Structured Adaptive Control for Poorly Modeled Nonlinear Dynamical Systems

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Abstract: Model reference adaptive control formulations are presented that rigorously impose the dynamical structure of the state space descriptions of several distinct large classes of dynamical systems. Of particular interest, the formulations enable the imposition of exact kinematic differential equation constraints upon the adaptation process that compensates for model errors and disturbances at the acceleration level. Other adaptive control formulations are tailored for redundantly actuated and constrained dynamical systems. The utility of the resulting structured adaptive control formulations is studied by considering examples from nonlinear oscillations, aircraft control, spacecraft control, and cooperative robotic system control. The theoretical and computational results provide new insights and provide a basis for optimism regarding practical adoption of adaptive control methodology in advanced implementations.

1 Introduction

Over the past two decades, adaptive control formulations have evolved and have been studied as candidates for controlling low- to moderate-dimensioned uncertain dynamical systems. A number of obstacles have become evident that have limited the degree to which adaptive control methodology has found a home in practical applications. One pervasive qualitative obstacle stems from the truth that adaptive control theory has been developed with a relatively high level of mathematical abstraction that limits access and understanding of the formulations to a small fraction of the engineers involved in the target applications. However, the low level of adoption is not merely a consequence of abstraction, there are several technical limitations that need to be addressed in order for adaptive control methods to more adequately accommodate:

1. High state space dimensionality;

2. Actuator saturation and actuator dynamics;

3. Over- and Under-actuation, as regards the number of available actuators compared to the number of acceleration states of the system;

4. Unusual features of a particular system's state space description that are not adequately captured in a generic adaptive control formulation;

5. Relevant stability proofs for a candidate adaptive control approach, as regards applicability to a particular system and operational environment;

6. Output feedback, as opposed to full-state feedback, to comply with the demands of a particular system and available sensors;

7. Multi-mode systems with drastic variations in local input/output characteristics; and finally,

8. Means for tuning of various matrices that parameterize, and therefore govern, the behavior of the resulting adaptive control laws.

These and related issues have been addressed to some extent in the literature, and the present paper is an effort to bring some of the recent results together, extend and particularize them in a coherent way, in order to study their utility through applications to several aerospace and robotic system examples.

2 Structured Dynamical Systems

The most common starting point for formulating adaptive control theory for nonlinear dynamical systems is to begin with a vector first order differential equation of the form

$$\dot{x} = f(t,x,p) + [B(x,p)]u + d \tag{1}$$

where, x is an n vector of state variables, d is an n vector of unknown (but bounded) disturbances, u is an m vector of control inputs and p is an r vector of uncertain parameters. We have found that exploiting more particularized structure in the state variable models is highly desirable. Perhaps the most important structural feature: A subset of the system differential equations is often exact (do not depend on uncertain parameters) and therefore should be imposed as exact differential equation constraints in the adaptation process. For the purpose of formulating adaptive control formulations, several alternatives to Eq. 1 are considered, based upon functional description of inherent structure present in the description of system's governing differential equations. One motivation of the present paper is to enable rigorous, exact differential equation constraints to be imposed in the adaptive control theory, and

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to demonstrate that the theoretical and computational developments are very attractive. The state-space differential equations governing the motion of most systems of interest can be partitioned into categories treated in the following sections.

2.1 Structured First Order Form

The differential equations for a large family of physical systems can be neatly partitioned into two subsets of differential equations, one governing exact kinematic relationships and the other describing the typically uncertain momentum dynamics of the system. The set of differential equations governing the kinematics is often mathematically and physically exact once a "judicious" choice of position and velocity coordinates is made. For example, to describe the dynamics of an aircraft, one could use the position coordinates as three translational positions and any set from an infinite choice of attitude (rotational position) descriptions [Junkins and Kim (1995); Shuster (1993); Junkins and Turner (1986)]. Similarly the velocity coordinates could be either be the traditional stability axes description or the body axes description. Once the position and velocity coordinate choices are made, certain time derivative relationships between the positions and the velocities can be written which are unique and exact. An infinite family of such systems is included in a nonlinear state space model of the following generic structure:

$$\dot{\sigma} = f(\sigma, \omega) \quad \text{(exact kinematics)} \tag{2}$$

$$\dot{\omega} = G(\sigma, \omega, p) + B(\sigma, \omega, p)u + d \quad \text{(uncertain dynamics)} \tag{3}$$

$$y = Y(\sigma, \omega, p) \quad \text{(uncertain measurement model)} \tag{4}$$

where,

$\sigma \in R^n$ is a vector of position coordinates

$\omega \in R^n$ is a vector of velocity coordinates

$d \in R^n$ is a vector of unknown disturbance accelerations

$f(\sigma, \omega) \in R^n$ is a known vector function of exact kinematic relationships

$p \in R^k$ is a parameter vector describing force and moment influences, e.g. Aerodynamic coefficients, Propulsive influences, Inertias etc.

$G(\sigma, \omega, p) \in R^n$ are control independent force terms

$B(\sigma, \omega, p) \in R^{n \times n}$ is a matrix of control dependent force terms

$u \in R^m$ is a control vector

$y \in R^s$ ($s \leq m$) is the output vector

All vector functions in the above are assumed smooth and twice differentiable functions of all arguments. We now present a methodology to derive output tracking control laws for systems, which conform to Eq. 2-4. First consider a further specialized class of problems, which is contained in the more general description in Eq. 2-4. We assume that the momentum level equations in Eq. 2-4 can be particularized to the following mathematical structure.

$$\dot{\omega} = Ag(\sigma, \omega) + Bu + H(\sigma, \omega) \tag{5}$$

where
$A \in R^{n \times p}$ is the assumed constant matrix containing all uncertain system model parameters (in lieu of the parameter vector p in Eq. 1).

$g(\sigma, \omega) \in R^p$ is a vector of basis functions whose amplitudes are the uncertain A matrix

$H(\sigma, \omega) \in R^n$ is a vector of known functions which do not depend on unknown model parameters.

In the case where outputs are the position coordinates $y = \sigma$, we can derive elegant control laws that drive $(y - y_r) \to 0$, where $y_r(t)$ is the trajectory output from a reference model or a pre-computed maneuver trajectory. Let the reference trajectory satisfy a differential equation with the same functional structure as Eq. 2-4, with known reference values for all parameters, and $\sigma_r(t)$, $\omega_r(t)$ describe the trajectory that is to be tracked. We will show that a control law based on dynamic *model inversion* can be derived that seeks to drive $(y - y_r) \to 0$ and $(\dot{y} - \dot{y}_r) \to 0$ (in this case: $(\sigma - \sigma_r) \to 0$ and $(\omega - \omega_r) \to 0$).

2.2 Ideal Model Inversion

Let us denote $e = \sigma - \sigma_r$ the error between the actual trajectory and the reference trajectory position coordinates. The control objective therefore is to drive 'e' to zero. Consider the following differential equation governing the *ideal position tracking error dynamics*.

$$\ddot{e} + C\dot{e} + Ke = 0 \tag{6}$$

where $C \in R^{n \times n}$ s.t. $x^T C x \geq 0 \ \forall x \neq 0$ and $K \in R^{n \times n}$ s.t. $x^T K x > 0 \ \forall x \neq 0$.

It can be seen clearly that Eq. 6 and the above positivity conditions on 'C' and 'K' guarantee exponential convergence of $e(t)$ to zero for all initial conditions. In the developments below, we exploit the above structure, to constrain the tracking dynamics to obey Eq. 6 and leave the matrices 'C' and 'K' to be chosen from the admissible set to satisfy appropriate design criteria. In particular, these matrices directly dictate the desired time constants and thus could be designed using pole-placement or eigenstructure assignment methods.

The ideal inversion control law is derived as follows. Let the position tracking error be defined as:

$$e \stackrel{\Delta}{=} \sigma - \sigma_r \tag{7}$$

Differentiating Eq. 7 twice, we obtain the desired acceleration

$$\ddot{\sigma} = \ddot{\sigma}_r - C(\dot{\sigma} - \dot{\sigma}_r) - K(\sigma - \sigma_r) \tag{8}$$

The non-linear equations describing the kinematics, Eq. 1 can be differentiated once to yield the following exact kinematic relationship

$$\ddot{\sigma} = \left[\frac{\partial f}{\partial \sigma}\right]\dot{\sigma} + \left[\frac{\partial f}{\partial \omega}\right]\dot{\omega} \tag{9}$$

Now substituting for $\dot{\omega}$ from Eq. 5 in Eq. 9 and replacing $\ddot{\sigma}$ by the desired acceleration of Eq. 8, we obtain the following acceleration constraint on the control 'u'.

$$\left[\frac{\partial f}{\partial \omega}\right][Ag(\sigma,\omega)+Bu+H(\sigma,\omega)] = \ddot{\sigma}_r - \left[\frac{\partial f}{\partial \sigma}\right]f(\sigma,\omega) -$$
$$C\left(f(\sigma,\omega)-\dot{\sigma}_r\right) - K(\sigma-\sigma_r) \tag{10}$$

which for the case of a square full rank 'B' matrix leads to the nonlinear feedback control law

$$u = -B^{-1}\left(Ag(\sigma,\omega)-\Psi\right) \tag{11}$$

where

$$\Psi \equiv -H(\sigma,\omega)+\left[\frac{\partial f}{\partial \omega}\right]^{-1}\left\{\ddot{\sigma}_r - \left[\frac{\partial f}{\partial \sigma}\right]f(\sigma,\omega) - \right.$$
$$\left. C\left(f(\sigma,\omega)-\dot{\sigma}_r\right) - K(\sigma-\sigma_r)\right\} \tag{12}$$

Equation 11 is the *ideal model inversion control law*. Worthy of note are the following: The control influence matrix 'B' is assumed full rank i.e. rank$(B)=n$. If this is not the case then the ideal control law obviously cannot guarantee perfect tracking as prescribed (an arbitrary desired acceleration cannot be achieved). Secondly the implementation requires calculation of two matrix inverses, We assume B^{-1} exists for the present discussion, but later in the adaptive version of the above control law, we show how to implement the inverse directly rather than estimating the matrix and then taking it's inverse at each instant of time. Finally, note the truth that for a 'proper' choice of position and velocity coordinates and a 'well-designed' reference trajectory, the Jacobian matrix $[\partial f/\partial \omega]$ almost always has an inverse. The inverse of the kinematic description of any rigid body orientation exists (and has a non-singular Jacobian matrix), so long as the coordinates remain in a singularity free volume. Moreover, the above control law guarantees exponential convergence of the tracking errors so long as the matrices 'A' and 'B' are known precisely. The proof of stability is trivial and can be established from Eq. 6. What to do when A and B are poorly known?

2.3 Structured Adaptive Model Inversion (SAMI)

The matrices 'A' and 'B' for the typical case of an aircraft or a missile are obtained through experimentation and empirical results and seldom known with high precision. Computational fluid dynamics and other tools are also used to estimate the aerodynamic and propulsive influences. These matrices are typically only locally valid in the neighborhood of a specified flight condition. As a consequence, these matrices virtually always have significant unknown errors. Further, if a severe maneuver is to be tracked or in the event of battle damage, these influences may drastically change, thereby altering the system input/output dynamics in a difficult to anticipate way. To maintain and/or recover stability and to provide improved robustness to changes in operating conditions, we present an adaptive version of the above control law, where 'A' and 'B' are estimated on the fly. Eq. 11 gives us the desired control law relation,

$$\ddot{e}+C\dot{e}+Ke = \frac{\partial f}{\partial \omega}[Ag(\sigma,\omega)+Bu-\Psi] \tag{13}$$

Of course, if A and B are exactly known, then the right hand side of Eq. 13 vanishes exactly. Let $\hat{A}$ and $\hat{B}$ be the estimates of 'A' and 'B' obtained in real time from some (to-be-specified) adaptation law, then the control law that's implemented is given by the following relation

$$u = -\hat{B}^{-1}\left(\hat{A}g(\sigma,\omega)-\Psi\right) \tag{14}$$

Re-arranging the above yields the control identity

$$\hat{B}u+\hat{A}g(\sigma,\omega)-\Psi = 0 \tag{15}$$

Substituting Eq. 14 into Eq. 5, the result into Eq. 9, and following straightforward algebraic manipulations, we are led to the closed-loop error dynamics:

$$\ddot{e}+C\dot{e}+Ke = \frac{\partial f}{\partial \omega}\left[\tilde{A}g(\sigma,\omega)+\tilde{B}u\right] \tag{16}$$

where we have made use of $\tilde{A}=A-\hat{A}$; $\tilde{B}=B-\hat{B}$.

Equation 14 can further be written compactly as the first order system

$$\dot{\varepsilon} = A_m\varepsilon + \left[\begin{array}{c} 0 \\ \frac{\partial f}{\partial \omega}\tilde{Q}^T\Phi \end{array}\right] \tag{17}$$

where,

$$\varepsilon = \left\{\begin{array}{c} e \\ \dot{e} \end{array}\right\}; \quad \tilde{Q} = \left[\begin{array}{c} \tilde{A} \\ \tilde{B} \end{array}\right]; \quad \Phi = \left[\begin{array}{c} g(\sigma,\omega) \\ u \end{array}\right];$$
$$A_m = \left[\begin{array}{cc} 0 & I \\ -K & -C \end{array}\right].$$

Now, we define a Lyapunov function as the positive definite function

$$V = \varepsilon^T P\varepsilon + \text{Tr}\left(\tilde{Q}^T\Gamma^{-1}\tilde{Q}\right) \tag{18}$$

where $\text{Tr}()$, defines the trace of a matrix () and $\Gamma=\Gamma^T>0$, is a user-chosen positive definite symmetric adaptation gain matrix and the unspecified matrix must also satisfy $P=P^T>0$. Obviously V is a positive definite measure of tracking and learning errors with a global minimum at $\{\varepsilon=0, \quad \tilde{Q}=0\}$. Differentiating '$V$' with respect to time and using Eq. 17, we obtain

$$\dot{V} = -\varepsilon^T R\varepsilon + \varepsilon^T\left[\begin{array}{cc} P_1 & P_{12} \\ P_{12} & P_2 \end{array}\right]\left[\begin{array}{c} 0 \\ \frac{\partial f}{\partial \omega}\tilde{Q}^T\Phi \end{array}\right] +$$
$$\left[\begin{array}{c} 0 \\ \frac{\partial f}{\partial \omega}\tilde{Q}^T\Phi \end{array}\right]^T\left[\begin{array}{cc} P_1 & P_{12} \\ P_{12} & P_2 \end{array}\right]\varepsilon +$$
$$\text{Tr}\left(\tilde{Q}^T\Gamma^{-1}\dot{\tilde{Q}}+\dot{\tilde{Q}}^T\Gamma^{-1}\tilde{Q}\right) \tag{19}$$

where P is now found as the solution of the algebraic Lyapunov equation $PA_m + A_m^T P = -R$ for a chosen $R = R^T > 0$. The Lyapunov equation always has a solution (i.e. a positive definite P, for any chosen $R = R^T > 0$) since A_m is Hurwitz. Equation 19 is now manipulated to cause all but the $-\varepsilon^T R\varepsilon$ term to vanish. This approach yields the following adaptation law for the estimate of 'Q',

$$\dot{\hat{Q}} = \dot{\tilde{Q}} = \Gamma \left[\frac{\partial f}{\partial \omega}\right]^T \dot{e}\Phi^T \tag{20}$$

and also the A, B adaptive estimates are

$$\dot{\hat{A}} = \Gamma_1 \left[\frac{\partial f}{\partial \omega}\right]^T \dot{e} g(\sigma, \omega)^T, \quad \dot{\hat{B}} = \Gamma_1 \left[\frac{\partial f}{\partial \omega}\right]^T \dot{e} u^T \tag{21}$$

The above choice of adaptation laws leads us to

$$\dot{V} = -\varepsilon^T R\varepsilon \le 0 \tag{22}$$

and guarantees bounded tracking of all the trajectory error states. To guarantee convergence of parameter estimates to their true values, on the other hand, one needs persistent excitation in the regression vector and for this parameter estimation problem, one requires further analysis. To prove asymptotic convergence of trajectory tracking errors however, one only needs to show that all the signals are bounded and from Barbalat's lemma[Ioannou and Sun (1995)], using the absolute continuity arguments[Ioannou and Sun (1995)] one can conclude that $\varepsilon \to 0$, as $t \to \infty$. The following lemma establishes asymptotic convergence of the tracking errors.

Lemma Consider the kinematic description of a dynamical system as elicited in Eq. 2-4, the momentum level description as the specialized class in Eq. 5 and a prescribed reference trajectory generated from a system with similar description in Eq. 2-4. Then control law in Eq. 14 together with the adaptation laws in Eq. 21 guarantees asymptotic convergence of the tracking errors ε i.e $\varepsilon \to 0$, as $t \to \infty$.

Proof We define the following, $x(t) \in L_\infty$ if $||x(t)||_\infty \overset{\Delta}{=} \overset{sup}{t \ge 0}$ $|x(t)| < \infty$ and $x(t) \in L_p$ if $||x(t)||_p \overset{\Delta}{=} [\int_0^\infty |x(\tau)|^p d\tau]^{1/p} < \infty$, for $p \in [1, \infty)$. From Eqs. 18 and 22, we conclude that $\varepsilon \in L_\infty$, $\tilde{Q} \in L_\infty$ and hence $\tilde{A}, \tilde{B} \in L_\infty$. Further integrating Eq. 22 between the limits $t = 0$ and $t = \infty$, we obtain $V(\infty) - V(0) = \int_0^\infty \varepsilon(\tau)^T R\varepsilon(\tau) d\tau < \infty$ (Using the fact that $\dot{V} \le 0, V > 0$). Thus we conclude that $\varepsilon \in L_2$ and hence $\varepsilon \in L_2 \cap L_\infty$.

Consider Eq. 17, where we have the boundedness of the first term on the right hand side (A_m is Hurwitz) established. We now proceed to establish the boundedness of the second term. We already know that $\tilde{Q}$ is bounded and for a coordinate choice that's away from singularities, $\partial f/\partial \omega$ is bounded. We only need to show the boundedness of Φ to conclude the boundedness of $\dot{\varepsilon}$.

Now, $\Phi = \left[g^T(\sigma, \omega) u^T\right]^T$ and since $e, \dot{e} \in L_\infty$, and $\sigma_r, \dot{\sigma}_r \in L_\infty$, we conclude that $\sigma, \dot{\sigma} \in L_\infty$ which implies that $g(\sigma, \omega) \in L_\infty$.

(In particular, we restrict $g(\sigma, \omega)$ to not have terms of the form $g_1(\sigma, \omega)/g_2(\sigma, \omega)$ where $g_2(\sigma, \omega)$ could vanish locally). Similarly looking at the expression for u, we conclude $u \in L_\infty$ and therefore $\Phi \in L_\infty$. We have finally established that $\dot{\varepsilon}$ is bounded (i.e. $\dot{\varepsilon} \in L_\infty$). Now, we are in a position to use Barbalat's Lemma [Ioannou and Sun (1995)] which implies that $\varepsilon \to 0$, as $t \to \infty$, since $\varepsilon \in L_2 \cap L_\infty$ and $\dot{\varepsilon} \in L_\infty$ and therefore $e, \dot{e} \to 0$ as $t \to \infty$. This concludes the proof of asymptotic stability.

Example 1. We illustrate the above control methodology on a simple example. Consider a reference trajectory generated by by the following reference differential equation

$$\ddot{x}_r + c\dot{x}_r + kx_r = u_r \tag{23}$$

Let the actual system be exactly modeled by the modified Duffing differential equation

$$\ddot{x} + a_1\dot{x} + a_2x + a_3x^3 = u \tag{24}$$

The control objective is to force the actual non-linear system of Eq. 24 to track a reference trajectory generated by the linear Eq. 23. This example provides a simple situation to evaluate the robustness of the Structured Adaptive Model Inversion (SAMI) controller to parameter variations, un-modeled dynamics, and initial condition errors. We follow this simple example with higher dimensional studies. The reference trajectory is represented by x_r, $\dot{x}_r$, which according to our notation are designated σ_r and ω_r respectively. Re-casting the system in Eq. 24 into the generic state space form described in Eqs. 1, 2-4, we obtain the specializations

$\sigma = x,$
$\omega = \dot{x},$
$A = \left[\begin{array}{ccc} -a_3 & -a_2 & -a_1 \end{array}\right]$
$g(\sigma, \omega) = \left[\begin{array}{ccc} x^3 & x & \dot{x} \end{array}\right]^T$
$B = 1,$
$H(\sigma, \omega) = 0$
$f(\sigma, \omega) = \omega$
$\left[\frac{\partial f}{\partial \sigma}\right] = 0$
$\left[\frac{\partial f}{\partial \omega}\right] = 1$

Control Law:

$$u = -\hat{B}^{-1}\left(\hat{A}g(\sigma, \omega) - \Psi\right)$$

Adaptation Laws:

$$\dot{\hat{A}} = \gamma_1(\dot{x} - \dot{x}_r)\left[\begin{array}{c} x^3 \\ x \\ \dot{x} \end{array}\right]^T,$$

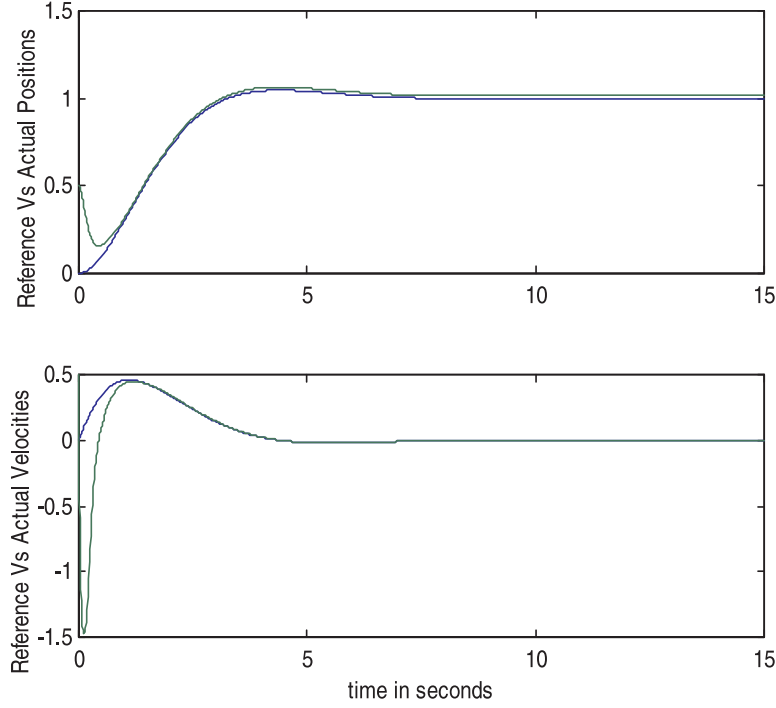$$\dot{\hat{B}} = \gamma_2(\dot{x} - \dot{x}_r)u^T$$

Figure 1 : Reference trajectory and the actual trajectory for example 1

Table 1 : System and Control Gain Parameters for Example 1

True system parameters:
$$A = [1.5 \quad -1 \quad 0.5], \quad B = b = 1$$
Model system parameters:
$$A_r = [1 \quad 1.4 \quad], \quad B_r = 1, \quad u_r = 1$$
$$\hat{A}(t=0) = (1 + 0.25 * RN) * A$$
$$\hat{B}(t=0) = (1 + 0.25 * RN) * B$$
$$RN = \text{Random number between } -1 \text{ and } 1$$
Desgin parameters:
$$C_1 = 30, \quad K_1 = 120, \quad \gamma_1 = \gamma_2 = 1$$
Initial conditions:
True system: $[0.5 \quad 0.5]$
Model system: $[0 \quad 0]$

where $\Psi = \{\ddot{x}_r - C_1(\dot{x} - \dot{x}_r) - K_1(x - x_r)\}$, $\hat{A} = [-\hat{a}_3 \quad -\hat{a}_2 \quad -\hat{a}_1]$, $\hat{B} = 1/\hat{b}$, and γ_1 and γ_2 are chosen scalar adaptation gains and K_1 and C_1 are selected to design the desired tracking error dynamics. Note that we can define $M = \hat{B}^{-1}$ and make use of the identity $d(\hat{B}\hat{B}^{-1})/dt = 0$, leading to the adaptation law $\dot{M} = -\gamma_2 M [(\dot{x} - \dot{x}_r)u^T] M$ to be used in lieu of the $\dot{\hat{B}}$ equation. Thus $u = -M(\hat{A}g - \Psi)$ and we avoid the necessity of computing $\hat{B}(t)$ and $\hat{B}^{-1}$. It is easy to verify that a large family of numerical values for $\{K_1, C_1, \gamma_1, \gamma_2\}$ lead to a corresponding family of stable closed loop dynamic response characteristics. The values for $\{K_1, C_1\}$ can be selected to achieve the desired eigenstructure for the closed loop response, whereas $\{\gamma_1, \gamma_2\}$ can be chosen based upon a simulation study considering a family of representative model errors and disturbances. The numerical values used for simulation in the example detailed above are given in Tab. 1.

It is trivial to numerically verify that this control law is very robust to initial condition errors, modeling errors and parameter uncertainties. In addition, observe that the true system in this case is unstable. Thus the adaptive control law achieves essentially perfect tracking in addition to stabilizing this unstable system. The actual trajectory is plotted over the reference trajectory in Fig. 1 while the position and velocity tracking errors are plotted in Fig. 2. The control effort is plotted in Fig. 3.

2.4 Structured Adaptive Model Inversion for Aircraft Trajectory Tracking

Having developed the control methodology and considered a simple example, we now explore the application of SAMI based controllers to different dynamical systems, each having it's own peculiarity in the system dynamics. Though the above approach is fairly general, encompassing a very large class of problems, sometimes it is required/desired to make minor modifications to further tailor the procedure for particular problems. The examples studied in the next few sections highlight this truth.
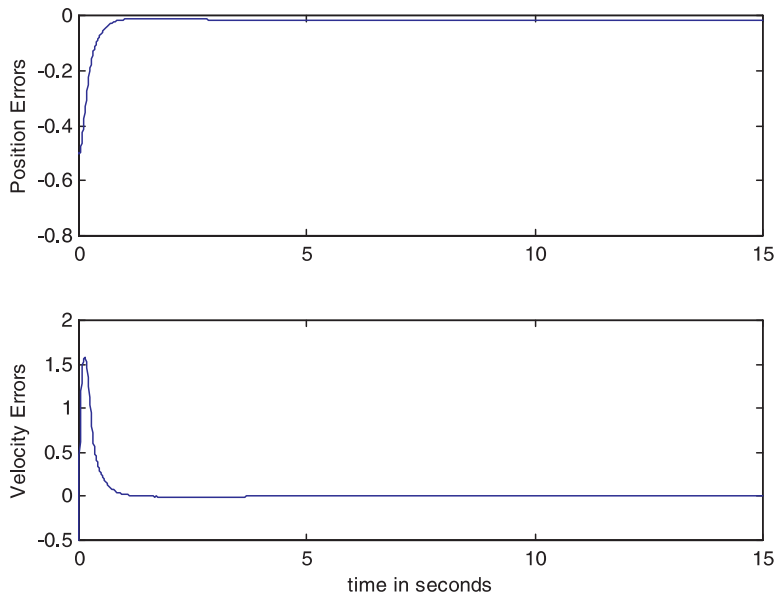
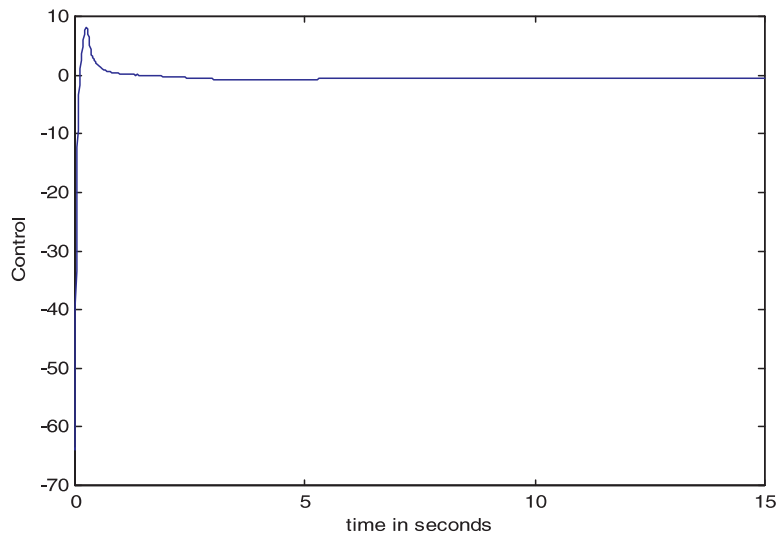**Figure 2** : Position and velocity tracking errors for example 1

Figure 3 : Control effort for example 1

2.4.1   Longitudinal Dynamics for Vertical Takeoff and Landing (VTOL) Aircraft

With reference to Fig. 4, consider the dynamics of a Harrier class aircraft. In particular, consider the low-speed transition from vertical take-off and landing (VTOL) to forward wing-borne flight. The momentum level equations for the three degree of freedom longitudinal dynamics are modeled by the fol-

lowing three differential equations:

$$m\dot{u} = -mqw + F_x(\delta e, \delta n, T, \alpha, V) - mg\sin\theta \qquad (25)$$

$$m\dot{w} = mqu + F_z(\delta e, \delta n, T, \alpha, V) + mg\cos\theta \qquad (26)$$

$$I\dot{q} = M(\delta e, \delta rcs, \delta n, T, \alpha, V) \qquad (27)$$

where m and I are the instantaneous mass and pitch inertia, velocity coordinates $\omega(t)$ consist of linear velocities and angular velocity (u, w, q) with vector components taken in the
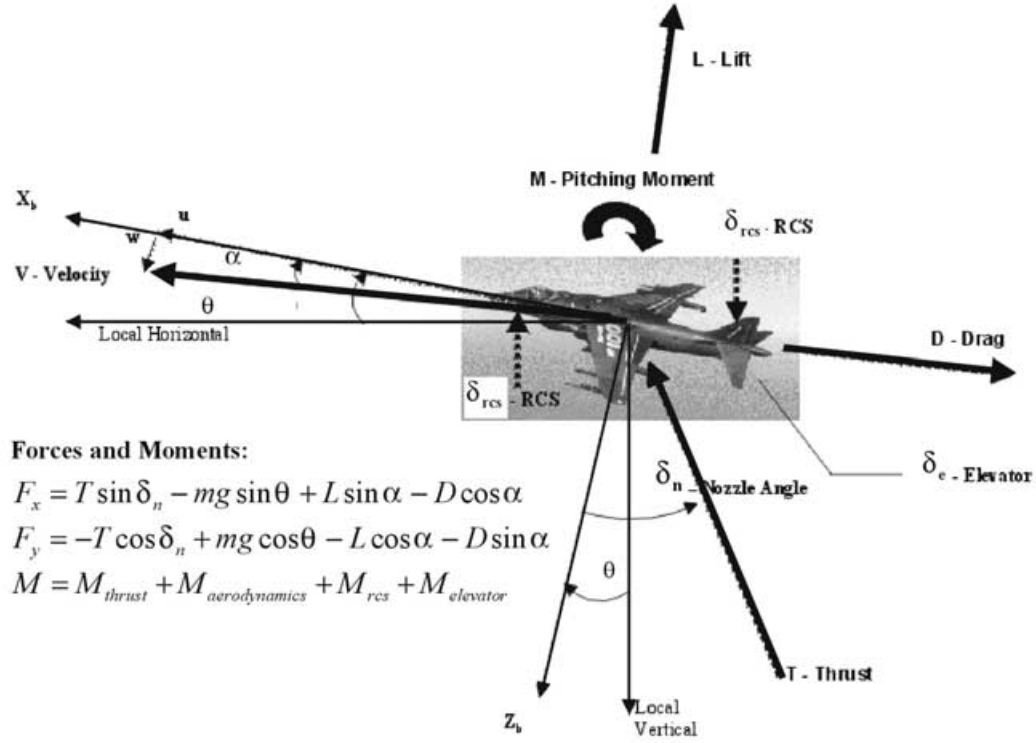
Figure 4 : Harrier vertical takeoff and landing aircraft forces and moments

body axes system. The complicated non-linear function dependence of F_x, F_z, M on $\{\delta_e, \delta_{rcs}, \delta_n, T, \alpha, V\}$ is simply noted; these functional dependencies are not given in detail here (see http://aero.tamu.edu/ucav). The dependencies are often approximated for simulation by multi-dimensioned tables with local interpolation. We are interested in the transition from low-velocity thruster-supported and controlled flight to higher velocity aerodynamically supported flight. F_x, F_z are the total external forces acting on the a/c, while 'M' is the moment acting along the y-axis of the body axes, g is the acceleration due to gravity. Nominally the aerodynamic forces and moment are predominantly functions of angle of attack 'α' and the total velocity 'V', defined as

$$\alpha(t) = \tan^{-1}\left(\frac{w}{u}\right), \quad V(t) = \left(u^2 + w^2\right)^{1/2} \tag{28}$$

however, in the low speed transition region, these forces and moments are also strongly dependent on the thrust vector magnitude and direction, which alters the flow field about the airplane and therefore modifies drastically the aerodynamic forces and moments. This nonlinear coupling is very difficult to model accurately. The nonlinear aerodynamic force and moment models for most simulations are based on experimental and computational sudies and are stored for real-time interpolation in a multidimensional lookup table. The set $\{\delta e, \delta n, \delta rcs, T\}$ represents the four-element control vector, consisting of the elevator δe, the thrust vector gimbal angle δn, the reaction control δrcs and the thrust T. The position coordinates $\sigma(t)$ consist of (x, H, θ) where (x, H) represent the location of aircraft cg in ground axes (x-location and altitude) and θ is the pitch angle. The exact kinematic differential equations in terms of $h(t)$, $x(t)$ and $\theta(t)$ are given as

$$\dot{H}(t) = u(t)\sin\theta(t) - w\cos\theta(t) \tag{29}$$
$$\dot{x}(t) = u(t)\cos\theta(t) + w\sin\theta(t) \tag{30}$$
$$\dot{\theta} = q(t) \tag{31}$$

Let the tracking error in position coordinates be denoted $\sigma - \sigma_r = [\Delta H \quad \Delta x \quad \delta\theta]^T$. We are not interested in tracking x-location of the aircraft, although we want to track ground velocity $\dot{x}$, along with other position coordinates H and θ. The ideal model tracking error dynamics for this example are specified as

$$\left\{ \begin{array}{c} \Delta\ddot{H} \\ \Delta\ddot{\theta} \end{array} \right\} + \left[\begin{array}{cc} c_1 & 0 \\ 0 & c_3 \end{array} \right] \left\{ \begin{array}{c} \Delta\dot{H} \\ \Delta\dot{\theta} \end{array} \right\}$$
$$+ \left[\begin{array}{cc} k_1 & 0 \\ 0 & k_3 \end{array} \right] \left\{ \begin{array}{c} \Delta H \\ \Delta\theta \end{array} \right\} \tag{32}$$
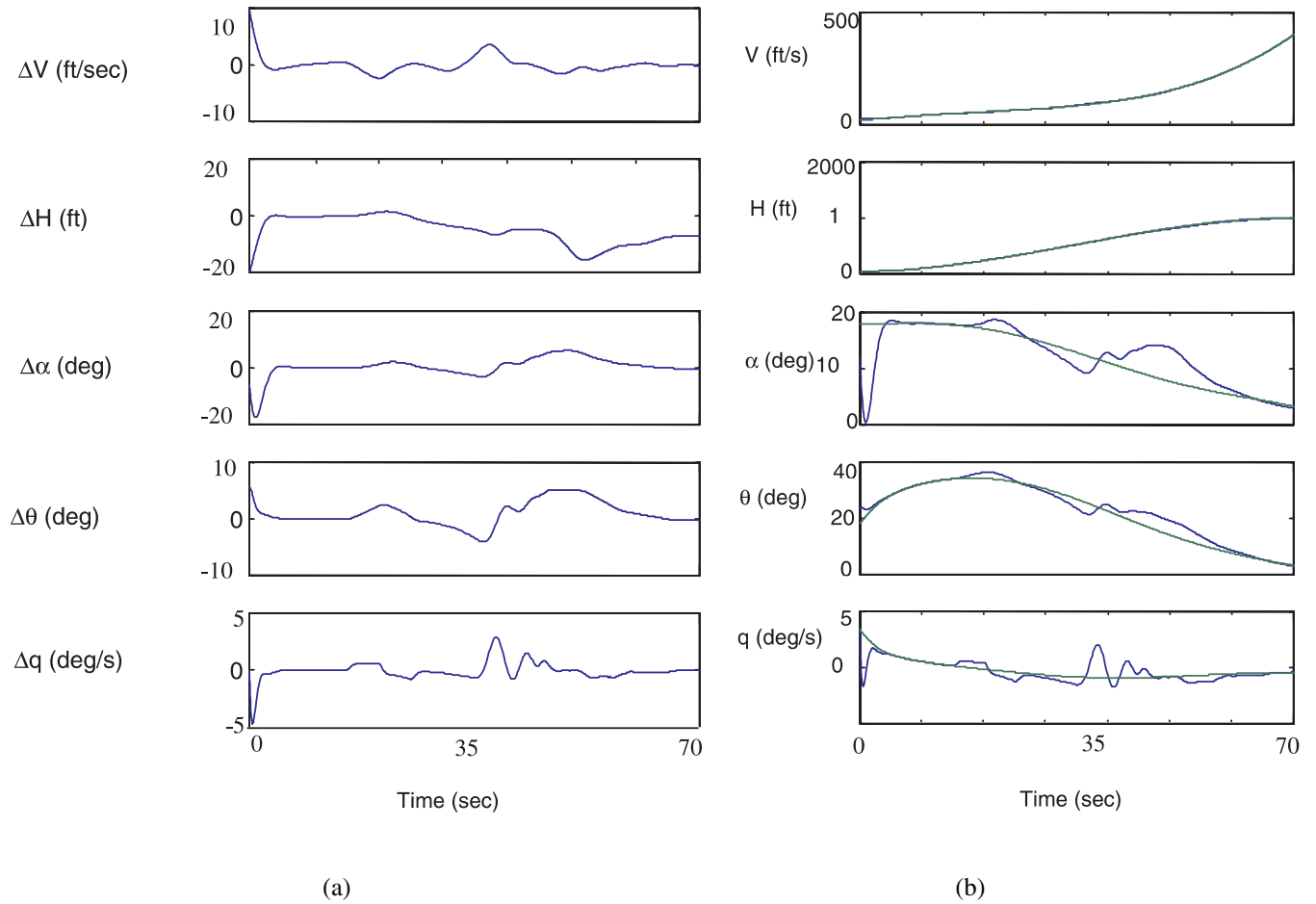$$\Delta\ddot{x} + c_2\dot{x} = 0 \tag{33}$$

Figure 5 : (a) Error states; (b) References vs actual Trajectory

For this system we identify that

$$f(\sigma,\omega) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ w \\ q \end{Bmatrix} \quad (34)$$

$$H(\sigma,\omega) = \begin{Bmatrix} -qw - g\sin\theta \\ qu + g\cos\theta \\ 0 \end{Bmatrix} \quad (35)$$

$$g(\sigma,\omega,p,U) = \begin{Bmatrix} F_x \\ F_z \\ M \end{Bmatrix} \quad (36)$$

$$\frac{\partial f}{\partial\omega} = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (37)$$

The uncertain terms $g(\sigma,\omega,p,U)$ are the aerodynamic and propulsive forces and moments given by (F_x, F_z, M). Suppose that the forces and moments have been linearized at a family of

equilibrium points (through out the trajectory given by (H,V) as

$$g(\sigma,\omega,p,U) = g(\sigma_r,\omega_r,U_r) + A(H,V)\Delta x + B(H,V)\Delta u \quad (38)$$

Note that (H,V) are time varying for the maneuver that has been considered. Hence, the assumption that unknown matrices A and B are constant is no longer true. But we can typically assume that A and B are varying sufficiently slowly and the adaptive laws continuously modify A and B fast enough to guarantee stable tracking. We find this two time scale approach is usually valid, but is not universally true, of course. The ideal model inversion control law is same as given by Eq. 11. The adaptive control law (SAMI) adopted is given by Eqs. 14 and 21.

Example 2. A trajectory was generated for a transition maneuver using the approach in Verma and Junkins (1999, 2000) for the Harrier VTOL aircraft. The maneuver starts with
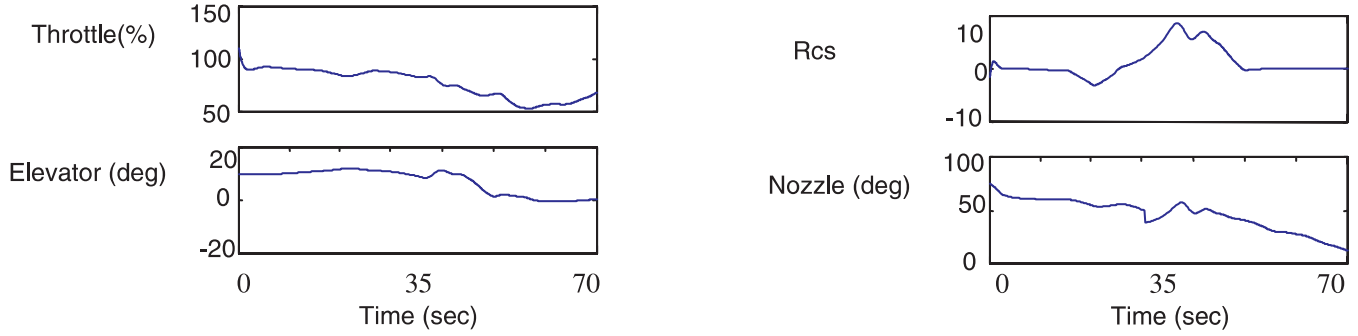
Figure 6 : Control requirements

thrust-borne flight at 20 *ft/sec* and 50 *ft* altitude and transitions to aerodynamic flight at 400 *ft/sec* and 1000 *ft* altitude. Notice, during the course of this maneuver, the control influence matrix changes in a drastic way due to the velocity squared dependence and propulsive/ aerodynamic coupling effects. An adaptive controller is presented to track this transition trajectory, taking into consideration this wide variation in control effectiveness. Modeling errors of up to 15% in all system parameters were introduced in system matrices (aerodynamic derivatives etc.). The aircraft is also subjected to an (unknown to the control system) external disturbance of triangular vertical gust starting at 15 *sec.* and ending at 25 *sec.* The aircraft response is plotted over the reference trajectory in Fig. 5b. The tracking error in trajectory states is plotted in Fig. 5a. The control requirements are plotted in Fig. 6. It can be seen that admirable tracking has been achieved, keeping the errors very small within desirable bounds. The control requirement is reasonable and is seen to be well within saturation limits. While technically the four actuators are redundant for this three degree of freedom vehicle, the velocity dependence of the control influence matrix means the aerodynamic input δe has very limited effectiveness at low velocity. Finally we desire not to utilize thrust vectoring at high velocity so the system would become under-actuated at high velocity flight as is usual for aircraft control.

2.5 Structured Adaptive Model Inversion for Tracking Spacecraft Maneuvers

The rotational equations of motion[Junkins and Turner (1986)] of a spacecraft in any chosen reference frame can be structured to follow the same form as in Eq. 1.

$$\dot{\sigma} = \frac{1}{4}B(\sigma)\omega \quad \text{(exact kinematics)} \tag{39}$$

$$[I]\dot{\omega} + [\tilde{\omega}][I]\omega = u + d \quad \text{(uncertain dynamics)} \tag{40}$$

where,

$$B(\sigma) = \left[(1 - \sigma^T \sigma)I_{3\times 3} + 2[\tilde{\sigma}] + 2\sigma\sigma^T\right] \tag{41}$$

$[I]$ is the Inertia Matrix
d is the unknown bounded external disturbance vector and
$$[\tilde{x}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \text{ is the skew symmetric vector}$$
cross product operator

The modified Rodrigues Parameter vector[Shuster (1993)] σ is adopted as a rigid body attitude measure relative to the inertially fixed reference frame. Note that the vector σ contains information about both the principal rotation axis $\hat{e}$ and the principal rotation angle Φ, since they are related through the geometric condition

$$\sigma = \hat{e}\tan\frac{\Phi}{4} \tag{42}$$

Therefore, $\sigma \to 0$ corresponds to zero angular error and for $\sigma \to \infty$, the orientation corresponds to $\Phi \to \pm 360$ which means that the MRP vector set goes singular for large tumbling motions. For all rotations $-180° < \Phi < 180°$, it is evident that $\sigma^T \sigma < 1$. It has been shown that it is possible to map the original MRP vector set to its corresponding shadow (or image) counterpart σ^S through the transformation

$$\sigma^S = -\frac{1}{\sigma^2}\sigma \tag{43}$$

where the notation $\sigma^2 = \sigma^T \sigma$ is used. It can be shown that σ and σ^S both satisfy the same differential equation Eq. 39. By choosing to switch the MRPs; using Eq. 43 whenever $\sigma^T \sigma = \sigma^2 > 1$, the MRP vector remains bounded within a unit sphere, even for the case of tumbling motion. Switching when the $\sigma^2 = 1$ surface is penetrated also results in the corresponding MRPs always measuring the shortest rotational distance back to the origin. (i.e., less than 180°). As mentioned earlier, it is

assumed that the given (desired) reference trajectory, denoted by $\sigma_r(t)$ is twice differentiable. The trajectory tracking error $\varepsilon(t)$ can now be defined as

$$\varepsilon(t) = \sigma(t) - \sigma_r(t) \tag{44}$$

For this case, we specify an ideal tracking error dynamics as the PID form

$$\ddot{\varepsilon} + C\dot{\varepsilon} + K\varepsilon + K_i \int \varepsilon \, dt = 0 \tag{45}$$

Note an integral feedback term has been added to the desired tracking error dynamics. We could still retain the stability properties by a proper choice of the matrices C, K and K_i, as well as further optimize the closed loop response.

Note, as an alternative to the MRP vector, any attitude or position vector could have been used. The MRP vector however has important advantages for large motions. We note this approach (imposing a sliding error surface dynamics) is closely related to the developments in Ref. [19], however the use of MRPs and especially, the important extensions to allow adaptive trajectory tracking are novel extensions of Paielli's idea.

Carrying on with the developments mentioned earlier in this paper we obtain

$$\left[\frac{\partial f}{\partial \omega}\right]\dot{\omega} = \ddot{\sigma}_r - \left[\frac{\partial f}{\partial \sigma}\right]f(\sigma, \omega) - C\left(f(\sigma, \omega) - \dot{\sigma}_r\right) -$$
$$K(\sigma - \sigma_r) - K_i \int (\sigma - \sigma_r)dt \tag{46}$$

Thus we calculate the desired acceleration vector $\dot{\omega}$ which would enforce linear tracking error dynamics thereby achieving the objective of dynamic inversion. Using this desired $\dot{\omega}$ one can impose an algebraic constraint that determines the required control torque by model inversion. For this spacecraft maneuver problem, let us now evaluate the required quantities to solve for the controls explicitly. Differentiating Eq. 39, we obtain

$$\ddot{\sigma} = \frac{1}{4}B\dot{\omega} + \frac{1}{4}\dot{B}\omega \tag{47}$$

and we can make use of the explicit expression of the matrix inverse of B which is

$$B^{-1} = \frac{1}{(1 + \sigma^2)^2}B^T \tag{48}$$

The above expression is readily verified by using it to confirm that $B^{-1}B = I_{3 \times 3}$. Since for $|\sigma| \leq 1$, the matrix B is obviously always invertible (and this is another manifestation of the advantages of the MRPs as attitude co-ordinates). The product $\dot{B}\omega$ using the vector product definitions Eq. 41 for 'B' can be expressed as

$$\dot{B}\omega = \sigma^T\omega(1 - \sigma^2)\omega - (1 + \sigma^2)\frac{\omega^2}{2}\sigma -$$
$$2\sigma^T\omega[\tilde{\omega}]\sigma + 2(\sigma^T\omega)^2\sigma \tag{49}$$

where the shorthand notation $\omega^2 = \omega^T\omega$ is used. The expression in Eq. 49 is obtained using the algebraic identities $[\tilde{a}]a = 0$ and $[\tilde{a}][\tilde{a}] = aa^T - a^TaI_{3 \times 3}$, $\forall a \in R^3$. If we denote the right hand side of Eq. 49 as Ψ, we can rewrite Eq. 47 as

$$\ddot{\sigma} = \frac{1}{4}B\dot{\omega} + \Psi \tag{50}$$

Observing that $[\partial f/\partial \omega] = B$ in this case, and using Eqs. 46 and 50 we obtain the desired angular acceleration vector $\dot{\omega}$ as follows

$$\dot{\omega} = \frac{4B^T}{(1 + \sigma^2)^2}\left\{ \ddot{\sigma}_r - \Psi - C(\dot{\sigma} - \dot{\sigma}_r) - K(\sigma - \sigma_r) - \right.$$
$$\left. K_i \int (\sigma - \sigma_r)dt \right\} \tag{51}$$

Let us denote this acceleration constraint on $\dot{\omega}$ as $\dot{\omega} = \Theta$, where Θ represents the right hand side of Eq. 51. Using Eq. 40, we then have the required torque

$$u = [\tilde{\omega}][I]\omega + [I]\Theta - d \tag{52}$$

The above control law will ensure that the tracking error dynamics is linear and of the form expressed in Eq. 45. Observe, that even though Eqs. 51, 52 constitute nonlinear feedback laws, the ideal tracking dynamics are implicitly constrained to be a linear differential Eq. 45 with C, K and K_i left free as the control design parameters.

It is mentioned here that some of these developments were motivated by those in Schaub, Akella, and Junkins (1999); Paielli and Bach (1993). Paielli's nice paper used the vector part of the quaternion as attitude coordinates and did not consider any adaptive control issues. In our work, we eliminated the $\pm 180°$ singularities implicit in the Paielli formulation and also introduced adaptive control methods to enable stable control with large model errors. We also observe that the control law contains the inertia matrix linearly. When the inertia matrix is unknown, of course we cannot directly implement Eq. 52 exactly. In the following section we introduce an adaptive controller for such situations.

An attractive component of this methodology when dealing with *known* system parameters is that the structure of the tracking error dynamics can be easily modified using standard linear control theory techniques, which lead to appropriate choice of the constants C, K and K_i.

2.5.1 Structured Adaptive Model Inversion

While the desired acceleration vector Θ is a kinematic quantity depending only on the position σ (and the derivatives thereof) in order to compute the proper control vector to enforce linearized tracking error dynamics, the system inertia matrix $[I]$ and the external torque vector d must be known precisely. In the following development it is assumed that only crude estimates of the two are known. In this case, it is evident that

the desired acceleration vector Θ is no longer equal to the actual acceleration $\dot{\omega}$. The following adaptive control law formulation approach requires that the unknown parameters appear linearly in the control formulation. Therefore we rewrite Eq. 52 as

$$u = [L^*]g + [M^*]\Theta - d^* \tag{53}$$

where the matrices $[L^*]$ and $[M^*]$ are defined as follows

$$[L_1] = \begin{bmatrix} 0 & I_{23} & -I_{23} \\ -I_{13} & 0 & I_{13} \\ I_{12} & I_{12} & 0 \end{bmatrix},$$

$$[L_2] = \begin{bmatrix} I_{13} & I_{33} - I_{22} & -I_{12} \\ -I_{23} & I_{12} & I_{11} - I_{33} \\ I_{22} - I_{11} & -I_{13} & I_{23} \end{bmatrix},$$

$$[L^*] = [L_1 \vdots L_2],$$
$$[M^*] = [I]$$

The vector d^* is the true external torque and the 6×1 vector g is defined as

$$g \equiv [\omega_1^2 \quad \omega_2^2 \quad \omega_3^2 \quad \omega_1\omega_2 \quad \omega_2\omega_3 \quad \omega_3\omega_1]^T$$

The control vector expression in Eq. 53 is rewritten by introducing a 3×10 matrix $[Q^*]$

$$[Q^*] = [L^* \vdots M^* \vdots d^*] \tag{54}$$

and the 10×1 vector of basis functions:

$$x \equiv \begin{bmatrix} g \\ \Theta \\ -1 \end{bmatrix} \tag{55}$$

into a compact form for the adaptive feedback law:

$$u = [Q^*]x \tag{56}$$

Note that direct implementation using Eq. 56, would require that all system parameters are perfectly known. Now assume that the inertia matrix and the external disturbance torque vector are not known precisely. Using Eq. 56 the system is over-parameterized because we have a 3×10 matrix parameterizing all the uncertain parameters, however the actual number of uncertain parameters are only 12 (nine components of the Inertia matrix and the 3×1 disturbance torque vector. This version of the control law is a direct version as we are obviously not learning the true parameters but some equivalent representation of the system; were we concerned with system identification, then this redundant parameterization would have to be avoided. The actual control vector u, which is implemented in the adaptive approach, is given by the adaptive feedback law

$$u = [Q(t)]x \tag{57}$$

where $[Q(t)] = [L(t) \vdots M(t) \vdots d(t)]$ contains the time varying adaptive estimates of the poorly known system parameters. The difference between the adaptive estimates and true system parameters is expressed through the error matrix

$$[\tilde{Q}] = [Q(t)] - [Q^*] \tag{58}$$

The desired linear tracking dynamics can be compactly written as

$$\dot{z} = Az + b \tag{59}$$

where

$$z \equiv \left\{ \begin{array}{c} \int \varepsilon dt \\ \varepsilon \\ \dot{\varepsilon} \end{array} \right\}, \tag{60}$$

$$A \equiv \begin{bmatrix} 0 & I_{3\times3} & 0 \\ 0 & 0 & I_{3\times3} \\ -K_i I_{3\times3} & -KI_{3\times3} & -CI_{3\times3} \end{bmatrix}, \tag{61}$$

$$b \equiv \left\{ \begin{array}{c} 0 \\ 0 \\ \xi \end{array} \right\}, \quad \text{and} \tag{62}$$

$$\xi \equiv \frac{1}{4}B[I]^{-1}[\tilde{Q}]x \tag{63}$$

Let us now define a positive definite Lyapunov function V as a positive measure of the tracking error (z) and the adaptation errors $\tilde{Q}$.

$$V = z^T Pz + \text{Tr}\left[\tilde{Q}^T \Gamma^{-1} \tilde{Q}\right] \tag{64}$$

where P and Γ are yet to be determined positive definite gain matrices. Differentiating the above with respect to time yields

$$\dot{V} = z^T (PA + A^T P)z + 2z^T Pb + 2\text{Tr}\left(\tilde{Q}^T \Gamma^{-1} \dot{\tilde{Q}}\right) \tag{65}$$

Since $[A]$ is a stable matrix, Lyapunov's stability theorem for linear systems states that for any symmetric positive definite matrix $[R]$, we are guaranteed that there exists a corresponding symmetric, positive definite matrix $[P]$ such that $PA + A^T P = -R$.

Therefore

$$\dot{V} = -z^T Rz + 2z^T Pb + 2\text{Tr}\left(\tilde{Q}^T \Gamma^{-1} \dot{\tilde{Q}}\right) =$$
$$-z^T Rz + 2z^T P_3 \frac{1}{4}B[I]^{-1}\tilde{Q}x + 2\text{Tr}\left(\tilde{Q}^T \Gamma^{-1} \dot{\tilde{Q}}\right) \tag{66}$$

where P_3 is the 9×3 sub-matrix formed out of $P = [P_1 \vdots P_2 \vdots P_3]$. Further we can show that

$$\frac{1}{4}z^T P_3 B[I]^{-1}\tilde{Q}x = \frac{1}{4}\text{Tr}\left(x^T \tilde{Q}^T [I]^{-1}B^T P_3^T z\right)$$
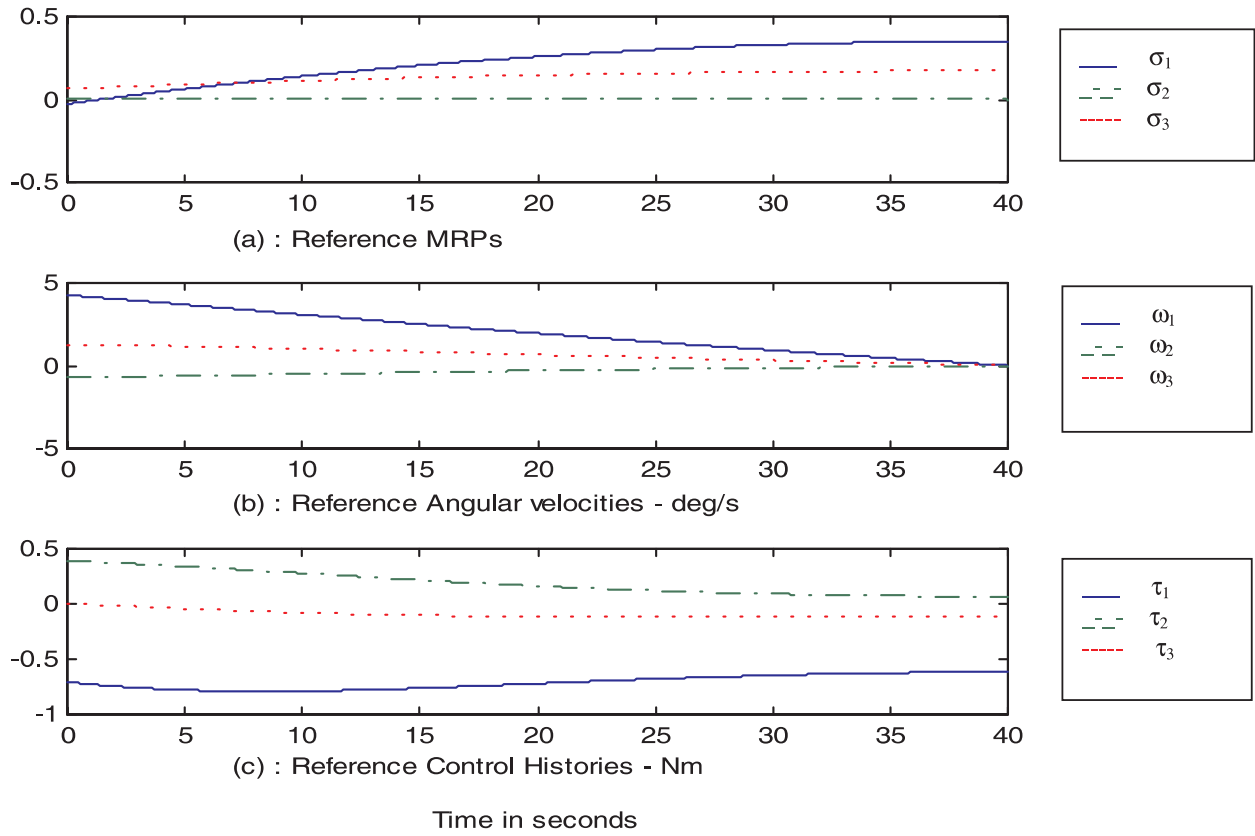$$\frac{1}{4}\text{Tr}\left(\tilde{Q}^T [I]^{-1}B^T P_3^T zx^T\right) \tag{67}$$

Figure 7 : Control of spacecraft tracking maneuvers with large disturbances

Therefore:

$$\dot{V} = -z^T R z + 2\text{Tr}\left(\frac{1}{4}\tilde{Q}^T[I]^{-1}B^T P_3^T z x^T + \tilde{Q}\Gamma^{-1}\dot{\tilde{Q}}\right) \qquad (68)$$

Thus we choose the adaptive update law to be of the form

$$\dot{Q} = \dot{\tilde{Q}} = -\frac{1}{4}\Gamma[I]^{-1}B^T P_3^T z x^T \qquad (69)$$

This choice nulls the second term of Eq. 68 and we obtain:

$$\dot{V} = -z^T R z \leq 0 \qquad (70)$$

Choosing $\Gamma = [I]$, we can eliminate both Γ and the unknown inertia matrix $[I]$ and the adaptive law becomes simply

$$\dot{Q} = -\frac{1}{B}^T P_3^T z x^T \qquad (71)$$

Thus we achieve bounded tracking of all error states and bounded parameter errors using the control law Eq. 57 in conjunction with the adaptation law Eq. 71.

2.5.2 Numerical Simulations

Two different simulations were studied to explore the adequacy of the control law proposed in the previous sections.

Example 3 *Robust Tracking for Spacecraft Attitude Maneuvers*

This example deals with de-tumbling a rigid spacecraft and requiring it to track a smooth prescribed reference trajectory. Another way of looking at this example would be to view it from a 'formation flight task point of view'. Imagine two rigid spacecraft in formation flight, one being the chief and the other a deputy. The commanded reference trajectory for the deputy could be derived from the ideal or desired relative motion and relayed to the deputy by the chief. The deputy now tries to follow the chief's commanded trajectory. The reference maneuver in this case was simply designed to orient a spacecraft at rest from 3-1-3 Euler angles (-20, 15, 4 deg) to the angles (40, 35, 40) with a zero final angular velocity. This illustrative reference trajectory for the MRPs corresponding to the above specified Euler angles was chosen to be cubic splines functions of time. The maneuver is by no means an optimal maneuver satisfying some performance index, it is simply an illustration
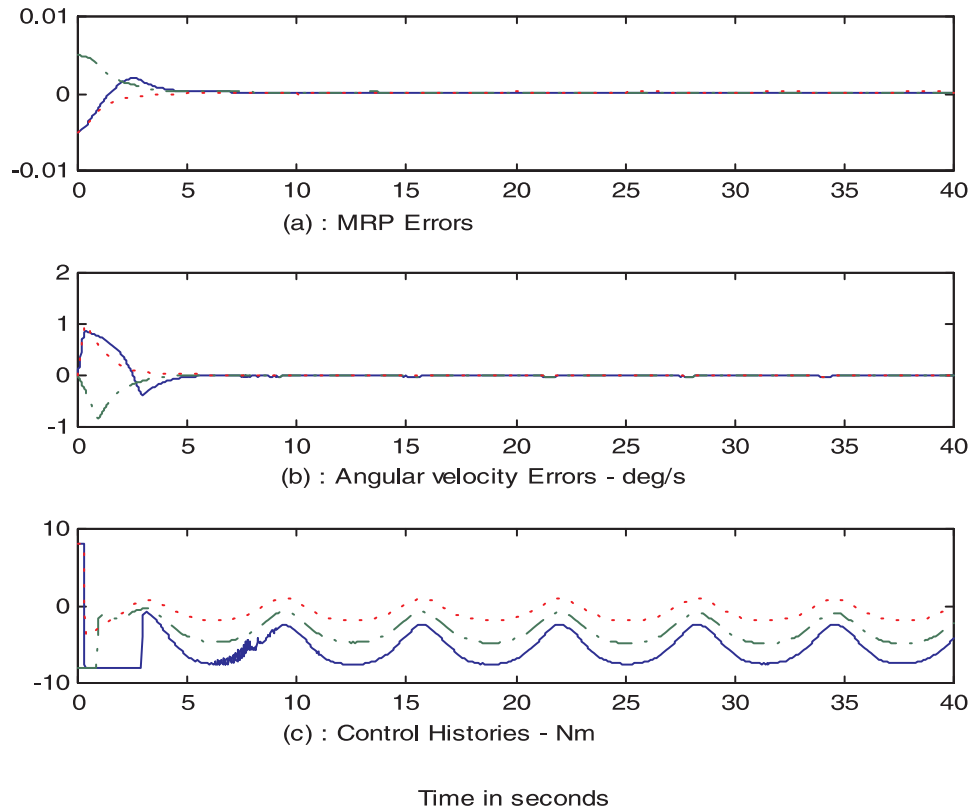
Figure 8 : Control of spacecraft tracking maneuvers with large disturbances

to evaluate the effectiveness of the control law derived in earlier sections. The spacecraft properties are taken from Akella, Junkins, and Robinett (1998).

A large and persistent disturbance of magnitude $d(t) = 0.02 \sin[\log(2 + \cos t)][1, 1, 1]^T \ rad/s^2$ (unknown to the adaptive controller), is applied to the system throughout it's maneuvering phase The control law compensates for all errors and is able to track the simple reference maneuver (Fig. 7(a) (b)) very well, in spite of all these uncertainties and disturbances. The time histories of the MRP and the angular velocity errors can be seen in Fig. 8(a) (b). Note very high control torques are demanded to correct for the errors initially. This is mainly because of the large initial condition errors and the large external disturbances. These control torque demands are not physically achievable on a spacecraft mission. However the saturated control case in Fig. 8(c) shows that the tracking is near perfect after a few seconds even when there is a drastic torque reduction imposed, to restrict the magnitude to an achievable torque bound. Further it is mentioned that the reference maneuver and control gains do not correspond to any optimal strategy and hence the control demands must be viewed in this context. The control histories in Fig. 8(c) shows a periodic oscillatory behavior, required to cancel the external disturbance.

2.6 Constrained Second Order Dynamical Systems

Consider the class of dynamical systems whose behavior is governed by the classical discrete coordinate version[Junkins and Kim (1995)] of Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q \tag{72}$$

where the Lagrangian L is defined in the classical form $L = T - V$ and Q is the generalized force. The kinetic and potential energy functions have the forms $T = T(q, \dot{q}, t), V = V(q)$. A modest generalization using the Lagrange multiplier approach allows Eq. 72 to be applied to a significant class of redundant coordinate, constrained systems. The constraints resulting due to redundant coordinates can be formulated as kinematic non-holonomic constraints of the Pfaffian type and expressed as

$$A\dot{q} + a_0 = 0 \tag{73}$$

For an n coordinate system with m redundancy, Lagrange's equations are modified with the additional constraint forces as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q + A^T \lambda \tag{74}$$

where $A = A(q)$ is an $m \times n$ continuous, differentiable matrix function, $a_0(q)$ is a smooth, $m \times 1$ vector function, and λ is an $m \times 1$ vector of Lagrange multipliers.

For natural multi-body systems, the kinetic energy can be written as a symmetric quadratic form in the generalized velocities

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}(q) \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{q}^T M \dot{q} \tag{75}$$

It is convenient to collect the mass matrix $M = M(q)$ using velocity level kinematics, before the differentiation implied by Lagrange's equations are carried out. The equations of motion then follow[Junkins and Kim (1995)] from Eq. 74 as the following system of second order equations

$$M\ddot{q} + \frac{\partial V}{\partial q} + G = Q + A^T \lambda \tag{76}$$

where $\partial V / \partial q$ is the $n \times 1$ vector gradient of the potential energy function. The $n \times 1$ vector $G = G(q, \dot{q})$ is given as

$$G = \begin{bmatrix} \dot{q} C^1 \dot{q} & \cdots & \dot{q}^T C^N \dot{q} \end{bmatrix}^T \tag{77}$$

$$c^{(i)}_{jk} = \frac{1}{2} \left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{jk}}{\partial q_i} \right), \tag{78}$$

where m_{ik} are the elements of the mass matrix, c^i_{jk} generates an element of the $n \times n$ symmetric matrix $C^i = C^i(q)$ and is known as the *Christoffel operator*. Clearly the C^i and therefore G vanishes identically if the simplest case for which the mass matrix is configuration invariant. For the case wherein the non-conservative forces are generated by a $m \times 1$ vector of control inputs u, we typically have $Q = Bu$ and Eq. 76 becomes

$$M(q)\ddot{q} + \frac{\partial V}{\partial q} + G(q, \dot{q}) = Bu + A(q)^T \lambda \tag{79}$$

To obtain the solution for this redundant coordinate dynamical system it is convenient to differentiate the kinematic constraints in Eq. 73 obtain the constraints at momentum level, given as

$$A\ddot{q} + \dot{A}\dot{q} + \dot{a}_0 = 0 \tag{80}$$

One standard approach for solving $n + m$ unknowns in the vectors $q(t)$ and $\lambda(t)$ is to solve Eqs. 79 and 80 simultaneously[Ahmad and Zribi (1991); Krishnan (1992)]. The resulting generalized constraint forces $A^T \lambda$ and the dynamics of the system can be given as

$$A^T \lambda = F_1 + F_2 u \tag{81}$$

$$M\ddot{q} + \frac{\partial V}{\partial q} + G' = B' u \tag{82}$$

where $F_1 = A^T (AM^{-1}A^T)^{-1} \left[AM^{-1} (G + \partial V / \partial q) - (\dot{A}\dot{q} + \dot{a}_0) \right]$, $F_2 = -A^T (AM^{-1}A^T)^{-1} AM^{-1}B$, and $G' = G - F_1$, $B' = B - F_2$.

Note that, though Eq. 82 represents the n coupled differential equations governing the dynamics of the system, not all of them are independent. Also note the presence of the inverse of mass matrix in the differential equation governing the dynamics of the system.

Another way of approaching the solution of this system is to project the dynamics of the system onto the null space of the constraints or null space of matrix A. Let us first componentiate the n-dimensional vector space into two orthogonal spaces defined by range space of $S_1 \in R^{n \times m}$, and $S_2 \in R^{n \times (n-m)}$. Further, S_1 is chosen such that $(AS_1)^{-1}$ exists. Let the acceleration vector $\ddot{q}(t)$ be given as

$$\ddot{q}(t) = S_1 \xi(t) + S_2 \eta(t) \tag{83}$$

Then from Eq. 80, we get

$$\xi(t) = -(AS_1)^{-1} (AS_2 \eta + \dot{A}\dot{q}). \tag{84}$$

If we define $N = \text{Null}(A)$, then we have $AN = 0$. Using Eqs. 76 and 84, the projected dynamics of the system on null space of constraints is given as

$$N^T M (I - S_1 (AS_1)^{-1} A) S_2 \eta - N^T M S_1 (AS_1)^{-1} \dot{A}\dot{q}$$
$$= N^T \left(Bu - G - \frac{\partial V}{\partial q} \right) \tag{85}$$

Let coordinates q be arranged such that matrix A is defined as

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \tag{86}$$

and $A_1 \in R^{m \times m}$ is full rank. It can be easily verified that the null space of matrix of A can be written

$$N = \begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix} \tag{87}$$

Let us also choose the matrix S_1 and S_2 as

$$S_1 = \begin{bmatrix} I_{m \times m} \\ 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 \\ I_{(n-m) \times (n-m)} \end{bmatrix}. \tag{88}$$

Hence if $\ddot{q}(t)$ is partitioned as $\ddot{q}(t) = \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix}$, then $\ddot{q}_1(t) = \xi(t)$ and $\ddot{q}_2(t) = \eta(t)$. Using Eqs. 86, 87 and 88, the Eq. 85 can be simplified to give the dynamics as

$$N^T M N \ddot{q}_2 = N^T \left(M \begin{bmatrix} A_1^{-1} \dot{A} \\ 0 \end{bmatrix} \dot{q} + Bu - G - \frac{\partial V}{\partial q} \right). \tag{89}$$

In many systems, the mass matrix M can be block partitioned as $M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$.

Substituting the mass matrix M and the matrix N from Eq. 87 and defining $H_1 = A_1^{-1} A_2$, $H_2 = A_1^{-1} \dot{A}$ and $H_3 = \begin{bmatrix} -H_1^T & I_{(n-m) \times (n-m)} \end{bmatrix}$, Eq. 89 is further simplified as

$$(\tilde{M}_1 + M_2) \ddot{q}_2 + \tilde{G} = H_3 Bu \tag{90}$$

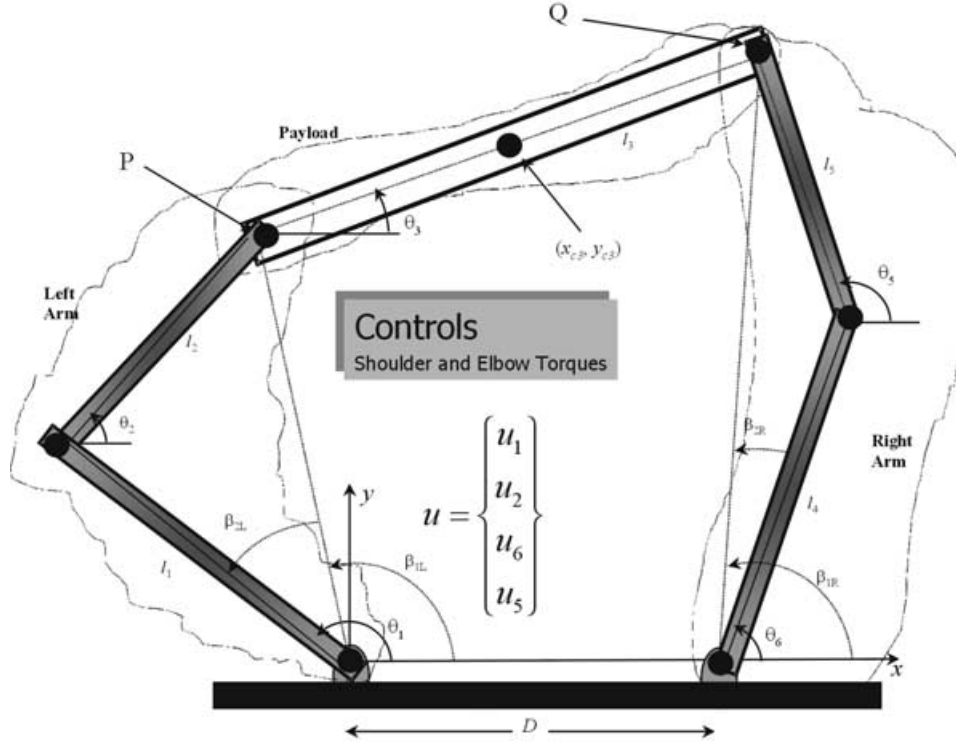where $\tilde{M}_1 = H_1^T M_1 H_1$, $\tilde{G} = H_1^T M_1 H_2 \dot{q} + H_3 G$.

Figure 9 : Dual robot cooperative manipulation example

2.6.1 Dynamics of a Dual Robot System

Consider the pair of robot arms moving a payload[Junkins and Kim (1995); Sanyal, Verma, and Junkins (2000)] as shown in Fig. 9. For simplicity, we assume that there are four active joints, namely, the shoulder and elbow joints of the left and right robot arms. The wrist joints are considered free. We assume the manipulator to be composed of rigid links, the payload to be a rigid body, and the entire system to undergo planar motion. We also assume that there are no conservative forces and hence, neglect gravitational potential energy (V) in this example. In this example, the configuration coordinate vector naturally partitions into left (L), right (R), and payload (P) configuration coordinates as:

$$q = \left\{ \begin{array}{c} q_L \\ q_R \\ q_P \end{array} \right\} = \begin{bmatrix} \theta_1 & \theta_2 & \vdots & \theta_3 & \theta_4 & \vdots & \theta_5 & x_{c3} & y_{c3} \end{bmatrix}^T$$

The 7×7 system mass matrix has the block diagonal structure[Junkins and Kim (1995); Sanyal, Verma, and Junkins (2000)] given as $M(q) = \text{diag}[M_L, \quad M_R, \quad M_p]$. Introducing the elbow angles $\theta_{ij} = \theta_j - \theta_i$ the partitioned mass matrices

can be compactly written as:

$$M_L = \begin{bmatrix} I_1 + \frac{1}{4}m_1 l_1^2 + m_2 l_1^2 & \frac{1}{4}m_2 l_1 l_2 \cos\theta_{12} \\ \frac{1}{4}m_2 l_1 l_2 \cos\theta_{12} & I_2 + \frac{1}{4}m_2 l_2^2 \end{bmatrix}, \tag{91}$$

$$M_R = \begin{bmatrix} I_4 + \frac{1}{4}m_5 l_5^2 + m_4 l_4^2 & \frac{1}{4}m_4 l_5 l_4 \cos\theta_{65} \\ \frac{1}{4}m_4 l_5 l_4 \cos\theta_{65} & I_4 + \frac{1}{4}m_4 l_4^2 \end{bmatrix}, \tag{92}$$

$$M_p = \text{diag}[I_3, \quad m_3, \quad m_3]. \tag{93}$$

The nonlinear vector $G(q, \dot{q})$ has the form:

$$G(q, \dot{q}) = \left\{ \begin{array}{c} G_L \\ G_R \\ 0 \end{array} \right\}, \tag{94}$$

$$G_L = \frac{1}{2} \left\{ \begin{array}{c} -m_2 \dot{\theta}_2 l_1 l_2 \sin\theta_{12} \\ -m_2 \dot{\theta}_1 l_1 l_2 \sin\theta_{12} \end{array} \right\}, \tag{95}$$

$$G_R = \frac{1}{2} \left\{ \begin{array}{c} -m_4 \dot{\theta}_5 l_4 l_5 \sin\theta_{65} \\ -m_4 \dot{\theta}_6 l_4 l_5 \sin\theta_{65} \end{array} \right\} \tag{96}$$

The control vector containing the four shoulder and elbow torques is

$$u = \begin{bmatrix} u_1 & u_2 & u_6 & u_5 \end{bmatrix}^T \tag{97}$$

and, we can readily establish that the control influence matrices are

$$B = \begin{bmatrix} B_L & 0 \\ 0 & B_R \\ 0 & 0 \end{bmatrix}, \quad B_L = B_R = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad (98)$$

Taking the origin for an inertial coordinate system (x, y) at the shoulder joint of the left arm, the geometric constraints arising out of the fixing of left and right robot wrists to the payload at points Q and P are:

$$l_1 \cos\theta_1 + l_2 \cos\theta_2 + \frac{1}{2}l_3 \cos\theta_3 - x_{c3} = 0 \quad (99)$$

$$l_1 \sin\theta_1 + l_2 \sin\theta_2 + \frac{1}{2}l_3 \sin\theta_3 - y_{c3} = 0 \quad (100)$$

$$l_5 \cos\theta_6 + l_4 \cos\theta_5 - \frac{1}{2}l_3 \cos\theta_3 - x_{c3} + D = 0 \quad (101)$$

$$l_5 \sin\theta_6 + l_4 \sin\theta_5 - \frac{1}{2}l_3 \sin\theta_3 - y_{c3} = 0 \quad (102)$$

The above four constraints effectively reduce the seven degrees of freedom of the system to only three degrees of freedom. On differentiating Eqs. 99-102 with respect to time, yields a kinematic constraint of the Pfaffian form like Eq. 73 with $a_0 = 0$ and with

$$A(q) = \begin{bmatrix} A_L & 0 & A_{LP} \\ 0 & A_R & A_{RP} \end{bmatrix}, \quad (103)$$

where

$$A_L = \begin{bmatrix} -l_1 \sin\theta_1 & -l_2 \sin\theta_2 \\ l_1 \cos\theta_1 & l_2 \cos\theta_2 \end{bmatrix}, \quad (104)$$

$$A_R = \begin{bmatrix} -l_5 \sin\theta_6 & -l_4 \sin\theta_5 \\ l_5 \cos\theta_6 & l_4 \cos\theta_5 \end{bmatrix}, \quad (105)$$

$$A_{LP} = \begin{bmatrix} -\frac{1}{2}l_3 \sin\theta_3 & -1 & 0 \\ \frac{1}{2}l_3 \cos\theta_3 & 0 & -1 \end{bmatrix}, \quad (106)$$

$$A_{RP} = \begin{bmatrix} \frac{1}{2}l_3 \sin\theta_3 & -1 & 0 \\ -\frac{1}{2}l_3 \cos\theta_3 & 0 & -1 \end{bmatrix}. \quad (107)$$

To represent the dynamics of the system as described by Eq. 90, we need to identify the following variables:

$$M_1(q) = \begin{bmatrix} M_L(q) & \\ & M_R(q) \end{bmatrix}_{4 \times 4}, \quad (108)$$

$$M_2 = [M_P]_{3 \times 3} = \mathrm{diag} \begin{bmatrix} I_3, & m_3, & m_3 \end{bmatrix} \quad (109)$$

$$A_1(q) = \begin{bmatrix} A_L(q) & \\ & A_R(q) \end{bmatrix}_{4 \times 4}, \quad (110)$$

$$A_2(q) = \begin{bmatrix} A_{LP}(q) \\ A_{RP}(q) \end{bmatrix}_{4 \times 3}. \quad (111)$$

2.6.2 Ideal Model Inversion

Assume that we are given a reference trajectory q_r, which is twice differentiable. This implies that at each point of time we have all the reference position, velocity and acceleration coordinates $(q_r, \dot{q}_r, \ddot{q}_r)$ available. Defining error as $e = q_2 - q_{2,r}$, the error dynamics can be written as

$$[\tilde{M}_1(q) + M_2]\ddot{e} = -\tilde{G}(q, \dot{q}) + H_3(q)Bu + [\tilde{M}_1(q) + M_2]\ddot{q}_{2,r} \quad (112)$$

We define the model error dynamics as

$$[\tilde{M}_1(q) + M_2]\ddot{e} + \left[C + \frac{1}{2}\dot{\tilde{M}}_1(q)\right]\dot{e} + Ke = 0, \quad (113)$$

where matrices C and K are positive definite. It can be seen that if we chose Lyapunov function as $V = \frac{1}{2}\dot{e}^T [\tilde{M}_1(q) + M_2]\dot{e} + \frac{1}{2}e^T Ke$, the time derivative of Lyapunov function is

$$\dot{V} = \dot{e}^T [\tilde{M}_1(q) + M_2]\ddot{e} + \frac{1}{2}\dot{e}\dot{\tilde{M}}_1(q)\dot{e} + e^T K\dot{e} \quad (114)$$

or $\dot{V} = -\dot{e}^T C\dot{e} \leq 0$.

By taking higher time derivatives of V, it can be easily shown that the dynamics of Eq. 113 are asymptotically[Mukherjee and Chen (1992, 1993)]. The ideal inversion controller for achieving the model error dynamics of Eq. 113, results in the following non-linear feedback control law:

$$u = (H_3 B)^\dagger (M_2 \ddot{q}_{2,r} - \psi) \quad (115)$$

where

$$\psi = -\tilde{G} - \tilde{M}_1 \ddot{q}_{2,r} + \left(C + \frac{1}{2}\dot{\tilde{M}}_1\right)\dot{e} + Ke \quad (116)$$

and $()^\dagger$ denote the pseudo inverse of the quantity in parentheses. As seen in earlier examples, for perfect tracking of a trajectory using inverse dynamics control law, the rank of the control influence matrix should at least be equal to the degree of freedom of the system. When the system is over-actuated, the pseudo inverse ensures the minimum norm solution for the control vector. Due to uncertainties in the payload parameters, the payload mass matrix M_2 is poorly known and hence the controller in Eq. 115 can not be implemented exactly. We also assume that the control influence matrix B is uncertain. As is the case in this example, it is assumed that both uncertain matrices M_2 and B are constant. In the next section we develop the structured adaptive model inversion control law algorithm to provide robustness in the presence of these uncertainties.

2.6.3 Cooperative Adaptive Control of the Dual Robot System

Let $\hat{M}_2$ and $\hat{\beta}$ be the time varying estimates of the respective true (but unknown) constant matrices M_2 and B, where the estimates are obtained in real time from some adaptation law. Then, the following relation gives the control law that's implemented

$$u = \left(H_3 \hat{\beta}\right)^\dagger (\hat{M}_2 \ddot{q}_{2,r} - \psi) \quad (117)$$

Re-arranging the above yields the control identity

$$H_3 \hat{B} u - \hat{M}_2 \ddot{q}_{2,r} + \psi = 0. \tag{118}$$

Note that the true error dynamics of Eq. 112 can be re written as

$$\left[\tilde{M}_1(q) + M_2 \right] \ddot{e} + \left[c + \frac{1}{2} \dot{\tilde{M}}_1(q) \right] \dot{e} + Ke$$
$$= H_3(q) Bu - M_2 \ddot{q}_{2,r} + \psi \tag{119}$$

From Eq. 119 and Eq. 118 we obtain the error dynamics in the form

$$\left[\tilde{M}_1(q) + M_2 \right] \ddot{e} + \left[C + \frac{1}{2} \dot{\tilde{M}}_1(q) \right] \dot{e} + Ke$$
$$= H_3(q) \Delta B u - \Delta M_2 \ddot{q}_{2,r}, \tag{120}$$

where, $\Delta M_2 = M_2 - \hat{M}_2$ and $\Delta B = B - \hat{B}$. Now we define a Lyapunov function as

$$V = \frac{1}{2} \dot{e}^T \left[\tilde{M}_1(q) + M_2 \right] \dot{e} + \frac{1}{2} e^T K e +$$
$$\frac{1}{2} \mathrm{Tr} \left(\Delta M_2^T \Gamma_m^{-1} \Delta M_2 + \Delta B^T \Gamma_b^{-1} \Delta B \right), \tag{121}$$

where Γ_m and Γ_b are user-defined positive definite symmetric adaptation gain matrices. The time derivative of Lyapunov function is obtained as

$$\dot{V} = -\dot{e}^T C \dot{e} + \mathrm{Tr} \left[\Delta M_2^T \left(\Gamma_m^{-1} \Delta \dot{M}_2 - \dot{e} \ddot{q}_{2,r}^T \right) + \right.$$
$$\left. \Delta B^T \left(\Gamma_b^{-1} \Delta \dot{B} + H_3^T \dot{e} u^T \right) \right] \tag{122}$$

The adaptive laws for $\hat{M}_2$ are $\hat{B}$ obtained from the above equation so as to force the terms in square bracket to go to zero. The resulting adaptive laws are:

$$\dot{M}_2 = -\Gamma_m \dot{e} \ddot{q}_{2,r}^T, \quad \dot{B} = \Gamma_b H_3^T \dot{e} u^T \tag{123}$$

The resulting time derivative of the Lyapunov function is

$$\dot{V} = -\dot{e}^T C \dot{e} \tag{124}$$

Since $\dot{V}$ is semi negative definite, it ensures that the trajectory tracking dynamics is stable and all error states are bounded. However, this does not ensure that the error dynamics is asymptotically stable. The proof for asymptotic stability requires similar arguments as those described in the above Lemma. Eq. 117 along with Eq. 123 represents the complete structure of the control law. The design parameters for this controller are the matrices C, K, which define the model error dynamics as in Eq. 113, the adaptive gain matrices Γ_m, Γ_b. The payload matrix $\hat{M}_2(0)$ and control influence matrix $\hat{B}(0)$ has to be initialized with the best possible estimates available. For the numerical simulation, first a smooth reference trajectory was generated using the approach as outlined in Junkins
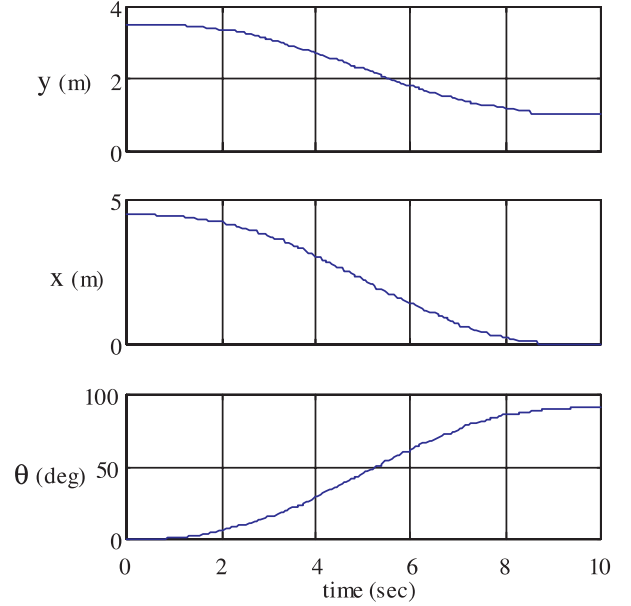

Figure 10 : Reference trajectory for the payload

and Kim (1995); Verma and Junkins (1999). The initial and final boundary conditions for the payload trajectory are chosen as [0, 4.5, 3.5] and [$\pi/2$, 0, 1] respectively, where the first coordinate is the angular displacement in radians and the other two coordinates are the (x, y) location of the payload. Fig. 10 shows the reference trajectory of the payload for a rest to rest maneuver.

Each arm length for the robot is chose as 5 m and the separation between the two shoulders is chosen as 6 m. The true payload was chosen as having a mass of 8 kg and inertia of 10 $kg - m^2$. Large uncertainty was assumed in the payload. The initial uncertain estimate of the payload is chosen to have a mass of 3 kg (62% error) and inertia of 5 $kg - m^2$ (50% error). An initial error was introduced in position as well as velocity coordinates. Fig. 11 shows the response of error states for ideal case (when no uncertainty present), and the cases with uncertainty including both, the adaptive and non-adaptive cases. The figure shows the advantage and the effectiveness of adaptive controls by suppressing the errors more effectively in the trajectory states. We note that small e_x, e_y offsets are evident in Fig. 11. Of course, these can be eliminated by including integral feedback in Eq. 113, analogous to Eq. 45.

3 Conclusions

We presented through various examples from aerospace and robotic systems, an elegant methodology to derive control laws, based on Structured Adaptive Model Inversion. The highlight of this exercise is the ability to enforce the exact kinematic relationships on the system under considera-
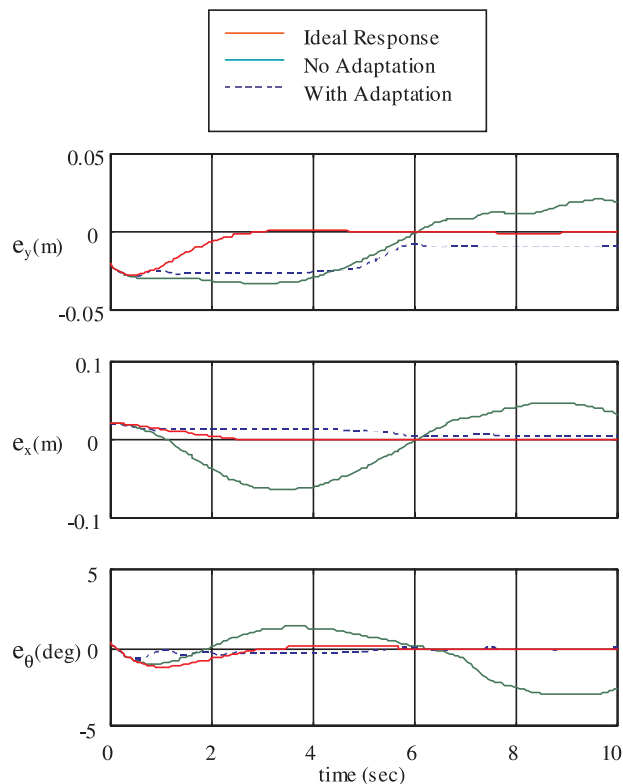
Figure 11 : Comparative payload trajectory error plots for ideal case, uncertain case with and without adaptation

tion and be able to derive a robust trajectory-tracking controller. The controller is shown to be robust under a variety of structured and unstructured perturbations like external disturbances, modeling errors, initial condition errors and parametric uncertainties with and without actuator saturation. The trajectory tracking in all cases is admirable and the controls are within acceptable limits.

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