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On Deformation of an Euler-Bernolli Beam Under Terminal Force and Couple

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Abstract: The paper studies the behavior of a spatial Euler-Bernoulli beam loaded by a terminal thrusting force and a couple. The classical Clebsch-Kirchhoff equilibrium equations are written by using appropriate angular coordinates describing the finite rotations of the local frames attached to each cross-sections of the beam with respect to a fixed system. When we have geometric boundary conditions at one end and dynamic boundary conditions (a force and a couple) at the other the set of equilibrium equations form and initial value probem which can easily be solved with standard Runge-Kutta method.

keyword: spatial Euler-Bernoulli beam, finite rotation, terminal thrust and couple.

1 Introduction

Buckling of slender beams under a terminal thrusting force is a classical stability problem of structural mechanics [Euler (1749)]. Quite similar phenomenon happens when the rod is loaded by a couple [Kovari (1969)]. The critical force and moment can be obtained by using analytic method for linearized equilibrium equations, but if also nonlinear terms are included we find essentially different behaviors of the two cases at the critical loading [Kovari (1969)]. When the rod is loaded by a force the result of the application of static bifurcation theory [Atanackovic (1997), Béda, Steindl, Troger (1992)] leads to a so-called supercritical pitchfork bifurcation. In such case at the loss of stability and uniqueness of the trivial straight line solution stable buckled shapes appear near to it. However, at critical moment the way of the loss of stability is a subcritical pitchfork. Then there are no nearby stable equilibrial shapes, if the trivial one loses stability. One of our previous papers [Béda, Steindl, Troger (1992)] studied the case of the combined loading [Greenhill (1883)]. It was an analytic nonlinear study thus the investigation was restricted into a small neighbourhood of the trivial straight line solution. In that paper we could determine the transition from the supercritial to the subcritical bifurcation and we could even detect a secondary buckling for some values of the critical loading force and couple. The next question is: what happes beyound these critical loads. Such studies are generally performed by using finite element method for the same load [Hsiao, Lin (2003)] or for various types of tip moment loadings [Gotou, Kuwataka, Nishihara, Iwakuma (2003)] or even for the more sophisticated Timoshenko beam model [Atluri, Iura and Vasudevan(2001); Iura, Suetake, Atluri (2003)]

This paper aims to formulate the spatial equilibrium equations in a form wich is suitable for numerical analysis and then perform the calculations and find shapes of the elastica for postcritical loads. We would also like to compare the result of this numerical study with our analytic one [Béda, Steindl, Troger (1992)].

The starting point is the set of classical Clebsch-Kirchhoff equilibrium equations of the spatial Euler elastica. A fine classical descrition of the model and the system of equilibrium equations can be found in [Love (1927)], for a more contemporary study see [Atanackovic (1997)]. In that model we assume that the axis of the rod is inextensible and and the cross-sections remain perpendicular to that axis. To describe the rotations of the cross-sections along the center line of the beam we use local frames x, y, z attached to them, which suffer finite rotations on loading. These are described by three angular variables $\varphi_1(s), \varphi_2(s), \varphi_3(s)$ which are functions of the arc length s . Assume that one end (point P_1) of the rod is clamped and other one is free (point P_2). The loading force $\mathbf{Q}$ and moment $\mathbf{W}$ acts at P_2. Then unknown functions $\varphi_1(s), \varphi_2(s), \varphi_3(s)$ can be calculated by solving a set of ordinary differential equations with the independent variable s . Because of the special geometric boundary conditions (geometric at P_1 dynamic one at P_2) we have an initial value problem to solve.

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2 Coordinate systems

Firstly, we define the global X, Y, Z and local frames x, y, z in Fig.1.

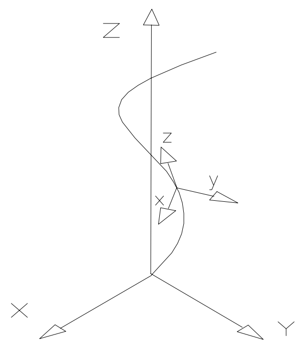

Figure 1 : Global and local frames of the spatial elastica

The relative angular position of the local coordinate system with respect to X, Y, Z can be described by three rotational angles $\varphi_1(s), \varphi_2(s), \varphi_3(s)$. Firstly, a rotation with angle φ_1 is performed around axis X. It moves axes Y into Y' and Z into Z'. Then we rotate frame X, Y', Z' around axis Y' with angle φ_2 and get frame X'', Y', Z'' and the last rotation is around axis Z'' with angle φ_3 to obtain the local system $x = X''', y = Y''', z = Z''$. In Figs.2,3,4. these rotations are shown. Each of them can be given by appropriate orthogonal matrices $\mathbf{R}_i(\varphi_i)$, $(i = 1, 2, 3)$, for example the first transformation matrix is

$$\mathbf{R}_1(\varphi_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & \sin\varphi_1 \\ 0 & -\sin\varphi_1 & \cos\varphi_1 \end{bmatrix}.$$

Thus a vector $\mathbf{v}_{[X,Y,Z]}$ given by coordinates in system X, Y, Z can be transformed into the local frame x, y, z as

$$\mathbf{v}_{[x,y,z]} = \mathbf{T}\mathbf{v}_{[X,Y,Z]} \tag{1}$$

where $\mathbf{T} = \mathbf{R}_3(\varphi_3)\mathbf{R}_2(\varphi_2)\mathbf{R}_1(\varphi_1)$ thus

$$\mathbf{T} = \begin{bmatrix} c_3c_2 & s_3c_1 - c_3s_2s_1 & s_3s_1 + c_3s_2c_1 \\ -s_3c_2 & c_3c_1 + s_3s_2s_1 & c_3s_1 - s_3s_2c_1 \\ -s_2 & -c_2s_1 & c_2c_1 \end{bmatrix},$$

and abbrevations $c_i = \cos\varphi_i$, $s_i = \sin\varphi_i$, $i = 1, 2, 3$ are used. The inverse transformation $\mathbf{T}^{-1}$ is also very important if we need the tangent vector

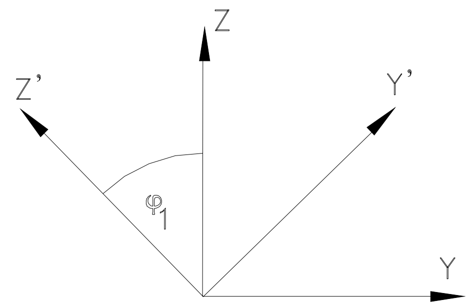

Figure 2 : Rotation on axis X

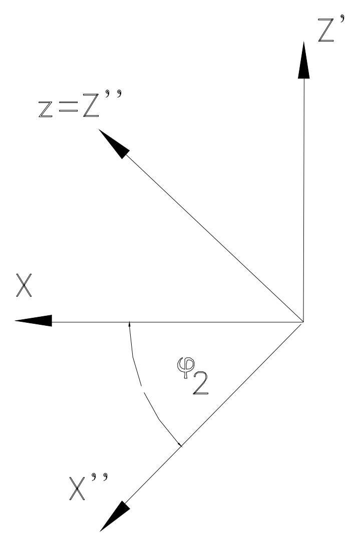

Figure 3 : Rotation on axis Y

$$\mathbf{t}_{[X,Y,Z]} = \frac{d}{ds}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{2}$$

or in the local frame

$$\mathbf{t}_{[x,y,z]} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3}$$

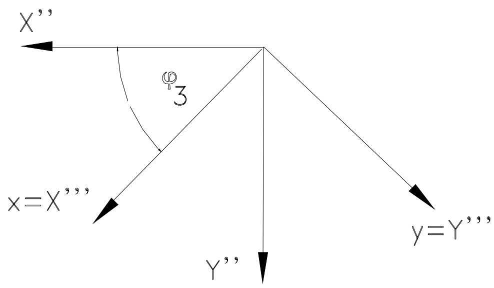

Figure 4 : Rotation on axis Z

While $\mathbf{t}_{[X,Y,Z]} = \mathbf{T}^{-1}\mathbf{t}_{[x,y,z]}$ we have

$$\frac{dX}{ds} = \sin\varphi_2, \quad \frac{dY}{ds} = -\sin\varphi_1\cos\varphi_2, \quad \frac{dZ}{ds} = \cos\varphi_1\cos\varphi_2. \tag{4}$$

3 The equilibrium equations

Let us use curvatures p,q and twist r to describe the deformation of the rod. In the local system x,y,z from [Goldstein (1980)] we find that

$$\begin{aligned} p &= \varphi_1'\cos\varphi_2\cos\varphi_3 + \varphi_2'\sin\varphi_3, \\ q &= -\varphi_1'\cos\varphi_2\sin\varphi_3 + \varphi_2'\cos\varphi_3, \\ r &= \varphi_3' + \varphi_1'\sin\varphi_2, \end{aligned} \tag{5}$$

where prime denotes derivative $\frac{d}{ds}$ of the lenght s of the axis of the rod considered to be inextensible in Euler-Bernoulli beam model. The equilibrium equations of the rod can be obtained in form [Béda, Steindl, Troger (1992)]

$$\begin{aligned} &EJ_1p' - (EJ_2 - GJ_T)\,qr = Q_2 \\ &EJ_2q' - (GJ_T - EJ_1)\,rp = -Q_1 \\ &GJ_Tr' - E\,(J_1 - J_2)\,pq = 0, \end{aligned} \tag{6}$$

where E,G are elasticity moduli, J_1,J_2,J_T are moments of inertia in the local system x,y,z of principal axes of each cross-sections. By using transformation (1) the local components of the loading force

$$\mathbf{Q} = \begin{bmatrix} 0 \\ 0 \\ -Q \end{bmatrix}_{[X,Y,Z]}$$

are

$$\begin{aligned} Q_1 &= -Q\,(\sin\varphi_1\sin\varphi_3 - \cos\varphi_1\sin\varphi_2\cos\varphi_3), \\ Q_2 &= -Q\,(\sin\varphi_1\cos\varphi_3 + \cos\varphi_1\sin\varphi_2\sin\varphi_3). \end{aligned} \tag{7}$$

In order to be able to perform an analytic study [Béda, Steindl, Troger (1992)] we had to assume that the cross-section has a circular ellipse of inertia

$$J_1 = J_2\,(\equiv J). \tag{8}$$

We keep this assumption even now, because we would like to compare results of the two different investigations. Remark that it is not a necessary restriction in our numerical treatment. We could do almost the same calculation if (8) is not satisfied.

From the third equation of (6)

$$GJ_Tr = const. \tag{9}$$

Assume that the direction of twisting moment **W** is of the local axis z at the end of the rod $(s = \ell)$. Then the value of the constant in (9) is W thus

$$r = \frac{W}{GJ_T}. \tag{10}$$

Now by using (8) and (10) the system of three equilibrium equations (6) can be reduced to

$$\begin{aligned} p' - \frac{W}{GJ_T}\frac{(EJ - GJ_T)}{EJ}q &= \frac{Q_2}{EJ}, \\ q' + \frac{W}{GJ_T}\frac{(EJ - GJ_T)}{EJ}p &= -\frac{Q_1}{EJ}. \end{aligned} \tag{11}$$

To substitute angles $\varphi_1,\varphi_2,\varphi_3$ into (11) we need derivatives p',q'

$$\begin{aligned} p' &= (\varphi_1''\cos\varphi_2 + \varphi_2'\varphi_3' + \varphi_1'\varphi_2'\sin\varphi_2)\cos\varphi_3 \\ &\quad + (\varphi_2'' - \varphi_1'\varphi_3'\cos\varphi_2)\sin\varphi_3, \\ q' &= -(\varphi_1''\cos\varphi_2 + \varphi_2'\varphi_3' + \varphi_1'\varphi_2'\sin\varphi_2)\sin\varphi_3 \\ &\quad + (\varphi_2'' - \varphi_1'\varphi_3'\cos\varphi_2)\cos\varphi_3. \end{aligned} \tag{12}$$

Then from (6), (11), (12) and (7) the equations of motion in angular coordinates are

$$\begin{aligned} &(\varphi_1''\cos\varphi_2 + \varphi_2'\varphi_3' + \varphi_1'\varphi_2'\sin\varphi_2)\cos\varphi_3 \\ &+ (\varphi_2'' - \varphi_1'\varphi_3'\cos\varphi_2)\sin\varphi_3 \\ &- \frac{A-C}{AC}W\,(-\varphi_1'\cos\varphi_2\sin\varphi_3 + \varphi_2'\cos\varphi_3) \\ &+ \frac{Q}{A}(\sin\varphi_1\cos\varphi_3 + \cos\varphi_1\sin\varphi_2\sin\varphi_3) = 0 \end{aligned} \tag{13}$$

and

$$\begin{aligned}&-\left(\varphi_1''\cos\varphi_2+\varphi_2'\varphi_3'+\varphi_1'\varphi_2'\sin\varphi_2\right)\sin\varphi_3\\&+\left(\varphi_2''-\varphi_1'\varphi_3'\cos\varphi_2\right)\cos\varphi_3\\&+\frac{A-C}{AC}W\left(\varphi_1'\cos\varphi_2\cos\varphi_3+\varphi_2'\sin\varphi_3\right)\\&-\frac{Q}{A}\left(\sin\varphi_1\sin\varphi_3-\cos\varphi_1\sin\varphi_2\cos\varphi_3\right)=0.\end{aligned}\tag{14}$$

From (13) and (14) φ_1'' and φ_2'' can be expressed as

$$\varphi_1''=\frac{-\varphi_3'\varphi_2'+\varphi_1'\varphi_2'\sin\varphi_2+W\frac{A-C}{AC}\varphi_2'-\frac{Q}{A}\sin\varphi_1}{\cos\varphi_2},\tag{15}$$

$$\varphi_2''=\varphi_1'\varphi_3'\cos\varphi_2-W\frac{A-C}{AC}\varphi_1'\cos\varphi_2-\frac{Q}{A}\sin\varphi_2\cos\varphi_1,$$

where $A=EJ$ and $C=GJ_T$.

Remark that the first equation of (15) is singular at $\cos\varphi_2=0$ and it restricts the use of such formulation to small angles φ_2. In the numerical analysis below we study the behavior of the rod near to its trivial form $\varphi_1\equiv 0,\varphi_2\equiv 0,\varphi_3\equiv 0$ thus this singularity does not appear. Unfortunately, (15) contains three variables thus we need one more equation. Let us return to the third equation of (6). Previously it was used to eliminate variable r from the other two. Here we use (10) to obtain an equation for φ_3'. From (10) and (5)

$$\varphi_3'=\frac{W}{C}-\varphi_1'\sin\varphi_2\tag{16}$$

Now equations (4), (15) and (16) are suitable to calculate the shape of the elastica under force Q and moment W.

We need also boundary conditions to perform calculations. At one end (point P_1) a clamped boundary is assumed, thus we should prescribe $\varphi_1(s=0),\varphi_2(s=0),\varphi_3(s=0)$, for the derivatives $\varphi_1'(s=0),\varphi_2'(s=0)$, and $x(s=0),y(s=0),z(s=0)$. The other remains unconstrained, but the load (force **Q** and moment **W**) is present as dynamic boundary condition. That is, we treat the problem as an initial value problem , the dynamic boundary conditions at the free end are already included into the equilibrium equations.

4 Numerical analysis

The system of equations (4), (15) and (16) can be solved by a standard Runge-Kutta method. In our analytic work [3] the post-buckling of the rod was studied by using static bifurcation theory [1], that is, we searched for the appearance of a non-trivial solution near to the trivial one $\varphi_1\equiv 0,\varphi_2\equiv 0,\varphi_3\equiv 0$ by quasistatic change of the loads. While this trivial solution is always a solution for all vales of loading parameters Q,W to search for nontrivial shapes in numerical calculation a small initial imperfection should be added otherwise we obtain always solution $\varphi_1\equiv 0,\varphi_2\equiv 0,\varphi_3\equiv 0$. In our calculations we selected imperfection in initial values $\varphi_1(s=0)=1\cdot 10^{-5},\varphi_1'(s=0)=1\cdot 10^{-5}$ and zeros for all the others. We set the ratio $\frac{A}{C}=\frac{3}{2}$ and use parameters $\rho=\frac{Q}{A},w=\frac{W}{C}$ for loading.

Some of the solutions obtained are shown in the following figures. Figs. 5,6 show the initial state, when there is no load at all. Remark that because of the imperfect initial conditions Fig.6 is not really a straight line, but note that the values of axis Y are multiplied by 10^{-3}.

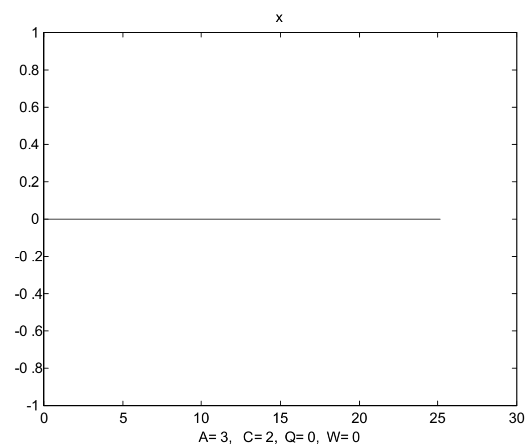

Figure 5 : X(Z) no load

If we add pure twist we find the formation of a helical shape Fig. 7, 8

When the load is only a thrusting force a buckling happens Figs. 9,10,11 Fig.10 shows that the rod is in the plane Y,Z while in Fig.11 we can see the buckled shape in coordinates X,Y,Z..

Now let us see, what happens when a coexistent thrustig force and twisting couple is applied Figs. 12,13. The comparison of Figs. 10,12 shows that additional twist effects a jumps out of plane Y,Z.

Let us now inrease twist Figs. 14,15. In Fig.14 the curve of the elastica "shrinks" due to the increased thurst. If we compare Fig.15 with Fig.8 at pure twist we find a definite decrease of the extension in directions X,Z and

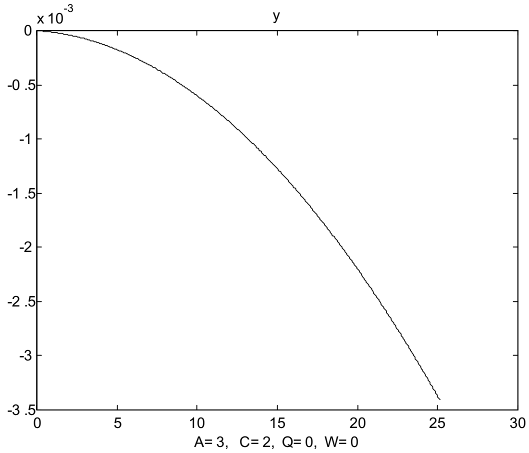

Figure 6 : Y(Z) no load

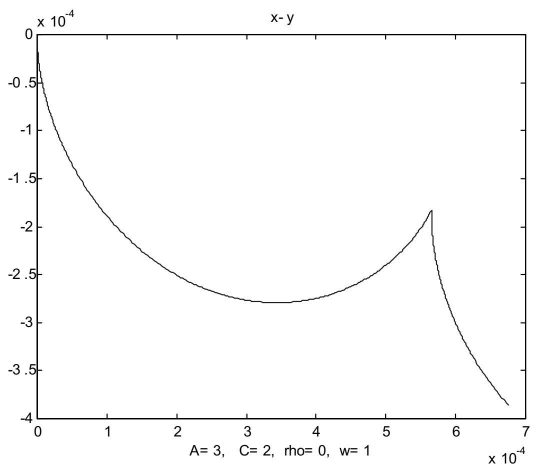

Figure 7 : Y(X) at pure twist $w = 1$

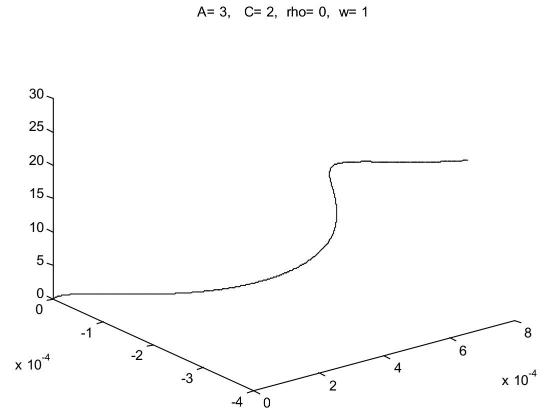

Figure 8 : 3D (X,Y,Z) plot at pure twist $w = 1$

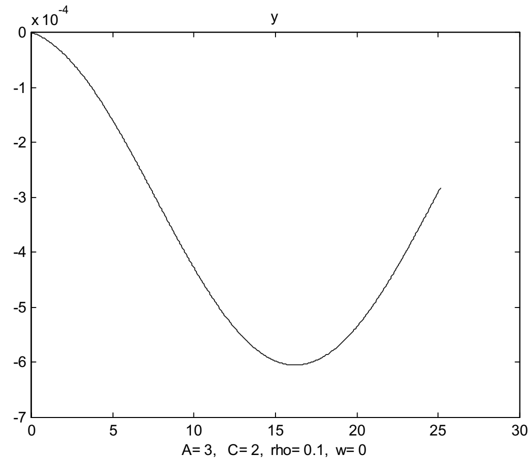

Figure 9 : Y(Z) at pure twist $\rho = 0.1$

a secondary looping.

5 Concluding remarks

In the paper by using a fixed and a local coordinate systems. This second was connected to the cross-sections of the Euler-Bernoulli beam and suffered finite rotations by applying a combined load of a terminal thrust and couple. By using angular coordinates to describe such rotations we could derive a system of differential equations, which describes the shape of the rod.

We prescribed quite special boundary conditions and it enabled us to study the problem by solving ordinary differential equations numerically. As a result we find the same as the classical analytical studies [Beck (1955), Euler (1749), Greenhill (1883), Kovari (1969), Love (1927)], both simple thrust and twist may lead to a buckled shape of the rod. Moreover, when a buckled (by pure thrust) rod is additionally loaded by a twisting couple a secondary buckling happens and the elastica jumps into a looped helical form.

6 Acknowledgements

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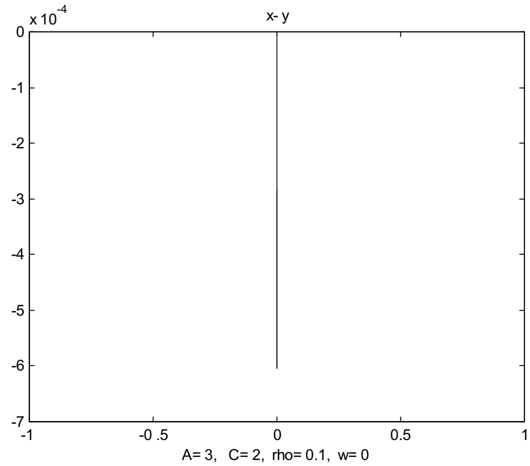

Figure 10 : Y(X) at pure thrust $\rho = 0.1$

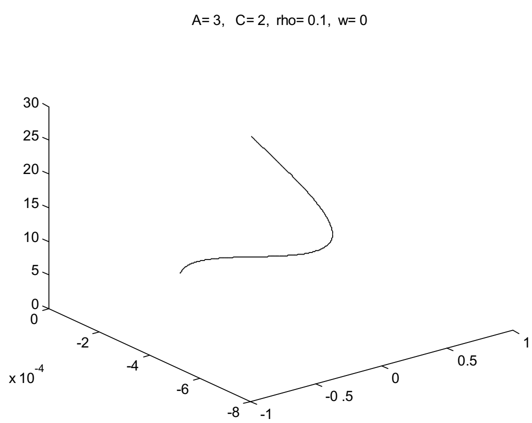

Figure 11 : 3D plot at pure thrust $\rho = 0.1$

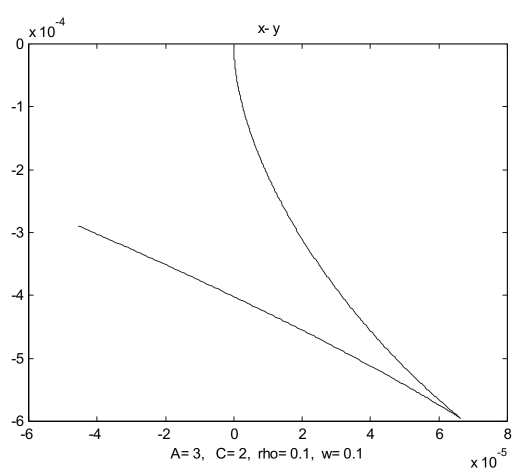

Figure 12 : Y(X) (couple and force) $\rho = 0.1, w = 0.1$

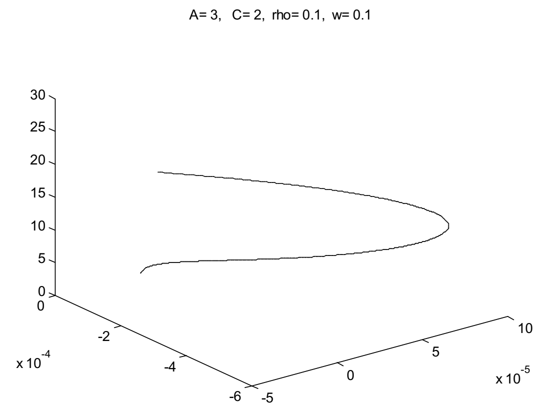

Figure 13 : 3D plot (couple and force) $\rho = 0.1, w = 0.1$

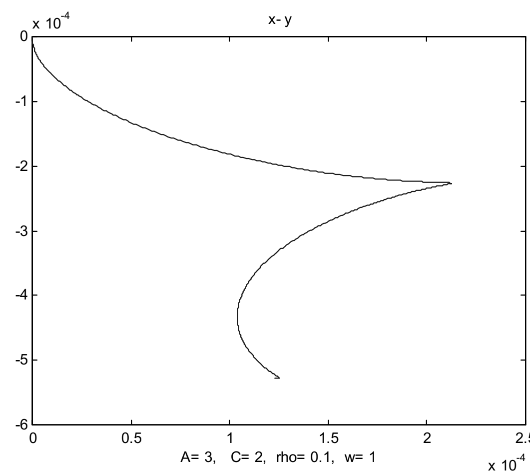

Figure 14 : Y(X) (couple and force) $\rho = 0.1, w = 1$

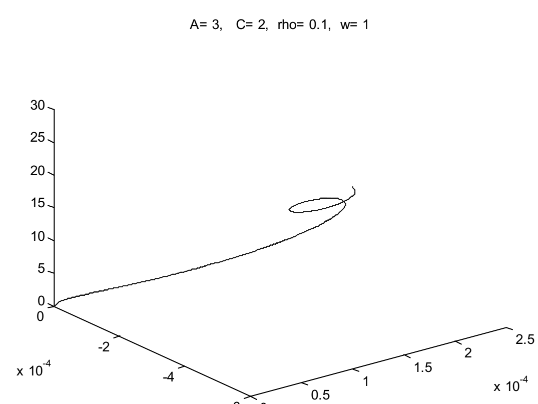

Figure 15 : 3D plot (couple and force) $\rho = 0.1, w = 1$

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