# Boundary Element Stress Analysis of Thick Reissner Plates in Bending under Generalized Loading 

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#### Abstract

In a recent publication, the author has introduced boundary integral equations for thick plate bending problems, for cases with generalized types of loading. Internal bending moments and shear forces, required for stress analysis, were calculated by means of a finite difference procedure, which requires fine boundary element meshes to achieve an acceptable degree of accuracy. In this paper, boundary integral equations for internal bending moments and shear forces are presented for thick Reissner plates in bending. Domain loading terms in those boundary integral equations have also been simplified for a variety of loading types including concentrated loading, linearly-distributed loading, and line bending moments and shear forces acting on arbitrary curves defined on the plate surface. A number of case studies, with different loading and boundary conditions, have been analysed and boundary element results have been compared with corresponding analytical solutions. It is clear that the boundary integral equations, presented in this work, for internal bending moments and shear forces, have led to very accurate results for plate bending problems with generalized types of loading.


## 1 Introduction

The boundary element analysis of thick Reissner plates in bending was first introduced in 1982 by Weeën (1982). Since then, several papers [Antes (1984); Karam and Telles (1988); Long, Brebbia, and Telles (1988); ElZafrany, Debbih, and Fadhil (1995); El-Zafrany, Fadhil, and Debbih (1995); El-Zafrany and Fadhil (1997); El-Zafrany (1998a)] have appeared in the literature, with some interesting developments on Weeën's original work. In a recent publication [El-Zafrany (1998b)], the author has introduced boundary integral equations for thick plate bending problems, for cases with generalized types of loading. Internal bending moments and shear
forces, required for stress analysis, were calculated by means of a finite difference procedure, which requires fine boundary element meshes to achieve an acceptable degree of accuracy.
In this paper, boundary integral equations for internal bending moments and shear forces are presented for thick Reissner plates in bending. Domain loading terms in those boundary integral equations have been reduced to single terms for cases with concentrated shear forces and bending moments, and they have also been reduced to boundary integrals for cases with linearly-distributed loading, and line bending moments and shear forces acting on arbitrary curves defined on the plate surface. The new derivations have been implemented in a computer program for the analysis of thick Reissner plates in bending, and several case studies were analysed.

## 2 Review of governing equations

For a plate of uniform thickness $h$, the equilibrium equations, over the plate thickness, derived according to Reissner's [Reissner (1945)] theory, can be written in the following form:
$M_{x x, x}+M_{x y, y}-Q_{x}=0$
$M_{y x, x}+M_{y y, y}-Q_{y}=0$
$Q_{x, x}+Q_{y, y}+q=0$
where bending moments per unit length ( $M_{x x}, M_{y y}, M_{x y}$ ) and shear forces per unit length $\left(Q_{x}, Q_{y}\right)$ are defined in terms of the lateral deflection $w$ and the average slope angles $\left(\theta_{x}, \theta_{y}\right)$ by means of the following generalized equations:

$$
\begin{align*}
M_{\alpha \beta} & =\frac{1}{2}(1-v) D\left(\theta_{\alpha, \beta}+\theta_{\beta, \alpha}\right)+\delta_{\alpha, \beta} \\
& \quad\left[D v\left(\theta_{1,1}+\theta_{2,2}+\zeta q\right]\right.  \tag{4}\\
Q_{\beta} & =\frac{1}{2}(1-v) D \lambda^{2}\left(\theta_{\beta}+w_{, \beta}\right) \tag{5}
\end{align*}
$$

where $D$ is the flexural rigidity of the plate, $v$ is its Poisson's ratio, $\lambda^{2}=10 / h^{2}, q$ is the domain loading intensity defined as shear force per unit area of the plate surface, and
$\zeta=\frac{v}{(1-v) \lambda^{2}}, \quad f_{\beta, \alpha} \equiv \frac{\partial f_{\beta}}{\partial x_{\alpha}}, \quad \alpha=1,2, \beta=1,2$,
with $\left(x_{1}, x_{2}\right) \equiv(x, y),()_{1} \equiv()_{x},()_{2} \equiv()_{y}$, etc.
Using a weighted-residual approach, and Cauchy's pricipal value theorem, the boundary integral equations for thick plates can be obtained as follows [El-Zafrany and Fadhil (1997)]:

$$
\begin{align*}
& c_{11} \theta_{x}\left(x_{i}, y_{i}\right)+c_{12} \theta_{y}\left(x_{i}, y_{i}\right)+c_{13} w\left(x_{i}, y_{i}\right) \\
& \quad+\oint_{\Gamma}\left(T_{11} \theta_{n}+T_{21} \theta_{t}+T_{31} w\right) d \Gamma  \tag{13}\\
& \quad=\oint_{\Gamma}\left(U_{11} M_{n}+U_{21} M_{n t}+U_{31} Q_{n}\right) d \Gamma+\iint_{\Omega} L_{1} q d \Gamma  \tag{6}\\
& c_{21} \theta_{x}\left(x_{i}, y_{i}\right)+c_{22} \theta_{y}\left(x_{i}, y_{i}\right)+c_{23} w\left(x_{i}, y_{i}\right) \\
& \quad+\oint_{\Gamma}\left(T_{12} \theta_{n}+T_{22} \theta_{t}+T_{32} w\right) d \Gamma  \tag{14}\\
& \quad=\oint_{\Gamma}\left(U_{12} M_{n}+U_{22} M_{n t}+U_{32} Q_{n}\right) d \Gamma+\iint_{\Omega} L_{2} q d \Gamma  \tag{7}\\
& c_{31} \theta_{x}\left(x_{i}, y_{i}\right)+c_{32} \theta_{y}\left(x_{i}, y_{i}\right)+c_{33} w\left(x_{i}, y_{i}\right)  \tag{15}\\
& \quad+\oint_{\Gamma}\left(T_{13} \theta_{n}+T_{23} \theta_{t}+T_{33} w\right) d \Gamma \\
& \quad=\oint_{\Gamma}\left(U_{13} M_{n}+U_{23} M_{n t}+U_{33} Q_{n}\right) d \Gamma+\iint_{\Omega} L_{3} q d \Gamma
\end{align*}
$$

$$
\begin{aligned}
& B_{\alpha \beta j}=\frac{1}{2} D(1-v)\left(U_{j \alpha, \beta}+U_{j \beta, \alpha}\right)+ \\
& \quad \delta_{\alpha \beta} D v\left(U_{j 1,1}+U_{j 2,2}\right) \\
& p_{\alpha \beta}=\frac{1}{2} D(1-v)\left(L_{\alpha, \beta}+L_{\beta, \alpha}\right)+ \\
& \quad \delta_{\alpha \beta} D v\left(L_{1,1}+L_{2,2}\right)
\end{aligned}
$$

where $j=1,2,3$.
Similarly by substituting from Eqs. 9-11 into Eq. 5, the boundary integral equations of shear forces, per unit length, at an internal source point $\left(x_{i}, y_{i}\right)$ can be expressed as follows:

$$
\begin{align*}
& Q_{\beta}\left(x_{i}, y_{i}\right)=\oint_{\Gamma}\left(\phi_{\beta 1} \theta_{n}+\phi_{\beta 2} \theta_{t}+\phi_{\beta 3} w\right) d \Gamma-\iint_{\Omega} \Lambda_{\beta} q d \Gamma \\
& \quad-\oint_{\Gamma}\left(\psi_{\beta 1} M_{n}+\psi_{\beta 2} M_{n t}+\psi_{\beta 3} Q_{n}\right) d \Gamma \tag{16}
\end{align*}
$$

where
$\phi_{\beta j}=\frac{1}{2} D \lambda^{2}(1-v)\left(T_{j 3, \beta}-T_{j \beta}\right)$
$\psi_{\beta j}=\frac{1}{2} D \lambda^{2}(1-v)\left(U_{j 3, \beta}-U_{j \beta}\right)$
$\Lambda_{\beta}=\frac{1}{2} D \lambda^{2}(1-v)\left(L_{3, \beta}-L_{\beta}\right)$
Using the expressions of kernel functions of displacement boundary integral equations given in Appendix A, explicit expressions for the moment and shear kernel functions can be obtained as listed in Appendix B.

## 4 Analysis of loading domain integrals

Using the approach presented in El-Zafrany (1998b), the loading domain integrals in different boundary integral equations presented in this work can be simplified and reduced for different types of loading as will be discussed in this section.

### 4.1 Case with concentrated loading

Consider a case where a concentrated shear force $F$ and bending moments $T_{x}$, and $T_{y}$ are acting at a point $\left(x_{l}, y_{l}\right)$, in the $z, x$, and $y$ directions, respectively. Using the properties of the Dirac delta function [El-Zafrany (1998b)], the corresponding domain loading terms in Eqs. 6-11 can be reduced as follows:

$$
\begin{align*}
& \iint_{\Omega} L_{j} q d x d y \\
& \quad \equiv \iint_{\Omega} L_{j}\left(-T_{x} \frac{\partial}{\partial y}+T_{y} \frac{\partial}{\partial x}+F\right) \delta\left(x-x_{l}, y-y_{l}\right) d x d y \\
& \quad=\left(T_{x} \frac{\partial}{\partial y}-T_{y} \frac{\partial}{\partial x}+F\right) L_{j} \quad \text { at } x=x_{l}, y=y_{l} \\
& \quad \equiv-Q_{j 1} T_{y}+Q_{j 2} T_{x}+Q_{j 3} F \tag{20}
\end{align*}
$$

where $Q_{j \beta}=\frac{\partial}{\partial x_{\beta}} L_{j}, Q_{j 3}=L_{j}$ at $x=x_{l}, y=y_{l}$. Hence, the domain loading term in Eq. 12 can be simplified as follows:

$$
\begin{equation*}
\iint_{\Omega} p_{\alpha \beta} q d x d y \equiv-C_{\alpha \beta 1} T_{y}+C_{\alpha \beta 2} T_{x}+C_{\alpha \beta 3} F \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{\alpha \beta j}=\frac{1}{2} D(1-v)\left(Q_{\alpha j, \beta} Q_{\alpha j, \beta}\right)+\delta_{\alpha \beta} D v\left(Q_{1 j, 1}+Q_{2 j, 2}\right) \tag{22}
\end{equation*}
$$

which can be defined explicitly with respect to the source point $\left(x_{i}, y_{i}\right)$ by means of the following equations:

$$
\begin{align*}
& C_{\alpha \beta \gamma}=\frac{1-v}{4 \pi r_{l}}\left\{2 \frac{\partial r_{l}}{\partial x_{\alpha}} \frac{\partial r_{l}}{\partial x_{\beta}} \frac{\partial r_{l}}{\partial x_{\gamma}}+\left[1-\frac{2 \alpha_{1}}{r_{l}^{2}}\right]\left[\frac{\partial r_{l}}{\partial x_{\alpha}} \delta_{\beta \gamma}+\right.\right. \\
& \left.\left.\quad \frac{\partial r_{l}}{\partial x_{\beta}} \delta_{\gamma \alpha}+\frac{\partial r_{l}}{\partial x_{\gamma}} \delta_{\alpha \beta}-4 \frac{\partial r_{l}}{\partial x_{\alpha}} \frac{\partial r_{l}}{\partial x_{\beta}} \frac{\partial r_{l}}{\partial x_{\gamma}}\right]\right\}+\frac{\delta_{\alpha \beta} \nu}{2 \pi r_{l}} \frac{\partial r_{l}}{\partial x_{\gamma}}  \tag{23}\\
& C_{\alpha \beta 3}=\frac{1}{8 \pi}\left\{2(1+v) \delta_{\alpha \beta} \log z_{l}-(1-v)\left[1-\frac{2 \alpha_{1}}{r_{1}^{2}}\right]\right. \\
& \left.\quad\left[\delta_{\alpha \beta}-2 \frac{\partial r_{l}}{\partial x_{\alpha}} \frac{\partial r_{l}}{\partial x_{\beta}}\right]\right\} \tag{24}
\end{align*}
$$

where $\alpha=1,2, \beta=1,2, \gamma=1,2$,
$r_{l}=\sqrt{\left(x_{l}-x_{i}\right)^{2}+\left(y_{l}-y_{i}\right)^{2}}$
$\frac{\partial r_{l}}{\partial x}=\frac{x_{l}-x_{i}}{r_{l}}, \quad \frac{\partial r_{l}}{\partial y}=\frac{y_{l}-y_{i}}{r_{l}}$,
$z_{l}=\lambda r_{l}$, and $\alpha_{1}$ is as defined in Appendix A.
Similarly, the domain loading term in Eq. 16 can be reduced as follows:
$\iint_{\Omega} \Lambda_{\beta} q d x d y \equiv-C_{\beta 1}^{s} T_{y}+C_{\beta 2}^{s} T_{x}+C_{\beta 3}^{s} F$
where
$C_{\beta j}^{s}=\frac{1}{2} D(1-v) \lambda^{2}\left(Q_{3 j, \beta}-Q_{\beta j}\right)$
or explicitly:
$C_{\beta \alpha}^{s}=-\frac{1}{2 \pi r_{l}^{2}}\left[\delta_{\alpha \beta}-2 \frac{\partial r_{l}}{\partial x_{\alpha}} \frac{\partial r_{l}}{\partial x_{\beta}}\right]$
$C_{\beta 3}^{s}=-\frac{1}{2 \pi r_{l}} \frac{\partial r_{l}}{\partial x_{\beta}}$

### 4.2 Case of line loading on an arbitrary curve

Consider a loaded curve $\Gamma_{l}$, inside the domain $\Omega$ of the plate midplane, with line loadings $Q, M_{n}, M_{t}$, which represent shear force, and normal and tangential bending moments per unit length along $\Gamma_{l}$. Using an approach similar to that given in El-Zafrany (1998b), it can be shown that:

$$
\begin{align*}
& \iint_{\Omega} L_{j} q d x d y=\int_{\Gamma_{l}}\left(M_{n} \frac{\partial}{\partial t}-M_{t} \frac{\partial}{\partial n}+Q\right) L_{j} d \Gamma \\
& \quad \equiv \int_{\Gamma_{l}}\left(-q_{j 1} M_{t}+q_{j 2} M_{n}+q_{j 3} Q\right) d \Gamma \tag{31}
\end{align*}
$$

where $q_{j 3}=L_{j}, q_{j \beta}=\frac{\partial}{\partial n_{\beta}} L_{j}$
Hence it can be reduced for this case of loading that:
$\iint_{\Omega} p_{\alpha \beta} q d x d y \equiv \int_{\Gamma_{l}}\left(-\chi_{\alpha \beta 1} M_{t}+\chi_{\alpha \beta 2} M_{n}+\chi_{\alpha \beta 3} Q\right) d \Gamma$
where
$\chi_{\alpha \beta j}=\frac{1}{2} D(1-v)\left(q_{\alpha j, \beta}+q_{\beta j, \alpha}\right)+\delta_{\alpha \beta} D v\left(q_{1 j, 1}+q_{2 j, 2}\right)$
which can be written explicitly as follows:
$\chi_{\alpha \beta \gamma}=\frac{1-v}{4 \pi r}\left\{2 \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}} \frac{\partial r}{\partial n_{\gamma}}+\left[1-\frac{2 \alpha_{1}}{r^{2}}\right]\left[\frac{\partial r}{\partial x_{\alpha}}\left(\hat{n}_{\gamma} \cdot \hat{i}_{\beta}\right)+\right.\right.$
$\left.\left.\frac{\partial r}{\partial x_{\beta}}\left(\hat{n}_{\gamma} \cdot \hat{i}_{\alpha}\right)+\frac{\partial r}{\partial n_{\gamma}} \delta_{\alpha \beta}-4 \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}} \frac{\partial r}{\partial n_{\gamma}}\right]\right\}$
$+\frac{\delta_{\alpha \beta} \vee}{2 \pi r} \frac{\partial r}{\partial n_{\gamma}}$
$\chi_{\alpha \beta 3} \equiv \frac{1}{8 \pi}\left\{2(1+v) \delta_{\alpha \beta} \log z-(1-v)\left[1-\frac{2 \alpha_{1}}{r^{2}}\right]\right.$
$\left.\left[\delta_{\alpha \beta}-2 \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}}\right]\right\}$
with $z$ and $r$ and its derivatives as defined in Appendix A. Similarly, it can be deduced that:

$$
\begin{equation*}
\iint_{\Omega} \Lambda_{\beta} q d x d y \equiv \int_{\Gamma_{l}}\left(-\chi_{\beta 1}^{s} M_{t}+\chi_{\beta 2}^{s} M_{n}+\chi_{\beta 3}^{s} Q\right) d \Gamma \tag{36}
\end{equation*}
$$

where $\chi_{\beta j}^{s}=\frac{1}{2} D \lambda^{2}(1-v)\left(q_{3 j, \beta}-q_{\beta, j}\right)$ or explicitly
$\chi_{\beta \alpha}^{s}=-\frac{1}{2 \pi r^{2}}\left[\left(\hat{n}_{\alpha} \cdot \hat{i}_{\beta}\right)-2 \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}}\right]$
$\chi_{\beta 3}^{s} \equiv-\frac{1}{2 \pi r} \frac{\partial r}{\partial x_{\beta}}$

### 4.3 Case with distributed loading

From the definitions of loading kernels [El-Zafrany, Debbih, and Fadhil (1995)], it can be shown that:
$L_{\alpha}=\left[1+\frac{v}{(1-v) \lambda^{2}} \nabla^{2}\right] \frac{\partial \phi^{*}}{\partial x_{\alpha}}$,
$L_{3}=\left[1-\frac{2-v}{(1-v) \lambda^{2}} \nabla^{2}\right] \phi^{*}$
where $\phi^{*}=\frac{r^{2}}{8 \pi D}(\log z-1)$
Substituting from Eq. 39 and 40 into Eq. 15, it can be proved that

$$
\begin{equation*}
p_{\alpha \beta}=\left[(1-v) \frac{\partial^{2}}{\partial x_{\alpha} x_{\beta}}+\delta_{\alpha \beta} v \nabla^{2}\right] \psi^{*} \tag{41}
\end{equation*}
$$

where

$$
\begin{align*}
\psi^{*} & =D\left[1+\frac{v}{(1-v) \lambda^{2}} \nabla^{2}\right] \phi^{*} \\
& \equiv \frac{1}{2 \pi}\left[\frac{r^{2}}{4}(\log z-1)+\frac{v \log z}{(1-v) \lambda^{2}}\right] \tag{42}
\end{align*}
$$

Assuming a linearly-distributed loading defined in terms of the following intensity:
$q(x, y)=a_{0}+a_{1} x+a_{2} y$
where $\left(a_{0}, a_{1}, a_{2}\right)$ are given constants, then by using integration by parts [El-Zafrany (1993)], it can be proved that:

$$
\begin{align*}
& \iint_{\Omega} p_{\alpha \beta} d x d y= \\
& \quad\left[(1-v) \oint_{\Gamma} l_{\alpha} \frac{\partial \psi^{*}}{\partial x_{\beta}} q d \Gamma+\delta_{\alpha \beta} \nu \oint_{\Gamma} \frac{\partial \psi^{*}}{\partial n} q d \Gamma\right] \\
& \quad-\left[(1-v) \oint_{\Gamma} l_{\beta} a_{\alpha} \psi^{*} d \Gamma+\delta_{\alpha \beta} \nu \oint_{\Gamma} \frac{\partial q}{\partial n} \psi^{*} d \Gamma\right] \tag{43}
\end{align*}
$$

where $\left(l_{1}, l_{2}\right)$ are as defined in Appendix A.
Changing the order of integration, it can also be proved that:

$$
\begin{align*}
& \iint_{\Omega} p_{\alpha \beta} d x d y= \\
& \quad\left[(1-v) \oint_{\Gamma} l_{\beta} \frac{\partial \psi^{*}}{\partial x_{\alpha}} q d \Gamma+\delta_{\alpha \beta} \nu \oint_{\Gamma} \frac{\partial \psi^{*}}{\partial n} q d \Gamma\right] \\
& \quad-\left[(1-v) \oint_{\Gamma} l_{\alpha} a_{\beta} \psi^{*} d \Gamma+\delta_{\alpha \beta} \nu \oint_{\Gamma} \frac{\partial q}{\partial n} \psi^{*} d \Gamma\right] \tag{44}
\end{align*}
$$

For symmetric expressions we take the average of Eq. 43 and 44 , leading to
$\iint_{\Omega} p_{\alpha \beta} q(x, y) d x d y=\oint_{\Gamma} \rho_{\alpha \beta} q d \Gamma-\oint_{\Gamma} \Sigma_{\alpha \beta} d \Gamma$
where
$\rho_{\alpha \beta}=\frac{1}{2}(1-v)\left[l_{\alpha} \frac{\partial \psi^{*}}{\partial x_{\beta}}+l_{\beta} \frac{\partial \psi^{*}}{\partial x_{\alpha}}\right]+\delta_{\alpha \beta} v \frac{\partial \psi^{*}}{\partial n}$
and
$\Sigma_{\alpha \beta}=\left[\frac{1}{2}(1-v)\left(a_{\alpha} l_{\beta}+a_{\beta} l_{\alpha}\right)+\delta_{\alpha \beta} v \frac{\partial q}{\partial n}\right] \psi^{*}$
which can be expressed as follows:
$\rho_{\alpha \beta}=\frac{r}{8 \pi}\left[(1-v)\left(l_{\alpha} \frac{\partial r}{\partial x_{\beta}}+l_{\beta} \frac{\partial r}{\partial x_{\alpha}}\right)+2 \delta_{\alpha \beta} \vee \frac{\partial r}{\partial n}\right] \times$

$$
\begin{equation*}
\left[(\log z-0.5)+\frac{\alpha_{1}}{r^{2}}\right] \tag{48}
\end{equation*}
$$

$\Sigma_{\alpha \beta}=\frac{1}{8 \pi}\left[(1-v)\left(l_{\alpha} a_{\beta}+l_{\beta} a_{\alpha}\right)+2 \delta_{\alpha \beta} v \frac{\partial q}{\partial n}\right] \times$

$$
\begin{equation*}
\left[\frac{r^{2}}{2}(\log z-1)+\alpha_{1} \log z\right] \tag{49}
\end{equation*}
$$



Figure 1 : Deflection distribution of a simply-supported square plate


Figure 2 : Distribution of moment $M_{x}$ for a simply-supported square plate

Similarly, the domain loading terms in the boundary integral equations of shear forces can be reduced as follows:
$\iint_{\Omega} \Gamma_{\beta} q(x, y) d x d y=\oint_{\Gamma} \rho_{\beta}^{s} q d \Gamma-\oint_{\Gamma} \Sigma_{\beta}^{s} d \Gamma$
where
$\rho_{\beta}^{s}=-\frac{1}{4 \pi}\left[l_{\beta}(\log z-0.5)+\frac{\partial r}{\partial n} \frac{\partial r}{\partial x_{\beta}}\right]$
$\Sigma_{\beta}^{s}=-\frac{r}{4 \pi}(\log z-0.5) \frac{\partial r}{\partial x_{\beta}} \frac{\partial q}{\partial n}$

## 5 Case studies

The previous derivations were implemented in a computer program for the analysis of thick Reissner plates in bending. Several case studies were analyzed, and some of the results will be reviewed next.

### 5.1 Simply-supported square plate

A simply-supported square plate of thickness 0.5 m , and side length $=20 \mathrm{~m}$ was analyzed. The plate centre is the


Figure 3 : Radial distribution of deflection $w$ for a clamped circular plate under concentric line force


Figure 4 : Radial distribution of bending moment $M_{r}$ for a clamped circular plate under concentric line force
origin of the Cartesian coordinates, and its intersecting sides are parallel to the $x$ and $y$ axes. The Young's modulus of the plate material is $2.05 \times 10^{11} \mathrm{~N} / \mathrm{m}^{2}$, and its Poisson's ratio $=0.3$. Two types of loading were tested with this case: a uniformly-distributed loading with intensity $q=3.5 \times 10^{5} \mathrm{~N} / \mathrm{m}^{2}$, and a concentrated force $F=4.8 \times 10^{7} N$ acting at the plate centre. Linear and quadratic boundary element meshes with a total of 24 nodes equally spaced on the boundary were employed
in the analysis. The distributions of deflection $w$ and moment $M_{x}$, along the central line, which is parallel to the $x$ axis, were plotted versus corresponding analytical solutions, as shown in Figures 1 and 2, respectively. Those figures demonstrate a good agreement between boundary element results and analytical solutions with quadratic boundary elements leading to more accurate results than those obtained from linear elements with the same number of boundary nodes.


Figure 5 : Radial distribution of shear force $Q_{r}$ for a clamped circular plate under concentric line force


Figure 6 : Radial distribution of deflection $w$ for a clamped circular plate under concentric line moment

### 5.2 Clamped circular plate under concentric line loading

In this case, a clamped solid circular plate, with outer radius $=1 \mathrm{~m}$, and thickness $=0.1 \mathrm{~m}$ is analyzed. The plate material has a Young's modulus $=2.1 \times 10^{11} \mathrm{~N} / \mathrm{m}^{2}$, and a Poisson's ratio $=0.3$. Two types of line loading were attempted. For the first type, four cases of concentric line forces were tested, with a total shear force $F=$ $3.1416 \times 10^{5} \mathrm{~N}$, acting uniformly along concentric circles
with radii $R_{0}=0,0.25,0.50,0.75 \mathrm{~m}$. For the second type, three cases of concentric line moment were considered, where a total normal bending moment $M_{n}=3.1416 \times 10^{5}$ Nm , was applied uniformly along concentric circles with radii $=0.25,0.50,0.75 \mathrm{~m}$. The radial distributions of lateral deflection $w$, bending moment $M_{r}$ and shear force $Q_{r}$ as obtained from boundary element analysis, for the first type of line loading, were plotted against corresponding analytical solutions, as shown in Figures 3, 4 and 5, re-


Figure 7 : Radial distribution of bending moment $M_{r}$ for a clamped circular plate under concentric line moment


Figure 8 : Radial distribution of deflection $w$ for a simply-supported circular plate under concentric line force
spectively, whilst the radial distributions of lateral deflection w , and bending moment $M_{r}$ for the second type of line loading, were plotted against corresponding analytical solutions, as shown in Figures 6 and 7, respectively. It is clear from those figures that there is a very good agreement between boundary element results and analytical solutions. Figure 4 shows the discontinuities in the shear force distributions at the points of application of line forces, whilst Figure 7 indicates the discontinuities in the bending moment distributions at the points of ap-
plication of line moments.

### 5.3 Simply-supported circular plate under concentric line loading

This case study represents a simply-supported circular plate which has the same dimensions, material properties and cases of loading similar to the previous case, The radial distributions of lateral deflection $w$, bending moment $M_{r}$ and shear force $Q_{r}$ as obtained from boundary element analysis, for the first type of line loading,


Figure 9 : Radial distribution of bending moment $M_{r}$ for a simply-supported circular plate under concentric line force


Figure 10 : Radial distribution of shear force $Q_{r}$ for a simply-supported circular plate under concentric line force
were plotted against corresponding analytical solutions, as shown in Figures 8, 9 and 10, respectively and the radial distributions of lateral deflection $w$, and bending moment $M_{r}$ for the second type of line loading, were plotted against corresponding analytical solutions, as shown in Figures 11 and 12, respectively. It is clear from the figures that an excellent agreement between boundary element results and analytical solutions has been obtained.

## 6 Conclusions

The results of the previous case studies have proved that the new derivations presented in this work have been validated. Boundary integral equations for bending moments and shear forces, per unit length, have led to very accurate boundary element results for different cases with generalized loading. Discontinuous distributions of in-


Figure 11 : Radial distribution of deflection $w$ for a simply-supported circular plate under concentric line moment


Figure 12 : Radial distribution of bending moment $M_{r}$ for a simply-supported circular plate under concentric line moment
ternal shear forces and bending moments, as expected for cases with line forces and line moments, have been accurately evaluated with the new boundary element derivations.

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$$
\begin{align*}
& \text { Boundary element analysis of thick Reissner plates in } U_{33}=-\frac{1}{\pi D(1-v) \lambda^{2}}\left[\log z-\frac{1-v}{8} z^{2}(\log z-1)\right]  \tag{61}\\
& \text { bending. Engineering Analysis, vol. 14, pp. 159-169. }
\end{align*}
$$

$$
\begin{align*}
U_{\alpha 3} & =-\frac{r}{4 \pi D} \frac{\partial r}{\partial n_{\alpha}}\left(\log z-\frac{1}{2}\right)  \tag{59}\\
U_{3 \beta} & =-\frac{r}{4 \pi D} \frac{\partial r}{\partial x_{\beta}}\left(\log z-\frac{1}{2}\right)  \tag{60}\\
U_{33} & =-\frac{1}{\pi D(1-v) \lambda^{2}}\left[\log z-\frac{1-v}{8} z^{2}(\log z-1)\right]
\end{align*}
$$

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$$
\begin{aligned}
T_{1 \beta} & =-\frac{\lambda}{\pi(1-v)}\left\{(1+v) \frac{\partial r}{\partial x_{\beta}} C(z)+g \frac{\partial r}{\partial x_{\beta}} F(z)\right. \\
& \left.+\left[l_{\beta} \frac{\partial r}{\partial n}+v\left(\hat{t} \cdot \hat{i}_{\beta}\right) \frac{\partial r}{\partial t}\right] G(z)\right\}
\end{aligned}
$$

$$
T_{2 \beta}=-\frac{\lambda}{2 \pi}\left\{\left[\left(\hat{t} \cdot \hat{i}_{\beta}\right) \frac{\partial r}{\partial n}+l_{\beta} \frac{\partial r}{\partial t}\right] G(z)+\right.
$$

El-Zafrany, A.; Fadhil, S.; Debbih, M. (1995): An efficient approach for boundary element bending analysis

$$
\begin{equation*}
\left.2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial t} \frac{\partial r}{\partial x_{\beta}} F(z)\right\} \tag{63}
\end{equation*}
$$ of thin and thick plates. International Journal of Computers and Structures, vol. 56, pp. 565-576.

$$
\begin{equation*}
T_{3 \beta}=-\frac{\lambda^{2}}{2 \pi}\left\{\frac{\partial r}{\partial n} \frac{\partial r}{\partial x_{\beta}} A(z)-l_{\beta} B(z)\right\} \tag{64}
\end{equation*}
$$

Karam, V. J.; Telles, J. C. F. (1988):
boundary elements for Reissner's plate theory. Engineering Analysis, vol. 5, pp. 21-27.

$$
\text { On } T_{13}=-\frac{1}{4 \pi}\left[(1+v)\left(\log z-\frac{1}{2}\right)+g\right]
$$

$$
T_{23}=-\frac{1-v}{4 \pi} \frac{\partial r}{\partial n} \frac{\partial r}{\partial t}
$$

Long, S. Y.; Brebbia, C. A.; Telles, J. C. F. (1988):
Boundary element bending analysis of moderately thick plates. Engineering Analysis, vol. 5, pp. 64-74.
Reissner, E. (1945): The effect of transverse shear deReissner, E. (1945): The effect of transverse shear de-
formation on the bending of elastic plates. ASME J. Applied Mechanics, vol. 12, pp. A69-A77.

Weeën, F. V. (1982): Application of the boundary integral equation method to Reissner's plate model. Int. J. Numer. Methods Engng., vol. 18, pp. 1-10.
$\begin{array}{ll}\text { Appendix A: }: & \begin{array}{l}\text { Parameters and kernel functions for } \\ \text { displacement boundary integral equa- } \\ \text { tions }\end{array} \\ & B(z)=\frac{1}{2}\left[A(z)+K_{0}(z)+\frac{2}{z} K_{1}(z)-\frac{2}{z^{2}}\right.\end{array}$
$\theta_{n}=l \theta_{x}+m \theta_{y}$
(53)
$\theta_{t}=-m \theta_{x}+l \theta_{y}$
$M_{n}=l^{2} M_{x x}+m^{2} M_{y y}+2 l m M_{x y}$
$M_{n t}=\operatorname{lm}\left(M_{y y}-M_{x x}\right)+\left(l^{2}-m^{2} M_{x y}\right)$
$Q_{n}=l Q_{x}+m Q_{x}$
$U_{\alpha \beta}=-\frac{1}{\pi D(1-v)}\left\{\frac{\partial r}{\partial n_{\alpha}} \frac{\partial r}{\partial x_{\beta}}\left[A(z)+\frac{1-v}{4}\right]\right.$
$\left.-\left(\hat{n}_{\alpha} \cdot \hat{i}_{\beta}\right)\left[B(z)-\frac{1-v}{4}\left(\log z-\frac{1}{2}\right)\right]\right\}$

$$
\begin{align*}
& C(z)=\frac{A(z)}{z}+\frac{1-v}{4 z}  \tag{54}\\
& G(z)=-F(z)-\frac{2 A(z)}{z}  \tag{55}\\
& F(z)=-\frac{4 A(z)}{z}-\left[K_{1}(z)-\frac{1}{z}\right]-\frac{3-v}{2 z}  \tag{56}\\
& \left(x_{1}, x_{2}\right) \equiv(x, y)  \tag{57}\\
& \left(\hat{i}_{1}, \hat{i}_{2}\right) \equiv(\hat{i}, \hat{j}) \\
& \left(l_{1}, l_{2}\right) \equiv(l, m)
\end{align*}
$$

$\hat{n}_{1} \equiv \hat{n}=l \hat{i}+m \hat{j}$, representing a unit vector in the outward normal direction to the boundary.
$\hat{n}_{2} \equiv \hat{t}=-m \hat{i}+l \hat{j}$, representing a unit vector in the tangential direction to the boundary.

$$
\begin{align*}
& \frac{\partial r}{\partial n_{1}} \equiv \frac{\partial r}{\partial n}=(\nabla r) \cdot \hat{n}  \tag{78}\\
& \frac{\partial r}{\partial n_{2}} \equiv \frac{\partial r}{\partial t}=(\nabla r) \cdot \hat{t}  \tag{79}\\
& g=\left(\frac{\partial r}{\partial n}\right)^{2}+v\left(\frac{\partial r}{\partial r}\right)^{2}  \tag{80}\\
& \alpha_{1}=\frac{2 v}{(1-v) \lambda^{2}}  \tag{81}\\
& \alpha_{2}=\frac{2-v}{(1-v) \lambda^{2}} \tag{82}
\end{align*}
$$

Appendix B: Kernel functions for internal moment and shear boundary integral equations

$$
\begin{align*}
& A_{\alpha \beta 1}=-\frac{D \lambda^{2}}{2 \pi}\left\{\frac{2 \delta_{\alpha \beta}}{z}[(1+v) C(z)+g F(z)-\right.  \tag{90}\\
& \left.\quad \frac{v(2 g-1-v)}{2 z}\right] \frac{2 G(z)}{z}\left[l_{\alpha} l_{\beta}+v\left(\hat{t} \cdot \hat{i}_{\alpha}\right)\left(\hat{t} \cdot \hat{i}_{\beta}\right)\right]  \tag{92}\\
& \quad+\left(\frac{4 F(z)}{z}-\frac{2}{z^{2}}-A(z)\right)\left(\frac{\partial r}{\partial x_{\alpha}}\left[l_{\beta} \frac{\partial r}{\partial n}+v\left(\hat{t} \cdot \hat{i}_{\beta}\right) \frac{\partial r}{\partial t}\right]\right. \\
& \left.\quad+\frac{\partial r}{\partial x_{\beta}}\left[l_{\alpha} \frac{\partial r}{\partial n}+v\left(\hat{t} \cdot \hat{i}_{\alpha}\right) \frac{\partial r}{\partial t}\right]\right)+2 \frac{\partial r}{\partial x_{\alpha}} \\
& \left.\frac{\partial r}{\partial x_{\beta}}\left(g\left[A(z)+\frac{1+v}{z^{2}}\right]-(6 g-1-v) \frac{F(z)}{z}\right)\right\} \tag{83}
\end{align*}
$$

$$
\left(\frac{\partial r}{\partial x_{\alpha}}\left[l_{\beta} \frac{\partial r}{\partial t}+\left(\hat{t} \cdot \hat{i}_{\beta}\right) \frac{\partial r}{\partial n}\right]+\frac{\partial r}{\partial x_{\beta}}\left[l_{\alpha} \frac{\partial r}{\partial t}+\left(\hat{t} \cdot \hat{i}_{\alpha}\right) \frac{\partial r}{\partial n}\right]\right)
$$

$A_{\alpha \beta 3}=-\frac{D \lambda^{3}(1-v)}{4 \pi}\left\{2 \frac{\partial r}{\partial n} \frac{A(z)}{z}\left[\delta_{\alpha \beta}-\frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}}\right]+\right.$
$\phi_{\beta 3}=\frac{(1-v) D \lambda^{4}}{4 \pi}\left[\frac{\partial r}{\partial n} \frac{\partial r}{\partial x_{\beta}} A(z)-l_{\beta} B(z)\right]$
$\psi_{\beta \alpha}=\frac{\lambda^{2}}{2 \pi}\left[\frac{\partial r}{\partial n_{\alpha}} \frac{\partial r}{\partial x_{\beta}} A(z)-\left(\hat{n}_{\alpha} \cdot \hat{i}_{\beta}\right) B(z)\right]$
$\psi_{\beta 3}=-\frac{1}{2 \pi r} \frac{\partial r}{\partial x_{\beta}}$

$$
A_{\alpha \beta 2}=-\frac{D \lambda^{2}(1-v)}{4 \pi}\left\{\frac{4 \delta_{\alpha \beta}}{z} \frac{\partial r}{\partial n} \frac{\partial r}{\partial n}\left[F(z)+\frac{v}{z}\right]\right.
$$

$$
\frac{2 G(z)}{z}\left[l_{\alpha}\left(\hat{t} \cdot \hat{i}_{\beta}\right)+l_{\beta}\left(\hat{t} \cdot \hat{i}_{\alpha}\right)\right]+\left(\frac{4 F(z)}{z}-\frac{2}{z^{2}}-A(z)\right)
$$

$$
\begin{equation*}
\left.+4 \frac{\partial r}{\partial n} \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}}\left[A(z)+\frac{1+v}{z^{2}}-6 \frac{F(z)}{z}\right]\right\} \tag{84}
\end{equation*}
$$

$$
\left.\left[G(z)-\frac{1-v}{2 z}\right]\left[l_{\alpha} \frac{\partial r}{\partial x_{\beta}}+l_{\beta} \frac{\partial r}{\partial x_{\alpha}}-2 \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}} \frac{\partial r}{\partial n}\right]\right\}
$$

