An Iterative Boundary Element Method for the Solution of a Cauchy Steady State Heat Conduction Problem

N.S. Mera, L. Elliott, D.B. Ingham, D. Lesnic[1]

Abstract: In this paper the iterative algorithm proposed by [Kozlov and Maz'ya (1990)] for the backward heat conduction problem is extended in order to solve the Cauchy steady state heat conduction problem and the accuracy, convergence and stability of the numerical algorithm are investigated. The numerical results which are obtained confirm that this new iterative BEM procedure is accurate, convergent and stable with respect to increasing the number of boundary elements and decreasing the amount of noise which is added into the input data.

keyword: BEM, heat conduction, Cauchy problem, iterative algorithm

1 Introduction

At present there are various approaches to solving the Cauchy problem for elliptical equations and these can be divided into three major groups. The first group comprises methods based on bringing the problem into the class of well-posedness in the Tikhonov's sense, e.g. [Lavrent'ev, Romanov and Vasil'ev (1969)], whilst the second group consists of those methods that use universal regularization algorithms which can be obtained with the aid of the Tikhonov parametric functionals or related versions, e.g. [Tikhonov and Arsenin (1977)]. Finally, the most recently developed group includes iterative direct solution methods, e.g. [Bakushinsky and Goncharsky (1994)]. This latter group has some advantages, which include the simplicity of the computational schemes, the similarity of the schemes for problems with linear and nonlinear operators, the high accuracy and stability of the solutions, etc. The iterative methods also allow any restriction upon the solution that may be essential for the problem, for example the requirements that the solution be non-negative, monotonic, etc., to be easily taken directly into account into the iterative algorithm scheme. However, one of the possible disadvantages of using iterative algorithms is that large numbers of iterations may be required in order to achieve convergence but then relaxation algorithms may be adopted to improve the rate of convergence.
In this study an iterative boundary element method is developed for the solution of the Cauchy problem for the steady state heat conduction. The algorithm is a new extension of the iterative procedure proposed by [Kozlov and Maz'ya (1990)] for the backward heat conduction problem and consists of obtaining successive solutions of well-posed mixed boundary value problems of the original equation. The intermediate well-posed problems are discretised using a classic BEM since it is nowadays well-known that the BEM performs better than other domain discretisation methods, such as finite differences or finite elements, for solving linear partial differential equations. The iterative algorithm proposed is of a general character and therefore it may be applied to various other ill-posed problems. The numerical results obtained show that the iterative BEM produces a convergent, stable and consistent numerical solution with respect to increasing the number of boundary elements and decreasing the amount of noise added into data.

[1] Department of Applied Mathematics, University of Leeds, Leeds, LS2 9JT, UK

2 Mathematical formulation

Consider an isotropic medium in an open bounded domain $\Omega \subset R^2$ and assume that Ω is bounded by a surface Γ which may consist of several segments, each being sufficiently smooth in the sense of Liapunov. We also assume that the boundary consists of four distinct parts, $\partial\Omega = \Gamma = \Gamma_0 \bigcup \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$. In this study we refer to steady heat conduction applications in isotropic media and we assume that heat generation is absent. Hence the function T, which denotes the temperature distribution in Ω, satisfies the Laplace equation

$$\nabla^2 T(x) = 0, \quad x \in \Omega \tag{1}$$

but similar algorithms may be developed for the steady state heat conduction equation in an anisotropic medium or other elliptic equations.
In the direct problem formulation, if the temperature and/or the heat flux on the boundary Γ is given then the temperature distribution in the domain can be calculated provided that the temperature is specified at least at one point. However, many experimental impediments may arise in measuring or enforcing a complete boundary specification over the whole boundary Γ. For example, the temperature or the heat flux measurement may be seriously affected by the presence of a sensor and hence there is a loss of accuracy in the measurement, or, more simply, a part of the surface of the body may be unsuitable for attaching a sensor. Instead some other boundary information may be given elsewhere. For example we can assume that the boundary Γ_0 is underspecified, i.e. both the temperature $T|_{\Gamma_0}$ and the heat flux $\frac{\partial T}{\partial \nu}|_{\Gamma_0}$ are unknown and have to be retrieved. We also can assume that the heat flux is prescribed on $\Gamma_1 \bigcup \Gamma_3$

while the boundary Γ_2 is overspecified by prescribing both the temperature and the heat flux. Thus the Cauchy problem to be solved becomes

$$\nabla^2 T = 0 \quad \text{in} \quad \Omega \tag{2}$$

$$\frac{\partial T}{\partial \nu} = q \quad \text{on} \quad \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3 \tag{3}$$

$$T = f \quad \text{on} \quad \Gamma_2 \tag{4}$$

where f and q are given functions of the angular polar coordinate θ.

This problem, termed the Cauchy Problem, is much more difficult to solve both analytically and numerically than is the direct problem since the solution does not satisfy the general conditions of well-posedness. Although the problem may have a unique solution, it is well known, see for example [Hadamard (1923)], that this solution is unstable with respect to small perturbations in the data on Γ_2. Thus the problem is ill-posed and we cannot use a direct approach, e.g. the Gaussian elimination method, to solve the system of linear equations which arise from discretising the partial differential equation Eq. 1 or Eq. 2 and the boundary conditions Eq. 3−Eq. 4 since such an approach would produce a highly unstable numerical solution. Instead we use an iterative algorithm, which we describe in the next section, in order to accurately retrieve the temperature and the heat flux on the unspecified boundary Γ_0.

In order to illustrate the iterative method we consider an isotropic medium in a circular plane domain of radius R, $\Omega = \{(x,y)|x^2+y^2 < R\}$ with its boundary divided into four parts as follows:

$$\Gamma_0 = \{z \in \Omega | \alpha \le \theta(z) < 2\pi\} \tag{5}$$

$$\Gamma_1 = \{z \in \Omega | 0 \le \theta(z) < \alpha - \pi\} \tag{6}$$

$$\Gamma_2 = \{z \in \Omega | \alpha - \pi \le \theta(z) < \pi\} \tag{7}$$

$$\Gamma_3 = \{z \in \Omega | \pi \le \theta(z) < \alpha\} \tag{8}$$

where $\theta(z)$ is the angular polar coordinate of the point z and $\alpha \in (\pi, 2\pi)$ is a given angle.

It should be noted that the boundary Γ has been divided into four distinct parts given by Eq. 5-Eq. 8 such that the underspecified part of the boundary Γ_0 and the overspecified part Γ_2 are symmetric with respect to the origin $O(0,0)$. This condition is necessary in order to formulate a marching condition similar to that proposed in the algorithm of [Kozlov and Maz'ya (1990)] for the backward heat conduction problem. For various other formulations of the Cauchy problem, which do not satisfy this condition, a different iterative algorithm must be applied, see for example [Mera, Elliott, Ingham and Lesnic (1999)].

3 Description of the algorithm

The highly ill-posed Cauchy problem Eq. 2-Eq. 4 may be reduced to a sequence of well-posed problems as follows:

Step 1

Specify an initial guess u_0 for the temperature on the underspecified part of the boundary $T|_{\Gamma_0}$.

Step 2

If u_k has been constructed, solve the mixed well-posed problem

$$\nabla^2 T^{(k)} = 0 \quad \text{in} \quad \Omega \tag{9}$$

$$\frac{\partial T^{(k)}}{\partial \nu} = q \quad \text{on} \quad \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3 \tag{10}$$

$$T^{(k)} = u_k \quad \text{on} \quad \Gamma_0 \tag{11}$$

to determine the k-th approximation $T^{(k)}$ for the temperature distribution inside the solution domain and evaluate the temperature on the boundary Γ_2, $\mu_k = T^{(k)}|_{\Gamma_2}$.

Step 3

Construct u_{k+1} as given by the equation

$$u_{k+1}(\theta) = u_k(\theta) - \gamma[\mu_k(\pi - \theta + \alpha) - f(\pi - \theta + \alpha)] \tag{12}$$

where γ is a small positive parameter and θ is the angular polar coordinate.

Step 4

Repeat steps 2 and 3 until a prescribed stopping criterion is satisfied.

It was proved by [Kozlov and Maz'ya (1990)] that a similar algorithm produces a convergent and stable solution for the backward heat conduction problem. It is the purpose of this study to introduce an iterative BEM numerical implementation of this modified algorithm and to investigate the numerical convergence and stability for the Cauchy problem for the Laplace equation Eq. 2-Eq. 4.

In the boundary conditions formulation given by Eq. 3-Eq. 4 it can be seen that, by prescribing both the temperature f and the heat flux q, the boundary Γ_2 is overspecified, while the boundary Γ_0 is underspecified since both the temperature $T|_{\Gamma_0}$ and the heat flux $\frac{\partial T}{\partial \nu}|_{\Gamma_0}$ are unknown and have to be determined. On the remainder of the boundary the heat flux q was prescribed but various other boundary conditions formulations may be considered. For example one may assume that the temperature is specified on one or both of the boundaries Γ_1 and Γ_3 and the heat flux is unknown. Moreover, if the temperature is specified on the boundary $\Gamma_1 \bigcup \Gamma_3$ then an initial guess for the temperature $T_{|\Gamma_0}$ which ensures the continuity of the temperature at the endpoints of Γ_0 can be constructed. For the boundary conditions formulation considered by Eq. 3-Eq. 4 such an initial guess cannot be constructed and an arbitrary guess must be specified. However, the numerical results for this formulation are more accurate than those obtained for other formulations, even if an initial guess which is far from the exact solution is specified. For other boundary condition formulations, if the temperature is specified and the heat flux is unknown on the boundary $\Gamma_1 \bigcup \Gamma_3$, then the errors in evaluating the heat flux may lead to a less accurate numerical solution. For the Cauchy problem considered by Eq. 2-Eq. 4 the iterative algorithm proposed leads to an accurate solution since it is known that the imposition of the heat flux on the boundary generally provides more stable information than the imposition of the temperature.

4 Numerical results and discussion

The purpose of this section is to introduce an iterative BEM implementation of the iterative algorithm described in the previous section and to investigate in more detail the numerical convergence and accuracy with respect to mesh size discretisation and the number of iterations.

We note that in order to pass from one iteration to the next the values of the temperature and of the heat flux are required only on the boundary. Thus, the boundary element method (BEM) is a very suitable technique for solving the intermediate well-posed problems Eq. 9-Eq. 11. Furthermore, the temperature inside the solution domain has to be evaluated only after the stopping criterion has been satisfied, thus saving a substantial amount of computational time. The BEM for solving Eq. 9-Eq. 11 is classical, see [Brebbia, Telles and Wrobel (1984)], and is based on the fundamental solution of the Laplace equation and Green's identities. Therefore the BEM technique is not described in detail.

In order to illustrate the performance of the numerical method proposed we have considered a typical bench-mark test example, namely the harmonic temperature to be retrieved is given by

$$T(x,y) = \cos(x)\cosh(y) + \sin(x)\sinh(y) \tag{13}$$

Numerous other examples have been investigated and the same conclusions as those obtained using the test example given by the Eq. 13 have been drawn. Therefore we only investigate in detail the numerical results obtained for the test example given by Eq. 13. In order to illustrate typical numerical results for a smooth domain we have considered a circle of radius $R = 1$ and the angle α was taken $3\pi/2$, although various other values of α may be prescribed. The Cauchy problem given by Eq. 2-Eq. 4 has been solved iteratively using the BEM to provide simultaneously the unspecified boundary temperature and the heat flux. The number of boundary elements used for discretising the boundary Γ was taken to be $N \in \{40, 80, 160\}$.

The convergence of the algorithm may be investigated by evaluating at every iteration the error

$$e_T = \|T_k - T_{an}\|_{L^2(\Gamma_0)} \tag{14}$$

where T_k is the numerical solution for the temperature on the boundary Γ_0 obtained after k iterations and T_{an} is the exact solution of the problem, given by Eq. 13. We note that the error e_T is an estimate of the accuracy in retrieving the temperature T on the boundary Γ_0. In a similar way we may evaluate the errors in retrieving the heat flux on the boundary Γ_0, or the temperature inside the solution domain, given by

$$e_{T'} = \|T'_k - T'_{an}\|_{L^2(\Gamma_0)} \tag{15}$$

$$e_{\Omega} = \|T_k - T_{an}\|_{L^2(\Omega)} \tag{16}$$

where T'_k and T_{an} are the numerical solution obtained after k iterations and the exact solution for the heat flux on the boundary Γ_0, respectively. Although not illustrated here it is reported that the errors e_T, $e_{T'}$ and e_{Ω} have a similar evolution with respect to increasing the number of iterations or the number of boundary elements. Therefore we investigate in detail only the error e_T. We note that if the error e_T indicates that the boundary temperature $T_{|\Gamma_0}$ has been accurately retrieved, then the heat flux on the boundary Γ_0 as well as the temperature inside the solution domain Ω are also accurately retrieved as numerical solutions of the mixed well posed problem given by Eq. 9-Eq. 11.

Fig. 1 shows the error e_T obtained using the iterative BEM described in the previous section with $N = 80$ boundary elements and $\gamma = 0.05$, as a function of the number of iterations. Similar results are obtained for various numbers of boundary elements and various values of the parameter γ. It can be seen that the errors e_T decreases rapidly over the first few iterations but the rate of convergence decreases as the number of iterations increases. Moreover, if the number of iterations is very large, due to the accumulation of the numerical noise, the error e_T may start to slowly increase. Thus the iterative process has to be stopped at a point where the error e_T, obtained by comparing the numerical solution with the analytical solution, stops decreasing. Various stopping criteria may be used to locate the point where the error e_T stops decreasing but further work should be performed in order to develop an optimal stopping criterion. It should be noted that if the effect of the numerical noise is reduced by using the numerical solution of the direct problem as data on the boundary Γ_2, then the process is convergent with respect to increasing the number of iterations and the error e_T keeps decreasing even for large numbers of iterations. In this case we may use a simple Cauchy type stopping criterion based on evaluating the difference between two consecutive approximations for the temperature on the boundary Γ_0 given by

$$e_k = \|T_{k+1} - T_k\|_{L^2(\Gamma_0)} \tag{17}$$

Indeed, if the error e_k is small enough then the numerical solution does not improve from one iteration to the next and thus the iterative process may be stopped.

The numerical solution for the Cauchy problem considered obtained by stopping the iterative process at the iteration where the error e_T attains its minimum value is presented in Fig. 2

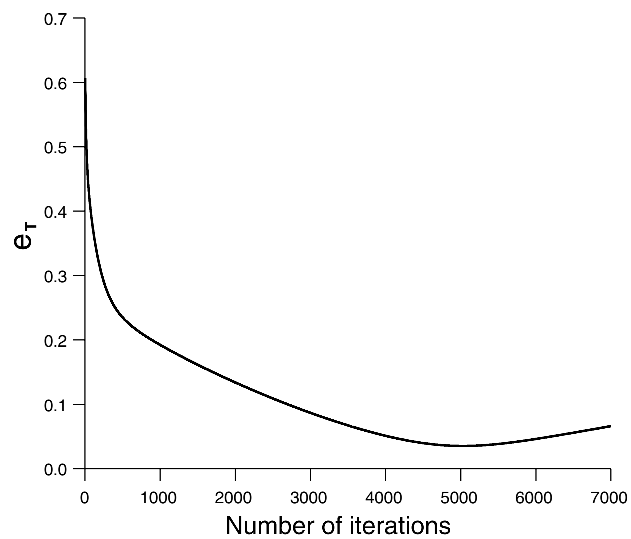

Figure 1 : The error e_T obtained with $N = 80$ boundary elements and $\gamma = 0.05$, as a function of the number of iterations, for the Cauchy problem given by Eq. 2-Eq. 4.

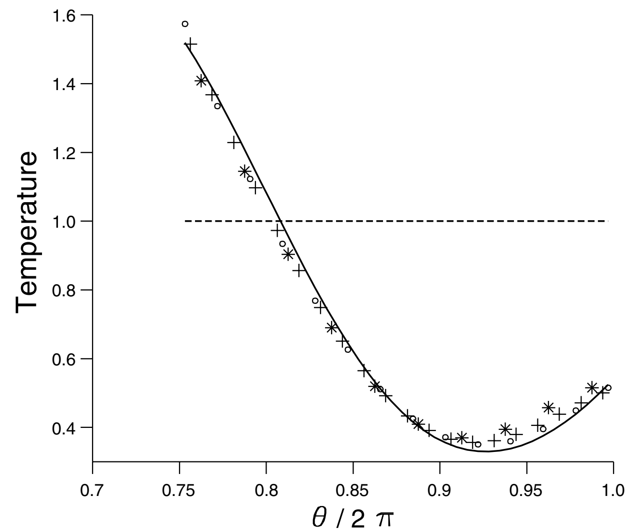

Figure 2 : The numerical solution for the temperature on the boundary Γ_0 obtained using the iterative BEM with $\gamma = 0.05$ for various numbers of boundary elements, namely, $N = 40$ ($*$), $N = 80$, (+) and $N = 160$ ($\circ$), the analytical solution (—) and the initial guess (- - -).

for various numbers of boundary elements and $\gamma = 0.05$. Also shown in this figure, for comparison, are the analytical solution given by Eq. 13 and the constant initial guess $u_0 \equiv 1$.

We note that the numerical solution for the temperature on the underspecified part of the boundary Γ_0 is a good approximation to the analytical solution given by Eq. 13. Thus we may conclude that the iterative BEM algorithm is convergent with respect to increasing the number of boundary elements used to discretise the boundary of the domain. It can be seen that for large numbers of boundary elements there is a good agreement between the numerical and the analytical solutions, even if the initial guess is a long way from the exact solution.

For the formulation considered in this section it is not possible to construct an initial guess that ensures the continuity of the temperature at the endpoints of the boundary Γ_0 and therefore a constant initial guess $u_0 \equiv 1$ was specified. If any minimum and/or maximum estimates for the temperature on the boundary Γ_0 are available, the initial guess should satisfy these estimates. The rate of convergence may be substantially increased by prescribing an initial guess which is not to far from the exact values of the temperature. Therefore, for the test example considered an guess $u_0 \equiv 1$ was prescribed since this value is close to the mid point of the interval $[0.4, 1.5]$ which contains all the values of the temperature on the boundary Γ_0.

However, if no estimates are available for the temperature on the boundary Γ_0, an arbitrary initial guess may be prescribed, even if the number of iterations necessary to obtain an accurate solution is substantially increased. Various values were specified for the initial guess and for all the examples considered the numerical results were found to be in good agreement with the analytical solution of the problem. Fig. 3 shows the numerical solution obtained using the described iterative BEM with $N = 80$ boundary elements and $\gamma = 0.05$ for various constant initial guesses. It can be seen that the results are accurate even if the guess is far from the exact solution.

In order to increase the rate of convergence of the iterative procedure one may apply a classical relaxation procedure if at the end of each iteration the approximation to the temperature u_{k+1} is modified as follows:

$$u_{k+1} = \alpha u_{k+1} + (1-\alpha)u_k \tag{18}$$

where α is a relaxation factor. If this relaxation condition is introduced in the marching scheme of the iterative algorithm we obtain a new marching condition given by

$$u_{k+1} = u_k - \alpha\gamma(\mu_k - f) \tag{19}$$

Thus, various relaxation procedures may be obtained by simply altering the value of the parameter γ. We may conclude that the parameter γ acts as a relaxation parameter. The numerical results presented in this section were obtained for $\gamma = 0.05$ but similar results may be obtained for various values of the relaxation parameter γ. Fig. 4 shows the numerical solution obtained for the temperature on the boundary Γ_0 with $N = 80$ boundary elements and the initial guess $u_0 \equiv 1.0$ for various

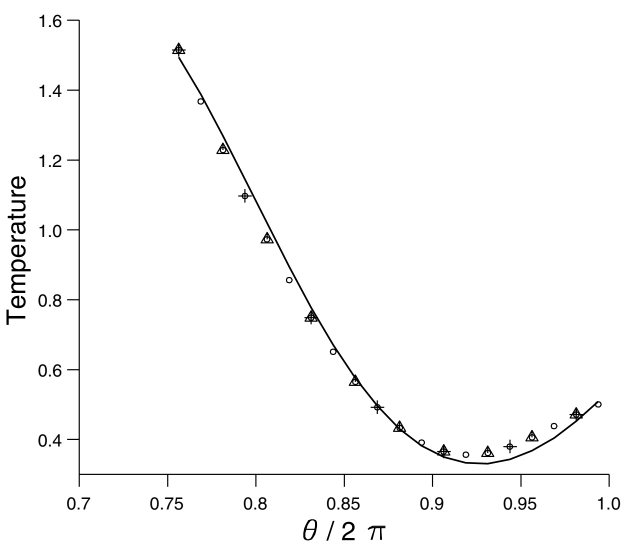

Figure 3 : The numerical solution for the temperature on the boundary Γ_0 obtained using the iterative BEM with $N = 40$ boundary elements for various initial guesses, namely $u_0 \equiv 0$ ($\circ$), $u_0 \equiv 2$ (+) and $u_0 \equiv 3$ ($\triangle$), in comparison with the analytical solution (—).

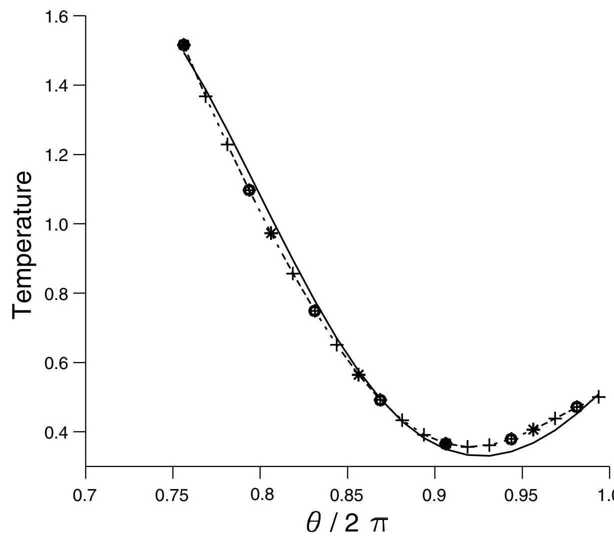

Figure 4 : The numerical solution for the temperature on the boundary Γ_0 obtained using the iterative BEM with $N = 80$ boundary elements for various values of the parameter γ, namely, $\gamma = 0.05$ (- - -), $\gamma = 0.5$ (+), $\gamma = 1.0$ ($\circ$) and $\gamma = 1.4$ ($*$) in comparison with the analytical solution (—).

values of the parameter γ, in comparison with the analytical solution given by Eq. 13. Similar results are obtained for any relaxation parameter $\gamma \in [0, 1.4]$.

5 Stability of the algorithm

Next, the stability of the iterative BEM proposed is investigated by perturbing the initial data $T|_{\Gamma_2}$ as follows:

$$\tilde{T}|_{\Gamma_2} = T|_{\Gamma_2} + \varepsilon, \tag{20}$$

$$\varepsilon = \text{G05DDF}(0, \sigma), \quad \sigma = \frac{s}{100} \max|T|_{\Gamma_2}| \tag{21}$$

where ε is a Gaussian random variable with mean zero and standard deviation σ, generated by the NAG routine G05DDF and s is the percentage of additive noise included in the input data $T|_{\Gamma_2}$ in order to simulate the inherent measurement errors. If a smooth solution is desired then it is useful to smooth the noisy data before using it as input data for the iterative algorithm. Fig. 5 presents the numerical solutions for the temperature on the boundary Γ_0 for various amounts of noise which is added into the given temperature on the boundary Γ_2, see Eq. 20.

It can be seen that as s decreases, the numerical solution better approximates the exact solution, whilst remaining stable. Numerous other examples have been investigated and it was found that the method described produces a stable solution with respect to decreasing the amount of noise.

6 Conclusions

In this paper an iterative BEM has been employed in order to reduce the Cauchy problem associated with the steady-state heat conduction to a sequence of well-posed problems.
The numerical convergence of the proposed algorithm was investigated for various numbers of boundary elements and various initial guesses. The numerical results obtained were found to be in good agreement with the exact solution. The stability of the numerical solution was also investigated and it was found that the numerical solution is stable with respect to small perturbations in the input data. Overall, from the numerical results obtained it can be concluded that the iterative BEM produces a convergent, stable and consistent numerical solution with respect to increasing the number of boundary elements and decreasing the amount of noise for the Cauchy steady state heat conduction problem.

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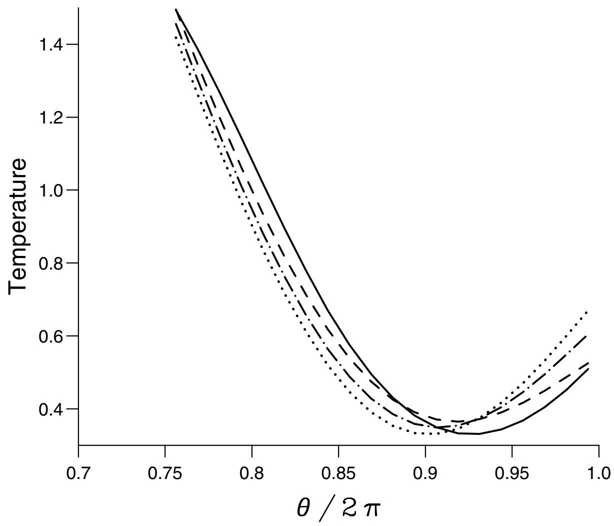

Figure 5 : The numerical solution for the temperature on the boundary Γ_0 obtained using the iterative BEM with $N = 80$ boundary elements for various amounts of noise, namely , $s =$ 1.0% (- - -), $s = 2.0\%$ ($- \cdot -$), $s = 3.0\%$ ($\cdots$) in comparison with the analytical solution (—).

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