

Meshless BEM for Three-dimensional Stokes Flows

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Abstract: This paper describes a combination of the dual reciprocity method (DRM) and the method of fundamental solution (MFS) as a meshless BEM (DRM-MFS) to solve three-dimensional Stokes flow problems by the velocity-vorticity formulation, where the DRM is based on the compactly supported, positive definite radial basis functions (CS-PD-RBF). In the velocity-vorticity formulation, both of the diffusion type vorticity equations and the Poisson type velocity equations are solved by DRM-MFS. Here a typical internal cubic cavity flow and an external flow past a sphere are presented. The results are acceptable. Furthermore, this paper provides a preliminary work for applications to the three-dimensional Navier-Stokes equations.

keyword: Velocity-vorticity formulation, Stokes flow, meshless, boundary element method, radial basis function, dual reciprocity method, method of fundamental solution

1 Introduction

In the past years, there has been an increasing interest in the idea of meshless numerical methods for solving partial differential equations (PDE). Generally speaking, such methods can be divided into three types. The first one is the so-called DRM-MFS, which combines the dual reciprocity method and the method of fundamental solution, the second one is the so-called Kansa's method [Kansa (1999a,b)], and the third one is the so-called meshless local Petrov-Galerkin (MLPG) and local boundary integral equation (LBIE) methods based on the integral equations [Wordelman, Aluru, Ravaioli

(2000); Lin, Atluri (2000); Kim, Atluri (2000); Lin, Atluri (2001); Atluri and Zhu (1998); Zhu, Zhang, Atluri (1998),]. All of them are based on the radial basis functions (RBF), [Atluri and Shen (2002) also discuss 4 other types of trial functions, besides the RBF] and it is convinced that researchers will finally find very solid meshless methods to solve the PDE with non-homogeneous terms. In this paper, we will adopt the DRM-MFS method to solve the 3D Stokes flow problems, including the internal and external flow fields.

The dual reciprocity method (DRM) was introduced by Nardini and Brebbia to approximate the particular solution of the equation in their 1982 pioneer work [Nardini and Brebbia (1982)]. Since then, many researches of meshless numerical methods have been carried out. Furthermore, the method of fundamental solution (MFS) is used to approximate the homogenous solution of the equation. More details about MFS can be found in the excellent review papers [Goldberg, Chen (1998); Fairweather, Karageorghis (1998)]. The meshless BEM, which combines the DRM and MFS, has been used to solve many PDE in different areas successfully [Kansa (1999a); Goldberg, Chen (1998), Chen, Brebbia, Power (1999); Muleshkov, Goldberg, Chen (1999); Goldberg, Chen (2001)]. In the paper, we use the DRM-MFS to solve the three-dimensional internal and external Stokes flow problems.

There are three well-known formulations for the solution of the incompressible Navier-Stokes equations in terms of: primitive variables of pressure and velocity, velocity-stream function and velocity-vorticity. The first two formulations have been thoroughly investigated by various researchers for two and three-dimensional problems by using various numerical methods such as finite difference methods (FDM) [Anderson, Tannehill, Pletcher (1984)], finite element methods (FEM) [Gunzaburger (1987)], and boundary element methods (BEM) [Power, Wrobel (1995)]. The third formulation in terms of velocity and vorticity also has been explained in the last decade in two

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and three dimensions using various numerical schemes [Dennis, Ingham, Cook (1979); Gunzaburger, Peterson (1988); Skerget, Rek (1995); Young, Liu, Eldho (1999); Young, Yang, Eldho (2000)].

Stokes flow problems can be considered as a subprogram of the Navier-Stokes flow problems, in which the nonlinear convective terms are very small, thus can be neglected. For the solution of Stokes flows using the velocity-vorticity formulation, the governing equations have been written as a system of diffusion-type and Poisson-type equations for the components of the vorticity and velocity fields, respectively. The main advantage of this formulation is the numerical separation of the kinematic and kinetic aspects of the fluid flow.

In this paper, numerical experiments of Stokes flow about a cubic cavity internal flow and a flow around a sphere are investigated by using DRM-MFS. For further application to the three dimensional Navier-Stokes equations, this paper really provides a preliminary work. The advantage of mesh free can be used for large-scale industrial application problems. The disadvantages of full matrix can be circumvented by using compactly-support RBFs and some iterative schemes, which could result in an iterative scheme without the assembly of matrix [Young, Tsai, Eldho, Cheng (in press); Cheng, Young, Tsai (2000); Tsai, Young, Cheng (2001)].

2 Governing Equations

The governing equations of Stokes flow for the velocity-vorticity formulation can be derived from the Navier-Stokes equations in a non-dimensional form and written as [Currie (1993)]:

$$\begin{cases} \frac{\partial \vec{\omega}}{\partial t} = \nabla^2 \vec{\omega} \\ \nabla^2 \vec{u} = -\nabla \times \vec{\omega} \end{cases} \quad \text{in } \Omega \quad (1a, 1b)$$

$$\vec{u} = \vec{U} \quad \text{on } \Gamma \quad (2)$$

where \vec{u} is the velocity vector, $\vec{\omega}$ is the vorticity vector, \vec{U} is the known boundary velocity, and Ω as well as Γ are the domain and boundary, respectively. The vorticity vector $\vec{\omega}$ can be expressed as:

$$\vec{\omega} = \nabla \times \vec{u} \quad (3)$$

In equation (1), the governing equations are the diffusion and Poisson equations and can be solved by DRM-MFS to be described below.

3 Numerical Formulation

The governing equations (1) to be solved can be classified into two categories. The first is the diffusion equation and the second is the Poisson's equation. The numerical procedures involved for solving the two problems will be introduced as follow.

Poisson's equations

The Poisson's equations have the form:

$$\nabla^2 \Phi(\vec{x}) = b(\vec{x}) \quad \text{and} \quad \Phi(\vec{x}) = BC(\vec{x}) \quad \text{on } \Gamma \quad (4)$$

where Γ is the boundary of the problem. We decompose the solution into

$$\Phi(\vec{x}) = \Phi_h(\vec{x}) + \Phi_p(\vec{x}) \quad (5)$$

where the particular solution, $\Phi_p(\vec{x})$, satisfies

$$\nabla^2 \Phi_p(\vec{x}) = b(\vec{x}) \quad (6)$$

and the homogenous solution, $\Phi_h(\vec{x})$, satisfies

$$\nabla^2 \Phi_h(\vec{x}) = 0 \quad \text{and}$$

$$\Phi_h(\vec{x}) = BC(\vec{x}) - \Phi_p(\vec{x}) \quad \text{on } \Gamma \quad (7)$$

The particular solution corresponding to equation (6) can be approximated by the DRM [Nardini, Brebbia (1982); Chen, Brebbia, Power (1999); Goldberg, Chen (2001); Young, Tsai, Eldho, Cheng (in press); Cheng, Young, Tsai (2000); Tsai, Young, Cheng (2001)]. Let the right hand side of the Poisson's equation take the form

$$b(\vec{x}_j) = \sum_{i=1}^{n_d} \alpha_i f(r_{ij}) \quad (8)$$

where $f(r)$ is a radial basis function, $r_{ij} = |\vec{x}_i - \vec{x}_j|$ is the radial distance between a field point \vec{x}_j and the i -th collocation point \vec{x}_i , and n_d is the number of collocation nodes. The collocation nodes are typically distributed in the interior domain as well as on the boundary. The collocation points can be chosen arbitrary, regular or irregular mesh, depending on the convenience for computation.

And α_i are the collocation coefficients to be determined. If we let the functional values be equal at n_d collocation points, we result in a linear system with n_d unknowns, α_i , and n_d equations, which can be solved if the system is nonsingular. After α_i 's have been solved, we can find the particular solution which is of the form:

$$\Phi_p(\vec{x}_j) = \sum_{i=1}^{n_d} \alpha_i F(r_{ij}) \quad (9)$$

where $F(r)$ is the inverse Laplacian of the radial basis function $f(r)$, i.e. $\nabla^2 F(r) = f(r)$.

For the current three-dimensional problems, we choose the following CS-PD-RBF:

$$f(r) = \begin{cases} (1 - \frac{r}{a})^2 & r \leq a \\ 0 & r > a \end{cases} \quad (10)$$

The detail of CS-PD-RBF is referred to Tsai, Young, Cheng (2001). The particular solution corresponding to equation (10) is derived by Chen, Brebbia, Power (1999).

$$F(r) = \begin{cases} \frac{r^2(10a^2 - 10ar + 3ar^2)}{60a^2} & r \leq a \\ \frac{a^2}{12} - \frac{a^3}{30r} & r > a \end{cases} \quad (11)$$

Furthermore, the homogeneous solution corresponding to equation (7) can be solved by the MFS. Let the homogeneous solution in equation (7) the linear combination of the fundamental solution of the Laplace operator, i.e.:

$$\Phi_h(\vec{x}_j) = \sum_{i=1}^{m_d} \beta_i g(r_{ij}) \quad (12)$$

where $g(r) = -\frac{1}{4\pi r}$ is the fundamental solution of the Laplace operator, $r_{ij} = |\vec{\xi}_i - \vec{x}_j|$ is the distance between a field point \vec{x}_j and the i -th source point $\vec{\xi}_i$, and m_d is the number of source nodes. The source nodes are typically distributed in small distance away from the boundary to avoid the coincidence of \vec{x}_j and $\vec{\xi}_i$ and thus to avoid the singularity. Typically, it has been observed that the locations and the distances of the source points from the boundary depend on the type of the boundary condition [Balakrishman, Ramachandrn (1999)]. We also

found the irregular collocation points and source points are also possible. The regular collocation points and source points adopted here, which are called as mesh, are only for the convenient purpose of computation. And β_i are the coefficients to be determined. If we let the functional values be equal to the boundary condition (7) at m_d boundary points, we result in a linear system with m_d unknowns, β_i , and m_d equations, which can be solved if the system is nonsingular. After β_i 's have been solved, we can find the homogeneous solution.

After the homogeneous solution Φ_h and the particular solution Φ_p have been solved, we can apply the principle of superposition (equation (5) to get the solution.

Diffusion equations

The diffusion equations take the following form:

$$\frac{\partial \Phi(\vec{x})}{\partial t} = \nabla^2 \Phi(\vec{x}) \quad \text{and} \\ \Phi = \Phi_0(\vec{x}) \quad \text{in } \Omega \quad \text{at } t = 0, \quad \text{and}$$

$$\Phi(\vec{x}) = BC(\vec{x}) \quad \text{on } \Gamma \quad (13)$$

where $\Phi_0(\vec{x})$ and $BC(\vec{x})$ are the initial condition in the domain Ω and the boundary condition on the boundary Γ , respectively.

If we discretize the equation in time by finite difference method, we can get:

$$\nabla^2 \Phi^{(n+1)}(\vec{x}) - \frac{\Phi^{(n+1)}(\vec{x}) - \Phi^{(n)}(\vec{x})}{\Delta t} = -\frac{\Phi^{(n)}(\vec{x})}{\Delta t} \quad (14)$$

where Δt is the interval for time discretization.

Furthermore, if we set $\lambda = \sqrt{\frac{1}{\Delta t}} > 0$, rearranging the terms in equation (14) and combining with the BC of equation (13), we can get:

$$\nabla^2 \Phi^{(n+1)}(\vec{x}) - \lambda^2 \Phi^{(n+1)}(\vec{x}) = -\frac{\Phi^{(n)}(\vec{x})}{\Delta t} \quad \text{and}$$

$$\Phi^{(n+1)}(\vec{x}) = BC(\vec{x}) \quad \text{on } \Gamma \quad (15)$$

which is a modified Helmholtz equation with proper boundary condition. The source term of the modified Helmholtz equation is corresponding to the initial condition of the diffusion equation.

Now, we can apply the same procedure as we have undertaken in the Poisson's equation to equation (15) and get similar results.

We decompose the solution into

$$\Phi^{(n+1)}(\vec{x}) = \Phi_h^{(n+1)}(\vec{x}) + \Phi_p^{(n+1)}(\vec{x}) \quad (16)$$

where the particular solution, $\Phi_p^{(n+1)}(\vec{x})$, satisfies

$$\nabla^2 \Phi_p^{(n+1)}(\vec{x}) - \lambda^2 \Phi_p^{(n+1)}(\vec{x}) = -\frac{\Phi^{(n)}(\vec{x})}{\Delta t} \quad (17)$$

and the homogenous solution, $\Phi_h^{(n+1)}(\vec{x})$, satisfies

$$\nabla^2 \Phi_h^{(n+1)}(\vec{x}) - \lambda^2 \Phi_h^{(n+1)}(\vec{x}) = 0 \text{ and}$$

$$\Phi_h^{(n+1)}(\vec{x}) = BC(\vec{x}) - \Phi_p^{(n+1)}(\vec{x}) \text{ on } \Gamma \quad (18)$$

The particular solution corresponding to equation (17) can be approximated by the DRM, which have already been introduced in the Poisson's equation [Chen, Brebbia, Power (1999); Muleshkov, Goldberg, Chen (1999); Young, Tsai, Eldho, Cheng (in press); Cheng, Young, Tsai (2000); Tsai, Young, Cheng (2001)]. The process of the collocation procedure is exactly the same as before and the detail will be ignored here. Let

$$-\frac{\Phi^{(n)}(\vec{x})}{\Delta t} = \sum_{i=1}^{n_d} \alpha_i f(r_{ij}) \quad (19)$$

After α_i 's have been solved, we can find the particular solution which is of the form:

$$\Phi_p^{(n+1)}(\vec{x}_j) = \sum_{i=1}^{n_d} \alpha_i F(r_{ij}) \quad (20)$$

where $F(r)$ is the inverse modified Helmholtz operator of the radial basis function $f(r)$, i.e. $\nabla^2 F(r) - \lambda^2 F(r) = f(r)$.

If we choose the same CS-PD-RBF (equation (10)) as we have performed in Poisson's equation, we can get the required $F(r)$ are following [Goldberg, Chen, Ganesh (2000)]

$$F(r) = \begin{cases} \lambda(2B + q(0)) + q'(0) & r = 0 \\ \frac{Ae^{-\lambda r} + Be^{\lambda r} + q(r)}{r} & 0 < r \leq a \\ \frac{Ce^{-\lambda r}}{r} & r > a \end{cases} \quad (21)$$

where

$$\begin{cases} A = -(B + q(0)) \\ B = -\frac{e^{-\lambda a}(q'(a) + \lambda q(a))}{2\lambda} \\ C = B(e^{2a\lambda} - 1) + q(a)e^{a\lambda} - q(0) \\ q(r) = \frac{4a - r\lambda^2 a^2 - 6r + 2r^2 \lambda^2 a - r^3 \lambda^2}{\lambda^4 a^2} \end{cases} \quad (22)$$

Furthermore, the homogeneous solution corresponding to equation (18) can be solved by the MFS. Let the homogeneous solution to be the linear combination of the fundamental solution of the modified Helmholtz operator, i.e.:

$$\Phi_h^{(n+1)}(\vec{x}_j) = \sum_{i=1}^{m_d} \beta_i g(r_{ij}) \quad (23)$$

where $g(r) = -\frac{e^{-\lambda r}}{4\pi r}$ is the fundamental solution of the modified Helmholtz operator. The detail of the solution procedure is exact the same as we have introduced in the Poisson's equation. After β_i 's have been solved, we can find the homogeneous solution.

After the homogeneous solution Φ_h and the particular solution Φ_p have been solved, we can apply the superposition principle (equation (16)) to get the solution of the modified Helmholtz equation, which is also the solution of the original diffusion equation (13).

4 Solution procedures

The governing equation (equation (1)) of the Stokes flow problem involves three diffusion vorticity equations and three Poisson's velocity equations. If we can solve the diffusion vorticity equations and the Poisson velocity equations, we will be able to solve the Stokes flow problem only when we have appropriate vorticity boundary conditions and sources terms of velocity Poisson's equations. The computational procedure adopted here includes the following steps:

1. Choose suitable time interval $\Delta t \leq 0.5 * (L/d)^3$, where L is the large length scale and d is the minimum distance of the numerical spatial points.

2. Set $\begin{cases} \vec{u}^{(0)}(\vec{x}) = (u1^{(0)}(\vec{x}), u2^{(0)}(\vec{x}), u3^{(0)}(\vec{x})) = \vec{0} \\ \vec{\omega}^{(0)}(\vec{x}) = (\omega1^{(0)}(\vec{x}), \omega2^{(0)}(\vec{x}), \omega3^{(0)}(\vec{x})) = \vec{0} \end{cases}$

where $\vec{u}^{(n)} = \vec{u}(\vec{x}, n * \Delta t)$ and $\vec{\omega}^{(n)} = \vec{\omega}(\vec{x}, n * \Delta t)$, i.e. $\vec{u}^{(0)}$ and $\vec{\omega}^{(0)}$ are the initial conditions.

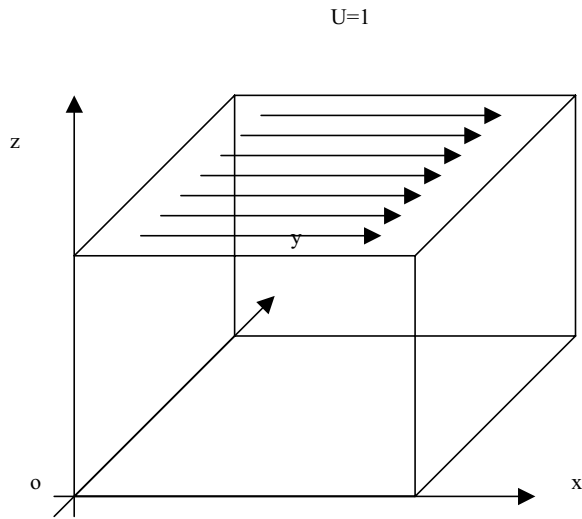


Figure 1 : Flow field configuration of cubic cavity

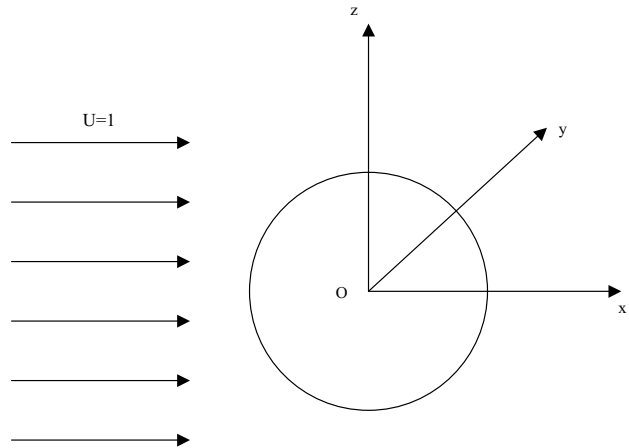


Figure 2 : Flow field configuration of flow passing a sphere

3. Solve the homogeneous velocity equation (1b) with the known boundary velocity condition. (Laplace equation)
4. Get the vorticity at boundary by $\vec{\omega} = \nabla \times \vec{u}$. The derivatives of the velocity can be got by direct differentiation of the series, i.e.

$$\frac{\partial \Phi}{\partial x_k} = \sum_{i=1}^{n_d} \alpha_i \frac{\partial F(r_{ij})}{\partial x_k} + \sum_{i=1}^{m_d} \beta_i \frac{\partial g(r_{ij})}{\partial x_k} \quad (24)$$

5. Solve the vorticity $\vec{\omega}^{(n+1)}$ by equation (1a) with the boundary condition in step 4.
6. Solve the velocity equation (1b) with the known boundary, where the source term is the derivative of the vorticity components got from step 5. The technique for differentiation is the same as described in step 4.
7. Repeat 4~6 until the solutions are convergent

In the procedure of taking the derivatives, we use the direct differentiation of the series (equation (24)). In equation (24), the m_d source points for the homogeneous solution should not be near the boundary. In the numerical experiments, we find the derivatives at the boundary points are not accurate if the source points are too close to the boundary.

5 Results and Discussions

In order to see if the present scheme works or not, two numerical experiments of unsteady Stokes flow problems in three-dimension have been done. One is the cubic cavity internal flow problem, which has the configuration shown in the Fig. 1, with a uniform velocity driving the top face of the cubic to the right. And the other is the problem of the flow passing a sphere, which has the configuration shown in the Fig. 2, with a uniform outflow in far field and the flow domain is extended to infinity. Using the method described above, we are able to get the following results.

For the cavity problem, the contours of the vorticity components at $x=0.5$, $y=0.5$ and $z=0.5$ are shown in Fig. 3~Fig. 8. The components, which are not shown, are zero horizontal planes. Also, the velocity fields at $y=0.5$ and $z=0.5$ are shown in Fig. 9~Fig. 10. The velocity components at $x=0.5$ are zeros in the case of the Stokes flow. The mesh size in these figures is $21 \times 21 \times 21$.

In order to check the accuracy of the method, a comparison with the results of Young et al. (1999) is performed. In the comparison (Fig. 11 and Fig. 12), the velocity distribution in the middle ($x=0.5$) of the main cut profile ($y=0.5$) is shown. Here, the same mesh size is used for the purpose of having same order of convergence. The result seems to be acceptable. Furthermore, A comparison for different mesh sizes ($9 \times 9 \times 9$, $11 \times 11 \times 11$ and $13 \times 13 \times 13$) has also been done (Fig. 13 and Fig. 14). As shown in the figures, the solution approaches the same

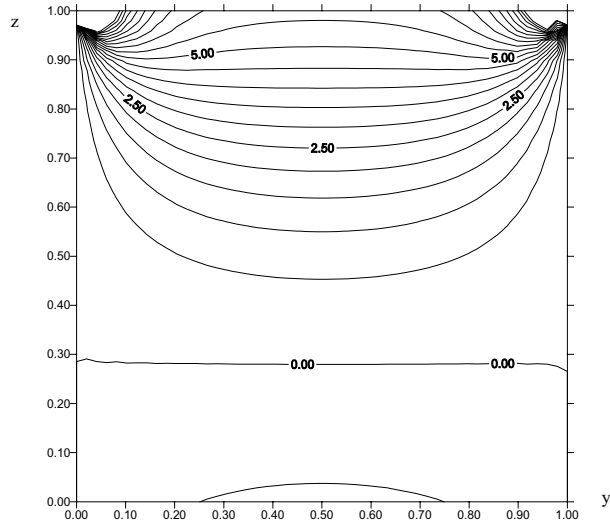


Figure 3 : Vorticity component y at x= 0.5

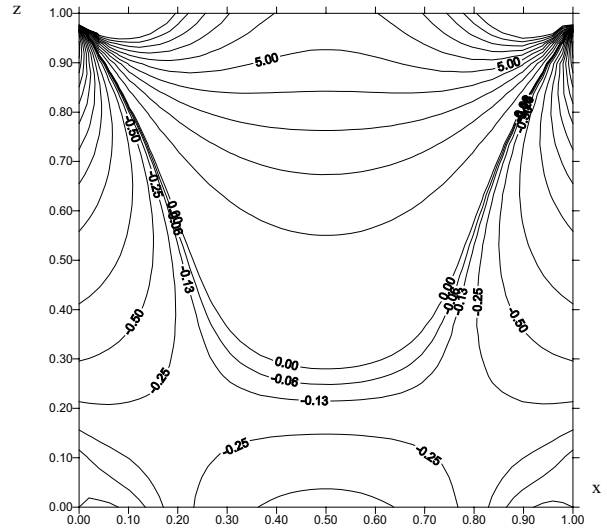


Figure 5 : Vorticity component y at y= 0.5

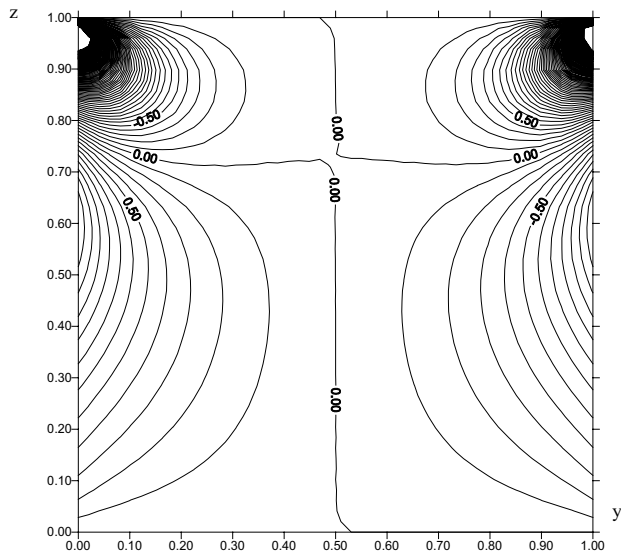


Figure 4 : Vorticity component z at x= 0.5

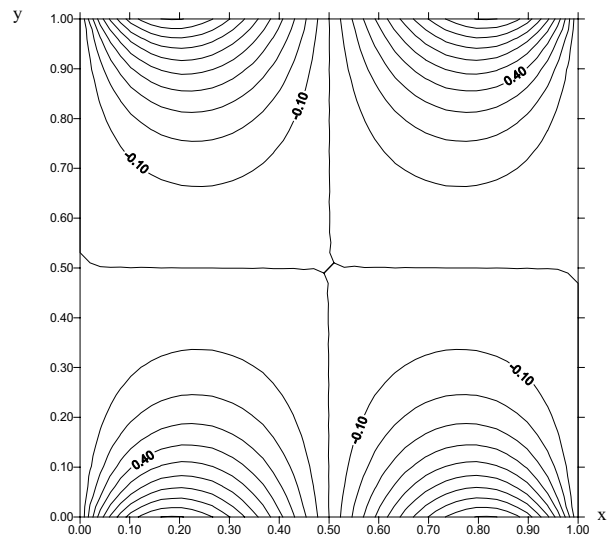


Figure 6 : Vorticity component x at z= 0.5

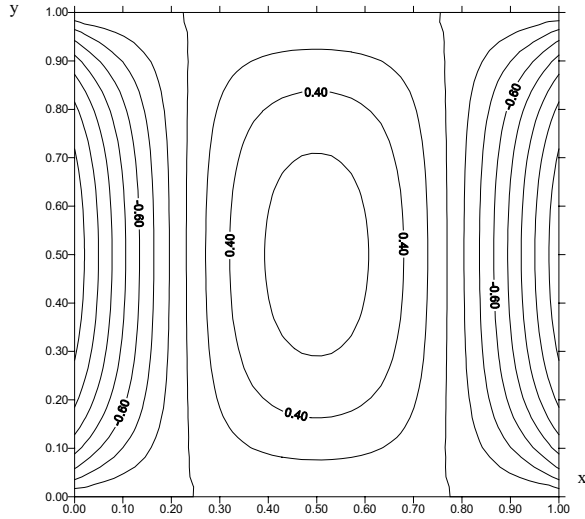


Figure 7 : Vorticity component y at z= 0.5

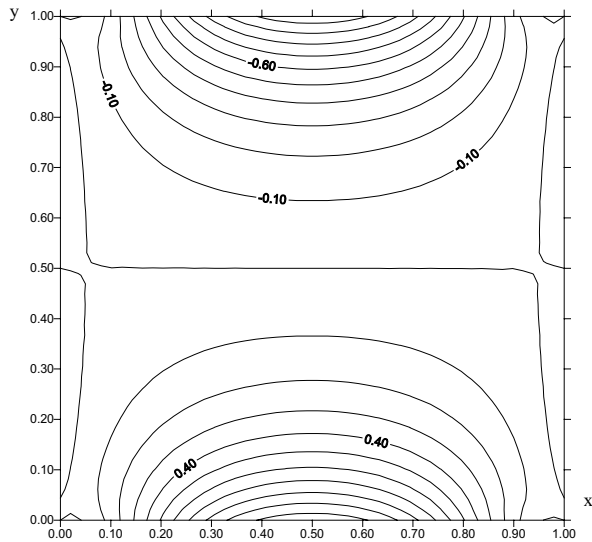


Figure 8 : Vorticity component z at z= 0.5

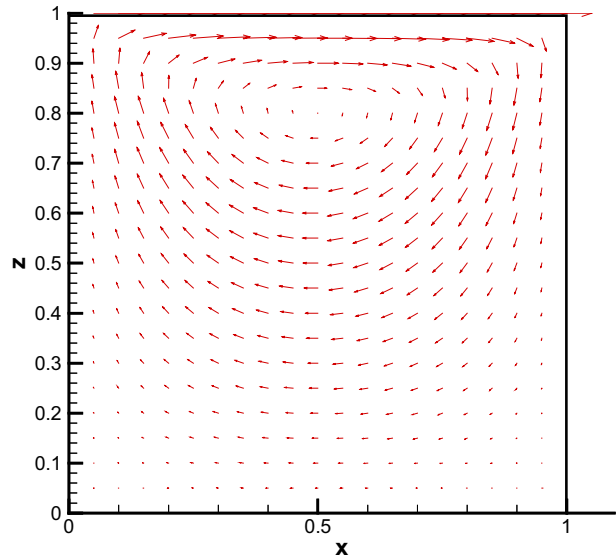


Figure 9 : Velocity profile at y=0.5

limitation as the mesh sizes increasing.

For the problem of uniform flow passing a sphere, there is an exact solution [Tritton (1988)] for comparison. The exact solution is

$$\begin{cases} u_r = U \cos \theta \left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right) \\ u_\theta = U \sin \theta \left(-1 + \frac{a^3}{4r^3} + \frac{3a}{4r}\right) \end{cases} \quad (25)$$

where U is the velocity of the uniform flow and a is the radius of the sphere. In our numerical experiment, both of the parameters are set to one. In the formula, the direction of $\theta = 0$ is corresponding to the direction of fluid flowing away.

Since the Stokes flow problem is a linear problem, the principle of superposition can be applied. The uniform flow passing a sphere can be decomposed to a motion of the sphere with negative unit velocity as well as a full field unit velocity. In case of the motion of the sphere with negative unit velocity, the far field boundary condition of velocity and vorticity are both zeroes. In our numerical experiments, we adopt the compact support radial basis function, which will automatically result in zeroes far field boundary values for the particular solutions in equation (6) and (17). Furthermore, the fundamental solutions for the homogeneous solutions are also zeroes, when the distances from the sphere go to infinity. The boundary condition corresponding to the sphere is

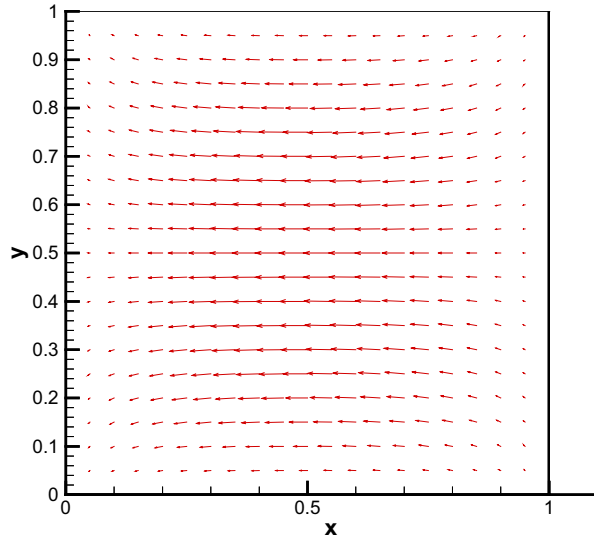


Figure 10 : Velocity profiles at z=0.5

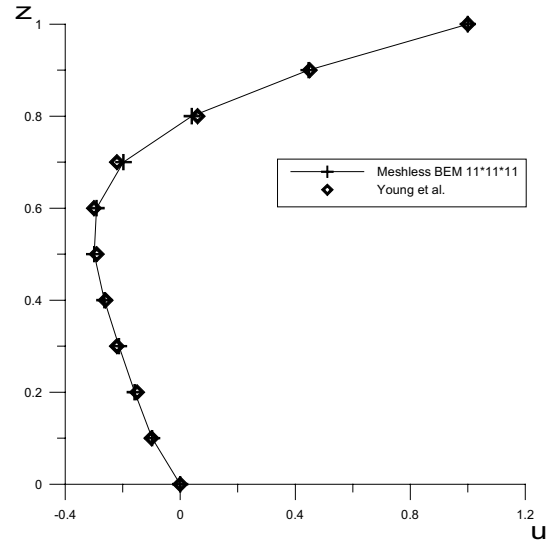


Figure 12 : Comparisons of the present and other methods (ul)

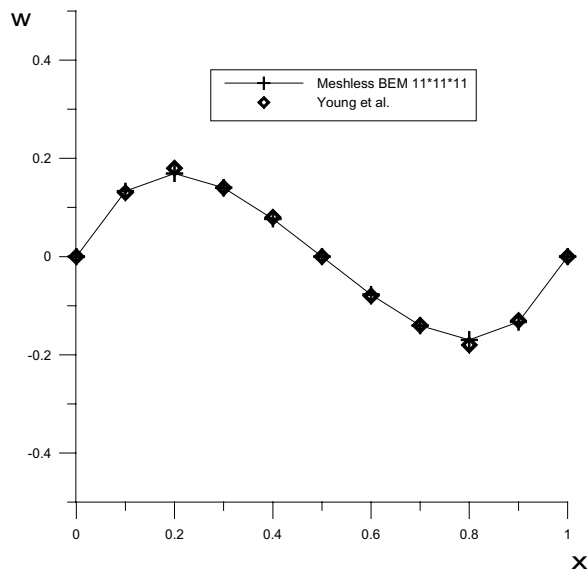


Figure 11 : Comparisons of the present and other methods (u3)

$\vec{u} = (-1, 0, 0)$. In our numerical experiments, three different meshes are tested. The meshes are 792 collocation points (144 boundary points), 1650 collocation points (220 boundary points), and 2652 collocation points (312 boundary points), respectively. The vorticity contour and the velocity profile for 2652 collocation points are shown in Fig. 15 and Fig. 16, respectively. These two figures show the qualitative agreements with the existing literature. The velocity field in Fig.16 in the near field of the sphere at $\theta = \frac{\pi}{2}$ has been magnified on Fig. 17. Also, a comparison for different mesh sizes with the exact solution of the u velocity component for the positions of $1 \leq \frac{r}{a} \leq 10$ and $\theta = \frac{\pi}{2}$ has been made (Fig. 17). The results show excellent agreements with the exact solutions. Since the components of vorticity and its gradients have non-zero values only near the sphere for the exterior problems, the numerical scheme adopted in our numerical experiments, which uses the compact support radial function to approximate the right hand side of the equation (1b), seems to be a reasonable way. Base on this reason, the same approach can be further applied to the full Navier-Stokes problems for any exterior problems. The meshless boundary element method possesses the advantages of the boundary element method, which uses the fundamental solution to catch the attribute of the governing equation, and also has the advantage of no need

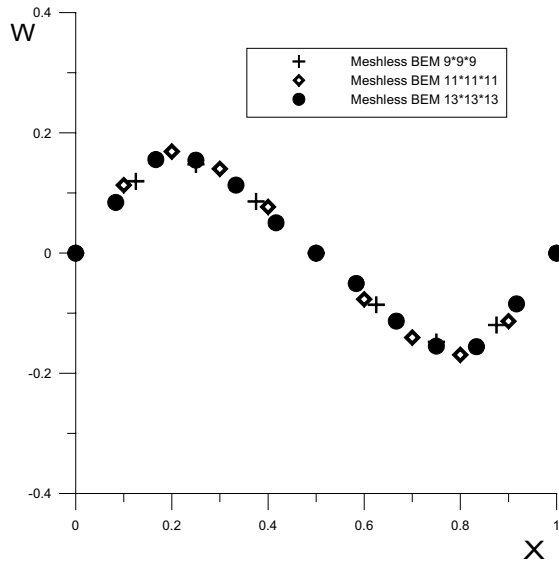


Figure 13 : Comparisons for different mesh sizes (u3)

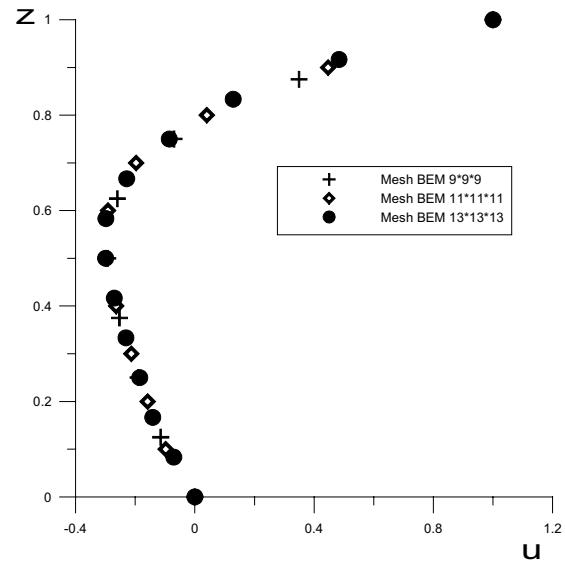


Figure 14 : Comparisons for different mesh sizes (u1)

to have a regular mesh. It is not necessary to perform the element connectivity, which is inevitable for the conventional finite element method. Besides, the present method is also not necessary to have the orthogonal properties, which is needed for the finite difference method. Actually, the only property we need for the solution procedure is the positions of collocation points and source points. Furthermore, the derivatives calculated by the direct differentiation of the solution series show good accuracy and can be applied to complex fluid dynamic problems even in a coarse mesh size.

For further application to the three-dimensional Navier-Stokes problem with low Reynolds number, this paper provides a preliminary work. Since the only differences between the Stokes and Navier-Stokes flow problem are the nonlinear convection term and the vortex-stretching term, $-\nabla \times (\vec{u} \times \vec{\omega})$, which can be directly imposed in the vorticity transport equation (equation (1a)). There is no any change at all as far as the computation of the velocity fields is concerned. Then, the resulted different procedure is an addition of the source terms in the diffusion equation (13), which is just the action of adding more source terms to the original source term of initial condition in the modified Helmholtz equation (15).

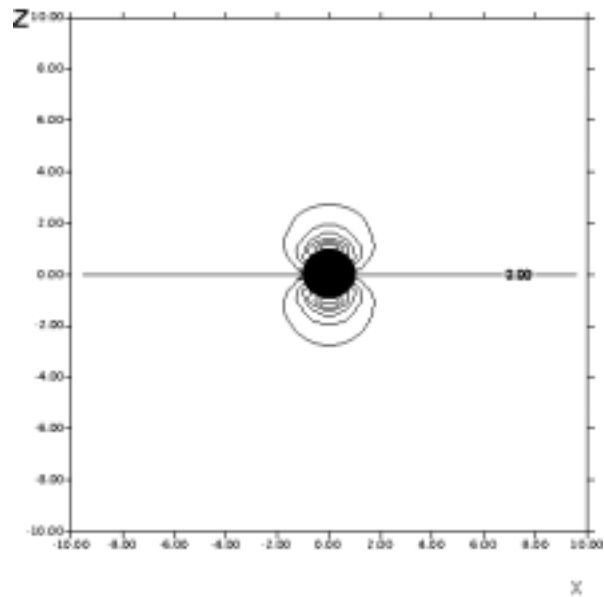


Figure 15 : Vorticity contour for flow passing a sphere

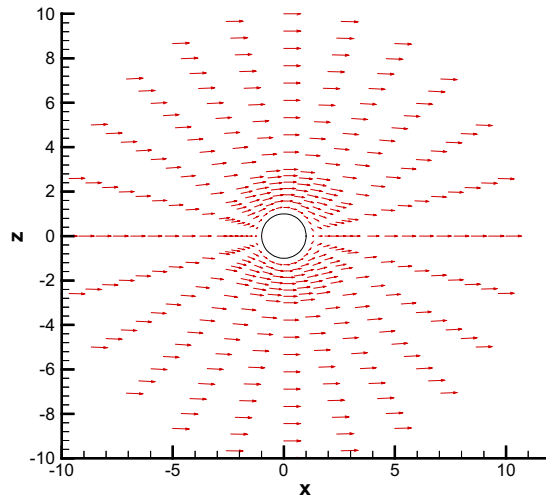


Figure 16 : Velocity profile for flow passing a sphere

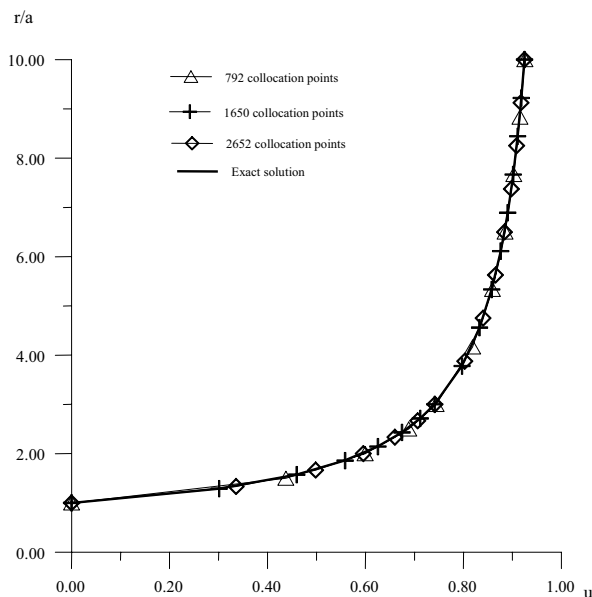


Figure 17 : Comparisons for the u velocity at the position of $1 \leq r/a \leq 10, \theta = \frac{\pi}{2}$ for different mesh sizes between the numerical and the exact solutions

6 Conclusions

A truly meshless boundary element method has been developed for solving the unsteady three-dimensional Stokes flow problem. The vorticity boundary conditions for the solution of vorticity transport equations are obtained directly from the derivatives term by term of the radial basis function series. Here, the result of cubic cavity problem is performed for different mesh sizes. The results show good convergence and generally in fairly close agreement with the results of other models. Also a numerical experiment of uniform flow passing a sphere, which has exact solution, has been performed. The result also shows good agreement. The computational results show that the meshless boundary element method presented here provides an efficient tool for the three-dimensional Stokes problems and is expected to be extended to the general incompressible viscous fluid flow problems which is governed by the Navier-Stokes equations.

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References

- Anderson, D.A.; Tannehill, J.C.; Pletcher, R.H.** (1984): *Computational fluid mechanics and heat transfer*, New York: McGraw-Hill, .
- Atluri, S.N. and Zhu, T.** (1998): A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. *Computational Mechanics*, vol. 22, pp. 117-127.
- Atluri, S.N. and Shen, S.** (2002): The Meshless Local Petrov-Galerkin (MLPG) Method: A simple & Less-Costly Alternative to the Finite Element and Boundary Element Methods. *CMES: Computer Modeling in Engineering and Sciences*, vol. 3, no. 1 pp. 11-51.
- Balakrishnan, K. and Ramachandrn, P.A.** (1999): A particular solution Trefftz method for non-linear Poisson problems in heat and mass transfer. *J. Comput. Phys.*, vol. 150, pp. 239-267.
- Chen, C.S.; Brebbia, C.A.; Power, H.** (1999): Dual reciprocity method using compactly supported radial ba-

- sis functions. *Commun. Numer. Meth. Engrg.*, vol. 15, pp. 137-150.
- Cheng, A.H.-D.; Young, D.L.; Tsai, C.C.** (2000): Solution of Poisson's equation by iterative DRBEM using compactly supported, positive definite radial basis function, *Engng. Anal. Bound. Elem.*, vol. 24, pp. 549-557.
- Currie, I.G.** (1993): *Fundamental mechanics of fluids*, McGraw-Hill Book Co.
- Dennis, S.C.R.; Ingham, D.B.; Cook, R.N.** (1979): Finite-difference methods for calculating steady incompressible flows in three dimensions, *J. Comput. Phys.*, vol. 33, pp. 325-339.
- Fairweather, G. and Karageorghis, A.** (1998): The method of fundamental solutions for elliptic boundary value problems. *Adv. Comput. Math.*, vol. 9, pp. 69-95.
- Golberg, M.A. and Chen, C.S.** (1998): The method of fundamental solutions for potential, Helmholtz and diffusion problems, in *Boundary Integral Methods – Numerical and Mathematical Aspects*, M.A. Golberg (Ed.), Computational Mechanics Publication, 99, pp. 103-176.
- Golberg, M.A. and Chen, C.S.** (2001): An efficient mesh-free method for nonlinear reaction-diffusion equations. *CMES: Computer Modeling in Engineering & Sciences*, Vol. 2(1), pp. 87-96.
- Golberg, M.A.; Chen, C.S.; Ganesh M.** (2000): Particular solutions of 3D Helmholtz-type equations using compactly supported radial basis functions, *Eng. Anal. Bound. Elem.*, vol. 24, pp. 539-547.
- Gunzburger, M.D.** (1987): Finite element theory and application, in D.L. Dwoyer, M.Y. Hussaini, R.G. Voigt (Ed.) *Proceedings of the ICASE/NASA Langley Workshop*, New York: Springer.
- Gunzburger, M.D. and Peterson, J.S.** (1988): On finite element approximations of the stream function-vorticity and velocity-vorticity equations. *Int. J. Numer. Meth. Fluids*, vol. 8, pp. 1229-1240.
- Kansa, E.J.** (1990a): Multiquadrics- a scattered data approximation scheme with applications to computational fluid dynamics-I. Surface approximations and partial derivative estimates. *Computers Math. Applic.*, vol. 19, pp. 127-145.
- Kansa, E.J.** (1990b): Multiquadrics- A scattered data approximation scheme with applications to computational fluid dynamics-II, *Computers Math. Applic.*, vol. 19, pp. 147-161.
- Kim, H.G. and Atluri, S.N.** (2000): Arbitrary placement of secondary nodes, and error control, in the meshless local Petrov-Galerkin (MLPG) method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1(3), pp. 11-32.
- Lin, H. and Atluri, S.N.** (2000): Meshless local Petrov-Galerkin (MLPG) method for convection-diffusion problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1(2), pp. 45-60.
- Lin, H. and Atluri, S.N.** (2001): The meshless local Petrov-Galerkin (MLPG) method for solving incompressible Navier-Stokes equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 2(2), pp. 117-142.
- Muleshkov, A.S.; Golberg, M.A.; Chen, C.S.** (1999): Particular solutions of Helmholtz-type operators using higher order polyharmonic splines. *Computational Mechanics*, vol. 23, pp. 411-419.
- Nardini, D. and Brebbia, C.A.** (1982): A new approach to free vibration analysis using boundary elements, in *Boundary Element Methods in Engineering*, C.A. Brebbia (Ed.), Springer-Verlag, Berlin.
- Power, H. and Wrobel, L.C.** (1995): *Boundary integral methods in fluid mechanics*. Southampton: Computational Mechanics Publication.
- Skerget, L. and Rek, Z.** (1995): Boundary-domain integral method using a velocity-vorticity formulation. *Engng. Anal. Bound. Elem.*, vol. 15, pp. 359-370.
- Tritton, D.J.** (1988): *Physical fluid dynamics*, Second edition, Oxford University Press.
- Tsai, C.C.; Young, D.L.; Cheng, A.H.-D.** (2001): An Iterative DRBEM for Three-Dimensional Poisson's Equation. *Boundary Element Technology, XIV*, pp. 323-332.
- Wordelman, C.J.; Aluru, N.R.; Ravaioli, U.** (2000): A meshless method for the numerical solution of the 2- and 3-D Semiconductor Poisson equation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1(1), pp. 121-126.
- Young, D.L.; Liu, Y.H.; Eldho, T.I.** (1999): Three-dimensional Stokes flow solution using combined boundary element and finite element methods. *The Chinese Journal of Mechanics*, vol. 15, pp. 169-176.
- Young, D.L.; Yang, S.K.; Eldho, T.I.** (2000): The so-

lution of vorticity-velocity Navier-Stokes equations using Eulerian-Lagrangian boundary element method. *Int. J. Numer. Meth. Fluids*, vol. 34, pp. 627-650.

Young, D.L.; Tsai, C.C.; Eldho, T.I.; Cheng, A.H.-D. (in press): Solution of Stokes flow using an iterative DRBEM based on compactly-supported, positive-definite radial basis function. *Computers Math. Applic.*.

Zhu, T.; Zhang, J.; Atluri, S.N. (1998): A meshless local boundary integral equation (LBIE) method for solving nonlinear problems. *Computational Mechanics*, vol. 22(2), pp. 174-186.