On Simple Formulations of Weakly-Singular Traction & Displacement BIE, and Their Solutions through Petrov-Galerkin Approaches

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Abstract: Using the directly derived non-hyper singular integral equations for displacement gradients [as in Okada, Rajiyah, and Atluri (1989a)], simple and straightforward derivations of weakly singular traction BIE's for solids undergoing small deformations are presented. A large number of "intrinsic properties" of the fundamental solutions in elasticity are developed, and are used in rendering the tBIE and dBIE to be only weakly-singular, in a very simple manner. The solutions of the weakly singular tBIE and dBIE through either global Petrov-Galerkin type "boundary element methods", or, alternatively, through the meshless local Petrov-Galerkin (MLPG) methods, are discussed. As special cases, the Galerkin type methods, which lead to symmetric systems of equations, are also discussed.

1 Introduction

In the past 25 years, much has been written about the integral equation formulations for the displacement and traction vectors in a solid body. Much of this work has been concentrated on linear elastic, homogenous, and isotropic solids. The focus in these derivations is on the "fundamental solution" in a linear elastic isotropic solid, viz., the Kelvin solution for a unit point load applied at an arbitrary location, in an arbitrary direction, in an infinite linear elastic solid. The Kelvin solution is well-understood, and is "singular". For clarity, we denote the various levels of singularity: If "r" is the distance between any arbitrary point ($\boldsymbol{\xi}$) in the solid and ($\mathbf{x}$) (the point at which the unit load is applied in a 3-dimensional solid), we denote the $\left(\frac{1}{r}\right)$ type singularities as being "weakly-singular", the $\left(\frac{1}{r^2}\right)$ type singularities as being "strongly-singular", the $\left(\frac{1}{r^3}\right)$ type singularities as being "hyper-singular".

In the Kelvin solution, for a 3-D solid, it is well-known that the displacement-vector is "weakly-singular", and the stress-tensor is "strongly-singular". In the classical formulation, the integral equation for the displacement vector at any point [$\mathbf{x} \in \Omega$ or $\partial\Omega$] is "strongly singular". If the displacement integral equation at any point $\mathbf{x} \in \Omega$, is differentiated with respect to $\mathbf{x} \in \Omega$, one may obtain an integral equation for $\nabla\mathbf{x}$ [$\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}$] for any $\mathbf{x} \in \Omega$. From this equation for the displacement gradients, one may derive a traction boundary integral equation [tBIE] at any $\mathbf{x} \in \Omega$ or $\partial\Omega$. It is seen that this tBIE is "hyper-singular". Much has been written in the last 10∼15 years on the "regularization" of the tBIE [i.e., render the "hyper-singular" tBIE into a "weakly-singular" tBIE], through what appears to be laborious mathematical exercises and "manipulations". This literature is too large to discuss here, but excellent summaries may be found in [Cruse and Richardson (1996); Bonnet, Maier, Polizzotto (1998); Li and Mear (1998)].

In this paper, we revisit the Kelvin solution, and delineate certain "fundamental properties" of this solution. By using the global-weak-form [or the weighted-residual-equation"] of the momentum balance laws of linear elasticity, corresponding to a point load, on which the Kelvin solution is based, we derive an arbitrary number of these "fundamental properties", by simply using an arbitrary number of different types of "test functions" in writing the weak-forms. These fundamental properties of the Kelvin solution, which otherwise has a $\left(\frac{1}{r^2}\right)$ singularity for tractions, are shown to be the key in-gradients in any "regularization" of the tBIE, which is derived by differentiating the strongly-singular displacement integral equation.

On the other hand, as far back as 1989, Okada, Rajiyah, and Atluri (1989a,b, 1991) have proposed a way to *di-*

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This paper is dedicated to the memory of Dr. John W. Lincoln of the U.S. Air Force Research Laboratory.

rectly derive integral equations for $\nabla \mathbf{u}$ $[\nabla = \mathbf{e}_i \dfrac{\partial}{\partial x_i}]$, rather than first derive the dBIE, and then differentiate it with respect to $\mathbf{x}$ as is most common is literature. Thus, one may also, from Okada, Rajiyah and Atluri (1989a,b), *directly derive* a tBIE. Thus, the directly derived tBIE [Okada, Rajiyah and Atluri (1989a,b)] is only "strongly-singular", as opposed to being "hyper-singular". It is shown in the present paper, that by using the "fundamental properties" of the Kelvin solution [which are also derived in the present paper], one may "regularize" the directly derived tBIE of Okada, Rajiyah and Atluri (1989a,b) in a very *straight-forward and simple manner*. In a like manner, it is shown here that the dBIE can also be "regularized" in a very straight-forward and simple manner.

It is also shown in this paper that the fundamental Kelvin solution for the stress-tensor can be naturally split into 2 parts, which we denote here as ϕ_{ij}^{*p} and ψ_{ij}^{*p}, respectively, and write $\sigma_{ij}^{*p} = \psi_{ij}^{*p} - \phi_{ij}^{*p}$, where σ_{ij}^{*p} is the Kelvin solution for stresses, ψ_{ij}^{*p} is divergence free, and the divergence of ϕ_{ij}^{*p} is the Dirac function.

We also discuss the numerical solution, by discretization, of the directly derived, tBIE of Okada, Rajiyah and Atluri (1989a,b), as well as of the regularized, weakly-singular dBIE. We write the general Petrov-Galerkin types of weak-forms of these integral equations at $\partial\Omega$. Thus, we introduce an arbitrary test function $\mathbf{w}(\mathbf{x})$, $\mathbf{x} \in \partial\Omega$. If $\mathbf{w}(\mathbf{x})$ us a Dirac function, and if the trial functions $\mathbf{t}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ are interpolated in terms of their nodal values over a contiguous ("non-overlapping") set of elements at $\partial\Omega$ ["boundary elements"], one obtains the popular "Boundary Element Methods". On the other hand, if $\mathbf{w}(\mathbf{x})$ is chosen to be the same as the complementary [or energy-conjugate] trial function [i.e., in the tBIE, we use $\mathbf{u}(\mathbf{x})$ as the test function, and in the dBIE, we use $\mathbf{t}(\mathbf{x})$ as the test function], we obtain the so-called "Symmetric Galerkin Approaches" for dBIE and tBIE [Bonnet, Maier, Polizzotto (1998)]. On the other hand, one may leave $\mathbf{w}(\mathbf{x})$ as arbitrary, and formulate a general Petrov-Galerkin Approach. It is further shown that, in the "Symmetric Galerkin" approach to solving the directly derived tBIE [Okada, Rajiyah and Atluri (1989a,b)], using the natural split of σ_{ij}^{*p}, and the use of the Stokes' theorem at $\partial\Omega$ when $\mathbf{w}(\mathbf{x})$ is a continuous function, the resulting discrete formulation results in certain further algebraic conveniences. These end results are found to be somewhat similar to the results in Li and Mear (1998), but the present results are different from those in Li and Mear (1998) in terms of the attendant kernel functions. However, the present formulations of the Patrov-Galerkin approaches to solving the directly derived tBIE [Okada, Rajiyah and Atluri (1989a,b)] are simple and straight-forward.

The structure of the paper is as follows. In Section 1, we briefly discuss the well-known Galerkin vector potential for displacements is an elastic solid undergoing small displacements. In Section 2, we derive the fundamental solution σ_{ij}^{*p} for a point load in an infinite body, and point out how it can be split into ϕ_{ij}^{*p} and ψ_{ij}^{*p}. In Section 3, we derive, following Okada, Rajiyah and Atluri (1989a,b), the displacement equations (dBIE), and *directly derive* the tBIE without differentiating the dBIE. In deriving these, we use the notion of "unsymmetric weak-forms" of differential equations, as first noted in Atluri (1985). In Section 4, we derive a large number of basic properties of σ_{ij}^{*p}, by using the weak-forms [with different test functions], of the balance laws for σ_{ij}^{*p}. We point out how these methods can be used to derive the "basic properties" of the fundamental solutions for *any* problems of mathematical physics [fluid mechanics, acoustics, electromagnetism, etc]. In Section 5, we discuss the "regularization" of tBIE, using these properties. We also discuss the Petrov-Galerkin approach to discretize the "regularized" tBIE. In Section 6, we discuss the "regularization" of dBIE, and the Petrov-Galerkin schemes to discretize these regularized dBIE. In Section 7, we consider the important practical issue of the evaluation of displacements and stresses near the surface. In Section 8, we briefly mention the forthcoming papers of the authors, which build upon the present results and to develop Meshless Local Petrov Galerkin [MLPG] approaches [Atluri and Zhu (1998), Atluri and Shen (2002a,b)].

2 Galerkin Vector Potential for Displacements in an Elastic Solid Undergoing Small Deformations

Consider a linear elastic, homogeneous, isotropic body in a domain Ω, with boundary $\partial\Omega$. The Lame' constants of the linear elastic isotropic body are λ and μ; and the corresponding Young's modulus and Poisson's ratio are E and υ, respectively. We use Cartesian coordinates ξ_i, and the attendant base vectors $\mathbf{e}_i$, to describe the geometry in Ω. The solid is assumed to undergo infinitesimal deformations. The displacement vector, strain-tensor, and the stress-tensor in the elastic body are denoted as $\mathbf{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, respectively, with the corresponding dyadic representations, as follows:

$$\mathbf{u} = u_i\mathbf{e}_i \tag{1}$$

$$\boldsymbol{\varepsilon} = \varepsilon_{ij}\mathbf{e}_i\mathbf{e}_j \tag{2}$$

$$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i\mathbf{e}_j \tag{3}$$

The equations of balance of linear and angular momentum can be written as:

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{f}=\mathbf{0};\quad \boldsymbol{\sigma}=\boldsymbol{\sigma}^{\mathrm{t}};\quad \nabla=\mathbf{e}_i\frac{\partial}{\partial\xi_i} \tag{4}$$

$$\sigma_{ij,i}+f_j=0;\quad \sigma_{ij}=\sigma_{ji} \tag{5}$$

The strain-displacement relations are:

$$\boldsymbol{\varepsilon}=\frac{1}{2}\nabla\mathbf{u}+\mathbf{u}\nabla);\quad \varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{6}$$

The constitutive relations of an isotropic linear elastic homogeneous solid are:

$$\boldsymbol{\sigma}=\lambda\mathbf{I}(\nabla\cdot\mathbf{u})+2\mu\,\boldsymbol{\varepsilon} \tag{7a}$$

$$=\lambda\mathbf{I}(\nabla\cdot\mathbf{u})+\mu(\nabla\mathbf{u}+\mathbf{u}\nabla) \tag{7b}$$

$$=\mu[\frac{1}{A}\mathbf{I}(\nabla\cdot\mathbf{u})+\nabla\mathbf{u}+\mathbf{u}\nabla-2\mathbf{I}(\nabla\cdot\mathbf{u})] \tag{7c}$$

where

$$A=\frac{\mu}{\lambda+2\mu}=\frac{1-2\upsilon}{2(1-\upsilon)} \tag{8}$$

It is well known [Fung & Tong (2001)] that the displacement vector, which is a continuous function of $\boldsymbol{\xi}$, can be derived, in general, from the Galerkin vector potential $\boldsymbol{\varphi}$ such that:

$$\mathbf{u} = \nabla^2\boldsymbol{\varphi}-\frac{1}{2(1-\upsilon)}\nabla(\nabla\cdot\boldsymbol{\varphi}) \tag{9a}$$

$$= A\nabla\Phi+\nabla\times\boldsymbol{\Psi} \tag{9b}$$

$$= A\nabla(\nabla\cdot\boldsymbol{\varphi})-\nabla\times\nabla\times\boldsymbol{\varphi} \tag{9c}$$

$$= A\nabla(\nabla\cdot\boldsymbol{\varphi})+\nabla^2\boldsymbol{\varphi}-\nabla(\nabla\cdot\boldsymbol{\varphi}) \tag{9d}$$

$$= \mathbf{u}^\Phi+\mathbf{u}^\Psi \tag{9e}$$

where, by definition,

$$\mathbf{u}^\Phi=A\nabla\Phi=A\nabla(\nabla\cdot\boldsymbol{\varphi}) \tag{10}$$

$$\mathbf{u}^\Psi=\nabla\times\boldsymbol{\Psi}=-\nabla\times\nabla\times\boldsymbol{\varphi}=\nabla^2\boldsymbol{\varphi}-\nabla(\nabla\cdot\boldsymbol{\varphi}) \tag{11}$$

where Φ is a scalar potential, and $\boldsymbol{\Psi}$ is a vector potential, such that:

$$\Phi=\nabla\cdot\boldsymbol{\varphi} \tag{12}$$

$$\boldsymbol{\Psi}=-(\nabla\times\boldsymbol{\varphi}) \tag{13}$$

The displacements $\mathbf{u}^\Phi$ and $\mathbf{u}^\Psi$ have the properties,

$$\nabla\times\mathbf{u}^\Phi=A\nabla\times\nabla\Phi=0 \tag{14}$$

$$\nabla\cdot\mathbf{u}^\Psi=\nabla\cdot\nabla\times\boldsymbol{\Psi}=0 \tag{15}$$

Using (7c), (9c), (12)-(15) in (4)a, it is easily found (in the absence of body force $\mathbf{f}$) that:

$$\nabla\cdot\boldsymbol{\sigma}=\mu\nabla^2\nabla^2\boldsymbol{\varphi}=\mathbf{0}\quad\text{or}\quad\nabla^2\nabla^2\boldsymbol{\varphi}=\mathbf{0} \tag{16}$$

since

$$\nabla\mathbf{u}^\Phi+\mathbf{u}^\Phi\nabla-2\mathbf{I}\nabla\cdot\mathbf{u}^\Phi =A(\nabla^2\Phi+\nabla\Phi\nabla-2\mathbf{I}\nabla^2\Phi)=0 \tag{17}$$

For a curl-free solution, in which:

$$\mathbf{u}=\mathbf{u}^\Phi=A\nabla\Phi\quad\text{and}\quad\mathbf{u}^\Psi=\mathbf{0} \tag{18}$$

the balance equation (4) is simplified, as:

$$\nabla\nabla^2\Phi=\mathbf{0}\quad\text{or}\quad\nabla^2\Phi=C \tag{19}$$

3 Fundamental Solutions in a Linear Elastic Isotropic Homogeneous Infinite Medium

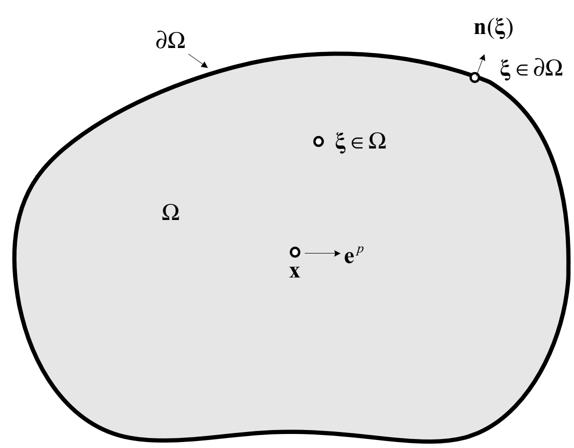

Figure 1 : A solution domain with source point $\mathbf{x}$ and target point $\boldsymbol{\xi}$

Consider a point unit load applied in an arbitrary direction $\mathbf{e}^p$ at a generic location $\mathbf{x}$ in a linear elastic isotropic homogeneous infinite medium as shown in Fig. 1. It is well-known [Fung and Tong (2001)] that the displacement solution corresponding to this unit point load is given by the Galerkin vector displacement potential:

$$\boldsymbol{\varphi}^{*p} = (1-\upsilon)F^*\mathbf{e}^p \tag{20}$$

where

$$F^* = \frac{r}{8\pi\mu(1-\upsilon)} \quad \text{for 3D problems} \tag{21a}$$

and

$$F^* = \frac{-r^2\ln r}{8\pi\mu(1-\upsilon)} \quad \text{for 2D problems} \tag{21b}$$

where $r = \|\boldsymbol{\xi}-\mathbf{x}\|$

Thus, $\boldsymbol{\varphi}^{*p}$ in (20) is the solution, in infinite space, to the differential equation (in the coordinates $\boldsymbol{\xi}$),

$$\begin{aligned} &\mu\nabla^2\nabla^2\boldsymbol{\varphi}^{*p} + \delta(\mathbf{x},\boldsymbol{\xi})\mathbf{e}^p = \mathbf{0} \\ &\nabla\cdot\boldsymbol{\sigma}(\mathbf{x},\boldsymbol{\xi}) + \delta(\mathbf{x},\boldsymbol{\xi})\mathbf{e}^p = \mathbf{0} \\ &\nabla = \mathbf{e}_i\frac{\partial}{\partial\xi_i} \end{aligned} \tag{22}$$

or

$$\mu(1-\upsilon)\nabla^2\nabla^2F^* + \delta(\mathbf{x},\boldsymbol{\xi}) = 0 \tag{23}$$

The corresponding displacements are derived from the Galerkin vector displacement potential, using (9a), as:

$$u_i^{*p}(\mathbf{x},\boldsymbol{\xi}) = (1-\upsilon)\delta_{pi}F^*_{,kk} - \frac{1}{2}F^*_{,pi} \tag{24}$$

The gradients of the displacements in (24) are:

$$u_{i,j}^{*p}(\mathbf{x},\boldsymbol{\xi}) = (1-\upsilon)\delta_{pi}F^*_{,kkj} - \frac{1}{2}F^*_{,pij} \tag{25}$$

The corresponding stresses in a linear elastic homogenous isotropic body are given by:

$$\begin{aligned} \sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) &\equiv E_{ijkl}\varepsilon_{kl}^{*p} \equiv E_{ijkl}u_{k,l}^{*p} \\ &= \mu[(1-\upsilon)(\delta_{pi}F^*_{,kkj} + \delta_{pj}F^*_{,kki}) + \upsilon\delta_{ij}F^*_{,pkk} - F^*_{,pij}] \\ &= \mu[(1-\upsilon)\delta_{pi}F^*_{,kkj} + \upsilon\delta_{ij}F^*_{,pkk} - F^*_{,pij}] \\ &\quad + \mu(1-\upsilon)\delta_{pj}F^*_{,kki} \end{aligned} \tag{26}$$

These stresses are seen to satisfy the balance laws:

$$\begin{aligned} &\sigma_{ij,i}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \mu(1-\upsilon)\delta_{pj}F^*_{,kkii} = -\delta_{pj}\delta(\mathbf{x},\boldsymbol{\xi}); \\ &\sigma_{ij}^{*p} = \sigma_{ji}^{*p} \end{aligned} \tag{27}$$

We define two functions ϕ_{ij}^{*p} and ψ_{ij}^{*p}, as

$$\phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) \equiv -\mu(1-\upsilon)\delta_{pj}F^*_{,kki} \tag{28a}$$

$$\begin{aligned} \psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) &\equiv \sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) + \phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) \\ &= \mu[(1-\upsilon)\delta_{pi}F^*_{,kkj} + \upsilon\delta_{ij}F^*_{,pkk} - F^*_{,pij}] \end{aligned} \tag{28b}$$

Then, from (27), (28a) and (28b), it can be seen that:

$$\begin{aligned} \phi_{ij,i}^{*p}(\mathbf{x},\boldsymbol{\xi}) &= -\sigma_{ij,i}^{*p}(\mathbf{x},\boldsymbol{\xi}) \\ &= -\mu(1-\upsilon)\delta_{pj}F^*_{,kkii} = \delta_{pj}\delta(\mathbf{x},\boldsymbol{\xi}) \end{aligned} \tag{29a}$$

$$\psi_{ij,i}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \mathbf{0} \quad [\text{ divergence of } \boldsymbol{\psi}^*(\mathbf{x},\boldsymbol{\xi}) = \mathbf{0}] \tag{29b}$$

Hence, as a divergence free tensor, $\boldsymbol{\psi}^*(\mathbf{x},\xi)$ must be a curl of another divergence free tensor. We choose to rewrite it in term of F^* from Eq. (28b), as:

$$\begin{aligned} \psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) &= \mu[(1-\upsilon)\delta_{pi}F^*_{,kkj} + \upsilon\delta_{ij}F^*_{,pkk} - F^*_{,pij}] \\ &= \mu[(1-\upsilon)(\delta_{pi}F^*_{,kkj} - \delta_{ij}F^*_{,pkk}) + \delta_{ij}F^*_{,pkk} - F^*_{,pij}] \\ &= \mu[(1-\upsilon)(\delta_{pi}\delta_{js}F^*_{,kks} - \delta_{ij}\delta_{ps}F^*_{,kks}) \\ &\quad - (\delta_{ki}\delta_{js}F^*_{,pks} - \delta_{ij}\delta_{ks}F^*_{,pks})] \\ &= \mu[(1-\upsilon)e_{tpj}e_{tis}F^*_{,kks} - e_{tkj}e_{tis}F^*_{,pks}] \\ &= \mu e_{tis}[(1-\upsilon)e_{tpj}F^*_{,kk} - e_{tkj}F^*_{,pk}]_{,s} \\ &\equiv e_{ist}G_{tj,s}^{*p} \end{aligned} \tag{30}$$

where, by definition,

$$G_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \mu[(1-\upsilon)e_{ipj}F_{,kk}^{*} - e_{ikj}F_{,pk}^{*}] \tag{31}$$

In a 3-dimensional linear elastic homogeneous body we can easily derive the derivatives of F^*, from (21a), as:

$$F_{,p}^{*} = \frac{r_{,p}}{8\pi\mu(1-\upsilon)} \tag{32a}$$

$$F_{,pi}^{*} = \frac{1}{8\pi\mu(1-\upsilon)r}(\delta_{pi} - r_{,p}r_{,i}) \tag{32b}$$

$$F_{,kk}^{*} = \frac{1}{4\pi\mu(1-\upsilon)r} \tag{32c}$$

$$F_{,pij}^{*} = -\frac{1}{8\pi\mu(1-\upsilon)r^2}(\delta_{pi}r_{,j} + \delta_{pj}r_{,i} + \delta_{ij}r_{,p} - 3r_{,p}r_{,i}r_{,j}) \tag{32d}$$

$$F_{,kki}^{*} = -\frac{1}{4\pi\mu(1-\upsilon)r^2}r_{,i} \tag{32e}$$

$$F_{,kkij}^{*} = -\frac{1}{4\pi\mu(1-\upsilon)r^3}(\delta_{ij} - 3r_{,i}r_{,j}) \tag{32f}$$

and for a 2-dimensional body,

$$F_{,p}^{*} = -\frac{1}{8\pi\mu(1-\upsilon)}(r + 2r\ln r)r_{,p} \tag{33a}$$

$$F_{,pi}^{*} = -\frac{1}{8\pi\mu(1-\upsilon)}[\delta_{pi}(1+2\ln r) + 2r_{,p}r_{,i}] \tag{33b}$$

$$F_{,kk}^{*} = -\frac{1}{2\pi\mu(1-\upsilon)}(1+\ln r) \tag{33c}$$

$$F_{,pij}^{*} = -\frac{1}{4\pi\mu(1-\upsilon)r}(\delta_{pi}r_{,j} + \delta_{pj}r_{,i} + \delta_{ij}r_{,p} - 2r_{,p}r_{,i}r_{,j}) \tag{33d}$$

$$F_{,kki}^{*} = -\frac{1}{2\pi\mu(1-\upsilon)r}r_{,i} \tag{33e}$$

$$F_{,kkij}^{*} = -\frac{1}{2\pi\mu(1-\upsilon)r^2}(\delta_{ij} - 2r_{,i}r_{,j}) \tag{33f}$$

Thus, from (28a), (28b) and (30), one may see that:

$$\begin{aligned}\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) &= \psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) - \phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) \\ &= \mu(1-\upsilon)\delta_{pj}F_{,kki}^{*} \\ &\quad + \mu e_{ist}[(1-\upsilon)e_{tpj}F_{,kk}^{*} - e_{tkj}F_{,pk}^{*}]_{,s}\end{aligned} \tag{34}$$

From (32b), (32c), (32), the singularity in each of the terms in Eq. (34) can be seen, for 3D problems, as:

$$F_{,kki}^{*} \propto O(\frac{1}{r^2}) \tag{35a}$$

$$F_{,kk}^{*} \propto O(\frac{1}{r}) \tag{35b}$$

$$F_{,pk}^{*} \propto O(\frac{1}{r}) \tag{35c}$$

and for 2D problems as:

$$F_{,kki}^{*} \propto O(\frac{1}{r}) \tag{36a}$$

$$F_{,kk}^{*} \propto O(\ln r) \tag{36b}$$

$$F_{,pk}^{*} \propto O(\ln r) \tag{36c}$$

We write the kernel functions for 3D problems, from Eq. (32), as:

$$u_i^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{16\pi\mu(1-\upsilon)r}[(3-4\upsilon)\delta_{ip} + r_{,i}r_{,p}] \tag{37}$$

$$G_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{8\pi(1-\upsilon)r}[(1-2\upsilon)e_{ipj} + e_{ikj}r_{,k}r_{,p}] \tag{38}$$

$$\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{8\pi(1-\upsilon)r^2}[(1-2\upsilon)(\delta_{ij}r_{,p} - \delta_{ip}r_{,j} - \delta_{jp}r_{,i}) - 3r_{,i}r_{,j}r_{,p}] \tag{39}$$

and for 2D plane strain problems, from Eq. (33), as:

$$u_i^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{8\pi\mu(1-\upsilon)}[-(3-4\upsilon)\ln r\,\delta_{ip} + r_{,i}r_{,p}] \tag{40}$$

$$G_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{4\pi(1-\upsilon)}[-(1-2\upsilon)\ln r\, e_{ipj} + e_{ikj}r_{,k}r_{,p}] \tag{41}$$

$$\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{4\pi(1-\upsilon)r}[(1-2\upsilon)(\delta_{ij}r_{,p} - \delta_{ip}r_{,j} - \delta_{jp}r_{,i}) - 2r_{,i}r_{,j}r_{,p}] \tag{42}$$

4 Displacement & Traction BIE: Derivations From Unsymmetric Weak Forms of Balance Laws in Elasticity

The governing equations of momentum balance in a solid undergoing small displacements are

$$\sigma_{ij,i}+f_j=0;\quad \sigma_{ij}=\sigma_{ji}\quad ()_{,i}\equiv\frac{\partial}{\partial\xi_i} \tag{43}$$

For the present, we ignore the body forces f_i (but include them later, when necessary). Thus, (43) is reduced to:

$$\sigma_{ij,i}=0\quad \text{in } \Omega \tag{44}$$

For a homogeneous linear elastic isotropic homogeneous solid, the constitutive equation is

$$\sigma_{ij}=E_{ijkl}\varepsilon_{kl}=E_{ijkl}u_{k,l} \tag{45}$$

where

$$\varepsilon_{kl}=\frac{1}{2}(u_{k,l}+u_{l,k}) \tag{46}$$

and

$$E_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{47}$$

with λ and μ being the Lame's constants.

Let u_i be the trial functions for displacements, to satisfy Eq. (44), in terms of u_i, when Eqs. (45)-(47) are used. Let $\overline{u}_j$ be the test functions to satisfy the momentum balance laws in terms of u_i, in a weak form. The weak form of the equilibrium Eq. (44) can then be written as,

$$\int_\Omega \sigma_{ij,i}\overline{u}_j d\Omega\equiv\int_\Omega (E_{ijmn}u_{m,n})_i\overline{u}_j d\Omega=0 \tag{48}$$

Applying the divergence theorem two times[2] in Eq. (48), we obtain:

$$\begin{aligned}0&=\int_\Omega E_{ijmn}u_{m,ni}\overline{u}_j\,d\Omega\\ &=\int_{\partial\Omega} n_iE_{ijmn}u_{m,n}\overline{u}_j\,dS-\int_\Omega E_{ijmn}u_{m,n}\overline{u}_{j,i}\,d\Omega\\ &=\int_{\partial\Omega} n_iE_{ijmn}u_{m,n}\overline{u}_j\,dS-\int_{\partial\Omega} n_nE_{ijmn}u_m\overline{u}_{j,i}\,dS\\ &\quad+\int_\Omega E_{ijmn}u_m\overline{u}_{j,in}\,d\Omega\\ &=\int_{\partial\Omega} n_iE_{ijmn}u_{m,n}\overline{u}_j\,dS-\int_{\partial\Omega} n_nE_{ijmn}u_m\overline{u}_{j,i}\,dS\\ &\quad+\int_\Omega u_m(E_{ijmn}\overline{u}_{j,i})_{,n}\,d\Omega\end{aligned} \tag{49}$$

Instead of the *scalar* weak form of Eq. (44), as in Eq. (48), we may also write a *vector* weak form of Eq. (44), by using the tensor test functions $\overline{u}_{i,k}$ [as originally proposed in Okada, Rajiyah, and Atluri (1988,1989)] as:

$$\int_\Omega \sigma_{ij,i}\overline{u}_{j,k}d\Omega=0\quad k=1,2,3 \tag{50}$$

By applying divergence theorem *three times* in Eq. (50), we may write:

$$\begin{aligned}0&=\int_\Omega E_{ijmn}u_{m,ni}\overline{u}_{j,k}\,d\Omega\\ &=\int_{\partial\Omega} n_iE_{ijmn}u_{m,n}\overline{u}_{j,k}\,dS-\int_\Omega E_{ijmn}u_{m,n}\overline{u}_{j,ki}\,d\Omega\\ &=\int_{\partial\Omega} n_iE_{ijmn}u_{m,n}\overline{u}_{j,k}\,dS-\int_{\partial\Omega} n_kE_{ijmn}u_{m,n}\overline{u}_{j,i}\,dS\\ &\quad+\int_\Omega E_{ijmn}u_{m,nk}\overline{u}_{j,i}\,d\Omega\\ &=\int_{\partial\Omega} n_iE_{ijmn}u_{m,n}\overline{u}_{j,k}\,dS-\int_{\partial\Omega} n_kE_{ijmn}u_{m,n}\overline{u}_{j,i}\,dS\\ &\quad+\int_{\partial\Omega} n_nE_{ijmn}u_{m,k}\overline{u}_{j,i}\,dS-\int_\Omega E_{ijmn}u_{m,k}\overline{u}_{j,in}\,d\Omega\\ &=\int_{\partial\Omega} n_iE_{ijmn}u_{m,n}\overline{u}_{j,k}\,dS-\int_{\partial\Omega} n_kE_{ijmn}u_{m,n}\overline{u}_{j,i}\,dS\\ &\quad+\int_{\partial\Omega} n_nE_{ijmn}u_{m,k}\overline{u}_{j,i}\,dS-\int_\Omega u_{m,k}(E_{ijmn}\overline{u}_{j,i})_{,n}\,d\Omega\end{aligned} \tag{51}$$

By taking the fundamental solution $u_i^{*p}(\mathbf{x},\boldsymbol{\xi})$ as the test function $\overline{u}_i(\boldsymbol{\xi})$, and with the consideration of its properties in Eq. (22), we re-write Eqs. (49), (51), respectively,

[2] If we use the divergence theorem only once in Eq. (48), we obtain the "symmetric" weak form:

$\int_\Omega \sigma_{ij}\overline{u}_{j,i}d\Omega-\int_{\partial\Omega} n_i\sigma_{ij}\overline{u}_j d\Omega=0$

Thus, in the symmetric weak form, both the trial functions u_i as well as the test functions $\overline{u}_j$ are only required to be once differ entiable. However, in the "unsymmetric weak form" of Eq. (49), the test functions $\overline{u}_j$ in Ω are required be twice-differentiable, while there is no differentiability requirement on u_i in Ω.

as,

$$u_p(\mathbf{x}) = \int_{\partial\Omega} n_i(\boldsymbol{\xi}) E_{ijmn} u_{m,n}(\boldsymbol{\xi}) u_j^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} n_n(\boldsymbol{\xi}) E_{ijmn} u_m(\boldsymbol{\xi}) u_{j,i}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS \equiv \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_j^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} u_m(\boldsymbol{\xi}) t_m^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS \quad (52)$$

$$-u_{p,k}(\mathbf{x}) = \int_{\partial\Omega} n_i(\boldsymbol{\xi}) E_{ijmn} u_{m,n}(\boldsymbol{\xi}) u_{j,k}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} n_k \boldsymbol{\xi}) E_{ijmn} u_{m,n}(\boldsymbol{\xi}) u_{j,i}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS + \int_{\partial\Omega} n_n(\boldsymbol{\xi}) E_{ijmn} u_{m,k}(\boldsymbol{\xi}) u_{j,i}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS \quad (53)$$

Eqs. (52) and (53) were originally given in [Okada, Rajiyah, and Atluri (1989a,b)], and the notion of using unsymmetric weak-forms of the differential equations, to obtain integral representations for displacements, was presented in [Atluri (1985)]. It should be noted that the integral equations for $u_p(\mathbf{x})$ and $u_{p,k}(\mathbf{x})$ as in Eqs. (52) and (53) are derived independently of each other. On the other hand, if we derive the integral equation for displacement-gradients, by directly differentiating $u_p(\mathbf{x})$ in Eq. (52), i.e. by differentiating,

$$u_p(\mathbf{x}) = \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_j^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} u_m(\boldsymbol{\xi}) t_m^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS$$

with respect to x_k, we obtain:

$$u_{p,k}(\mathbf{x}) = \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_{j,k}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} u_m(\boldsymbol{\xi}) t_{m,k}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS \quad (54)$$

Thus, Eq. (54) is hypersingular, since $t_{m,k}^{*p}(\mathbf{x},\boldsymbol{\xi})$ is of $O(r^{-3})$ for 3D problems. On the other hand, the directly derived integral equations for $u_{p,k}(\mathbf{x},\boldsymbol{\xi})$ as in Eq. (53) contain only singularities of $O(r^{-2})$.

Eq. (52) is the original displacement BIE (dBIE) in its strongly-singular form before any regularization. On the other hand, Eq. (53) are the non-hypersingular ("strongly-singular") integral equations for displacement gradients in a homogeneous linear elastic solid, as originally derived in Okada, Rajiyah, and Atluri (1989a,b). It is but a simple extension to derive a non-hypersingular integral equation for tractions in a linear elastic solid, from Eq. (53),

$$-E_{abpk} u_{p,k}(\mathbf{x}) = E_{abpk} \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_{j,k}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS + E_{abpk} \int_{\partial\Omega} [n_n(\boldsymbol{\xi}) u_{m,k}(\boldsymbol{\xi}) - n_k(\boldsymbol{\xi}) u_{m,n}(\boldsymbol{\xi})] \sigma_{nm}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS = E_{abpk} \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_{p,k}^{*j}(\mathbf{x},\boldsymbol{\xi})\, dS + E_{abpk} \int_{\partial\Omega} D_t u_m(\boldsymbol{\xi}) e_{nkt} \sigma_{nm}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS \quad (55)$$

where the surface tangential operator D_t is defined as,

$$D_t = n_r e_{rst} \frac{\partial}{\partial \xi_s} \quad (56)$$

Then Eq. (55) can be re-written as,

$$-\sigma_{ab}(\mathbf{x}) = \int_{\partial\Omega} t_q(\boldsymbol{\xi}) \sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\, dS + \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi}) e_{nlp} E_{abkl} \sigma_{nq}^{*k}(\mathbf{x},\boldsymbol{\xi})\, dS \equiv \int_{\partial\Omega} t_q(\boldsymbol{\xi}) \sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\, dS + \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi}) \Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS \quad (57)$$

where by definition,

$$\Sigma_{ijpq}^{*}(\mathbf{x},\boldsymbol{\xi}) = E_{ijkl} e_{nlp} \sigma_{nq}^{*k}(\mathbf{x},\boldsymbol{\xi}) \quad (58)$$

Contracting Eq. (57) with $n_a(\mathbf{x})$, we have

$$-t_b(\mathbf{x}) = \int_{\partial\Omega} t_q(\boldsymbol{\xi}) n_a(\mathbf{x}) \sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\, dS + \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi}) n_a(\mathbf{x}) \Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS \quad (59)$$

where the traction is defined as,

$$t_b(\mathbf{x}) = n_a(\mathbf{x}) \sigma_{ab}(\mathbf{x}) \quad (60)$$

5 Some Basic Properties of the Fundamental Solution

Consider a body of an infinite extent, subject to a point force at a generic location $\mathbf{x}$ in the direction $\mathbf{e}^p$, as shown in Fig. 1. The fundamental solution, in infinite space, of the stress field, denoted by $\boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi})$, at any point $\boldsymbol{\xi}$ due to this point load at $\mathbf{x}$, is generated by the balance law, from Eq. (22):

$$\nabla \cdot \boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi}) + \delta(\mathbf{x},\boldsymbol{\xi})\mathbf{e}^p = \mathbf{0} \tag{61}$$

We write the weak form of Eq. (61) over the domain, using a constant c as a test function, as

$$\int_\Omega \nabla \cdot \boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi})\, c\, d\Omega + \mathbf{e}^p c = \mathbf{0} \tag{62a}$$

or

$$\int_{\partial\Omega} \mathbf{n}(\boldsymbol{\xi}) \cdot \boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi}) dS + \mathbf{e}^p = \mathbf{0} \tag{62b}$$

or

$$\int_{\partial\Omega} n_i(\boldsymbol{\xi})\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) dS + \delta_{pj} = 0 \tag{62c}$$

or

$$\int_{\partial\Omega} t_j^{*p}(\mathbf{x},\boldsymbol{\xi}) dS + \delta_{pj} = 0 \quad \mathbf{x} \in \Omega \tag{62d}$$

Eq. (62d) is a "basic identity" of the fundamental solution $\boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi})$. Eq. (62d) is simply an affirmation of the force balance law for Ω: if the point load is applied at a point $\mathbf{x} \in \Omega$ when Ω is entirely embedded in an infinite space, the tractions exerted by the surrounding infinite body on the finite-body Ω should be equilibrated with the applied point force at $\mathbf{x}$ inside Ω.

Secondly, if we consider an arbitrary function $\mathbf{u}(\mathbf{x})$ in Ω as the test function, we may write the corresponding weak-form, from Eq.(61), as:

$$\int_\Omega [\nabla \cdot \boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi}) + \delta(\mathbf{x},\boldsymbol{\xi})\mathbf{e}^p] \cdot \mathbf{u}(\mathbf{x}) d\Omega = \mathbf{0} \tag{63a}$$

or

$$\int_{\partial\Omega} [\mathbf{n}(\boldsymbol{\xi}) \cdot \boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi})] \cdot \mathbf{u}(\mathbf{x}) dS + \mathbf{e}^p \cdot \mathbf{u}(\mathbf{x}) = \mathbf{0} \tag{63b}$$

or

$$\int_{\partial\Omega} \mathbf{t}^{*p}(\mathbf{x},\boldsymbol{\xi}) \cdot \mathbf{u}(\mathbf{x}) dS + \mathbf{e}^p \cdot \mathbf{u}(\mathbf{x}) = \mathbf{0} \quad \mathbf{x} \in \Omega \tag{63c}$$

or

$$\int_{\partial\Omega} t_j^{*p}(\mathbf{x},\boldsymbol{\xi}) u_j(\mathbf{x}) dS + u_p(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega \tag{63d}$$

Once the point $\mathbf{x}$ approaches a smooth boundary, i.e. $\mathbf{x} \in \partial\Omega$, the first term in Eq. (62d) can be written in a Cauchy Principal value (CPV) integral, denoted by $\int^{CPV}$, as,

$$\lim_{\mathbf{x}\to\partial\Omega} \int_{\partial\Omega} t_j^{*p}(\mathbf{x},\boldsymbol{\xi}) dS = \int_{\partial\Omega}^{CPV} t_j^{*p}(\mathbf{x},\boldsymbol{\xi}) dS - \frac{1}{2}\delta_{pj} \tag{64a}$$

and thus, one obtains:

$$\int_{\partial\Omega}^{CPV} t_j^{*p}(\mathbf{x},\boldsymbol{\xi}) dS + \frac{1}{2}\delta_{pj} = 0 \quad \mathbf{x} \in \partial\Omega \tag{64b}$$

The second term on the right hand-side of Eq. (64a) results from the principal value of the singular integral involving t_j^{*p}, which has a $O\left(\frac{1}{r^2}\right)$ singularity. Eq. (64b) may also be physically explained as below. σ_{ij}^{*p} (and thus t_j^{*p}) are solutions due to a point load applied in an infinite space. In reality, the point load can be assumed to be distributed over a small-sphere, of radius ε, in an infinite body. The tractions distributed over this sphere, that result in a point load, are of $O\left(\frac{1}{\varepsilon^2}\right)$; while the surface area of the sphere is $O(\varepsilon^2)$. As long as this sphere is inside Ω, and while Ω is a part of the infinite space, the load applied on Ω is still unity. Suppose $\mathbf{x} \to \hat{\mathbf{x}}$ at $\partial\Omega$ shown in Fig. 2, then the sphere of radius ε is centered at the boundary. As long as the boundary is smooth, only one-half of the sphere of radius ε is actually inside Ω, when $\mathbf{x} \to \hat{\mathbf{x}}$ at $\partial\Omega$. Thus while the load applied, in infinite space, on a sphere of radius ε at $\hat{\mathbf{x}} \in \partial\Omega$, is still unity, the actual load applied on Ω is only $\frac{1}{2}$. Thus we obtain Eq. (64b).

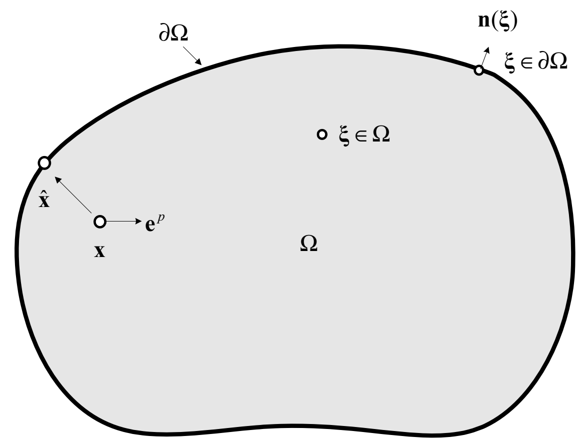

Figure 2 : A loading point **x** approaching the boundary

We can write Eqs. (62d) and (63d) for $\mathbf{x} \in \partial\Omega$, with Eq. (64), as:

$$\int_{\partial\Omega}^{CPV} t_j^{*p}(\mathbf{x},\boldsymbol{\xi}) dS + \frac{1}{2}\delta_{pj} = 0 \qquad \mathbf{x} \in \partial\Omega \tag{65}$$

$$\int_{\partial\Omega}^{CPV} t_j^{*p}(\mathbf{x},\boldsymbol{\xi}) u_j(\mathbf{x}) dS + \frac{1}{2}\delta_{pj} u_j(\mathbf{x}) = 0 \qquad \mathbf{x} \in \partial\Omega \tag{66}$$

From Eq. (29a), we also see that

$$-\int_{\partial\Omega}^{CPV} n_i(\boldsymbol{\xi})\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})dS = \int_{\partial\Omega}^{CPV} n_i(\boldsymbol{\xi})\phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})dS \tag{67a}$$

or

$$-\int_{\partial\Omega}^{CPV} t_j^{*p}(\mathbf{x},\boldsymbol{\xi})dS = \int_{\partial\Omega}^{CPV} n_i(\boldsymbol{\xi})\phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})dS \tag{67b}$$

for both $\mathbf{x}\in\Omega$ and $\mathbf{x}\in\partial\Omega$. Thus, we can write identities for ϕ_{ij}^{*p} which are similar to those in Eqs. (62d) and (63d) for σ_{ij}^{*p}, as:

$$\int_{\partial\Omega}^{CPV} n_i(\boldsymbol{\xi})\phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})dS - C\delta_{pj} = 0 \tag{68}$$

and

$$\int_{\partial\Omega}^{CPV} n_i(\boldsymbol{\xi})\phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})u_j(\mathbf{x})dS - Cu_p(\mathbf{x}) = 0 \tag{69}$$

where $C=\frac{1}{2}$ for $\mathbf{x}\in\partial\Omega$ and $C=1$ for $\mathbf{x}\in\Omega$.
The corresponding equations for Ψ_{ij}^{*p} can also be written as,

$$\int_{\partial\Omega} n_i(\boldsymbol{\xi})\Psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})dS = 0 \tag{70}$$

and

$$\int_{\partial\Omega} n_i(\boldsymbol{\xi})\Psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\cdot u_j(\mathbf{x})dS = 0 \tag{71}$$

Third, we consider the weak form of Eq. (61), and consider the test functions to be the gradients of an arbitrary function $\mathbf{u}(\boldsymbol{\xi})$ in Ω. This function $\mathbf{u}(\boldsymbol{\xi})$ is so chosen that it has constant gradients, as:

$$u_{j,k}(\boldsymbol{\xi}) = u_{j,k}(\mathbf{x}) \tag{72}$$

Then, the weak form of Eq. (61) may be written as:

$$\int_{\Omega} [\sigma_{ij,i}^{*p}(\mathbf{x},\boldsymbol{\xi}) + \delta(\mathbf{x},\boldsymbol{\xi})e_j^p]u_{j,k}(\mathbf{x})d\Omega = 0 \tag{73a}$$

Applying the divergence theorem, we obtain,

$$\int_{\partial\Omega} n_i(\boldsymbol{\xi})\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})u_{j,k}(\mathbf{x})d\Omega + u_{p,k}(\mathbf{x}) = 0 \tag{73b}$$

or

$$\int_{\partial\Omega} n_n(\boldsymbol{\xi})E_{ijmn}u_{j,i}^{*p}(\mathbf{x},\boldsymbol{\xi})u_{m,k}(\mathbf{x})d\Omega + u_{p,k}(\mathbf{x}) = 0 \tag{73c}$$

or

$$\int_{\partial\Omega} t_m^{*p}(\mathbf{x},\boldsymbol{\xi})u_{m,k}(\mathbf{x})d\Omega + u_{p,k}(\mathbf{x}) = 0 \quad \mathbf{x}\in\Omega \tag{73d}$$

In addition, we may observe that the first two terms in Eq. (53) have the following identity, as:

$$\begin{aligned}
&\int_{\partial\Omega} n_i(\boldsymbol{\xi})E_{ijmn}u_{m,n}(\mathbf{x})u_{j,k}^{*p}(\mathbf{x},\boldsymbol{\xi})\,dS \\
&\quad - \int_{\partial\Omega} n_k(\boldsymbol{\xi})E_{ijmn}u_{m,n}(\mathbf{x})u_{j,i}^{*p}(\mathbf{x},\boldsymbol{\xi})\,dS \\
&= \int_{\Omega} E_{ijmn}u_{m,n}(\mathbf{x})u_{j,ki}^{*p}(\mathbf{x},\boldsymbol{\xi})\,dS \\
&\quad - \int_{\Omega} E_{ijmn}u_{m,n}(\mathbf{x})u_{j,ik}^{*p}(\mathbf{x},\boldsymbol{\xi})\,dS = 0
\end{aligned} \tag{74}$$

By adding Eq. (74) into Eq. (73c), we obtain,

$$\begin{aligned}
&\int_{\partial\Omega} n_i(\boldsymbol{\xi})E_{ijmn}u_{m,n}(\mathbf{x})u_{j,k}^{*p}(\mathbf{x},\boldsymbol{\xi})\,dS \\
&\quad - \int_{\partial\Omega} n_k(\boldsymbol{\xi})E_{ijmn}u_{m,n}(\mathbf{x})u_{j,i}^{*p}(\mathbf{x},\boldsymbol{\xi})\,dS \\
&\quad + \int_{\partial\Omega} n_n(\boldsymbol{\xi})E_{ijmn}u_{m,k}(\mathbf{x})u_{j,i}^{*p}(\mathbf{x},\boldsymbol{\xi})d\Omega \\
&\quad + u_{p,k}(\mathbf{x}) = 0
\end{aligned} \tag{75a}$$

or

$$\begin{aligned}
&\int_{\partial\Omega} n_i(\boldsymbol{\xi})E_{ijmn}u_{m,n}(\mathbf{x})u_{j,k}^{*p}(\mathbf{x},\boldsymbol{\xi})\,dS \\
&\quad + \int_{\partial\Omega} e_{nkt}D_t u_m(\mathbf{x})E_{ijmn}u_{j,i}^{*p}(\mathbf{x},\boldsymbol{\xi})\,dS \\
&\quad + u_{p,k}(\mathbf{x}) = 0
\end{aligned} \tag{75b}$$

Multiplying Eq. (75b) by E_{abpk}, we obtain the following identity for the corresponding stresses $\boldsymbol{\sigma}(\mathbf{x})$, as:

$$\begin{aligned}
&\int_{\partial\Omega} n_p(\boldsymbol{\xi})\sigma_{pq}(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})dS \\
&\quad + \int_{\partial\Omega} D_p u_q(\mathbf{x})\Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\,dS + \sigma_{ab}(\mathbf{x}) = 0
\end{aligned} \tag{76}$$

where Σ_{abpq}^{*} is defined in Eq. (58).
Forth more, we use the displacement field $\mathbf{u}(\boldsymbol{\xi})$ in Ω as the test function and write the weak form of Eq. (61) as,

$$\int_{\Omega} [\sigma_{ij,i}^{*p}(\mathbf{x},\boldsymbol{\xi}) + \delta(\mathbf{x},\boldsymbol{\xi})e_j^p]u_j(\boldsymbol{\xi})d\Omega = 0 \tag{77a}$$

or

$$
\begin{aligned}
0 &= \int_{\partial\Omega} n_i(\boldsymbol{\xi})\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})u_p(\boldsymbol{\xi})dS \\
&\quad - \int_{\Omega} \sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})u_{j,i}(\boldsymbol{\xi})d\Omega + u_p(\mathbf{x}) \\
&= \int_{\partial\Omega} n_i(\boldsymbol{\xi})\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})u_p(\boldsymbol{\xi})dS \\
&\quad - \int_{\Omega} E_{ijmn}u_{m,n}^{*p}(\mathbf{x},\boldsymbol{\xi})u_{j,i}(\boldsymbol{\xi})d\Omega + u_p(\mathbf{x}) \\
&= \int_{\partial\Omega} n_i(\boldsymbol{\xi})\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})u_p(\boldsymbol{\xi})dS \\
&\quad - \int_{\partial\Omega} n_n(\boldsymbol{\xi})E_{ijmn}u_{m}^{*p}(\mathbf{x},\boldsymbol{\xi})u_{j,i}(\boldsymbol{\xi})dS \\
&\quad + u_p(\mathbf{x})
\end{aligned}
\tag{77b}
$$

By differentiating it with respect to x_k, one may obtain

$$
\int_{\partial\Omega} t_{j,k}^{*p}(\mathbf{x},\boldsymbol{\xi})u_j(\boldsymbol{\xi})dS - \int_{\partial\Omega} n_n(\boldsymbol{\xi})E_{ijmn}u_{m,k}^{*p}(\mathbf{x},\boldsymbol{\xi})u_{j,i}(\boldsymbol{\xi})dS + u_{p,k}(\mathbf{x}) = 0 \tag{78}
$$

Multiplying Eq. (78) by E_{abpk}, we have an identity for the hypersingular $t_{i,j}^{*p}(\mathbf{x},\boldsymbol{\xi})$, in terms of the non-hypersingular $\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})$, as:

$$
E_{abpq}\int_{\partial\Omega} t_{j,q}^{*p}(\mathbf{x},\boldsymbol{\xi})u_j(\boldsymbol{\xi})dS - \int_{\Omega} n_p(\boldsymbol{\xi})\sigma_{pq}(\boldsymbol{\xi})\sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})d\Omega + \sigma_{ab}(\mathbf{x}) = 0 \tag{79}
$$

Clearly, one may derive any number of "properties" of the fundamental solution in elasticity. Some of additional properties are derived as follows:

By taking $\mathbf{u}(\boldsymbol{\xi})$ as the "radial displacement" field, as:

$$
\mathbf{u}(\boldsymbol{\xi}) = \alpha\,\mathbf{r}(\mathbf{x},\boldsymbol{\xi}) \tag{80}
$$

where α is a constant. We obtain from Eq. (80):

$$
u_{j,k}(\boldsymbol{\xi}) = u_{j,k}(\mathbf{x}) = \alpha\delta_{jk} \tag{81}
$$

By substituting Eq. (81) into Eq. (73d), we have

$$
\int_{\partial\Omega} t_m^{*p}(\mathbf{x},\boldsymbol{\xi})\delta_{mk}d\Omega + \delta_{pk} = \int_{\partial\Omega} t_k^{*p}(\mathbf{x},\boldsymbol{\xi})d\Omega + \delta_{pk} = 0 \tag{82}
$$

which is the same as Eq. (62c) in the basic identity.

Applying this displacement field in Eq. (81) to Eq. (79), we have

$$
E_{abpq}\int_{\partial\Omega} t_{j,q}^{*p}(\mathbf{x},\boldsymbol{\xi})r_j(\mathbf{x},\boldsymbol{\xi})dS - (\lambda+2\mu)[\int_{\Omega} n_m(\boldsymbol{\xi})\sigma_{ab}^{*m}(\mathbf{x},\boldsymbol{\xi})d\Omega + \delta_{ab}] = 0 \tag{83}
$$

Now we take another special case in which $\mathbf{u}(\boldsymbol{\xi})$ is from a displacement field without volume strain, and contains no free rotation, as:

$$
\mathbf{u}(\boldsymbol{\xi}) = \omega\,[\mathbf{v}\times(\mathbf{v}\times\mathbf{r}(\mathbf{x},\boldsymbol{\xi})) + \frac{2}{3}\mathbf{r}(\mathbf{x},\boldsymbol{\xi})] \tag{84a}
$$

or

$$
\mathbf{u}(\boldsymbol{\xi}) = \omega\,[\mathbf{v}*\mathbf{r}(\mathbf{x},\boldsymbol{\xi}))\mathbf{v} - \frac{1}{3}\mathbf{r}(\mathbf{x},\boldsymbol{\xi})] \tag{84b}
$$

where ω is a constant and $\mathbf{v}$ is a constant vector with a unit length. A similar identity for such problems can be easily obtained.

All the identities for σ_{ij}^{*p}, t_j^{*p}, ϕ_{ij}^{*p} and Ψ_{ij}^{*p} are obtained from the weak form of the fundamental solution, with various test functions. They can be readily used in the regularization of the stongly-singular and hypersingular integrals in BIE equations.

It is clear that "properties" similar to the above, can be derived for the fundamental solutions for any set of partial differential equations, such as those that arise in fluid mechanics, acoustics, electromagnetism, etc. These will be presented in subsequent papers by the present authors.

6 Regularization of tBIE

Contracting Eq. (76) with $n_a(\mathbf{x})$, and using the resulting equation in Eq. (59), we can obtain the fully regularized form of Eq. (59), as

$$
\begin{aligned}
0 = &\int_{\partial\Omega} [t_q(\boldsymbol{\xi}) - n_p(\boldsymbol{\xi})\sigma_{pq}(\mathbf{x})]n_a(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\,dS \\
+ &\int_{\partial\Omega} [D_p u_q(\boldsymbol{\xi}) - (D_p u_q)(\mathbf{x})]n_a(\mathbf{x})\Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\,dS
\end{aligned}
\tag{85}
$$

We know that $t_q(\boldsymbol{\xi}) - n_p(\boldsymbol{\xi})\ \sigma_{pq}(\mathbf{x})$ and $D_p u_q(\boldsymbol{\xi}) - (D_p u_q)(\mathbf{x})$ become $O(r)$ when $\boldsymbol{\xi}\to\hat{\mathbf{x}}$ at $\partial\Omega$ and hence

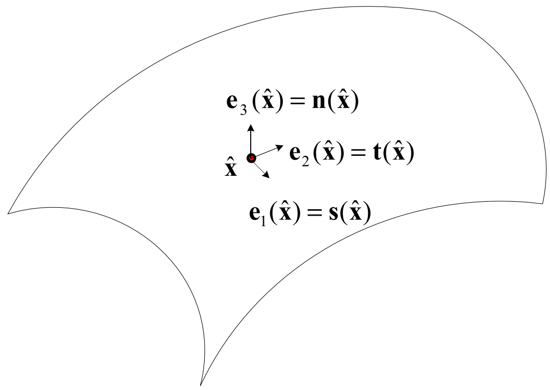

Figure 3 : Local coordinates at a boundary point $\hat{\mathbf{x}}$

Eq. (85) is weakly singular. Thus Eq. (85) can be applied as $\mathbf{x} \to \hat{\mathbf{x}} \in \partial\Omega$.

We define the local coordinates at point $\mathbf{x}$ as $\mathbf{e}_1(\mathbf{x}) = \mathbf{s}(\mathbf{x})$, $\mathbf{e}_2(\mathbf{x}) = \mathbf{t}(\mathbf{x})$ and $\mathbf{e}_3(\mathbf{x}) = \mathbf{n}(\mathbf{x})$, as shown in Fig. 3. From the strain-displacement relations in (6) and the stress-strain relations in Eq. (7), we have $\boldsymbol{\sigma}(\mathbf{x})$ in terms of $\mathbf{u}(\mathbf{x})$ and $\mathbf{t}(\mathbf{x})$, as:

$$\begin{aligned}
\sigma_{11}(\mathbf{x}) &= \frac{\lambda}{\lambda+2\mu} t_3(\mathbf{x}) + \frac{2\lambda\mu}{\lambda+2\mu}[u_{1,1}(\mathbf{x}) + u_{2,2}(\mathbf{x})] \\
&\quad + 2\mu u_{1,1}(\mathbf{x}) \\
\sigma_{22}(\mathbf{x}) &= \frac{\lambda}{\lambda+2\mu} t_3(\mathbf{x}) + \frac{2\lambda\mu}{\lambda+2\mu}[u_{1,1}(\mathbf{x}) + u_{2,2}(\mathbf{x})] \\
&\quad + 2\mu u_{2,2}(\mathbf{x}) \\
\sigma_{33}(\mathbf{x}) &= t_3(\mathbf{x}) \\
\sigma_{12}(\mathbf{x}) &= \sigma_{21}(\mathbf{x}) = \mu[u_{1,2}(\mathbf{x}) + u_{1,2}(\mathbf{x})] \\
\sigma_{13}(\mathbf{x}) &= \sigma_{31}(\mathbf{x}) = t_1(\mathbf{x}) \\
\sigma_{23}(\mathbf{x}) &= \sigma_{32}(\mathbf{x}) = t_2(\mathbf{x})
\end{aligned} \tag{86}$$

Thus, we can re-write Eq. (85) as:

$$\begin{aligned}
0 = \int_{\partial\Omega} &\{t_j(\boldsymbol{\xi}) - t_j(\mathbf{x}) \\
&+ [n_i(\mathbf{x}) - n_i(\boldsymbol{\xi})]\sigma_{ij}(\mathbf{x})\} n_a(\mathbf{x})\sigma_{ab}^{*j}(\mathbf{x},\boldsymbol{\xi})\, dS \\
+ \int_{\partial\Omega} &[D_p u_q(\boldsymbol{\xi}) - (D_p u_q)(\mathbf{x})] n_a(\mathbf{x}) \Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS
\end{aligned} \tag{87a}$$

or

$$\begin{aligned}
0 &= \int_{\partial\Omega} [t_q(\boldsymbol{\xi}) - t_q(\mathbf{x})] n_a(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\, dS \\
&\quad + \sigma_{pq}(\mathbf{x}) \int_{\partial\Omega} [n_p(\mathbf{x}) - n_p(\boldsymbol{\xi})] n_a(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\, dS \\
&\quad + \int_{\partial\Omega} [D_p u_q(\boldsymbol{\xi}) - (D_p u_q)(\mathbf{x})] n_a(\mathbf{x}) \Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS
\end{aligned} \tag{87b}$$

With Eq. (86), Eq. (87) can be evaluated numerically as tBIE in its fully-regularized form.

Eq. (87) is the regularized traction BIE, derivable from the non-hypersingular integral representation for tractions, as reported in Okada, Rajiyah, and Atluri (1988,1989).

Eq. (85), may be satisfied in a *weak-form* at $\partial\Omega$, using a Petrov-Galerkin scheme, as:

$$\begin{aligned}
0 &= \int_{\partial\Omega} w_b(\mathbf{x}) dS_x \int_{\partial\Omega} [t_q(\boldsymbol{\xi}) - n_p(\boldsymbol{\xi})\sigma_{pq}(\mathbf{x})] n_a(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\
&+ \int_{\partial\Omega} w_b(\mathbf{x}) dS_x \int_{\partial\Omega} [D_p u_q(\boldsymbol{\xi}) - (D_p u_q)(\mathbf{x})] n_a(\mathbf{x}) \Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi
\end{aligned} \tag{88}$$

where $w_b(\mathbf{x})$ is a test function. If $w_b(\mathbf{x})$ is chosen as a Dirac delta function, i.e. $w_b(\mathbf{x}) = \delta(\mathbf{x}, \mathbf{x}_m)$ at $\partial\Omega$, we obtain the standard "collocation" traction boundary element method.

From Eqs. (7) and (26), Σ_{ijpq}^{*} can be written in terms of F^* as:

$$\begin{aligned}
\Sigma_{ijpq}^{*}(\mathbf{x},\boldsymbol{\xi}) &= E_{ijkl} e_{nlp} \sigma_{nq}^{*k}(\mathbf{x},\boldsymbol{\xi}) \\
&= \mu^2 \Big(\frac{2\upsilon}{1-2\upsilon}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\Big) \cdot \\
&\quad e_{nlp}[(1-\upsilon)(\delta_{kn}F^{*}_{,bbq} + \delta_{kq}F^{*}_{,bbn}) + \upsilon\delta_{nq}F^{*}_{,bbk} - F^{*}_{,knq}] \\
&= \mu^2 [(-e_{nip}F_{,jnq} - e_{njp}F_{,inq} + \delta_{iq}e_{jpn}F_{,bbn} + \delta_{jq}e_{ipn}F_{,bbn}) \\
&\quad + \upsilon(e_{qip}F_{bbj} + e_{qjp}F_{,bbi} + 2\delta_{ij}e_{qpn}F_{,bbn} \\
&\quad - \delta_{iq}e_{jpn}F_{,bbn} - \delta_{jq}e_{ipn}F_{,bbn})] \\
&= \mu^2 [(e_{inp}F_{,jqn} - e_{inp}\delta_{jq}F_{,bbn} + e_{int}e_{tqk}e_{jpm}F_{,kmn}) \\
&\quad + \upsilon(e_{inq}\delta_{jp}F_{,bbn} + e_{jnq}\delta_{ip}F_{,bbn})]
\end{aligned} \tag{89}$$

We also have the divergence of Σ^*_{ijpq} as:

$$\Sigma^*_{ijpq,i}(\mathbf{x},\boldsymbol{\xi})$$
$$= \mu^2[(e_{inp}F_{,jqni} - e_{inp}\delta_{jq}F_{,bbni} + e_{int}e_{tqk}e_{jpm}F_{,kmni})$$
$$+ \upsilon(e_{inq}\delta_{jp}F_{,bbni} + e_{jnq}\delta_{ip}F_{,bbni})]$$

$$= \mu^2\upsilon e_{jnq}F_{,bbnp}$$

$$\equiv \Lambda^*_{ijpq,i}(\mathbf{x},\boldsymbol{\xi}) \tag{90}$$

where, by definition,

$$\Lambda^*_{ijpq}(\mathbf{x},\boldsymbol{\xi}) = \mu^2\upsilon e_{jiq}F_{,bbp} \tag{91}$$

We observe that:

$$\int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})\Lambda^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})dS$$
$$= -\int_{\partial\Omega} u_q(\boldsymbol{\xi})D_p\Lambda^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})dS$$
$$= -\int_{\partial\Omega} u_q(\boldsymbol{\xi})D_p[\mu^2\upsilon e_{jiq}F_{,bb}]_{,p}dS = 0 \tag{92}$$

Then we have

$$\int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})\Sigma^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})\, dS$$
$$= \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})[\Sigma^*_{ijpq}(\mathbf{x},\boldsymbol{\xi}) - \Lambda^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})]\, dS$$
$$\equiv \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})\mathrm{K}^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})\, dS \tag{93}$$

where, by definition

$$\mathrm{K}^*_{ijpq}(\mathbf{x},\boldsymbol{\xi}) = \Sigma^*_{ijpq}(\mathbf{x},\boldsymbol{\xi}) - \Lambda^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})$$

$$= \mu^2[(e_{inp}F_{,jqn} - e_{inp}\delta_{jq}F_{,bbn} + e_{int}e_{tqk}e_{jpm}F_{,kmn})$$

$$+ \upsilon(e_{inq}\delta_{jp}F_{,bbn} + e_{jnq}\delta_{ip}F_{,bbn} - e_{jiq}F_{,bbp})]$$

$$= \mu^2 e_{int}[(\delta_{tp}F_{,jq} - \delta_{tp}\delta_{jq}F_{,bb} + e_{tqk}e_{jpm}F_{,km})$$

$$+ \upsilon(\delta_{tq}\delta_{jp}F_{,bb} + e_{tpm}e_{jmq}F_{,bb})]_{,n}$$

$$\equiv e_{int}H^*_{tjpq,n}(\mathbf{x},\boldsymbol{\xi}) \tag{94}$$

We have H^*_{ijpq}, by definition, as

$$H^*_{ijpq}(\mathbf{x},\boldsymbol{\xi}) = \mu^2[(\delta_{ip}F_{,jq} - \delta_{ip}\delta_{jq}F_{,bb} + e_{iqk}e_{jpm}F_{,km})$$

$$+ \upsilon(\delta_{iq}\delta_{jp}F_{,bb} + e_{ipm}e_{jmq}F_{,bb})]$$

$$= \mu^2[-\delta_{ij}F_{,pq} + 2\delta_{ip}F_{,jq} + \delta_{jq}F_{,ip} - \delta_{pq}F_{,ij} \tag{95}$$

$$- 2\delta_{ip}\delta_{jq}F_{,bb} + 2\upsilon\delta_{iq}\delta_{jp}F_{,bb} + (1-\upsilon)\delta_{ij}\delta_{pq}F_{,bb})]$$

in which, the following results are used:

$$e_{ipm}e_{jmq}F_{,bb} = (-\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{pj})F_{,bb} \tag{96a}$$

$$e_{iqk}e_{jpm}F_{,km} =$$
$$+ \delta_{ip}F_{,jq} + \delta_{jq}F_{,ip} - \delta_{ij}F_{,pq} - \delta_{pq}F_{,ij} \tag{96b}$$
$$+ \delta_{ij}\delta_{pq}F_{,bb} - \delta_{ip}\delta_{jq}F_{,bb}$$

With Eqs. (21a) and (32), we can write H^*_{ijpq} for 3D problems as:

$$H^*_{ijpq}(\mathbf{x},\boldsymbol{\xi}) = \frac{\mu}{8\pi(1-\upsilon)r}$$

$$[4\upsilon\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} - 2\upsilon\delta_{ij}\delta_{pq}$$
$$+ \delta_{ij}r_{,p}r_{,q} + \delta_{pq}r_{,i}r_{,j} - 2\delta_{ip}r_{,j}r_{,q} - \delta_{jq}r_{,i}r_{,p}] \tag{97}$$

and with Eq. (33) for 2D plain strain problems as:

$$H^*_{ijpq}(\mathbf{x},\boldsymbol{\xi}) = \frac{\mu}{4\pi(1-\upsilon)}$$
$$[-4\upsilon\ln r\delta_{iq}\delta_{jp} + \ln r\delta_{ip}\delta_{jq} + 2\upsilon\ln r\delta_{ij}\delta_{pq}$$
$$+ \delta_{ij}r_{,p}r_{,q} + \delta_{pq}r_{,i}r_{,j} - 2\delta_{ip}r_{,j}r_{,q} - \delta_{jq}r_{,i}r_{,p}] \tag{98}$$

From Eqs. (90) and (94), some properties of the kernel functions can be found as following:

$$\nabla\cdot\boldsymbol{\Sigma}^*(\mathbf{x},\boldsymbol{\xi}) = \nabla\cdot\boldsymbol{\Lambda}^*(\mathbf{x},\boldsymbol{\xi}) \tag{99a}$$
$$\nabla\cdot\mathbf{K}^*(\mathbf{x},\boldsymbol{\xi}) = \mathbf{0} \tag{99b}$$
$$\mathbf{K}^*(\mathbf{x},\boldsymbol{\xi}) = \nabla\times\mathbf{H}^*(\mathbf{x},\boldsymbol{\xi}) \tag{99c}$$

We may take $w_b(\mathbf{x})$ to be any continuous function in Eq. (88), and derive a Petrov-Galerkin boundary element method. If $w_b(\mathbf{x})$ is continuous, one may use Stokes' theorem, and write:

$$\begin{aligned}
-&\int_{\partial\Omega} w_b(\mathbf{x})t_b(\mathbf{x})dS_x \\
&= \int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} t_q(\boldsymbol{\xi})n_a(\mathbf{x})[\Psi^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi}) - \phi^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})]dS_\xi \\
&+ \int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})n_a(\mathbf{x})\mathrm{K}^*_{abpq}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi
\end{aligned} \tag{100}$$

in which, Eqs. (76) and (93) are applied.

With the fact that

$$\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial \xi_i}, \tag{101}$$

and with Eqs. (69) and (94), one may rewrite Eq. (100) as:

$$\begin{aligned}
-&\frac{1}{2}\int_{\partial\Omega} t_b(\mathbf{x})w_b(\mathbf{x})dS_x \\
&= -\int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} t_q(\boldsymbol{\xi})D_a G^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\
&- \int_{\partial\Omega} t_q(\boldsymbol{\xi})\, dS_\xi \int^{CPV}_{\partial\Omega} n_a(\mathbf{x})w_b(\mathbf{x})\phi^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})dS_x \\
&- \int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})D_a H^*_{abpq}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi
\end{aligned} \tag{102}$$

where G^{*q}_{ab} is defined in Eq.(31); ϕ^{*q}_{ab} is defined in Eq. (28a), and H^*_{abpq} is defined in Eq. (95).

As $w_b(\mathbf{x})$ is continuous, one may use Stokes' theorem, and re-write Eq. (102) as:

$$\begin{aligned}
-&\frac{1}{2}\int_{\partial\Omega} t_b(\mathbf{x})w_b(\mathbf{x})dS_x \\
&= \int_{\partial\Omega} D_a w_b(\mathbf{x})dS_x \int_{\partial\Omega} t_q(\boldsymbol{\xi})G^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\
&- \int_{\partial\Omega} t_q(\boldsymbol{\xi})\, dS_\xi \int^{CPV}_{\partial\Omega} n_a(\mathbf{x})w_b(\mathbf{x})\phi^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})dS_x \\
&+ \int_{\partial\Omega} D_a w_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})H^*_{abpq}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi
\end{aligned} \tag{103}$$

If the test function $w_b(\mathbf{x})$ is chosen to be identical to a function that is energy-conjugate to t_b, namely, the trial function $\hat{u}_b(\mathbf{x})$, we generate the symmetric Galerkin BEM as [Han and Atluri (2002); Nikishkov, Park, Atluri(2001)]:

$$\begin{aligned}
-&\frac{1}{2}\int_{\partial\Omega} t_b(\mathbf{x})\hat{u}_b(\mathbf{x})dS_x \\
&= \int_{\partial\Omega} D_a \hat{u}_b(\mathbf{x})dS_x \int_{\partial\Omega} t_q(\boldsymbol{\xi})G^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\
&- \int_{\partial\Omega} t_q(\boldsymbol{\xi})\, dS_\xi \int^{CPV}_{\partial\Omega} n_a(\mathbf{x})\hat{u}_b(\mathbf{x})\phi^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})dS_x \\
&+ \int_{\partial\Omega} D_a \hat{u}_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})H^*_{abpq}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi
\end{aligned} \tag{104}$$

The results in Eq. (103) are similar to those reported in [Li and Mear (1998)] but are different from those in [Li and Mear (1998)] in the kernel functions appearing in Eq. (104). However, here, we obtain these results in a very straightforward and simple manner.

7 Regularization of dBIE

In this section, we consider the regularization of the displacement BIE (52), in order to render it tractable for numerical implementation. We also consider the possibility of satisfying the dBIE, at $\partial\Omega$, in *a weak form*, through a general Petrov-Galerkin scheme.

We subtract Eq. (63d) from Eq. (52), and obtain,

$$\begin{aligned}
0 = &\int_{\partial\Omega} t_j(\boldsymbol{\xi})u^{*p}_j(\mathbf{x},\boldsymbol{\xi})\, dS \\
&- \int_{\partial\Omega} n_i(\boldsymbol{\xi})[u_j(\boldsymbol{\xi}) - u_j(\mathbf{x})]\sigma^{*p}_{ij}(\mathbf{x},\boldsymbol{\xi})\, dS
\end{aligned} \tag{105}$$

We know that $u_j(\boldsymbol{\xi}) - u_j(\mathbf{x})$ becomes $O(r)$ when $\boldsymbol{\xi} \to \mathbf{x}$ and Eq. (105) becomes weakly singular. Then it can be evaluated numerically, and applicable to point $\mathbf{x}$ on the boundary $\partial\Omega$. Eq. (105) is the well-known regularized dBIE equation.

We can also use a Petrov-Galerkin scheme to write a weak-form for Eq. (105) as:

$$0 = \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_j^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} n_i(\boldsymbol{\xi})[u_j(\boldsymbol{\xi}) - u_j(\mathbf{x})]\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS \tag{106}$$

where $w_p(\mathbf{x})$ is a test function. If $w_p(\mathbf{x})$ is chosen as a Dirac delta function, i.e. $w_p(\mathbf{x}) = \delta(\mathbf{x},\mathbf{x}_m)$ at $\partial\Omega$, we obtain the standard "collocation" displacement boundary element method.

Using (28b), i.e., $\sigma_{ij}^{*p} = \psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) - \phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})$ [with Ψ_{ij}^{*p} is defined in Eq.(30). and ϕ_{ij}^{*p} is defined in Eq. (28a)], we re-write Eq. (106), with Eq. (69), as:

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega} w_p(\mathbf{x}) u_j(\mathbf{x}) dS_x = &\int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_j^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\ &- \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} n_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) \psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\ &+ \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega}^{CPV} n_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) \phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi\end{aligned} \tag{107}$$

Applying Stokes' theorem to Eq. (107), we have

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega} w_p(\mathbf{x}) u_p(\mathbf{x}) dS_x = &\int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_j^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\ &+ \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} D_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) G_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\ &+ \int_{\partial\Omega} w_p(\boldsymbol{\xi}) dS_x \int_{\partial\Omega}^{CPV} n_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) \phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi\end{aligned} \tag{108}$$

where G_{ij}^{*p} is defined in Eq.(31)

If $w_p(\mathbf{x})$ is chosen to be identical to a function which is energy-conjugate to u_p, viz., the trial function $\hat{t}_p(\mathbf{x})$, we obtain the symmetric Galerkin dBEM, as

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega} \hat{t}_p(\mathbf{x}) u_p(\mathbf{x}) dS_x = &\int_{\partial\Omega} \hat{t}_p(\mathbf{x}) dS_x \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_j^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\ &+ \int_{\partial\Omega} \hat{t}_p(\mathbf{x}) dS_x \int_{\partial\Omega} D_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) G_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\ &+ \int_{\partial\Omega} \hat{t}_p(\mathbf{x}) dS_x \int_{\partial\Omega}^{CPV} n_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) \phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi\end{aligned} \tag{109}$$

8 Evaluation of displacements and stresses near the surface

After obtaining the displacements and the stresses on the boundary, we sometimes also need to evaluate the displacements and stresses inside the domain. It is well known that we need to evaluate the strongly singular integrals if Eqs. (52) and (53) are applied directly, if the point approaches the boundary.

Consider a generic domain point $\mathbf{x} \in \Omega$ which is close to a boundary point $\hat{\mathbf{x}}$. By replacing the displacements in Eq. (63d) with $u_p(\hat{\mathbf{x}})$, Eq. (52) is re-written as,

$$u_p(\mathbf{x}) = \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u_j^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} [u_m(\boldsymbol{\xi}) - u_m(\hat{\mathbf{x}})]\sigma_{nm}^{*p}(\mathbf{x},\boldsymbol{\xi})\, dS + u_p(\hat{\mathbf{x}}) \tag{110}$$

We know that $u_m(\boldsymbol{\xi}) - u_m(\hat{\mathbf{x}})$ becomes $O(r)$ when $\boldsymbol{\xi} \to \hat{\mathbf{x}}$ and Eq. (110) becomes weakly singular. Then it can be evaluated numerically.

Eq. (53) is used for stress calculation. As a part of the solution, we have the known displacements and tractions at point $\hat{\mathbf{x}}$ as $u_j(\hat{\mathbf{x}})$ and $t_j(\hat{\mathbf{x}})$, respectively. We first calculate the stress $\sigma_{ab}(\hat{\mathbf{x}})$ at point $\hat{\mathbf{x}}$ from Eq. (86).

From Eq. (76), it is ready to re-write Eq. (57) as,

$$-\sigma_{ab}(\mathbf{x}) = \int_{\partial\Omega} [t_q(\boldsymbol{\xi}) - n_p(\boldsymbol{\xi})\sigma_{pq}(\hat{\mathbf{x}})]\sigma_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\, dS + \int_{\partial\Omega} [D_p u_q(\boldsymbol{\xi}) - (D_p u_q)(\hat{\mathbf{x}})]\Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS - \sigma_{ab}(\hat{\mathbf{x}}) \tag{111}$$

Again, we know that the singularities of terms $t_q(\boldsymbol{\xi}) - n_p(\boldsymbol{\xi})\sigma_{pq}(\hat{\mathbf{x}})$ and $D_p u_q(\boldsymbol{\xi}) - (D_p u_q)(\hat{\mathbf{x}})$ become $O(r)$ when $\boldsymbol{\xi} \to \hat{\mathbf{x}}$. Then Eq. (111) becomes weakly singular and can be evaluated numerically.

9 Closure

1. We have presented simple and straight-forward formulations for weakly-singular traction as well as displacement integral equations in a linear elastic solid undergoing small displacements. Clearly, these formulations can be extended to finite elasticity, large-strains, and rate-formulations of elastic-plastic solids undergoing

large deformations, using the methodologies presented in Okada, Rajiyah, and Atluri (1989b).

2. The traditional traction boundary element, or displacement boundary element methods can be derived from Eqs. (102) and (108), respectively. In these methods, one uses a "mesh" at $\partial\Omega$, which consists of a set of contiguous (non-overlapping) "elements". In general, the trial functions u_b, t_b, and the test functions w_b at $\partial\Omega$ are interpolated in terms of their respective values at the nodes of the boundary elements. On the other hand, using the concepts of the meshless local Petrov-Galerkin methods (MLPG) developed in Atluri et al [1998, 2002a,b], one may develop "meshless local Petrov-Galerkin boundary integral equation approaches". Also, Eqs. (102) and (108) in general involve double integrals over $\partial\Omega$ on their right-hand sides, where $\partial\Omega$ is the entire global boundary. However, in using MLPG, the second integral over $\partial\Omega$ in Eqs. (102) and (108) may be replaced by a local integral over a sub-region of $\partial\Omega$ only. In evaluating the first integral over the global $\partial\Omega$, one may use "shadow-elements", or alternatively, one may also develop a truly meshless MLPG method for integral equations. They are the subjects of our forthcoming papers [Atluri, Han, and Shen (2003); Han and Atluri (2003)].

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