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Accuracy of Co-rotational Formulation for 3-D Timoshenko's Beam

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Abstract: An accuracy of finite element solutions for 3-D Timoshenko's beams, obtained using a co-rotational formulation, is discussed. The co-rotational formulation has often been used with an assumption that the relative deformations are small. A fundamental question, therefore, has been raised as to whether or not the numerical solutions obtained approach the solutions of the exact theory. In this paper, from theoretical point of view, we investigate the accuracy of the co-rotational formulation for 3-D Timoshenko's beam undergoing finite strains and finite rotations. It is shown that the use of the conventional secant coordinates fails to give satisfactory numerical solutions. We introduce a new local coordinate system in which a linear beam theory is used to construct the strain energy function. It is shown that the finite element solutions obtained converge to those of the exact beam theory as the number of element increases.

keyword: Co-rotational Formulation, Timoshenko's Beam, Finite Element Method, Finite Rotations.

1 Introduction

In a finite element analysis for large displacement problems of flexible beams, the total Lagrangian formulation together with a fixed global coordinate system has been used (see e.g. [Iura and Atluri(1988); Iura and Atluri(1989); Atluri, Iura and Vasudevan(2001), Reissner(1981); Beda(2003); Zupan and Saje(2003)]). In this formulation, a highly nonlinear beam theory is indispensable even if relative deformations are small.

According to the polar decomposition theorem, the total deformation is decomposed into the rigid body motion and the relative deformation (see e.g. [Malvern (1969)]). On the basis of this theorem, the relative deformation is described by using the local coordinate system. With the help of coordinate transformation, the relative deformation is expressed in terms of displacement components associated with a fixed global coordinate system. This formulation has often been used in the finite element analysis for nonlinear problems of flexible beams [Crisfield (1990); Goto *et al.* (2003), Hsio and Lin (2000); Ijima *et al.* (2002), Lin and Hsiao (2003), Oran (1973); Yoshida, Masuda, Morimoto and Hirosawa (1980); Wen and Rahimzadeh (1983)]. In this paper, this formulation is called the co-rotational formulation. The use of co-rotational formulation is motivated by the assumption that the relative deformation is small but the rigid body motion is finite. A linear beam theory or beam-column theory has often been used for describing the relative deformation. As a result, the stiffness matrix in the local coordinate system takes a simple form. The highly nonlinear terms are included in the coordinate transformation of the displacement components. The numerical studies have shown that, in spite of using the small strain assumption in the local coordinates, satisfactory numerical results have been obtained by increasing the number of elements. The accuracy of finite element solutions in the co-rotational formulation , however, has not fully been discussed from a theoretical point of view.

Goto, Hasegawa and Nishino (1984) were the first who investigated theoretically the accuracy of co-rotational formulation. They have shown for the planar Bernoulli-Euler's beam that, when a linear beam theory is used in the local coordinates, the numerical solutions do not converge to those of exact beam theory. Iura (1994) has presented another method for the co-rotational formulation and discussed the accuracy of the numerical solutions. It has been concluded by Iura (1994) that, even though a linear beam theory is used in the local coordinates, the numerical solutions converge to those of the exact beam theory. Although this conclusion is different from that of Goto, Hasegawa and Nishino (1984), the numerical experiments show the validity of Iura (1994). For planar

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Timoshenko's beam, Iura and Furuta (1995) have shown that, when the secant coordinate system together with a linear beam theory is used in the co-rotational formulation, the numerical solutions do not converge to those of the exact beam theory. They have introduced another local coordinates, in which the use of linear beam theory leads to the numerical solutions which converge to those of the exact beam theory.

It is well known that the rotation in a 3-D space does not belong to a linear space. Therefore, very few papers have been published to discuss the accuracy of co-rotational formulation for 3-D beam. Goto, Hasegawa, Nishino and Matsuura (1985) have extended their method for planar beam to 3-D Bernoulli-Euler's beam. They have concluded that the use of linear beam theory in the local coordinates does not yield the numerical solutions which converge to those of the exact beam theory. They have also pointed out that the accuracy of torsional moment and rotations can not be improved even in the case of the inextensional deformation of axis.

In this paper, we discuss an accuracy of numerical solutions for 3-D Timoshenko's beam undergoing finite strains and finite rotations. The method to investigate the accuracy is the same as that of Iura (1994). In the co-rotational formulation for 3-D beam, one of main issues is the definition of a local coordinate system. Most popular local coordinate system is the secant coordinate system where the local beam axis is defined by connecting both end-nodes of the deformed beam element. The in-plane coordinates are chosen in a proper way such that the local coordinates become orthogonal. It is shown herein that the use of the secant coordinate system leads to unsatisfactory numerical results due to the shearing deformations. A new local coordinate system is proposed in this paper. When we use a linear beam theory in the present local coordinate system, we obtain the finite element solutions which converge to the solutions of the exact beam theory.

2 Formulation

We explain the present co-rotational formulation developed by Iura (1994). When an attention is confined to configuration-independent loads, the total potential energy of the beam Π may be expressed as

$$\Pi = \Pi_s + \Pi_f \tag{1}$$

where Π_s is the strain energy function and Π_f the potential function of external forces. It should be noted that the strain energy function is an invariant value. This fact plays an important role in the present formulation. Let $\{u\}$ denote the displacement components referred to the local coordinates. When the co-rotational formulation is used, the strain energy function is, at first, expressed in term of $\{u\}$. In the existing literatures, a linear beam theory or a beam-column theory has often been used to construct the strain energy function. Let $\{U\}$ denote the displacement components referred to the fixed global coordinates. From geometrical consideration, we obtain the nonlinear relationships between $\{u\}$ and $\{U\}$. Substituting this relationship into the strain energy function, we obtain the strain energy function expressed in terms of $\{\mathrm{U}\}$. The potential function Π_f can be also expressed in terms of $\{U\}$. Finally, therefore, the total potential energy of the beam becomes a function of $\{U\}$.

Following a standard FE procedure, we obtain the discretized equilibrium equations of the beam, expressed as

$$\left\{ \frac{\partial \Pi}{\partial U_m} \right\} = 0, \tag{2}$$

where $\{U_m\}$ is the independent variables at each node, referred to the fixed global coordinates. When the Newton-Raphson method is used to solve the equilibrium equations, the tangent stiffness matrix $[T_G]$ and the residual force vector $\{\Delta R\}$ are written, respectively, as

$$[T_G] = [\frac{\partial^2 \Pi_s}{\partial U_m \partial U_n}], \qquad \{\Delta R\} = -\{\frac{\partial \Pi}{\partial U_m}\} \tag{3}$$

While the convergence rate of numerical solutions depend on the tangent stiffness matrix, the accuracy of numerical solutions depends on the residual force vector. It is shown from Eq.(3) that the accuracy of residual force vector depends on the total potential energy of the beam. The exact expression for the potential function of external forces can be easily obtained. Therefore, the accuracy of residual force vector depends on the strain energy function. As mentioned before, in the existing co-rotational formulation, a linear beam theory or a beam-column theory has often been used to construct the strain energy function. The resulting strain energy function is not exact but approximated one. Since the strain energy function is an invariant value, a comparison of the present strain energy function with that of the exact beam theory is enough for the present purpose. In this paper, the geometrically exact beam theory developed by Iura and Atluri (1988, 1989) is used as the exact beam theory.

3 Basic Equations

Let us explain briefly the geometrically exact beam theory, which has been developed on the basis of total Lagrangian formulation (see e.g. [Iura and Atluri (1988, 1989); Reissner (1981)] for more details). We assume, herein, that there exist no initial curvatures and twist of the beam; the beam in the reference state is straight. The strain energy function of the beam is expressed as

$$\Pi_s = \int [\, \frac{1}{2}GA(h_1)^2 + \frac{1}{2}GA(h_2)^2 + \frac{1}{2}EA(h_3)^2 + \frac{1}{2}EI(\kappa_1)^2 + \frac{1}{2}EI(\kappa_2)^2 + \frac{1}{2}GJ(\kappa_3)^2 \,]\, dx, \quad (4)$$

where GA is the shearing rigidity, EA the stretching rigidity, EI the bending rigidity, GJ the twisting rigidity, h_α the shearing strain, h_3 the stretching strain, κ_α the bending strain and κ_3 the twisting strain.

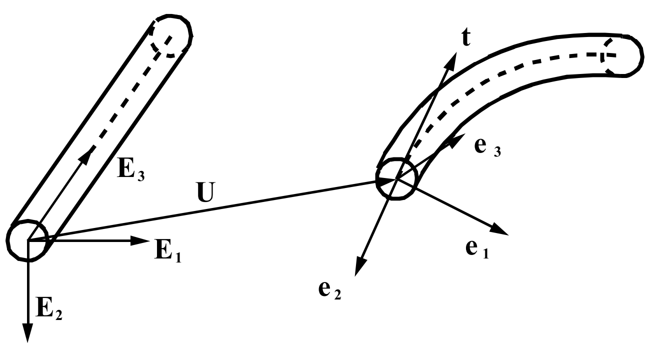

Figure 1 : Undeformed and Deformed Beam

Let $\boldsymbol{E}_m$ and $\mathbf{R}$ denote the undeformed base vectors of the beam and the finite rotation tensor, respectively. The base vectors $\boldsymbol{E}_m$ are transformed, due to the finite rotation tensor, such that $\boldsymbol{e}_m = \mathbf{R}\boldsymbol{E}_m$ (see Fig.1). Then the strain tensors are defined as

$$\begin{aligned} &h_1 = \boldsymbol{e}_1 \cdot \boldsymbol{t}, \quad h_2 = \boldsymbol{e}_2 \cdot \boldsymbol{t}, \quad h_3 = \boldsymbol{e}_3 \cdot \boldsymbol{t} - 1, \\ &\kappa_1 = \frac{1}{2}(\boldsymbol{e}_{2,3} \cdot \boldsymbol{e}_3 - \boldsymbol{e}_{3,3} \cdot \boldsymbol{e}_2), \quad \kappa_2 = \frac{1}{2}(\boldsymbol{e}_{3,3} \cdot \boldsymbol{e}_1 - \boldsymbol{e}_{1,3} \cdot \boldsymbol{e}_3), \\ &\kappa_3 = \frac{1}{2}(\boldsymbol{e}_{1,3} \cdot \boldsymbol{e}_2 - \boldsymbol{e}_{2,3} \cdot \boldsymbol{e}_1), \end{aligned} \quad (5)$$

where $(\ )_{,3}$ is the differentiation with respect to the axial coordinate, and $\boldsymbol{t}$ the deformed tangent vector defined by

$$\boldsymbol{t} = U_{1,3}\boldsymbol{E}_1 + U_{2,3}\boldsymbol{E}_2 + (1 + U_{3,3})\boldsymbol{E}_3. \quad (6)$$

in which $\boldsymbol{U}\,(= U_m\boldsymbol{E}_m)$ is the displacement vector at the beam axis and U_m the Lagrangian components of the displacement vector.

It is well known that the rigid body rotation is expressed by a single rotation of magnitude ω about an axis parallel to some unit vector $\boldsymbol{e}$, in which $\mathbf{R}\boldsymbol{e} = \boldsymbol{e}$. Even though alternate representations of the finite rotation vector are possible (see e.g. [Pietraszkiewicz (1979)]) here we assume for convenience that the finite rotation vector Ω has the form (see [Geradin and Cardona (1989); Iura and Atluri (1988)])

$$\Omega = 4tan\frac{\omega}{4}\boldsymbol{e} = \alpha^m\boldsymbol{E}_m. \quad (7)$$

where α^m is the rotational components referred to the fixed global coordinates. Then the Lagrangian components of $\mathbf{R}(= R_{mn}\boldsymbol{E}_m\boldsymbol{E}_n)$ are expressed by

$$R_{mn} = \frac{1}{(4-\alpha_0)^2} [(\alpha_0^2 - \alpha^k\alpha^k)\delta_{mn} + 2(\alpha^m\alpha^n - \varepsilon_{mnk}\alpha_0\alpha^k)], \quad (8)$$

where δ_{mn} is the Kronecker's delta, ε_{mnk} is the permutation tensor and

$$\alpha_0 = \frac{(16 - \alpha^k\alpha^k)}{8}. \quad (9)$$

4 Secant Coordinate System

The commonly used local coordinate system in the co-rotational formulation is the secant coordinate system, as shown in Fig. 2. The local beam axis at the current configuration is defined by passing through node i and node j, or end-nodes of the beam element. A variety of method have been proposed to define the in-plane coordinates. We shall not discuss this issue because the in-plane coordinates does not play an important role in the following discussion. In this paper, we shall show that the stretching strain defined by the secant coordinate system does not converge to the exact stretching strain.

Let $\boldsymbol{U}^{(i)}$ and $\boldsymbol{u}^{(i)}$ denote the displacement vector at the node i, referred to the fixed global coordinates and the secant coordinates, respectively. Let $\boldsymbol{a}_m$ be the base vector associated with the secant coordinate system, where $\boldsymbol{a}_3$ is the base vector associated with the beam axis and $\boldsymbol{a}_m \cdot \boldsymbol{a}_n = \delta_{mn}$. Because of the definition of the secant coordinate system, the displacement vector at node j in the

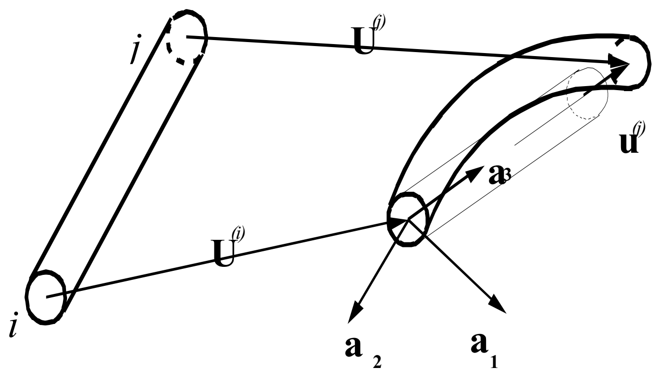

Figure 2 : Secant Coordinate System

local coordinates is expressed as $\boldsymbol{u}^{(j)} = u_3^{(j)}\boldsymbol{a}_3$. From geometrical consideration (see Fig.2), we have

$$\boldsymbol{U}^{(i)} + l\,\boldsymbol{a}_3 + u_3^{(j)}\boldsymbol{a}_3 = l\,\boldsymbol{E}_3 + \boldsymbol{U}^{(j)}, \tag{10}$$

where l is the length of the beam element in the reference state. It follows from Eq.(10) that

$$(1+\frac{u_3^{(j)}}{l})\boldsymbol{a}_3 = \boldsymbol{E}_3 + \frac{1}{l}(\boldsymbol{U}^{(j)} - \boldsymbol{U}^{(i)}). \tag{11}$$

It is common for FE analysis of Timoshenko's beam to interpolate the displacement into a linear function, expressed as

$$u_m = (1-\frac{x}{l})u_m^{(i)} + \frac{x}{l}u_m^{(j)}, \tag{12}$$

where x is the axial coordinate in the local coordinates. Note that, due to the definition of the secant coordinates, we have $u_m^{(i)} = 0$.When a linear beam theory is used in the local coordinates, the stretching strain is defined by

$$\varepsilon_3 = \frac{du_3}{dx} = \frac{u_3^{(j)}}{l}. \tag{13}$$

This is the expression for the stretching strain in the secant coordinate system.

Let $U_m^{(i)}$ denote the displacement component at the node i, defined by

$$\boldsymbol{U}^{(i)} = U_m^{(i)}\boldsymbol{E}_m. \tag{14}$$

Substituting Eqs.(13) and (14) into Eq.(11) and taking the scalar product of Eq.(11) with itself leads to

$$(1+\varepsilon_3)^2 = (\frac{U_1^{(j)} - U_1^{(i)}}{l})^2 + (\frac{U_2^{(j)} - U_2^{(i)}}{l})^2 + (1+\frac{U_3^{(j)} - U_3^{(i)}}{l})^2. \tag{15}$$

Solving Eq.(15) for the stretching strain, we have

$$\varepsilon_3 = \sqrt{(\frac{U_1^{(j)} - U_1^{(i)}}{l})^2 + (\frac{U_2^{(j)} - U_2^{(i)}}{l})^2 + (1+\frac{U_3^{(j)} - U_3^{(i)}}{l})^2} - 1. \tag{16}$$

This is the expression for the stretching strain in the fixed global coordinate system. The accuracy of the stretching strain ε_3 is investigated by comparing it with the exact stretching strain.

According to the exact beam theory, the deformed tangent vector is expressed as

$$\begin{aligned}\boldsymbol{t} &= h_1\boldsymbol{a}_1 + h_2\boldsymbol{a}_2 + (1+h_3)\boldsymbol{a}_3 \\ &= U_{1,3}\boldsymbol{E}_1 + U_{2,3}\boldsymbol{E}_2 + (1+U_{3,3})\boldsymbol{E}_3.\end{aligned} \tag{17}$$

Taking the scalar product of Eq.(17) with itself leads to

$$\begin{aligned}&(h_1)^2 + (h_2)^2 + (1+h_3)^2 \\ &= (U_{1,3})^2 + (U_{2,3})^2 + (1+U_{3,3})^2.\end{aligned} \tag{18}$$

It follows from Eq.(18) that

$$h_3 = \sqrt{(U_{1,3})^2 + (U_{2,})^2 + (1+U_{3,3})^2 - (h_1)^2 - (h_2)^2} - 1. \tag{19}$$

This is the expression for the exact stretching strain. It is common for finite element analysis of Timoshenko's beam to interpolate the shape function into a linear function (see e.g. [Hughes (1984)]), expressed as

$$U_m = (1-\frac{X}{l})U_m^{(i)} + \frac{X}{l}U_m^{(j)}, \tag{20}$$

where X is the axial coordinate in the fixed global coordinate system. The differentiation of displacement components with respect to the axial coordinate leads to

$$U_{m,3} = \frac{U_m^{(j)} - U_m^{(i)}}{l} \tag{21}$$

This expression can be obtained also by applying the forward difference to the first derivative $U^m{}_{,3}$. As the length of beam element l decreases or the number of element increases, Eq.(21) gives a good approximation of the first derivative (see e.g. [Atkinson (1978)]). Substituting Eq.(21) into Eq.(19), we have

$$h_3 = \sqrt{\frac{(\frac{U_1^{(j)}-U_1^{(i)}}{l})^2+(\frac{U_2^{(j)}-U_2^{(i)}}{l})^2}{+(1+\frac{U_3^{(j)}-U_3^{(i)}}{l})^2-(h_1)^2-(h_2)^2}} \; - \; 1. \quad (22)$$

By comparing Eq.(16) with Eq.(22), we may conclude that ε_3 is equal to h_3 only when $h_1 = h_2 = 0$. Since the shearing strains h_1 and h_2 do not become zero in general, the stretching strain ε_3 does not converge to the exact stretching strain even if the number of element increases. This fact shows that, when we use a linear beam theory in the secant coordinate system, the numerical solutions obtained does not approach those of the exact beam theory. The secant coordinate system might be used only when the shearing deformations can be neglected.

We consider, herein, the geometrical meaning of the strain defined by Ea.(16). Let $U^{(i)}$ and $U^{(j)}$ denote the displacement vectors of the element at nodes i and j , defined by

$$\boldsymbol{U}^{(i)} = U_m^{(i)}\boldsymbol{E}_m, \quad \boldsymbol{U}^{(j)} = U_m^{(j)}\boldsymbol{E}_m. \quad (23)$$

Then the engineering strain ε is defined by

$$\varepsilon = (\left\| U^{(i)} - U^{(j)} + lE_3 \right\| - l)/l \quad (24)$$

Substituting Eq.(23) into Eq.(24), we have $\varepsilon = \varepsilon_3$. This result shows that the strain defined by Eq.(13) is the engineering strain which is not conjugate with the present stress resultant.

5 New Coordinate System

In this chapter, we will present a new local coordinate system in place of the secant coordinate system. The origin of new coordinate system is taken at the node i, as shown in Fig. 3. The base vectors of the new coordinate system are defined by

$$\boldsymbol{b}_m = \mathbf{R}(\hat{\alpha}_k)\boldsymbol{E}_m, \qquad \hat{\alpha}_k = \frac{1}{2}(\alpha_k^{(i)} + \alpha_k^{(j)}), \quad (25)$$

where $\alpha_k^{(i)}$ is the rotational component of α_k at the node i. It is seen from Eq.(25) that $\hat{\alpha}_k$ denotes the mean value of the rotational components in the element. It should be noted that the base vectors b_m defined by Eq.(25) are orthogonal each other. This fact can be shown easily in the following. The scalar product between b_m and b_n is expressed as

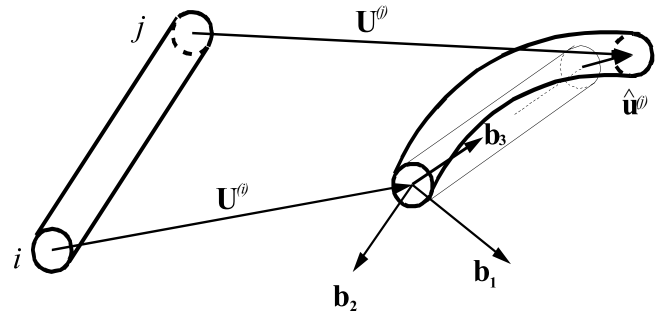

Figure 3 : New Coordinate System

$$\boldsymbol{b}_m \cdot \boldsymbol{b}_n = R_{km}R_{kn} \quad (26)$$

At first we consider the case where m=1 and n=2. Substituting Eq.(8) into Eq.(26) leads to

$$\begin{aligned}\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = & \frac{1}{(4-\hat{\alpha}_0^2)^2} \\ & [(\hat{\alpha}_0^2 - \hat{\alpha}_1^2 - \hat{\alpha}_2^2 - \hat{\alpha}_3^2 + 2\hat{\alpha}_1^2)(2\hat{\alpha}_1\hat{\alpha}_2 - 2\hat{\alpha}_0\hat{\alpha}_3) \\ & + (2\hat{\alpha}_2\hat{\alpha}_1 + 2\hat{\alpha}_0\hat{\alpha}_3)(\hat{\alpha}_0^2 - \hat{\alpha}_1^2 - \hat{\alpha}_2^2 - \hat{\alpha}_3^2 + 2\hat{\alpha}_2^2) \\ & + (2\hat{\alpha}_3\hat{\alpha}_1 - 2\hat{\alpha}_0\hat{\alpha}_2)(2\hat{\alpha}_3\hat{\alpha}_2 + 2\hat{\alpha}_0\hat{\alpha}_1)] \end{aligned} \quad (27)$$

where

$$\hat{\alpha}_0 = \frac{(16 - \hat{\alpha}_k\hat{\alpha}_k)}{8}. \quad (28)$$

A direct calculation shows that $b_1 \cdot b_2 = 0$. In a similar way, we can show that $b_m \cdot b_n = \delta_{mn}$.

The displacement vector at node j, referred to the local coordinates, can be expressed as (see Fig.3)

$$\hat{\boldsymbol{u}}^{(j)} = \hat{u}_m^{(j)}\boldsymbol{b}_m \quad (29)$$

From geometrical consideration, we have the following relation:

$$\boldsymbol{U}^{(i)} + \hat{u}_1^{(j)}\boldsymbol{b}_1 + \hat{u}_2^{(j)}\boldsymbol{b}_2 + \hat{u}_3^{(j)}\boldsymbol{b}_3 + l\,\boldsymbol{b}_3 = l\,\boldsymbol{E}_3 + \boldsymbol{U}^{(j)}. \quad (30)$$

We assume, herein, that the displacement components $\hat{u}_1^{(j)}$ and $\hat{u}_2^{(j)}$ are associated with the shearing deformations (see Fig.4) while the displacement component $\hat{u}_3^{(j)}$

is associated with the stretching deformation. Since we employ a linear theory in the local coordinates, we obtain the relationships between the displacements and the strains, expressed as

$$\hat{\gamma}_1 = \frac{\hat{u}_1^{(j)}}{l}, \qquad \hat{\gamma}_2 = \frac{\hat{u}_2^{(j)}}{l}, \qquad \hat{\varepsilon}_3 = \frac{\hat{u}_3^{(j)}}{l}, \tag{31}$$

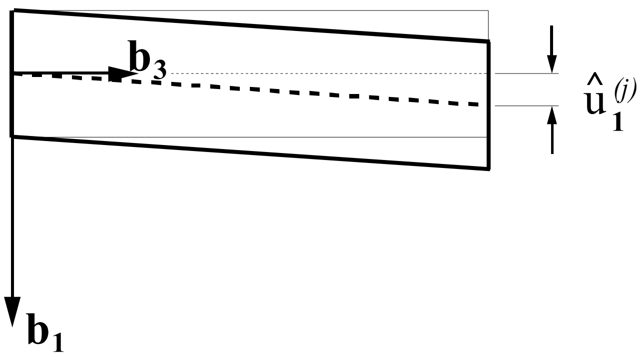

Figure 4 : Shearing Deformation

where $\hat{\gamma}_\alpha$ are the shearing strains and $\hat{\varepsilon}_3$ is the stretching strain. These are the expressions for the strain measures in the local coordinate system.

Since the base vectors $\boldsymbol{b}_m$ are orthogonal each other, it follows from Eqs.(30) and (31) that

$$\hat{\gamma}_\beta = \{\boldsymbol{E}_3 + \frac{1}{l}(\boldsymbol{U}^{(j)} - \boldsymbol{U}^{(i)})\} \cdot \boldsymbol{b}_\beta,$$
$$\hat{\varepsilon}_3 = \{\boldsymbol{E}_3 + \frac{1}{l}(\boldsymbol{U}^{(j)} - \boldsymbol{U}^{(i)})\} \cdot \boldsymbol{b}_3 - 1. \tag{32}$$

Substituting Eq.(14) into Eq.(32) and using Eq.(25), we have

$$\hat{\gamma}_\beta = \{(\frac{U_1^{(j)} - U_1^{(i)}}{l})\boldsymbol{E}_1 + (\frac{U_2^{(j)} - U_2^{(i)}}{l})\boldsymbol{E}_2 + (1 + \frac{U_3^{(j)} - U_3^{(i)}}{l})\boldsymbol{E}_3\} \cdot \mathbf{R}(\hat{\alpha}_k)\boldsymbol{E}_\beta,$$

$$\hat{\varepsilon}_3 = \{(\frac{U_1^{(j)} - U_1^{(i)}}{l})\boldsymbol{E}_1 + (\frac{U_2^{(j)} - U_2^{(i)}}{l})\boldsymbol{E}_2 + (1 + \frac{U_3^{(j)} - U_3^{(i)}}{l})\boldsymbol{E}_3\} \cdot \mathbf{R}(\hat{\alpha}_k)\boldsymbol{E}_3 - 1. \tag{33}$$

These are the expressions for the strain measures referred to the fixed global coordinate system.

Next we consider the strain measures of the exact beam theory. With the use of Eq.(17), the strain measures h_m of the exact beam theory can be written as

$$h_\beta = \{U_{1,3}\boldsymbol{E}_1 + U_{2,3}\boldsymbol{E}_2 + (1 + U_{3,3})\boldsymbol{E}_3\} \cdot \mathbf{R}\boldsymbol{E}_\beta,$$
$$h_3 = \{U_{1,3}\boldsymbol{E}_1 + U_{2,3}\boldsymbol{E}_2 + (1 + U_{3,3})\boldsymbol{E}_3\} \cdot \mathbf{R}\boldsymbol{E}_3 - 1. \tag{34}$$

It is common for FE analysis of Timoshenko's beam to use a linear shape function and the one-point Gauss integration rule. When a linear shape function is used for U_m, we have

$$U_{m,3} = \frac{(U_m^{(j)} - U_m^{(j)})}{l} \quad . \tag{35}$$

This expression can be obtained also by applying the forward difference to the first derivative $U_{m,3}$. Furthermore, when the one-point Gauss integration rule is used, we have $\mathbf{R} = \mathbf{R}(\hat{\alpha}_k)$. Then, Eq.(34) can be written as

$$h_3 = \{(\frac{U_1^{(j)} - U_1^{(i)}}{l})\boldsymbol{E}_1 + (\frac{U_2^{(j)} - U_2^{(i)}}{l})\boldsymbol{E}_2 + (1 + \frac{U_3^{(j)} - U_3^{(i)}}{l})\boldsymbol{E}_3\} \cdot \mathbf{R}(\hat{\alpha}_k)\boldsymbol{E}_\beta,$$

$$h_\beta = \{(\frac{U_1^{(j)} - U_1^{(i)}}{l})\boldsymbol{E}_1 + (\frac{U_2^{(j)} - U_2^{(i)}}{l})\boldsymbol{E}_2 + (1 + \frac{U_3^{(j)} - U_3^{(i)}}{l})\boldsymbol{E}_3\} \cdot \mathbf{R}(\hat{\alpha}_k)\boldsymbol{E}_3 - 1. \tag{36}$$

Comparing Eq.(33) with Eq.(36), we obtain

$$\hat{\gamma}_\beta = h_\beta, \qquad \hat{\varepsilon}_3 = h_3. \tag{37}$$

The strain measures in Eq.(36) are obtained by substituting a linear shape function and one-point Gauss integration rule into the exact strain measures. This procedure has often been used in the standard FE formulation. It has been established that the FE solutions obtained converge to the exact solutions. We may conclude ,therefore, that the shearing and stretching strains in the new coordinate system approaches the exact ones as the number of element increases.

Finally, let us consider the bending and twisting strains in the local coordinates. When we use a linear theory in the local coordinates, the bending strains are expressed as

$$\hat{\kappa}_1 = \frac{\hat{\phi}_1^{(j)} - \hat{\phi}_1^{(i)}}{l}, \qquad \hat{\kappa}_2 = \frac{\hat{\phi}_2^{(j)} - \hat{\phi}_2^{(i)}}{l}, \tag{38}$$

and the twisting strain is expressed as

$$\hat{\kappa}_3 = \frac{\hat{\phi}_3^{(j)} - \hat{\phi}_3^{(i)}}{l}. \tag{39}$$

Since we employ a linear theory in the local coordinate system, the rotations $\hat{\phi}_k^{(i)}$ referred to the local coordinates are also assumed to be small. Using the infinitesimal rotation vector $\hat{\boldsymbol{\phi}}^{(i)}(=\hat{\phi}_k^{(i)}\boldsymbol{b}_k)$ at node i, we have the relationship such that $\boldsymbol{e}_k^{(i)} = (\mathbf{I}+\hat{\boldsymbol{\phi}}^{(i)})\times\boldsymbol{b}_k$. With the help of rotational components, the base vectors at node i are written as

$$\begin{aligned}
\boldsymbol{e}_1^{(i)} &= \boldsymbol{b}_1 + \hat{\phi}_3^{(i)}\boldsymbol{b}_2 - \hat{\phi}_2^{(i)}\boldsymbol{b}_3,\\
\boldsymbol{e}_2^{(i)} &= -\hat{\phi}_3^{(i)}\boldsymbol{b}_1 + \boldsymbol{b}_2 + \hat{\phi}_1^{(i)}\boldsymbol{b}_3,\\
\boldsymbol{e}_3^{(i)} &= \hat{\phi}_2^{(i)}\boldsymbol{b}_1 - \hat{\phi}_1^{(i)}\boldsymbol{b}_2 + \boldsymbol{b}_3.
\end{aligned} \tag{40}$$

In a similar way, the base vectors at node j are obtained by changing superscript i into j in Eq.(40). According to Eq.(40), the rotational components at each node are written as

$$\begin{aligned}
\hat{\phi}_m^{(i)} &= \frac{1}{2}\varepsilon_{mnp}\boldsymbol{e}_n^{(i)}\cdot\boldsymbol{b}_p = \frac{1}{2}\varepsilon_{mnp}\boldsymbol{e}_n^{(i)}\cdot\mathbf{R}(\hat{\alpha}^k)\boldsymbol{E}_p,\\
\hat{\phi}_m^{(j)} &= \frac{1}{2}\varepsilon_{mnp}\boldsymbol{e}_n^{(j)}\cdot\boldsymbol{b}_p = \frac{1}{2}\varepsilon_{mnp}\boldsymbol{e}_n^{(j)}\cdot\mathbf{R}(\hat{\alpha}^k)\boldsymbol{E}_p.
\end{aligned} \tag{41}$$

Substituting Eq.(41) into Eqs.(38) and (39), we obtain the bending and twisting strains in the fixed global coordinate system, expressed as

$$\hat{\kappa}_m = \frac{1}{2}\varepsilon_{mnp}\frac{(\boldsymbol{e}_n^{(j)} - \boldsymbol{e}_n^{(i)})}{l}\cdot\mathbf{R}(\hat{\alpha}^k)\boldsymbol{E}_p. \tag{42}$$

According to the exact beam theory, the bending and twisting strains are expressed from Eq.(5) as

$$\kappa_m = \frac{1}{2}\varepsilon_{mnp}(\boldsymbol{e}_{n,3})\cdot(\mathbf{R}\boldsymbol{E}_p). \tag{43}$$

As mentioned before, it is common for FE analysis of Timoshenko's beam to use a linear shape function and one-point Gauss integration rule. When the one-point Gauss integration rule is used to integrate the strain energy function, we have $R = R(\hat{\alpha}^k)$. By comparing Eq.(42) with Eq.(43), we may conclude that the strain measures $\hat{\kappa}_m$ are equal to the exact strain measures κ_m when the following relation holds:

$$\boldsymbol{e}_{n,3} = \frac{1}{l}(\boldsymbol{e}_n^{(j)} - \boldsymbol{e}_n^{(i)}). \tag{44}$$

Equation (44) shows a forward difference of $\boldsymbol{e}_{n,3}$. When the number of element increases or the length of beam element decreases, the forward difference gives a good approximation of the first derivative (see e.g. [Atkinson (1978)]). Therefore, Eq.(44) may hold when the number of element increases. In such a case, we have $\hat{\kappa}_m = \kappa_m$: the exact strain measures κ_m are recovered when the new coordinate system with a linear theory is employed in the co-rotational formulation.

As shown before, even if we employ a linear theory in the new coordinate system, the strain measures in the local coordinate system approach the exact ones as the number of elements increases. We may conclude, therefore, the FE solutions obtained by the co-rotational formulation converge to those of the exact beam theory.

6 Numerical Examples

It is very hard to obtain the exact solutions for 3-D Timoshenko's beam undergoing finite strains and finite rotations. Goto, Yoshimitsu and Obata (1990) have obtained the exact solutions for plane elastica with axial and shear deformations. In this paper, therefore, planar Timoshenko's beam problems are solved to demonstrate the validity of the present theoretical results.

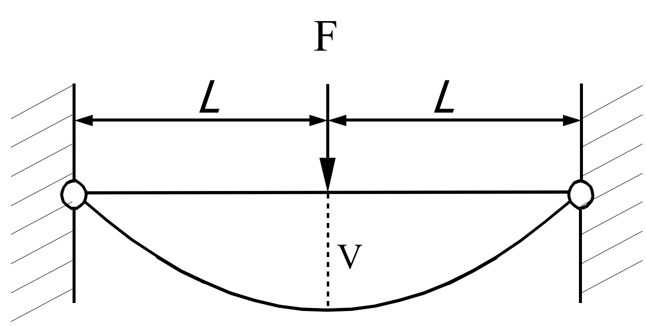

Figure 5 : Beam with Hinged Ends

The first problem is the beam with hinged ends as shown in Fig. 5. The concentrated force is applied at the center of the beam. The slenderness ratio of the beam is 5. Because of symmetry a half of the beam is discretized and 20 elements are used to obtain the converged solutions.

The parameter μ (=*EA*/*GA*) is taken as 0 and 10. When $\mu = 0$, the shear deformations are neglected. The FE solutions are compared with the exact solutions in Fig.6. The solid lines show the exact solutions of the exact beam theory. The circles and the squares show the FE solutions obtained by using the secant coordinate system and the new coordinate system, respectively. As shown in Fig.6, the FE solutions coincide with the exact solutions when $\mu = 0$. However, once the shear deformations can not be neglected, the FE solutions obtained by using the secant coordinate system are different from the exact solutions. When new coordinate system is used, the FE solutions coincide with the exact solutions even in the case of $\mu = 10$.

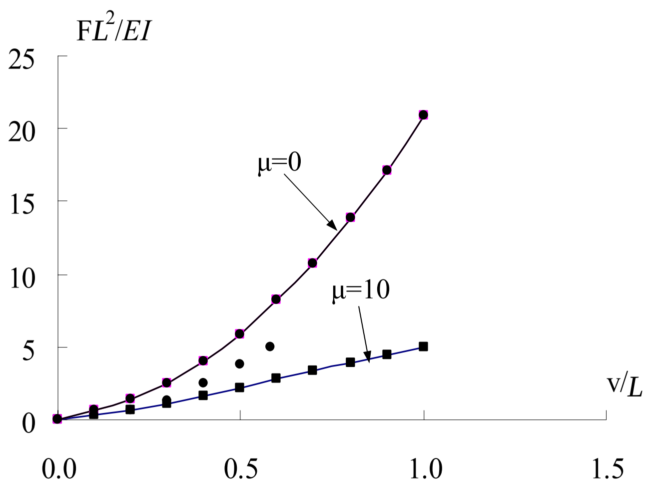

Figure 6 : Numerical Results

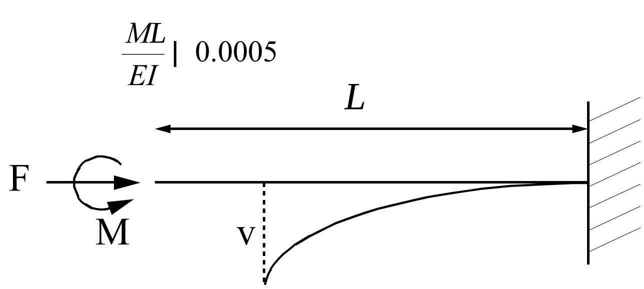

Figure 7 : Cantilever Beam

The second example is the cantilever beam subjected to an increasing compressive end force and a constant torque, as shown in Fig.7. The slenderness ratio of the beam is 4. The number of elements is 20. The parameter μ (=*EA*/*GA*) is taken as 0 and 10. The FE solutions are compared with the exact solutions in Fig. 8. Once again, the FE solutions coincide with the exact solutions when $\mu = 0$. In the case of $\mu = 10$, the FE solutions obtained using the secant coordinate system (circles) are different from the exact solutions. The use of new coordinate system leads to the complete agreement between FE and exact solutions.

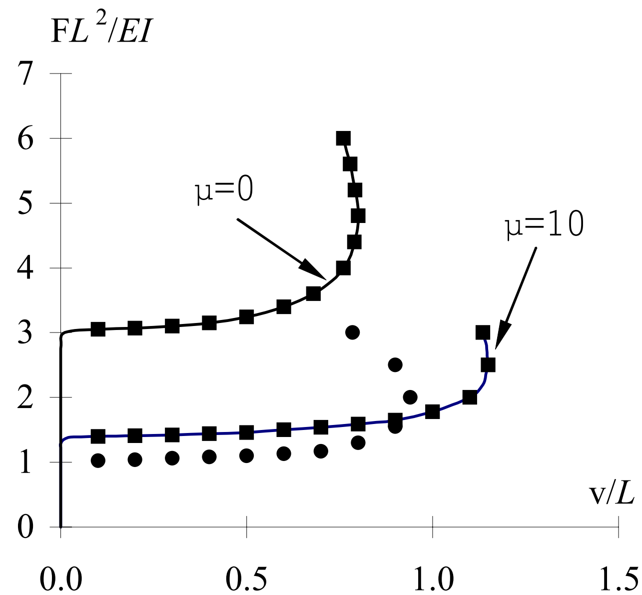

Figure 8 : Numerical Results

7 Concluding Remarks

We have discussed the accuracy of co-rotational formulation for 3-D Timoshenko's beam undergoing finite strains and finite rotations. It is common for the co-rotational formulation to use the secant coordinate system as the local coordinate system in which a linear theory or the beam-column theory is employed. It is shown herein that the use of the secant coordinate system together with a linear theory does not give the satisfactory numerical results. Instead of the secant coordinate system, we have introduced a new coordinate system where a linear theory is used. Then, using the exact coordinate transformation, we obtain the expressions for the strain measures referred to the fixed global coordinates. The resulting strain measures are compared with the exact ones. When a linear shape function and the one-point Gauss integration rule are substituted into the strain measures of the exact beam theory, a complete agreement between these two strain measures have been obtained. This fact shows that the FE solutions obtained by the co-rotational formulation converge to the solutions of the exact beam theory as the number of element increases.

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