPG) or node distribution (MLPG) with 11×11 nodes is used. Unless specified, the value of 2.1 times nodal distance h ($2.1h$) for radius of support is used.

5.2.1 *Convection skew to the mesh-I*

In this case, the source term is assumed to be zero. The flow is unidirectional and constant with velocity components:

$$u = cos(\pi/4), \quad v = sin(\pi/4). \tag{31}$$

See Fig.13 for the problem statement.

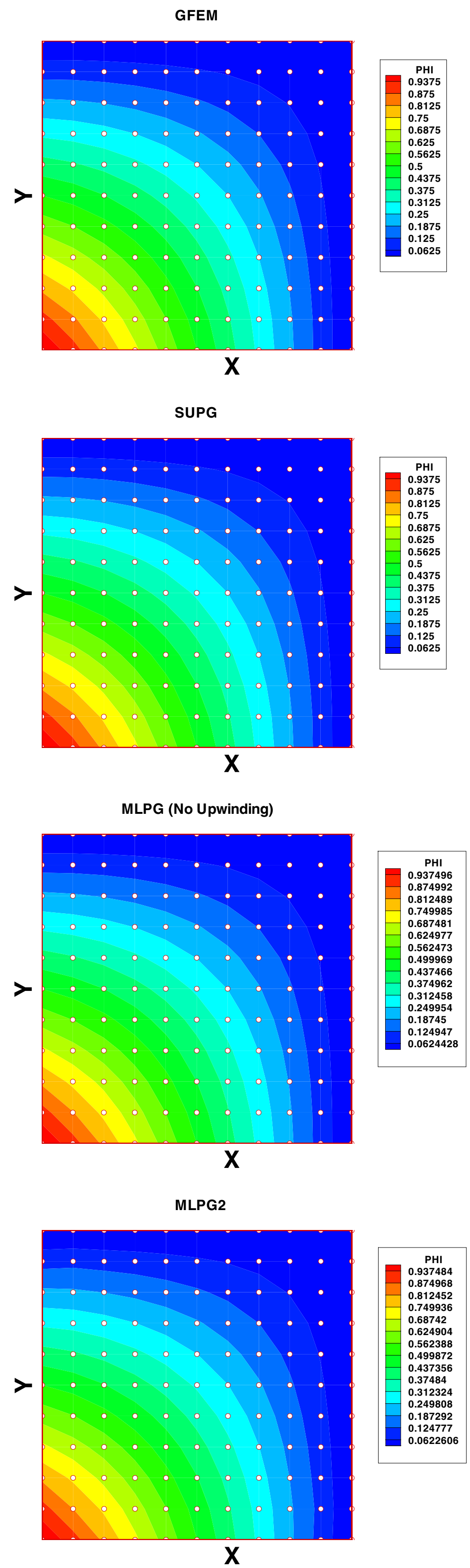


Figure 14 : Convection skew to the mesh-I: $Pe = 1$.

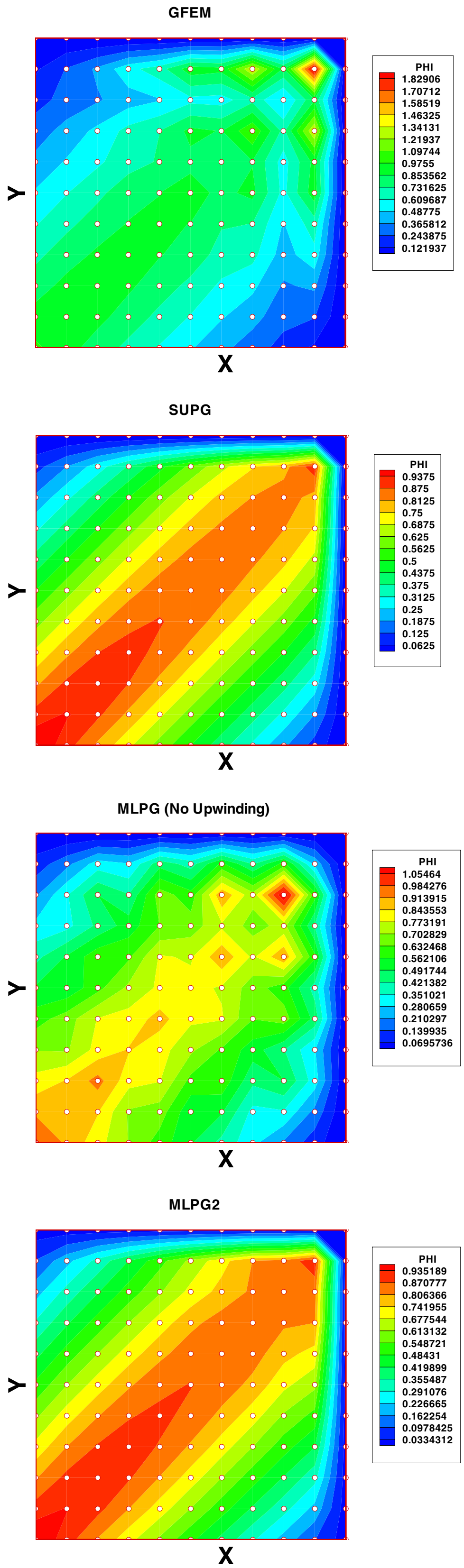


Figure 15 : Convection skew to the mesh-I: $Pe = 100$.

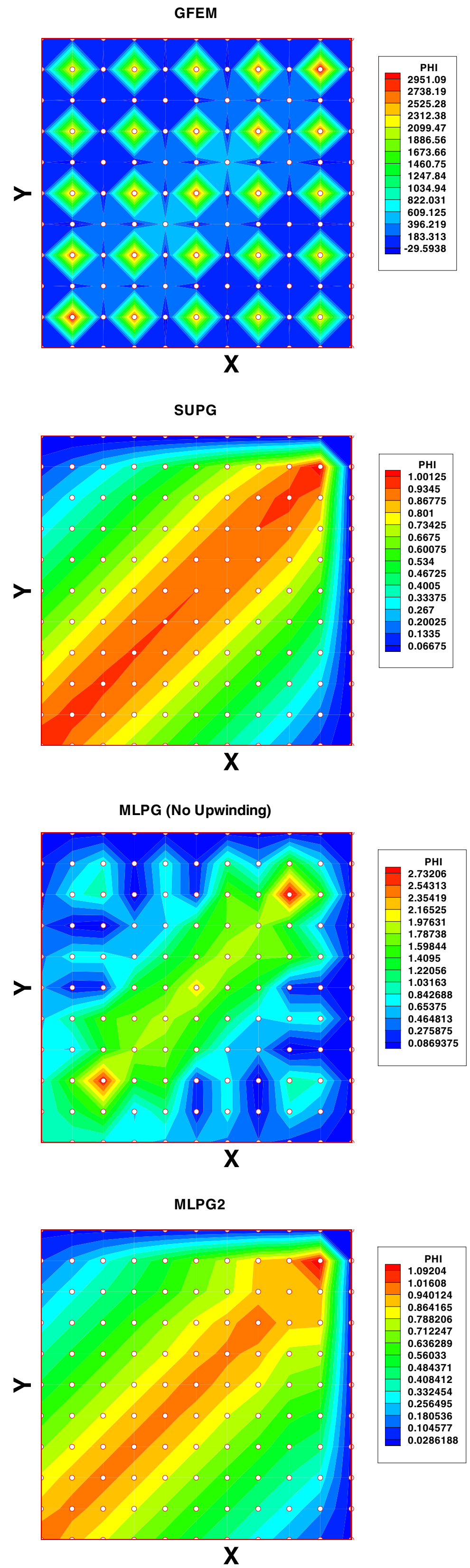


Figure 16 : Convection skew to the mesh-I: $Pe = 10^6$.

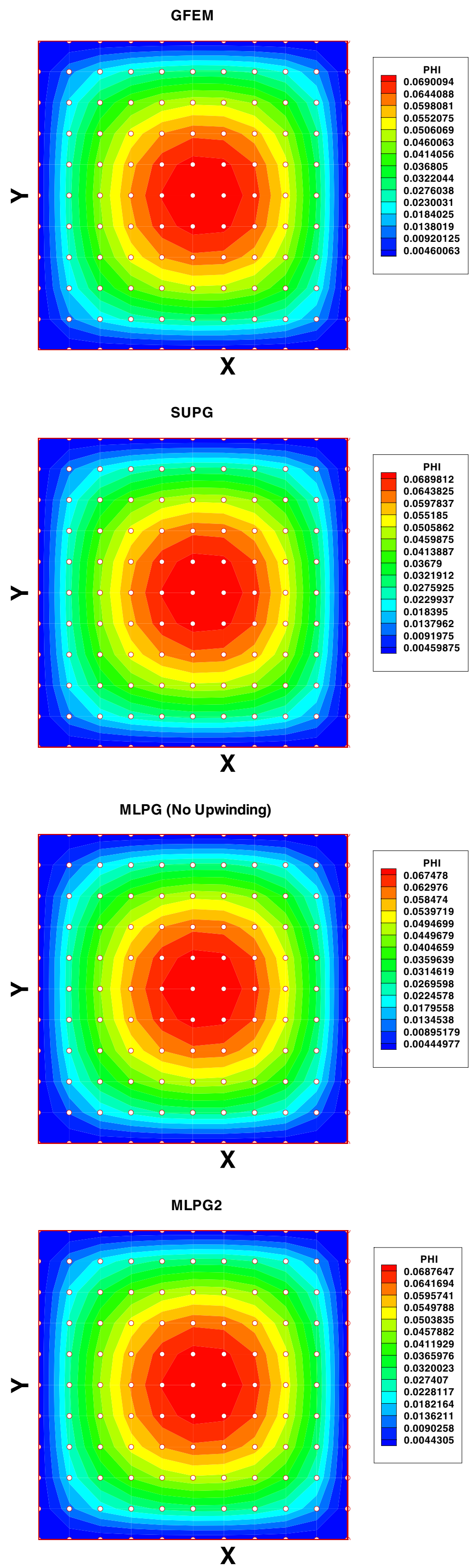


Figure 17 : Convection with a source term-I: $Pe = 1$.

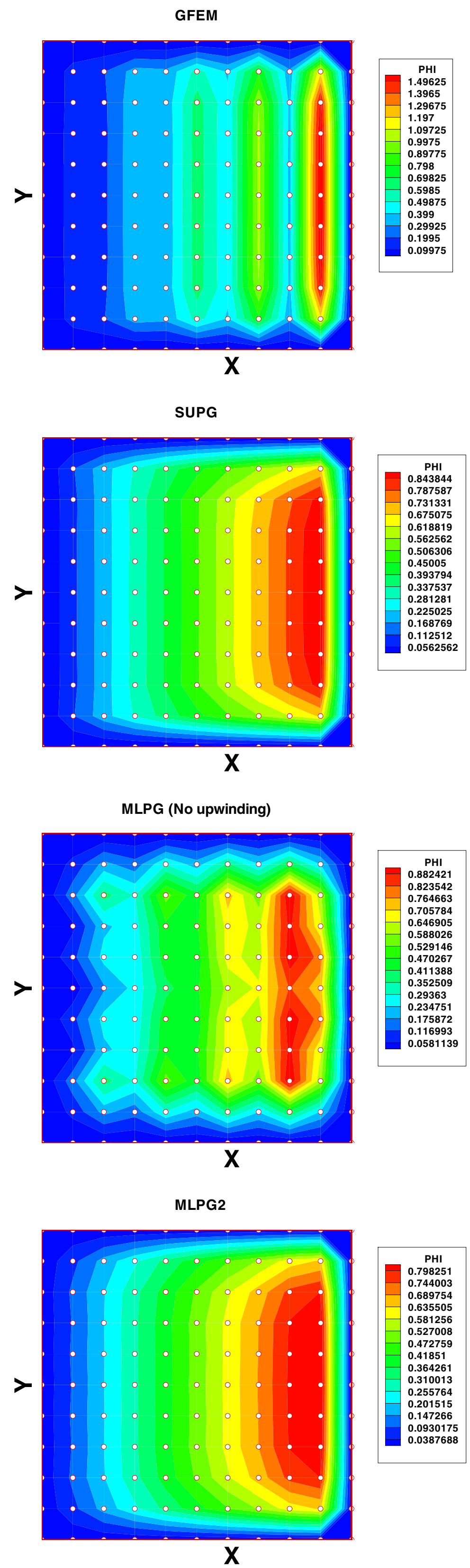


Figure 18 : Convection with a source term-I: $Pe = 100$.

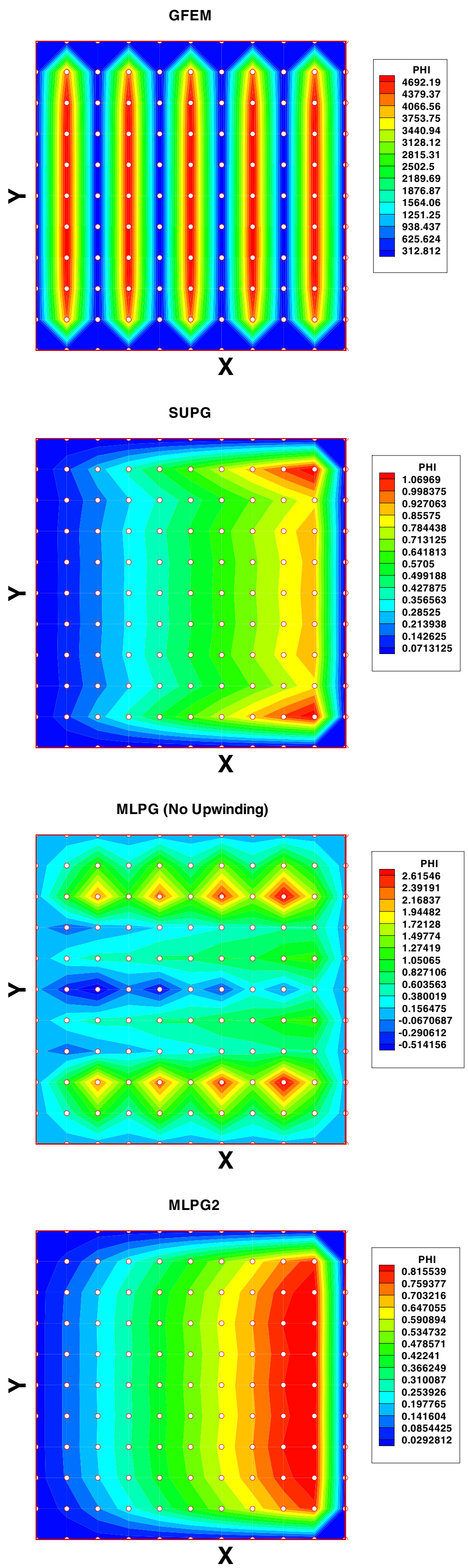


Figure 19 : Convection with a source term-I: $Pe = 10^6$.

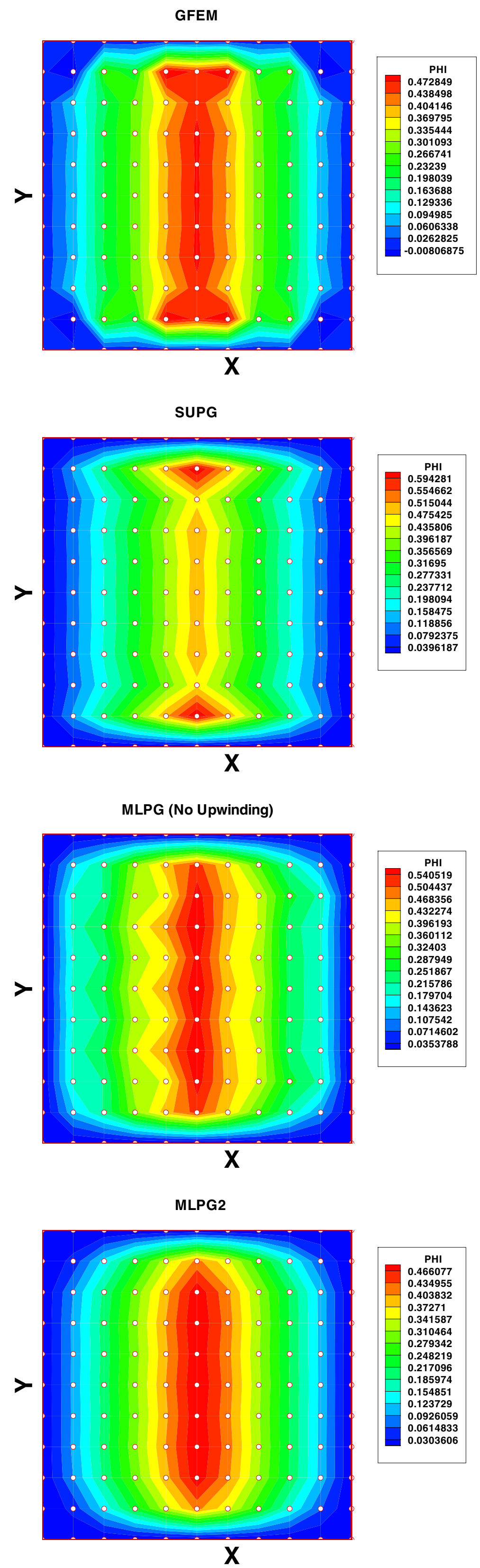


Figure 20 : Convection with a source term-II: $Pe = 10^6$.

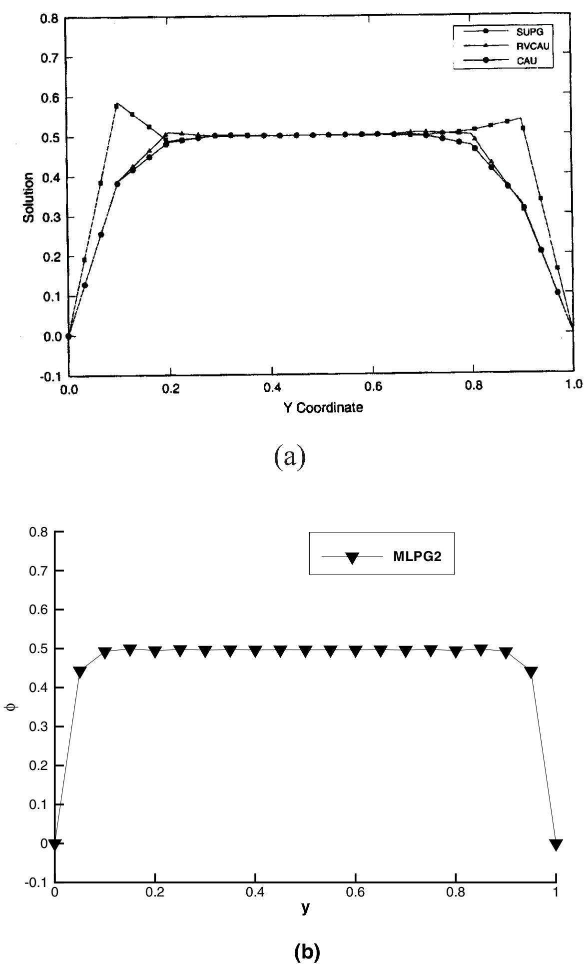


Figure 21 : Convection with a source term-II ($Pe = 10^8$): (a) results from Almeida and Silva (1997); (b) results for MLPG2.

K is varied in order to get different Peclet number Pe. Here, Pe is given by

$$Pe = \frac{||\mathbf{u}||L}{K} \tag{32}$$

where, $||\mathbf{u}|| = \sqrt{u^2+v^2}$. In this case, $||\mathbf{u}|| = 1$ and $L = 1$, so $Pe = 1/K$. It has smooth boundary conditions, so the penalty method is used for simplicity. Contours for ϕ are shown in Fig.14-Fig.16.

As in 1D problems, all methods give good results for low Peclet number. But, for high Peclet numbers, MLPG2 and SUPG are much superior to MLPG and GFEM. It also shows that MLPG2 is somewhat better than the currently most popular method, the SUPG.

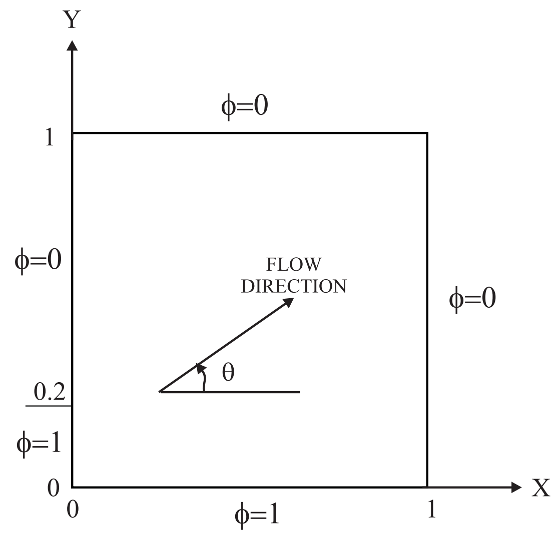


Figure 22 : Convection skew to the mesh-II: problem statement.

5.2.2 Convection with a source term-I

The problem is given by

$$\begin{aligned} &u = 1, \quad v = 0, \quad f = 1, \\ &\phi = 0 \quad \text{along the boundary.} \end{aligned} \tag{33}$$

K is also varied so as to get different Peclet number Pe, which is equal to $1/K$. Again, the penalty method is used to deal with the boundary conditions. Contours for ϕ are shown in Fig.17-Fig.19.

As before, all methods give good results for low Peclet number. But, for high Peclet numbers, MLPG2 and SUPG are much superior to MLPG and GFEM. Furthermore, it shows that MLPG2 gives better solutions than the SUPG, when Peclet number is very high.

5.2.3 Convection with a source term-II

The problem is given by

$$\begin{aligned} &u = 1, \quad v = 0, \quad f = \begin{cases} 1, & (0 < x \leq \frac{1}{2}) \\ -1, & (\frac{1}{2} < x < 1) \end{cases}, \\ &\phi = 0 \quad \text{along the boundary.} \end{aligned} \tag{34}$$

The diffusivity K is 10^{-6} or Peclet number Pe is 10^6. Again, the penalty method is used to deal with the boundary conditions. Contours for ϕ are shown in Fig.20.

It shows that MLPG2 and SUPG are better than GFEM and MLPG, and MLPG2 is superior to SUPG. Actually, the solutions of MLPG2 are comparable (even better) to the results produced by Almeida and Silva (1997) (see Fig.21).

However, Almeida and Silva (1997) introduced a very complicated modification for SUPG of Hughes and his co-workers, to achieve better results.

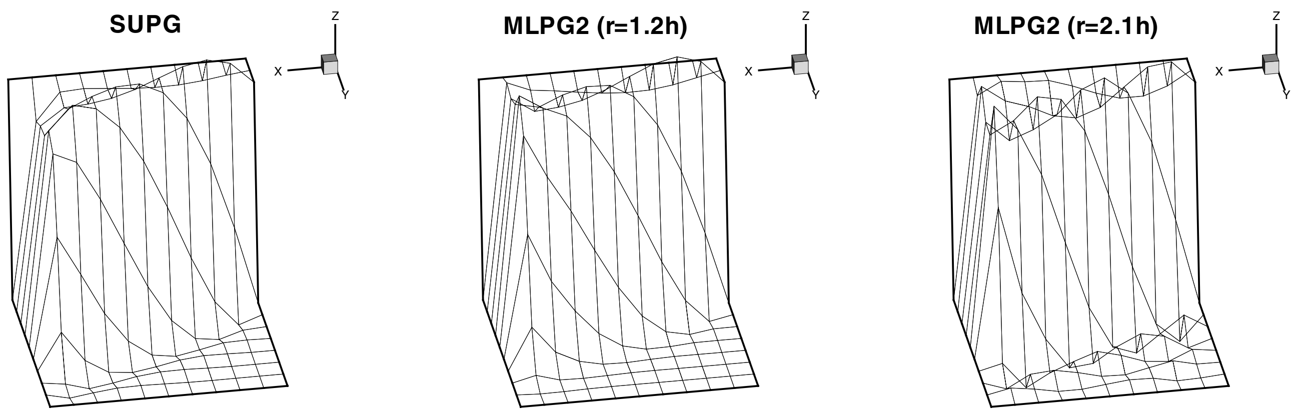


Figure 23 : Convection skew to the mesh-II: $Pe = 10^6$ and $\theta = \frac{\pi}{8}$.

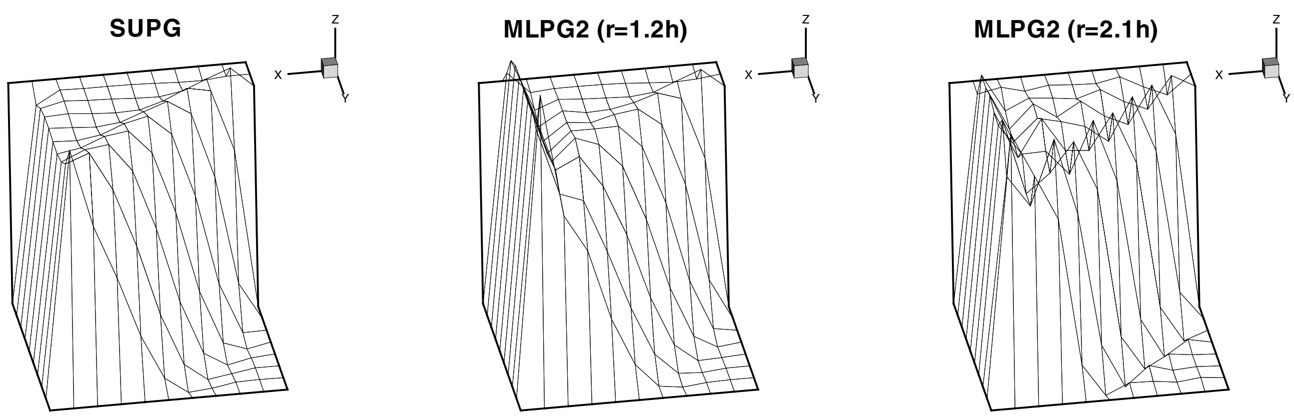


Figure 24 : Convection skew to the mesh-II: $Pe = 10^6$ and $\theta = \frac{\pi}{4}$.

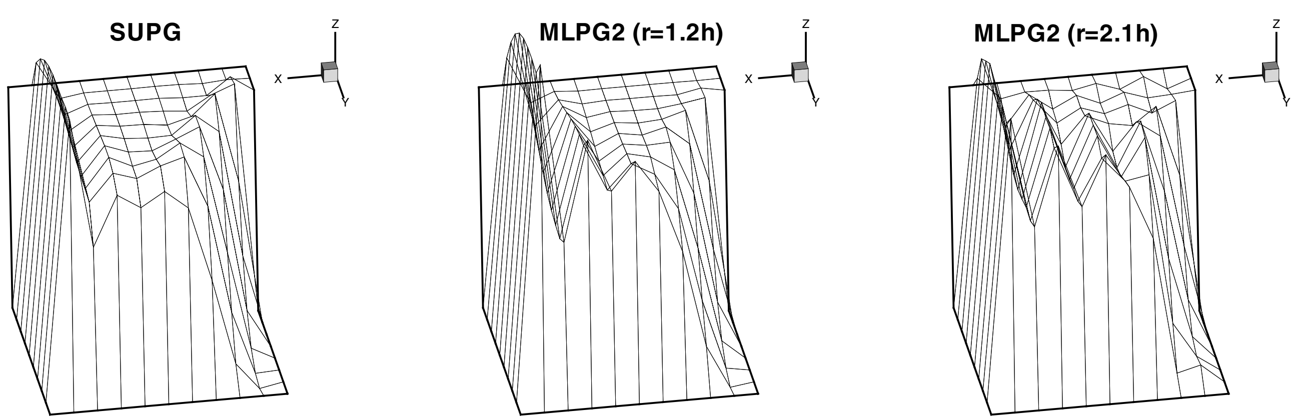


Figure 25 : Convection skew to the mesh-II: $Pe = 10^6$ and $\theta = \frac{3\pi}{8}$.

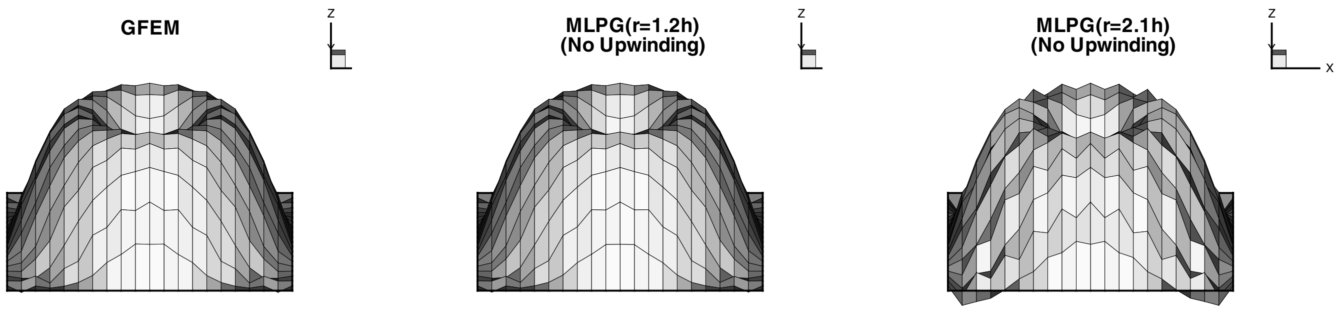


Figure 26 : Convection in a rotating flow field: $K = 10^{-6}$ (Continued).

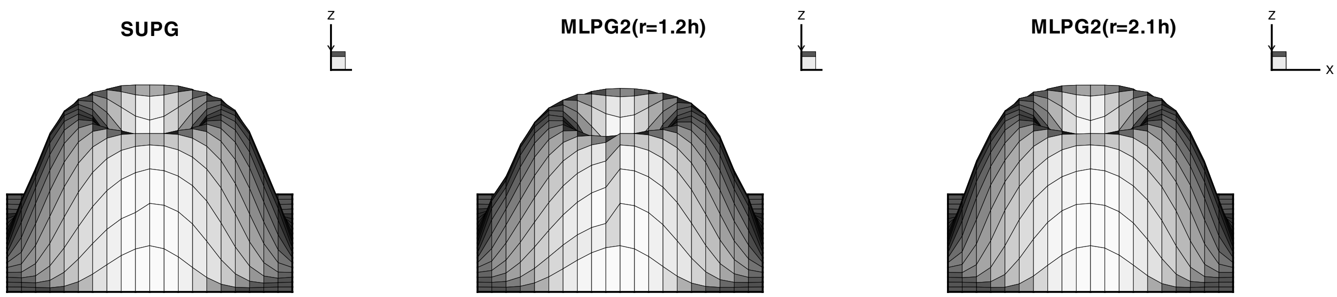


Figure 27 : Convection in a rotating flow field: $K = 10^{-6}$.

5.2.4 Convection skew to the mesh-II

Now we consider another case, which involves discontinuous boundary conditions and causes not only the sharp boundary layer but also an internal sharp layer. In this case, the source term is assumed to zero. The flow is unidirectional and constant with velocity components:

$$u = \cos\theta, \quad v = \sin\theta. \tag{35}$$

where, θ is the angle between the flow direction and the positive x direction. See Fig.22 for the problem statement.

Due to the discontinuous boundary conditions, it is difficult to impose the essential boundary conditions by using penalty method. (Actually it results in oscillation at boundary.) So we use the method developed by Atluri, Kim, and Cho (1999b). Just as before, for low Peclet numbers, all methods give good results. Therefore, only the plots of ϕ with $K = 10^{-6}$ or $Pe = 10^6$ and different flow directions are shown in Fig.23-Fig.25.

Here, the results for GFEM and MLPG are not shown due to their very large oscillations. Two different sizes of radius of support ($1.2h$ and $2.1h$) have been used for MLPG2. Generally, MLPG2 and SUPG gives much better solutions than GFEM and MLPG. Due to the internal discontinuity, it appears that a smaller size of support gives better solutions for MLPG2. This can be explained as following: as pointed out in Atluri, Cho, and Kim (1999a), larger size of support gives better accuracy for smooth solutions because it includes more points and works like higher order schemes, but when the solution is discontinuous, it is well known that higher order schemes produce more oscillatory results. Again,it seems that the MLPG2 with small size of support gives somewhat better solutions than SUPG at the internal discontinuity, but at the boundary, MLPG2 gives somewhat worse results than SUPG. This may be caused by the method of imposing boundary conditions, which is a very strong constraint.

5.2.5 Convection in a rotating flow field

In this case, the flow velocity components are given by

$$u = -y + 0.5, \quad v = x - 0.5 \tag{36}$$

and there is no source term $f = 0$. Along the external boundary $\phi = 0$ and along the internal boundary (AB),

$$\phi = \cos(2\pi(y - \frac{1}{4})) \quad \text{along} \quad x = 0.5, 0 \leq y \leq 0.5 \tag{37}$$

See Fig.28 for the problem statement. The diffusivity is chosen as $K = 10^{-6}$ and a uniform mesh or node distribution with $21 \times$

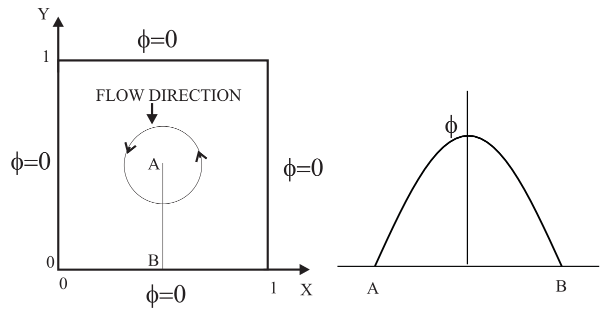


Figure 28 : Convection in a rotating flow field: problem statement.

21 nodes is employed. Due to the internal boundary condition, the method proposed by Atluri, Kim, and Cho (1999b) is used to cope with the boundary conditions. The solution for ϕ is plotted in Fig.26-Fig.27 for different methods.

It shows that both GFEM and MLPG give oscillationary results but SUPG and MLPG2 produce nonoscillatory solutions. Again, the effect of the size of support should be noted. For MLPG, when the size of support increases, the result becomes worse due to the oscillation; however, for MLPG2, larger size of support gives more accurate solutions and smaller size of support produces more spurious crosswind diffusion effect.

6 Concluding Remarks

The MLPG method has been extended to solve the convection-diffusion problems. Just as GFEM, MLPG works well for low Peclet number flows, but is not good for high Peclet number flow. To deal with convection-dominated flow, two kinds of natural upwinding schemes have been proposed. The Upwinding Scheme I (US-I) is very flexible. 1D numerical tests show it is possible to design a good scheme by using this idea. Both 1D and 2D examples show, however, that the Upwinding Scheme II (US-II) is very successful in dealing with convection-dominated flows.

Compared with SUPG, the computational cost of MLPG2 is still much higher, because cost-effective integration schemes are not yet found; but the concept of MLPG2 is very clear and it is very easy to implement it for multi-dimensional flow problems. Further, it may be argued that in a large-scale problem, the additional computational cost (due to the meshless integration) may be offset by the savings in human-resource cost (involved in gnerating the mesh) in the truly meshless MLPG method. Numerical tests also show that, in general, MLPG2 gives better solutions than SUPG. Furthermore, numerical examples show that the size of support plays a significant role in MLPG2. Larger sizes of support result in better accuracy for smooth solutions and smaller sizes of support produce larger spurious viscous effect, which sometimes is useful to get nonoscillatory solutions. This property makes MLPG2 very flexible to deal with different problems, such as problems with discontinuity. It is very easy to change the size of support so that it is possible that, for smooth part of solutions, high accuracy can be obtained, but for the discontinuous part, nonoscillatory solutions can be produced. Further research about this, and about boundary conditions, is needed.

The MLPG method with upwinding introduced in this paper is very general. It is very promising for more general fluid mechanics problems, such as Navier-Stokes equation, due to its simplicity and meshlessness. Further results will be published soon by the authors.

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References

Almeida, R. C.; Silva, R. S. (1997): A stable petrov-galerkin method for convection-dominated problems. *Comput. Methods Appl. Mech. Engrg.*, vol. 140, pp. 291–304.

Atluri, S. N. (1999): Truly meshless methods in computational mechanics. In Aravas, N.; Katsikadelis, J. T.(Eds): *Proc. 3rd Greek National Congress on Computational Mechanics*, pp. 3–25, 24-26 June, 1999, Vol. 1.

Atluri, S. N.; Cho, J. Y.; Kim, H. (1999a): Analysis of thin beams, using the meshless local petrov-galerkin method, with generalized moving least squares interpolations. *Comput. Mech.*, vol. 24, pp. 334–347.

Atluri, S. N.; Kim, H.; Cho, J. Y. (1999b): A critical assessment of the truly meshless local petrov-galerkin (mlpg), and local boundary integral equation (lbie) methods. *Comput. Mech.*, vol. 24, pp. 348–372.

Atluri, S. N.; Zhu, T. (1998a): A new meshless local petrov-galerkin (mlpg) approach in computational mechanics. *Comput. Mech.*, vol. 22, pp. 117–127.

Atluri, S. N.; Zhu, T. (1998b): A new meshless local petrov-galerkin (mlpg) approach to nonlinear problems in computer modeling and simulation. *Computer Modeling and Simulation in Engrg.*, vol. 3, pp. 187–196.

Atluri, S. N.; Zhu, T. (2000): New concepts in meshless methods. *Int. J. Num. Meth. Eng.*, vol. 47, pp. 537–556.

Babuška, I.; Melenk, J. (1997): The partition of unity method. *Int. J. Num. Meth. Eng.*, vol. 40, pp. 727–758.

Belytcshko, T.; Krongauz, Y.; Organ, D.; Fleming, M.; Krysl, P. (1996): Meshless methods: an overview and recent developments. *Comput. Methods Appl. Mech. Engrg.*, vol. 139, pp. 3–47.

Belytcshko, T.; Lu, Y. Y.; Gu, L. (1994): Element-free galerkin methods. *Int. J. Num. Meth. Eng.*, vol. 37, pp. 229–256.

Brooks, A. N.; Hughes, T. J. R. (1982): Streamline upwind petrov-galerkin formulations for convection-dominated flows with particular emphasis on the incompressible navier-stokes equations. *Comput. Methods Appl. Mech. Engrg.*, vol. 32, pp. 199–259.

Duarte, C. A.; Oden, J. T. (1996): An h-p adaptive method using clouds. *Comput. Methods Appl. Mech. Engrg.*, vol. 139, pp. 237–262.

Franca, L. P.; Frey, S. L.; Hughes, T. J. R. (1992): Stabilized finite element methods: I. application to the advective-diffusive model. *Comput. Methods Appl. Mech. Engrg.*, vol. 92, pp. 253–276.

Hughes, T. J. R.; Brooks, A. (1979): A muti-dimensional upwind scheme with no crosswind diffusion. In Hughes, T. J. R.(Ed): *Finite element methods for convection dominated flows, AMD Vol. 34*, ASME, New York.

Hughes, T. J. R.; Mallet, M.; Mizukami, A. (1986): A new finite element formulation for computational fluid dynamics: Ii. beyond supgs. *Comput. Methods Appl. Mech. Engrg.*, vol. 54, pp. 341–355.

Kelly, D. W.; Nakazawa, S.; Zienkiewicz, O. C.; Heinrich, J. C. (1980): A note of upwinding and anisotropic balancing dissipation in finite element approximations to convective diffusion problems. *Int. J. Num. Meth. Eng.*, vol. 15, pp. 1705–1711.

Leonard, B. (1979): A survey of finite difference of opinion on numerical muddling of the incomprehensible defective confusion confusion equation. In Hughes, T. J. R.(Ed): *Finite element methods for convection dominated flows, AMD Vol. 34*, ASME, New York.

Liu, W. K.; Jun, S.; Sihling, D. T.; Chen, Y.; Hao, W. (1997): Multiresolution reproducing kernel particle method for computational fluid mechanics. *Int. J. Num. Meth. Fluids*, vol. 24, pp. 1391–1415.

Liu, W. K.; Jun, S.; Zhang, Y. (1995): Reproducing kernel particle methods. *Int. J. Num. Meth. Fluids*, vol. 20, pp. 1081–1106.

Lucy (1977): A numerical approach to the testing of the fission hypothesis. *The Astro. J.*, vol. 8, pp. 1013–1024.

Naroles, B.; Touzot, G.; Villon, P. (1992): Generalizing the finite element method: diffuse approximation and diffuse elements. *Comput. Mech.*, vol. 10, pp. 307–318.

Oñate, E.; Idelsohn, S.; Zienkiewicz, O. C.; Taylor, R. L. (1996): A finite point method in computational mechanics. application to convective transport and fluid flow. *Int. J. Num. Meth. Eng.*, vol. 39, pp. 3839–3866.

Zhu, T.; Atluri, S. N. (1998): A modified collocation & a penalty formulation for enforcing the essential boundary conditions in the element free galerkin method. *Comput. Mech.*, vol. 21, pp. 211–222.

Zhu, T.; Zhang, J. D.; Atluri, S. N. (1998a): A local boundary integral equation (lbie) method in computational mechanics, and a meshless discretization approach. *Comput. Mech.*, vol. 21, pp. 223–235.

Zhu, T.; Zhang, J. D.; Atluri, S. N. (1998b): A meshless local boundary integral equation (lbie) method for solving nonlinear problems. *Comput. Mech.*, vol. 22, pp. 174–186.