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On a Meshfree Method for Singular Problems

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Abstract: Interests in meshfree (or meshless) methods have grown rapidly in the recent years in solving boundary value problems arising in mechanics, especially in dealing with difficult problems involving large deformation, moving discontinuities, etc. Rigorous error estimates of a meshfree method, the reproducing kernel particle method, for smooth solutions have been theoretically derived and experimentally tested in Han, Meng (2001). In this paper, we provide an error analysis of the meshfree method for solving problems with singular solutions. The results are presented in the context of one-dimensional problems. The error estimates are of optimal order and are supported by numerical results.

1 Introduction

The finite element method has been the dominant numerical method in computational mechanics for several decades. Since 1994, a new family of methods, collectively called *meshfree methods*, has attracted much interest in the community of computational mechanics. This new family of numerical methods is designed to inherit the main advantages of the finite element method such as compact supports of shape functions and function (polynomials, singular functions, etc.) reproducing properties, while at the same time, overcome the main disadvantages of the finite element method owing to the mesh-dependence. The meshfree methods share a common feature that no mesh is needed and shape functions are constructed from sets of particles, thus eliminating the need for time-consuming mesh generation. These methods can handle more effectively problems with large deformations, moving discontinuities, severe mesh distortions and other problems the finite element method experiences difficulty. The meshfree methods are hailed as numerical methods of the next generation (cf. Preface of Liu, Belytschko, Oden (1996)).

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A variety of numerical methods found in the literature belongs to the family of meshfree methods, e.g. Smooth Particle Hydrodynamics (SPH) methods (Lucy (1977); Monaghan (1982, 1988)), Diffuse Element Method (DEM) (Nayroles, Touzot, Villon (1992)), Element Free Galerkin Method (EFG) (Belytschko, Gu, Lu (1994); Belytschko, Lu, Gu (1994)), Reproducing Kernel Particle Method (RKPM) (Liu, Jun, Li, Adde (1995); Liu, Jun, Zhang (1995); Chen, Pan, Wu, Liu (1996)), Moving Least-Square Reproducing Kernel Method (Liu, Shaofan, Belytschko (1997); Li, Liu (1996)), *h*-*p*-Clouds (Duarte, Oden (1996a,b)), Partition of Unity Finite Element Method (Babuška, Melenk(1997)), Meshless Local Petrov-Galerkin Method (MLPG) (Atluri, Zhu (1998); Atluri, Kim, Cho (1999); Atluri, Zhu (2000)). In most of these methods, interpolation functions are constructed in a meshfree manner; however, background meshes are still needed for numerical integration in the construction of stiffness matrices and load vectors. MLPG is a method that is completely mesh independent.

A rigorous error analysis for the Reproducing Kernel Particle Method (RKPM) has been done recently in Han, Meng (2001). In that paper, conditions are identified for the method that lead to optimal order error estimates. The error estimates are comparable to those for the finite element method. Numerical results support the optimal order convergence. The error estimates in Han, Meng (2001) are stated and proved under the assumption of sufficient smoothness of the solution. Here, we will take one step further by deriving error estimates and showing numerical examples for boundary value problems with singular solutions. One-dimensional sample problems are used for our discussion. The results will be useful as an insight for analysis of meshfree methods in solving higher dimensional boundary value problems with geometry singularities. We notice that meshfree methods have been used for computer simulations of singular problems in engineering literature, e.g., in Kim, Atluri (2000); Ching, Batra (2001), MLPG is used to solve problems involving cracks and other singularities.

The paper is organized as follows. In the following section, we briefly review the method and some theoretical results. In the third section, we derive meshfree interpolation error estimates for singular functions. In Section 4, we derive some error estimates for meshfree approximations of one-dimensional boundary value problems with singular solutions. In the final section, we present some numerical results. The numerical examples are of two different types. The first type is designed for the purpose of demonstrating the theoretical error estimates. For the second type, we experiment on particle distributions in order to achieve better convergence order for singular solutions.

2 Reproducing Kernel Particle Approximation

In this section, we provide another point of view for the development of the reproducing kernel particle approximation and review some theoretical results on error estimates for smooth functions.

Let $\Omega \subset \mathbb{R}^d$ be a nonempty, open bounded set with a Lipschitz continuous boundary. In the one-dimensional case, $d=1$, we choose $\Omega=(0,L)$ for some $L>0$. A generic point in $\mathbb{R}^d$ is denoted by $\mathbf{x}=(x_1,\ldots,x_d)^T$ or $\mathbf{z}=(z_1,\ldots,z_d)^T$. We use the Euclidean norm to measure the vector length:

$$\|\mathbf{x}\|=\Big(\sum_{i=1}^{d}|x_i|^2\Big)^{1/2}.$$

For $\mathbf{x}\in\mathbb{R}^d$ and $r>0$, we use $B_r(\mathbf{x})$ for the closed ball centered at $\mathbf{x}$ with radius r in $\mathbb{R}^d$; in particular, B_1 is the closed unit ball centered at the origin in $\mathbb{R}^d$. Throughout the paper we use the multi-index notation for partial derivatives and indices. The symbol $P_p=P_p(\Omega)$ represents the space of the polynomials of degree less than or equal to p on Ω. The dimension of the space P_p is $N_p=(p+d)!/(p!d!)$.

Let $\{\mathbf{x}_i\}_{i=1}^I\subset\overline{\Omega}$ be a set of points, called *particles*. The idea of the particle approximation is to use particle function values for approximation:

$$u(\mathbf{x})\approx\sum_{i=1}^{I}\Psi_i(\mathbf{x})\,u(\mathbf{x}_i). \tag{1}$$

Here $\{\Psi_i\}_{i=1}^I$ are the shape functions associated with the particles $\{\mathbf{x}_i\}_{i=1}^I$. These functions can be constructed by a moving least-squares procedure (Belytschko, Gu, Lu (1994); Belytschko, Lu, Gu (1994)), or by a corrected reproducing kernel particle procedure (Liu, Jun, Li, Adde (1995); Liu, Jun, Zhang (1995)). We take a new point of view for the construction of these shape functions.

As we mentioned in Introduction, we want to keep the main advantages of the finite element method. The first requirement is then that each shape function should have a compact support. This requirement is satisfied by including a function of the form

$$\Phi_{r_i}(\mathbf{x}-\mathbf{x}_i)=\Phi\Big(\frac{\mathbf{x}-\mathbf{x}_i}{r_i}\Big)$$

as a factor for Ψ_i. The function Φ is called a *generating function* or a *window function*, and has the following properties:

$$\begin{cases}\Phi \text{ is continuous},\\ \operatorname{supp}\Phi=B_1,\\ \Phi(\mathbf{x})>0 \text{ for } \|\mathbf{x}\|<1.\end{cases}$$

A normalization condition

$$\int_{B_1}\Phi(\mathbf{x})\,dx=1$$

is usually used in the description of the derivation of reproducing kernel particle approximations, but this condition is not essential and is thus excluded from the outset. The number $r_i>0$ is small and represents the support size of the function Φ. For different particles, we may use different window functions. For example, a singularity can be introduced in the window functions for particles on the boundary in order to treat Dirichlet boundary values (see Chen, Wang (2000)).

There are infinite many possible choices for the generating function. We first list some generating functions in one dimension. A popular choice in engineering computations is the cubic spline, that has the smoothness C^2. Another popular choice is

$$\Phi(z)=\begin{cases}e^{1/(z^2-1)}, & |z|<1,\\ 0, & |z|\ge 1.\end{cases}$$

This function is infinitely smooth. One family of generating functions is given by the formula

$$\Phi_l(z)=\begin{cases}(1-z^2)^l, & |z|\le 1,\\ 0, & |z|>1.\end{cases}$$

We observe that $\Phi_l\in C^{l-1}$.

Any one-dimensional generating function $\Phi(z)$ can be used to create a d-dimensional generating function either in the form $\Phi(\|\mathbf{z}\|)$ or by a tensor product $\prod_{i=1}^{d}\Phi(z_i)$.

The second requirement on the shape function is that the approximation (1) should be able to reproduce polynomials (and other special functions) in order to achieve convergence with optimal orders. For definiteness, here we consider the polynomial reproducing property only. Thus we are inclined to choose a polynomial as another factor for Ψ_i. Moreover, we choose the coefficients of the polynomial to be the same for all the shape functions $\{\Psi_i\}_{i=1}^{I}$ but allow the coefficients to depend on $\mathbf{x}$. As a result, we use the following form for the shape functions $\{\Psi_i\}_{i=1}^{I}$:

$$\Psi_i(\mathbf{x}) = \Phi_{r_i}(\mathbf{x}-\mathbf{x}_i)\sum_{|\alpha|\le p}(\mathbf{x}-\mathbf{x}_i)^{\alpha}b_{\alpha}(\mathbf{x}), \quad 1\le i\le I. \tag{2}$$

Here $\alpha=(\alpha_1,\ldots,\alpha_d)$, $\alpha_i\ge 0$ integer, is a multi-index. The quantity $|\alpha|=\sum_{i=1}^{d}\alpha_i$ is the length of α. The expression $\mathbf{x}^{\alpha}$ stands for $x_1^{\alpha_1}\cdots x_d^{\alpha_d}$. Since the domain Ω is assumed to be Lipschitz continuous, it is locally on one side of the boundary. In case the particle $\mathbf{x}_i$ lies on or close to the boundary so that $B_{r_i}(\mathbf{x}_i)\cap\partial\Omega\neq\emptyset$, we redefine the function value $\Phi_{r_i}(\mathbf{x}-\mathbf{x}_i)$ to be zero outside that side of Ω on which the particle $\mathbf{x}_i$ lies. This is implicitly assumed throughout the paper.

Imposing the polynomial reproducing conditions on the formula (1),

$$u(\mathbf{x}) = \sum_{i=1}^{I}\Psi_i(\mathbf{x})\,u(\mathbf{x}_i) \quad \forall u\in P_p, \tag{3}$$

we have

$$\sum_{|\alpha|\le p} m_{\alpha+\beta}(\mathbf{x})\,b_{\alpha}(\mathbf{x}) = \delta_{|\beta|,0}, \quad |\beta|\le p, \tag{4}$$

where

$$m_{\alpha}(\mathbf{x}) = \sum_{i=1}^{I}\Phi_{r_i}(\mathbf{x}-\mathbf{x}_i)\,(\mathbf{x}-\mathbf{x}_i)^{\alpha}, \quad |\alpha|\le p, \tag{5}$$

are the *discrete moment functions*. The conditions (4) can be written as a consistency condition for the shape functions $\{\Psi_i(\mathbf{x})\}$:

$$\sum_{i=1}^{I}\Psi_i(\mathbf{x})\,(\mathbf{x}-\mathbf{x}_i)^{\beta} = \delta_{|\beta|,0}, \quad |\beta|\le p. \tag{6}$$

Denote the discrete moment matrix from the system (4) by $M(\mathbf{x})$. Then

$$M(\mathbf{x}) = \sum_{i=1}^{I}\Phi_{r_i}(\mathbf{x}-\mathbf{x}_i)\,\mathbf{h}(\mathbf{x}-\mathbf{x}_i)\,\mathbf{h}(\mathbf{x}-\mathbf{x}_i)^{T}, \tag{7}$$

where

$$\mathbf{h}(\mathbf{z}) = (\mathbf{z}^{\alpha})_{|\alpha|\le p}\in\mathbb{R}^{N_p}.$$

To describe conditions under which the method is defined (i.e. the system (4) is uniquely solvable for any $\mathbf{x}\in\overline{\Omega}$), we bring in a definition.

Definition 2.1 *A point $\mathbf{x}\in\overline{\Omega}$ is said to be covered by m shape functions if there are m indices $i_1,\ldots,i_m$ such that*

$$\|\mathbf{x}-\mathbf{x}_{i_j}\| < r_{i_j}, \quad j=1,\ldots,m.$$

It can be shown that for any $\mathbf{x}\in\overline{\Omega}$, a necessary condition for $M(\mathbf{x})$ to be invertible is that $\mathbf{x}$ is covered by at least N_p shape functions. In the one-dimensional case, the discrete moment matrix $M(x)$ is invertible if and only if x is covered by at least $p+1$ shape functions.

For the method to work well, we need conditions stronger than the nonsingularity of the discrete moment matrix $M(\mathbf{x})$. The notion of an (r,p)-regular family of particle distributions to be introduced and discussed later is one such condition. The (r,p)-regularity leads immediately to sufficient conditions for the nonsingularity of the discrete moment matrix.

Assume $M(\mathbf{x})$ is nonsingular. Then the shape functions $\{\Psi_i\}_{i=1}^{I}$ are uniquely determined from (2) and the following properties hold:

1. The shape functions have compact supports: $\operatorname{supp}\Psi_i\subset B_{r_i}(\mathbf{x}_i)$.
2. The shape functions $\{\Psi_i\}_{i=1}^{I}$ form a partition of unity.
3. If $\Phi\in C^k$, then $\Psi_i\in C^k$, $i=1,\ldots,I$.
4. Assume $\Phi\in C^k$. Then

$$\sum_{i=1}^{I}D^{\alpha}\Psi_i(\mathbf{x})(\mathbf{x}-\mathbf{x}_i)^{\beta} = (-1)^{|\alpha|}\beta!\,\delta_{\alpha\beta} \quad \forall\,|\alpha|\le k,\ |\beta|\le p. \tag{8}$$

Here $\delta_{\alpha\beta}$ equals 1 if $\beta=\alpha$, and is zero otherwise.

So unlike the finite element method, in the meshfree method it is easy to construct shape functions of any degree of smoothness. Thus the solution of higher order differential equations does not present any special difficulty in the construction of conforming meshfree shape functions. The relations (8) are consistency relations of derivatives of the shape functions and are derived from the consistency condition (6).

2.1 Regularity of Particle Distributions

As in Han, Meng (2001), we consider the case of quasiuniform support sizes, i.e. there exist two constants $c_1, c_2 \in (0,\infty)$ such that

$$c_1 \le \frac{r_i}{r_j} \le c_2 \quad \forall\, i, j.$$

For such particle distributions, there exists a parameter $r > 0$ such that

$$\tilde{c}_1 \le \frac{r_i}{r} \le \tilde{c}_2 \quad \forall\, i.$$

Let us introduce the *scaled discrete moment matrix*

$$M_0(\mathbf{x}) = \sum_{i=1}^{I} \Phi\Big(\frac{\mathbf{x}-\mathbf{x}_i}{r_i}\Big)\, \mathbf{h}\Big(\frac{\mathbf{x}-\mathbf{x}_i}{r}\Big)\, \mathbf{h}\Big(\frac{\mathbf{x}-\mathbf{x}_i}{r}\Big)^T.$$

Definition 2.2 *A family of particle distributions* $\{\{\mathbf{x}_i\}_{i=1}^I\}$ *is said to be* (r,p)-regular *(or we simply say the particle distributions are* (r,p)*-regular) if there is a constant* L_0 *such that*

$$\max_{\mathbf{x}\in\overline{\Omega}} \|M_0(\mathbf{x})^{-1}\|_2 \le L_0$$

for all the particle distributions in the family.

Since on a finite dimensional space all norms are equivalent, the spectral norm $\|\cdot\|_2$ in the above definition can be replaced by any other matrix norm. We observe that the essential point is to have $M_0(\mathbf{x})^{-1}$ uniformly bounded, or equivalently, the vectors $\{\mathbf{h}((\mathbf{x}-\mathbf{x}_i)/r)\}$, for which $\Phi((\mathbf{x}-\mathbf{x}_i)/r) \ge c_0 > 0$, are "uniformly" independent. The next several results concerning the regularity of particle distributions are shown in Han, Meng (2001).

Proposition 2.3 *A family of particle distributions is* (r,p)*-regular if it is* $(r,p+1)$*-regular, but not conversely.*

Theorem 2.4 *Assume there exist two constants* $c_0 > 0$, $\sigma_0 > 0$ *such that for any* $x \in [0,L]$, *there are* $i_0 < i_1 < \cdots < i_p$ *with*

$$\min_{0\le j\le p} \Phi\Big(\frac{x - x_{i_j}}{r_{i_j}}\Big) \ge c_0 > 0 \tag{9}$$

and

$$\min_{j\ne k} \Big|\frac{x_{i_j} - x_{i_k}}{r}\Big| \ge \sigma_0 > 0. \tag{10}$$

Then the family of particle distributions $\{\{x_i\}_{i=1}^I\}$ *is* (r,p)*-regular, i.e. there exists a constant* $L(c_0,\sigma_0)$ *such that*

$$\max_{0\le x\le L} \|M_0(x)^{-1}\|_2 \le L(c_0,\sigma_0). \tag{11}$$

Notice that the first condition (9) is a strengthened version of the necessary condition that any point must be covered by $p+1$ shape functions. The condition (10) can be equivalently written as

$$\min_{0\le j\le p-1} \frac{x_{i_{j+1}} - x_{i_j}}{r} \ge \sigma_0 > 0.$$

A geometrical interpretation of the condition (10) is that in any local region, at least $p+1$ particles do not coalesce as the refinement goes (i.e. as $r \to 0$).

As a further remark, assume equal support size $r_1 = \cdots = r_I \equiv r$ and consider the situation where Φ is increasing on $[-1,0]$ and decreasing on $[0,1]$, and is symmetric with respect to 0, as is the case in actual computations. If for any x, we can find $i_{-1} < i_0 < \cdots < i_{p+1}$ such that

$$|x - x_{i_j}| \le r, \quad -1 \le j \le p+1$$

with

$$\min_{-1\le j\le p} \frac{x_{i_{j+1}} - x_{i_j}}{r} \ge \sigma_0 > 0,$$

then (9) is automatically satisfied with

$$c_0 \ge \Phi(1-\sigma_0).$$

Theorem 2.5 *A family of particle distributions* $\{\{\mathbf{x}_i\}_{i=1}^I\}$ *in* $\mathbb{R}^d$ *is* $(r,1)$*-regular if there exist two constants* $c_0, \tilde{c}_0 > 0$ *such that for any* $\mathbf{x} \in \overline{\Omega}$, *there are* $d+1$ *particles* $\mathbf{x}_{i_0}, \ldots, \mathbf{x}_{i_d}$ *satisfying*

$$\min_{0\le j\le d} \Phi\Big(\frac{\mathbf{x}-\mathbf{x}_{i_j}}{r}\Big) \ge c_0 > 0$$

and the d-simplex with the vertices $\mathbf{x}_{i_0}, \ldots, \mathbf{x}_{i_d}$ *has a volume larger than* $\tilde{c}_0 r^d$.

We have the following result for bounds on the shape functions and their derivatives.

Theorem 2.6 *Assume the particle distributions are (r,p)-regular and the generating function Φ is k-times continuously differentiable. Then there exists a constant c such that*

$$\max_{1\le i\le I}\max_{\beta:|\beta|=l}\|D^{\beta}\Psi_i\|_\infty \le \frac{c}{r^l}, \quad 0\le l\le k.$$

2.2 Interpolation Error Estimates for Smooth Functions

Assume $\Phi\in C^k$. Given a continuous function u on $\overline{\Omega}\subset\mathbb{R}^d$, we define its meshfree interpolant by the formula

$$u^I(\mathbf{x})=\sum_{i=1}^{I}u(\mathbf{x}_i)\,\Psi_i(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}.$$

Notice that in general, $u^I(\mathbf{x}_i)\neq u(\mathbf{x}_i)$, so u^I is an interpolant of u in a generalized sense.

Let us introduce the following hypothesis.

Hypothesis (H). *There is a constant integer I_0 such that for any $\mathbf{x}\in\overline{\Omega}$, there are at most I_0 of $\mathbf{x}_i$ satisfying the relation $\|\mathbf{x}-\mathbf{x}_i\|<r_i$, i.e. each point in $\overline{\Omega}$ is covered by at most I_0 shape functions.*

The hypothesis (H) is quite natural since otherwise as the number of shape functions covering a local area increases, the shape functions tend to be more and more dependent in the local area.

The following result is proved in Han, Meng (2001).

Theorem 2.7 *Assume the particle distributions are (r,p)-regular, $\Phi\in C^k$, and the hypothesis $(\mathbf{H})$ holds. Let $m\ge 0$, $q\in[1,\infty]$ with $(m+1)\,q>d$ if $q>1$, or $m+1\ge d$ if $q=1$. Then for any $u\in W^{m+1,q}(\Omega)$, we have the optimal order interpolation error estimates*

$$\|u-u^I\|_{W^{l,q}(\Omega)}\le c\,r^{\min\{m+1,p+1\}-l}|u|_{W^{\min\{m+1,p+1\},q}(\Omega)} \quad \forall\, l\le\min\{m+1,p+1,k\}. \tag{12}$$

Notice that when $m\ge p$ and $\Phi\in C^k$ is chosen so smooth that $k\ge p+1$, then the error estimate (12) reduces to

$$\|u-u^I\|_{W^{l,q}(\Omega)}\le c\,r^{p+1-l}|u|_{W^{p+1,q}(\Omega)} \quad \forall\, l\le p+1.$$

3 Interpolation Error Estimates for Singular Functions in One Dimension

We assume for the function u, there is a non-integer $\lambda>0$ such that

$$|u^{(k)}(x)|\le c_k(x^{\lambda-k}+1) \quad \text{for } k=0,1,\ldots, \tag{1}$$

where c_k is a constant depending on k. The assumption (1) mimics corner singularities of solutions to elliptic boundary value problems (cf. Grisvard (1985)).

We are interested in estimating the error for the interpolation

$$u^I(x)=\sum_{i=1}^{I}u(x_i)\,\Psi_i(x).$$

Recall the Taylor theorem for a real-valued function

$$f(x)=\sum_{j=0}^{n}\frac{(x-x_0)^j}{j!}f^{(j)}(x_0)+\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt. \tag{2}$$

Using the formula (2) we have

$$u(x_i)=\sum_{j=0}^{p_\lambda}\frac{(x_i-x)^j}{j!}u^{(j)}(x)+\frac{1}{p_\lambda!}\int_{x}^{x_i}(x_i-t)^{p_\lambda}u^{(p_\lambda+1)}(t)\,dt, \tag{3}$$

where $p_\lambda\le p$ is determined later. For an integer $l\ge 0$, let

$$e_l(u)(x)\equiv(u^I-u)^{(l)}(x)=\sum_{i=1}^{I}u(x_i)\,\Psi_i^{(l)}(x)-u^{(l)}(x)$$

denote the lth derivative of the interpolation error. Then using (3) and the consistency property of the shape functions we obtain

$$e_l(u)(x)=\frac{1}{p_\lambda!}\sum_{i=1}^{I}\Psi_i^{(l)}(x)\int_{x}^{x_i}(x_i-t)^{p_\lambda}u^{(p_\lambda+1)}(t)\,dt. \tag{4}$$

Theorem 3.1 *Consider the case of quasiuniform support sizes. Assume the family of particle distributions is (r,p)-regular, and the hypothesis $(\mathbf{H})$ is valid. For a function u with the behavior (1), we have the error estimates*

$$\|(u^I-u)^{(l)}\|_{L^q(0,1)}\le c\,r^{\min\{\lambda+1/q,\,p+1\}-l}. \tag{5}$$

where $q\in[1,\infty]$ and when $q=\infty$, we adopt the convention $1/q=0$.

PROOF. Since support sizes are quasiuniform, we have

$$\tilde{c}_1 r\le r_i\le\tilde{c}_2 r \quad \forall\, i.$$

Let us prove the error estimate (5) for the case $q\in[1,\infty)$. Let $p_\lambda=\min\{[\lambda+2/q]-1,p\}$, where $[x]$ denotes the largest integer less than x. We notice that

$$\min\{\lambda,p_\lambda+1\}+1/q\ge\min\{\lambda+1/q,\,p+1\}. \tag{6}$$

Because of the hypothesis (**H**), we derive from (4) that

$$|e_l(u)(x)|^q \le c\sum_{i=1}^{I}|\Psi_i^{(l)}(x)|^q\left|\int_x^{x_i}(x_i-t)^{p_\lambda}u^{(p_\lambda+1)}(t)\,dt\right|^q.$$

Using Hölder's inequality we get

$$|e_l(u)(x)|^q \le$$
$$c\sum_{i=1}^{I}|\Psi_i^{(l)}(x)|^q|x_i-x|^{q-1}\left|\int_x^{x_i}|x_i-t|^{p_\lambda q}|u^{(p_\lambda+1)}(t)|^q dt\right|.$$

Thus

$$\int_0^1|e_l(u)(x)|^q dx \le c\sum_{i=1}^{I}I(i), \tag{7}$$

where

$$I(i)=\int_{[0,1]\cap[x_i-r_i,x_i+r_i]}|\Psi_i^{(l)}(x)|^q|x_i-x|^{q-1}\left|\int_x^{x_i}|x_i-t|^{p_\lambda q}|u^{(p_\lambda+1)}(t)|^q dt\right|dx.$$

We first estimate $I(i)$ for those i with $x_i\in[0,2\tilde{c}_2 r]$. We have

$$I(i)\le I(i,1)+I(i,2),$$

where

$$I(i,1)=\int_0^{x_i}|\Psi_i^{(l)}(x)|^q(x_i-x)^{q-1}\int_x^{x_i}(x_i-t)^{p_\lambda q}|u^{(p_\lambda+1)}(t)|^q dt\,dx,$$
$$I(i,2)=\int_{x_i}^{x_i+r_i}|\Psi_i^{(l)}(x)|^q(x-x_i)^{q-1}\int_{x_i}^{x}(t-x_i)^{p_\lambda q}|u^{(p_\lambda+1)}(t)|^q dt\,dx.$$

These terms are bounded as follows.

$$\begin{aligned}I(i,1)&\le c r^{(p_\lambda-l)q}\int_0^{x_i}(x_i-x)^{q-1}\int_x^{x_i}|u^{(p_\lambda+1)}(t)|^q dt\,dx\\&=c r^{(p_\lambda-l)q}\int_0^{x_i}|u^{(p_\lambda+1)}(t)|^q\int_0^t(x_i-x)^{q-1}dx\,dt\\&=c r^{(p_\lambda-l)q}\int_0^{x_i}|u^{(p_\lambda+1)}(t)|^q[x_i^q-(x_i-t)^q]\,dt\\&\le c r^{(p_\lambda-l)q}\int_0^{x_i}r^{q-1}t\,|u^{(p_\lambda+1)}(t)|^q dt.\end{aligned}$$

Using the bound (1), we have

$$I(i,1)\le c r^{(p_\lambda+1-l)q-1}\int_0^{x_i}t\left(t^{(\lambda-p_\lambda-1)q}+1\right)dt.$$

Therefore,

$$I(i,1)\le c r^{(\min\{\lambda+2/q,\,p_\lambda+1+2/q\}-l)q}. \tag{8}$$

We also have

$$\begin{aligned}I(i,2)&\le\int_{x_i}^{x_i+r_i}c r^{(1-l)q-1}dx\int_{x_i}^{x_i+r_i}(t-x_i)^{p_\lambda q}|u^{(p_\lambda+1)}(t)|^q dt\\&=c r^{(1-l)q}\int_0^{r_i}s^{p_\lambda q}|u^{(p_\lambda+1)}(x_i+s)|^q ds\\&\le c r^{(1-l)q}\int_0^{r_i}s^{p_\lambda q}\left(s^{(\lambda-p_\lambda-1)q}+1\right)ds.\end{aligned}$$

Therefore,

$$I(i,2)\le c r^{(\min\{\lambda+1/q,\,p_\lambda+1+1/q\}-l)q}. \tag{9}$$

Combining (8) and (9), we see that if $x_i\in[0,2\tilde{c}_2 r]$, then

$$I(i)\le c r^{(\min\{\lambda+1/q,\,p_\lambda+1+1/q\}-l)q}.$$

Since the number of x_i in $[0,2\tilde{c}_2 r]$ is bounded from above by a constant, we have

$$\sum_{i:x_i\in[0,2\tilde{c}_2 r]}I(i)\le c r^{(\min\{\lambda+1/q,\,p_\lambda+1+1/q\}-l)q}. \tag{10}$$

Then we estimate those $I(i)$ with $x_i>2\tilde{c}_2 r$. We have

$$\begin{aligned}I(i)&=\int_{x_i-r_i}^{\min\{x_i+r_i,1\}}|\Psi_i^{(l)}(x)|^q|x_i-x|^{q-1}\left|\int_x^{x_i}|x_i-t|^{p_\lambda q}|u^{(p_\lambda+1)}(t)|^q dt\right|\\&\le\int_{x_i-r_i}^{\min\{x_i+r_i,1\}}c r^{(1-l)q-1}dx\int_{x_i-r_i}^{\min\{x_i+r_i,1\}}r^{p_\lambda q}|u^{(p_\lambda+1)}(t)|^q dt\\&=c r^{(p_\lambda+1-l)q}\int_{x_i-r_i}^{\min\{x_i+r_i,1\}}|u^{(p_\lambda+1)}(t)|^q dt.\end{aligned}$$

So

$$\sum_{i:x_i>2\tilde{c}_2 r}I(i)\le c r^{(p_\lambda+1-l)q}\sum_{i:x_i>2\tilde{c}_2 r}\int_{x_i-r_i}^{\min\{x_i+r_i,1\}}|u^{(p_\lambda+1)}(t)|^q dt.$$

Because of the hypothesis (**H**), we have

$$\sum_{i:x_i>2\tilde{c}_2 r}\int_{x_i-r_i}^{\min\{x_i+r_i,1\}}|u^{(p_\lambda+1)}(t)|^q dt\le c\int_{\tilde{c}_2 r}^{1}|u^{(p_\lambda+1)}(t)|^q dt.$$

Using the bound (1) we obtain

$$\sum_{i:x_i>2\tilde{c}_2 r}I(i)\le c r^{(\min\{\lambda+1/q,\,p_\lambda+1+1/q\}-l)q}. \tag{11}$$

We can combine (10) and (11),

$$||e_l(u)||_{L^q(0,1)} \le c\, r^{\min\{\lambda+1/q,\, p_\lambda+1+1/q\}-l}.$$

By (6), (5) follows from this estimate.

Taking the limit $q \to \infty$ in (5), we see that it holds for $q = \infty$ also.

4 Reproducing Kernel Particle Method and Error Analysis

The Reproducing Kernel Particle Method is a Galerkin method combined with the use of reproducing kernel particle spaces. To explain the method in a concrete problem setting, we take a linear elliptic boundary value problem as an example. It is equally fine to consider nonlinear elliptic BVPs if we wish. Since nonhomogeneous Dirichlet boundary conditions can be rendered homogeneous in a standard way (see Han, Reddy (1999)) or one of many texts on modern PDE), we will assume Dirichlet boundary conditions, if any, are homogeneous. The weak formulation is

$$u \in V: \quad a(u,v) = \ell(v) \quad \forall\, v \in V. \tag{1}$$

Here V is a Sobolev space. For Neumann boundary value problems, V is a complete Sobolev space without boundary condition constraints, e.g., $H^1(\Omega)$ for second-order problems, and $H^2(\Omega)$ for fourth-order problems, Ω being the spatial domain of the differential equation. Otherwise, V is a subspace of a complete Sobolev space (e.g. $H_0^1(\Omega)$). The bilinear form $a(\cdot,\cdot)$ is continuous and elliptic on V, and ℓ is a linear continuous form on V. By the Lax-Milgram lemma, the variational problem (1) has a unique solution $u \in V$.

On $\overline{\Omega}$, introduce a set of particles $\{\mathbf{x}_i\}_{i=1}^I$, some of the particles lie on the boundary. Also introduce $\{r_i\}_{i=1}^I$, $r_i > 0$, and construct functions $\{\Psi_i\}_{i=1}^I$ in the form of (2) where $\{b_\alpha(\mathbf{x})\}_{|\alpha|\le p}$ are computed from (4). The reproducing kernel paticle space is

$$V_R = \mathrm{span}\{\Psi_i,\ 1 \le i \le I\} \cap V.$$

Then the RKPM is

$$u^R \in V_R: \quad a(u^R,v) = \ell(v) \quad \forall\, v \in V_R. \tag{2}$$

This problem admits a unique solution $u^R \in V_R$, again following the Lax-Milgram lemma. For error estimates of the RKP solution $u^R \in V_R$ defined in (2), we have Céa's inequality

$$||u-u^R||_V \le c \inf_{v\in V_R} ||u-v||_V. \tag{3}$$

In the rest of the section, we assume the (r,p)-regularity and hypothesis (**H**). Then we can use the error estimates for RKP interpolants derived in the previous section.

4.1 Error Estimates for BVP without Dirichlet Condition

For a boundary value problem without Dirichlet boundary condition,

$$V_R = \mathrm{span}\{\Psi_i,\ 1 \le i \le I\}.$$

Assume the solution u is continuous. Then its reproducing kernel particle interpolant

$$u^I(\mathbf{x}) = \sum_{i=1}^I u(\mathbf{x}_i)\, \Psi_i(\mathbf{x})$$

is well defined and $u^I \in V_R$. Then from (3), we have

$$||u-u^R||_V \le c\, ||u-u^I||_V \tag{4}$$

and the question of error estimation for the RKP solution u^R is reduced to that for the RKP interpolant u^I. As a sample result, we can state the following result.

Theorem 4.1 *Let us employ the RKPM to solve the second-order BVP of the type* (1) *without Dirichlet boundary condition. Assume the solution u is continuous and has the behavior* (1). *Assume $\Phi \in C^1$, and the (r,p)-regularity and hypothesis* (**H**) *are valid. Then we have the error estimate*

$$||u-u^R||_{H^1(\Omega)} \le c\, r^{\min\{\lambda-1/2,\, p\}}. \tag{5}$$

4.2 Error Estimates for BVP with Dirichlet Condition

When the boundary value problem includes a Dirichlet condition, derivation of rigorous error estimates is much more difficult. Since in general $u^I \notin V_R$, and we need to replace (4) by

$$||u-u^R||_V \le c\, ||u-\tilde{u}^I||_V, \tag{6}$$

where $\tilde{u}^I \in V_R$ is a modification of u^I. This approach cannot be carried out in case $d \ge 2$, since a function from V_R

does not vanish on a part of the boundary even when it is zero at all the particles on that part of the boundary.

In one-dimensional case, though, it is possible to derive rigorous error estimates. In the following, we consider a general linear elliptic BVP on $[0,L]$ with Dirichlet boundary conditions

$$u(0) = u_0, \quad u(L) = u_L. \tag{7}$$

Let the weak form of the problem be: Find $u \in H^1(0,L)$ satisfying (7) such that

$$a(u,v) = \ell(v) \quad \forall v \in H_0^1(0,L). \tag{8}$$

Let the RKP space be

$$V_R = \text{span}\{\Psi_i,\ 1 \le i \le I\}.$$

Then the RKPM for the problem is: Find $u^R \in V_R$ satisfying such that $u^R(0) = u_0$, $u^R(L) = u_L$, and

$$a(u^R,v) = \ell(v) \quad \forall v \in V_R \cap H_0^1(0,L). \tag{9}$$

For an error estimate, Céa's inequality (3) is modified to

$$\|u-u^R\|_V \le c\,\inf\{\|u-v\|_V : v \in V_R,\ v(0)=u_0,\ v(L)=u_L\}. \tag{10}$$

For the RKP interpolant, we have

$$u^I(x) = u(x) + R_p(x),$$

where

$$R_p(x) = \frac{1}{p!}\sum_{i=1}^{I}\Psi_i(x)\int_x^{x_i}(x_i-t)^p u^{(p+1)}(t)\,dt.$$

In particular,

$$u^I(0)-u(0) = R_p(0) = \frac{1}{p!}\sum_{i=1}^{I}\Psi_i(0)\int_0^{x_i}(x_i-t)^p u^{(p+1)}(t)\,dt.$$

Then

$$|u^I(0)-u(0)| \le c\sum_{i:|x_i|\le r_i} r_i^p \int_0^{r_i}|u^{(p+1)}(t)|\,dt.$$

Since there are at most I_0 points x_i with $|x_i| \le r_i$, we have

$$|u^I(0)-u(0)| \le c\,r^{p+1/2}\|u^{(p+1)}\|_{L^2(0,L)}. \tag{11}$$

Similarly,

$$|u^I(L)-u(L)| \le c\,r^{p+1/2}\|u^{(p+1)}\|_{L^2(0,L)}. \tag{12}$$

Under the assumption $u^{(p+1)} \in L^\infty(0,L)$, the estimates (11) and (12) can be sharpened,

$$|u^I(0)-u(0)| + |u^I(L)-u(L)| \le c\,r^{p+1}\|u^{(p+1)}\|_\infty.$$

Define a corrected RKP interpolant,

$$\tilde{u}^I(x) = u^I(x) + \frac{L-x}{L}\left(u(0)-u^I(0)\right) + \frac{x}{L}\left(u(L)-u^I(L)\right).$$

We have $\tilde{u}^I(0) = u(0)$, $\tilde{u}^I(L) = u(L)$. Since linear functions can be reproduced, $\tilde{u}^I \in V_R$. By (11) and (12), we have

$$\begin{aligned}&\|\tilde{u}^I - u^I\|_{W^{l,q}(0,L)} \le c\,r^{p+1/2}\|u^{(p+1)}\|_{L^2(0,L)},\\ &l \ge 0 \text{ integer},\ q \in [1,\infty].\end{aligned} \tag{13}$$

The definition of the corrected RKP interpolant and the related error estimate can be easily modified to adapt to the case with a Dirichlet condition at only one end of the interval $[0,L]$. Then from (10), we have

$$\begin{aligned}&\|u-u^R\|_{H^1(0,L)} \le c\,\|u-\tilde{u}^I\|_{H^1(0,L)} \le\\ &c\left[\|u-u^I\|_{H^1(0,L)} + \|\tilde{u}^I-u^I\|_{H^1(0,L)}\right].\end{aligned} \tag{14}$$

Using (14), (13) and the estimate for $u-u^I$ from the previous section we get the next result.

Theorem 4.2 *Let us employ the RKPM to solve the second-order BVP* (8) *with a solution* u *with the behavior* (1). *Assume* $\Phi \in C^1$, *and the* (r,p)*-regularity and hypothesis* $(\mathbf{H})$ *are valid. Then we have the error estimate*

$$\|u-u^R\|_{H^1(\Omega)} \le c\,r^{\min\{\lambda-1/2,\,p\}}. \tag{15}$$

Applying the well-known Aubin-Nitsche's technique, we can show that under the conditions stated in Theorem 4.2, the following L^2-norm error estimate holds:

$$\|u-u^R\|_{L^2(\Omega)} \le c\,r^{\min\{\lambda-1/2,\,p\}+1}. \tag{16}$$

5 Numerical Results

In this section, we present some numerical results on convergence orders of RKPM. The numerical results support the theoretical error estimates presented in the previous sections.

The boundary value problem we solve is

$$\begin{cases} -u''(x) = -\lambda(\lambda-1)x^{\lambda-2}, & x \in (0,1), \\ u(0)=0, \quad u(1)=1. \end{cases}$$

The exact solution is

$$u(x) = x^{\lambda}.$$

The parameter λ is chosen to be larger than $1/2$ so that the solution $u \in H^1(0,1)$. The smaller the parameter λ, the stronger the singularity. For various values of λ, we will report numerical results on the L^2-norm errors of meshfree solutions and their derivatives, as well as the L^2-norm errors of meshfree interpolants and their derivatives.

We first use uniform particle distributions with equal support size. We divide the interval $[0,1]$ into $N = 20$, 30, 40, 50 and 60 equal parts, and let $h = 1/N$. We use $r = (p+2.1)\,h$ as the support size. This choice of the support size guarantees the satisfaction of both (r,p)-regularity and hypothesis (**H**). Since r is proportional to h, we show figures for errors compared against h (rather than r itself) in the log-log scale. Numerical results for $\lambda = 0.6$, 1.5 and 2.5 are shown in Figures 1–3. These results support the theoretical convergence order $\min\{\lambda - 1/2, p\}$ in the H^1-norm (see (16)) and order $\min\{\lambda-1/2, p\}+1$ in the L^2-norm (see (16)) for the meshfree solution and meshfree interpolation errors.

To increase the accuracy of the meshfree solutions in solving singular problems, it is natural to use more particles near the singularity points. To see the effects, we discuss a local particle enhancement technique for the 1D model problem. Given a natural number n and a parameter $a \in (0,1)$, we define a particle distribution as follows: First, we introduce the nodes $(1-n^{-a})^j$, $j = 0,1,\ldots,n$. Since $(1-n^{-a})^{n-1} - (1-n^{-a})^n < (1-n^{-a})^n$, we further divide $[0,(1-n^{-a})^n]$ into subintervals with lengths nearly equal to $(1-n^{-a})^{n-1} - (1-n^{-a})^n$. The support size of the shape function corresponding to a particle is chosen to be $(2.1+p)$ times the larger length of the two subintervals containing the particle. This condition

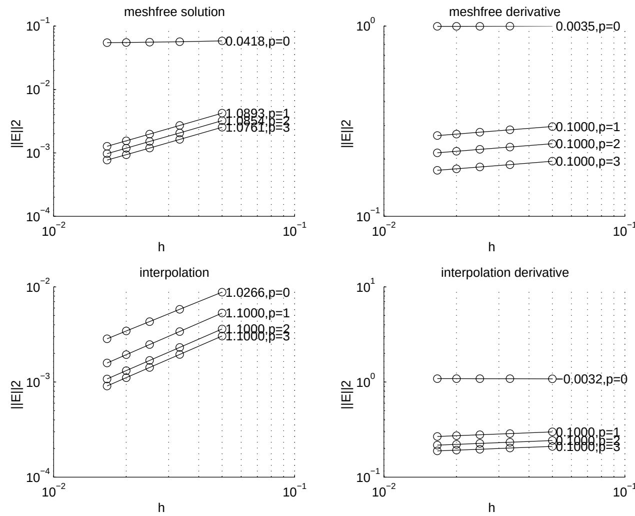

Figure 1 : Uniform particle distributions, $\lambda = 0.6$

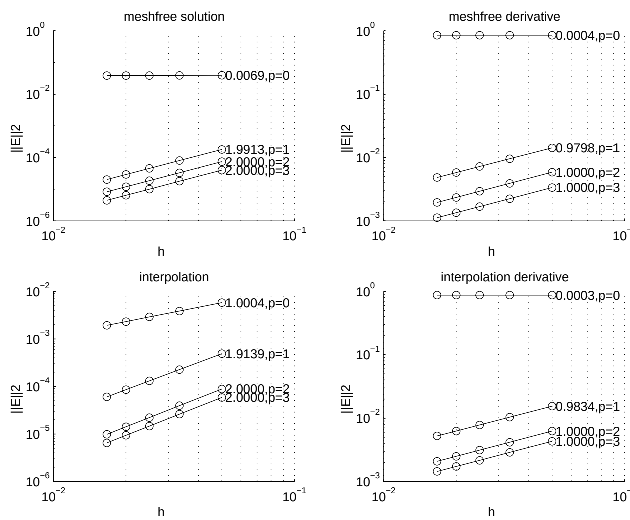

Figure 2 : Uniform particle distributions, $\lambda = 1.5$

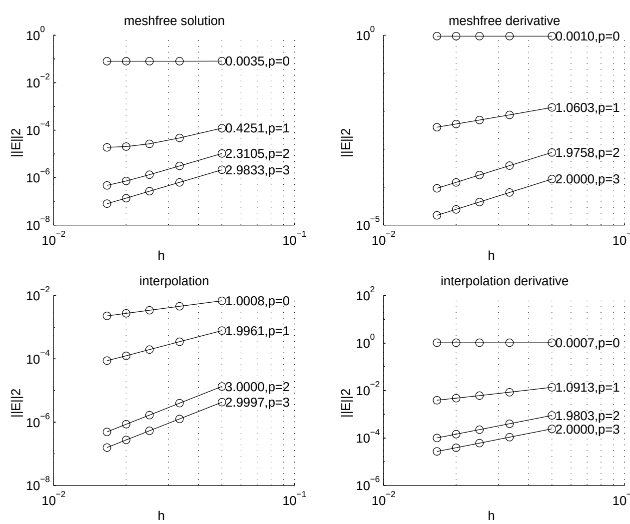

Figure 3 : Uniform particle distributions, $\lambda = 2.5$

guarantees the invertibility of the discrete moment matrix. The parameter a controls the strength of the particle enhancement for the singularity. A smaller value for a indicates a more dense particle distribution near the singularity.

The results of the particle enhanced meshfree solutions (indicated by "$\circ$") are compared with those with uniform particle distributions (indicated by "$+$"); see Figures 4–7. In these figures, N is the total degrees of freedom.

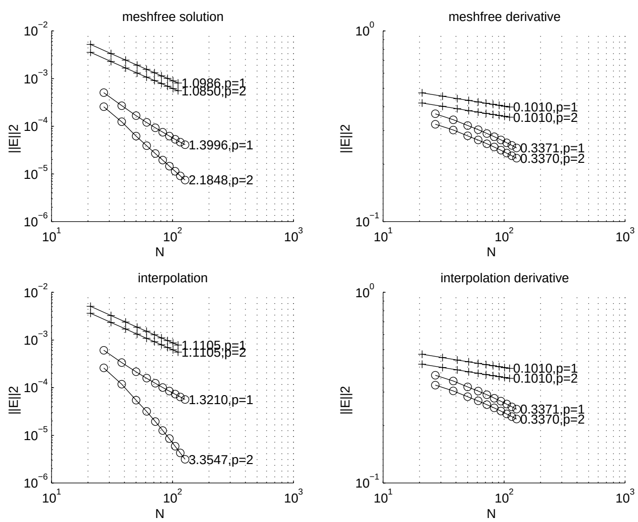

Figure 4 : Enhanced particle distributions with $a = 0.6$, $\lambda = 0.6$

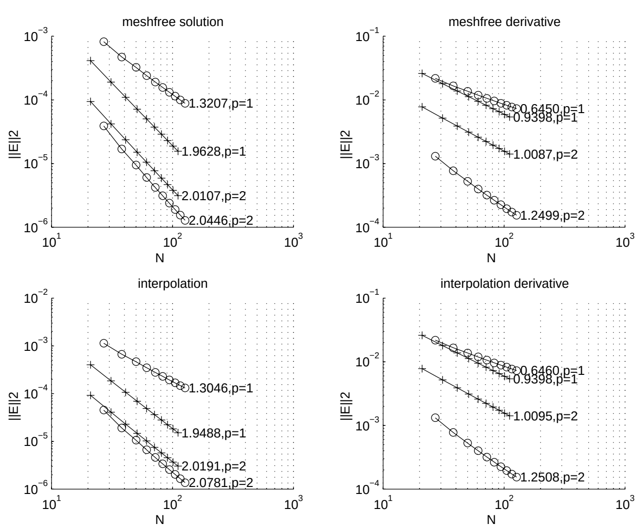

Figure 5 : Enhanced particle distributions with $a = 0.6$, $\lambda = 1.5$

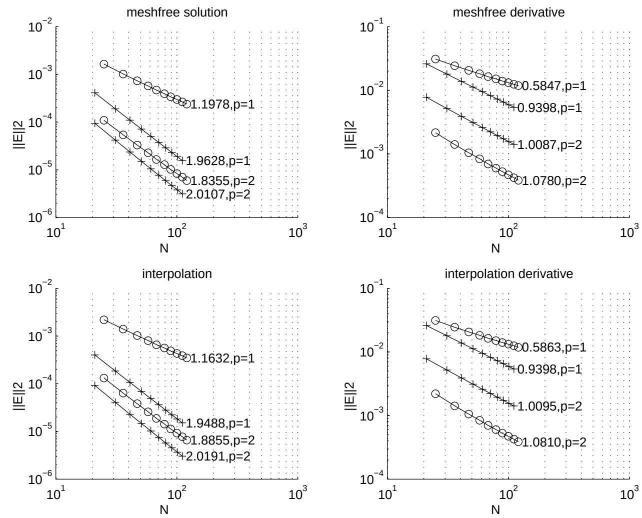

Figure 6 : Enhanced particle distributions with $a = 0.5$, $\lambda = 1.5$

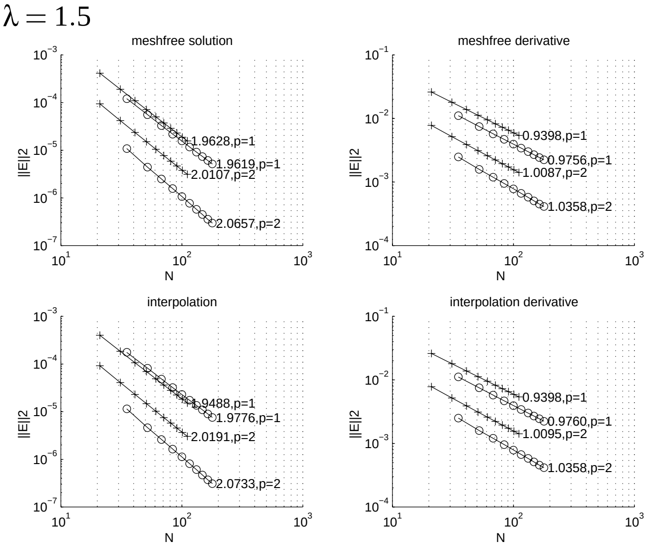

Figure 7 : Enhanced particle distributions with $a = 0.9$, $\lambda = 1.5$

Numerical results for using the meshfree method to solve elliptic boundary value problems on higher dimensional corner domains exhibit similar properties. As one such example, we solve a boundary value problem for the Laplace equation $-\Delta u = 0$ on the crack domain $(-1,1)^2 \backslash \{\{0\} \times [0,1)\}$. The solution of the problem is chosen to be

$$u(\mathbf{x}) = ||\mathbf{x}||^{1/2} \sin(\theta/2),$$

where θ is the angle variable ($\theta = 0$ corresponds to the x-axis). Due to the symmetry, we solve a half domain problem on the rectangle $(-1,1) \times (0,1)$. Homogeneous Neumann condition is specified on $(-1,0) \times \{0\}$, and

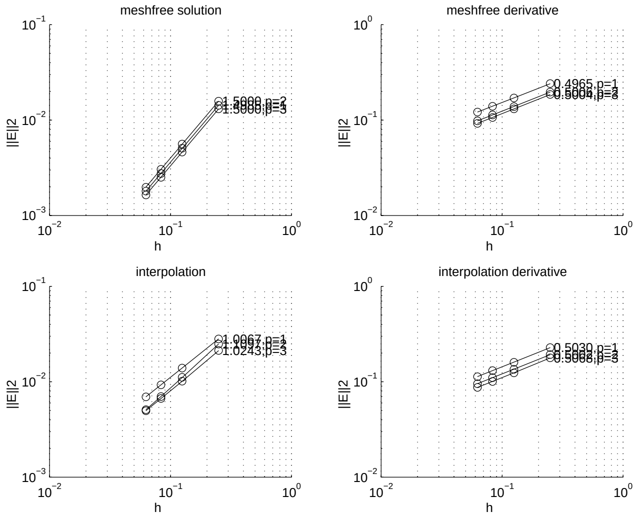

Figure 8 : Uniform particle distributions, crack problem

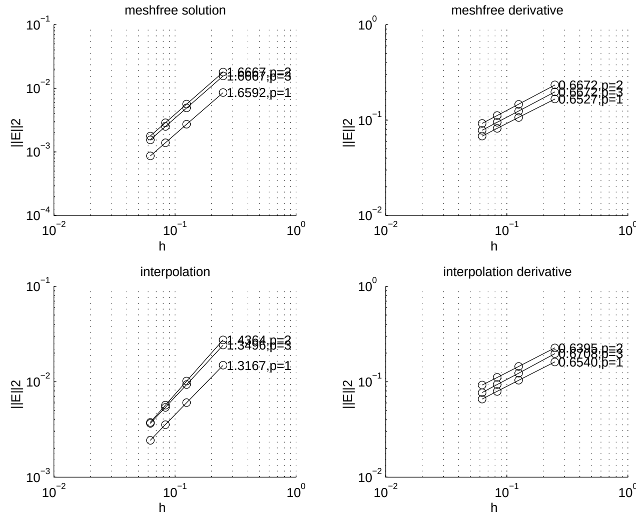

Figure 9 : Uniform particle distributions, L-shape problem

Dirichlet condition is specified on the rest of the boundary. Uniform particle distributions are used. The numerical results for the crack problem are shown in Figure 8. The numerical results suggest the following convergence order

$$\begin{aligned} \|\nabla(u-u^R)\|_{L^2(\Omega)} &= O(r^{1/2}), \\ \|u-u^R\|_{L^2(\Omega)} &= O(r^{3/2}), \end{aligned}$$

irregardless of value of the reproducing order p. The results also suggest that for the meshfree interpolation,

$$\begin{aligned} \|\nabla(u-u^I)\|_{L^2(\Omega)} &= O(r^{1/2}), \\ \|u-u^I\|_{L^2(\Omega)} &= O(r^1). \end{aligned}$$

Thus the convergence order for the interpolation in the $L^2(\Omega)$ norm is twice that in the $H^1(\Omega)$ norm, while the convergence order for the meshfree solution in the $L^2(\Omega)$ norm is one higher than that in the $H^1(\Omega)$ norm.

Similar numerical experiments are done for an L-shape domain problem. Again we solve a boundary value problem for the Laplace equation. This time, the domain is chosen to be $(-1,1)^2 \backslash \{[0,1)\times(-1,0]\}$. The data are chosen such that the exact solution has the form

$$u(\mathbf{x}) = \|\mathbf{x}\|^{2/3}\sin(2\theta/3).$$

Numerical results of the meshfree method corresponding to uniform particle distributions are shown in Figure 9. The numerical results suggest the following convergence order

$$\begin{aligned} \|\nabla(u-u^R)\|_{L^2(\Omega)} &= O(r^{2/3}), \\ \|u-u^R\|_{L^2(\Omega)} &= O(r^{5/3}), \\ \|\nabla(u-u^I)\|_{L^2(\Omega)} &= O(r^{2/3}), \\ \|u-u^I\|_{L^2(\Omega)} &= O(r^{4/3}). \end{aligned}$$

Again, we observe that the convergence order for the interpolation in the $L^2(\Omega)$ norm is twice that in the $H^1(\Omega)$ norm, while the convergence order for the meshfree solution in the $L^2(\Omega)$ norm is one higher than that in the $H^1(\Omega)$ norm.

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