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# Variational Formulation and Symmetric Tangent Operator for Shells with Finite Rotation Field

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**Abstract:** The objective of this paper is to examine the symmetry of the tangent operator for nonlinear shell theories with the finite rotation field. As well known, it has been stated that since the rotation field carries the Lie group structure, not a vector space one, the tangent operator incorporating the rotation field does not become symmetric. In this paper, however, it is shown that by adopting a rotation vector as a variable, the symmetry can be achieved in the Lagrangean (material) description. First, we present a general concept for the problem. Next, we adopt the finitely deformed thick shell problem as an example. We also present a tensor formula that plays a key role for the derivation of a symmetric tangent operator.

**keyword:** finite rotation, tangent operator, symmetry, shell theory, variational formulation.

## 1 Introduction

Many authors have discussed the symmetry and/or unsymmetry of the tangent operator for nonlinear solid mechanics. As well known, when the rotation field is introduced as an independent variable, the tangent operator for a nonlinear mechanical system is not always symmetric. The reason why the symmetry is lost is that a configuration space represented by a rotation tensor is the Lie group $SO(3)$ and therefore it does not carry a vector space structure.

Simo and Vu-Quoc (1986) show that the linearized tangent operator for 3-D finite-strain rods became unsymmetric at a non-equilibrium configuration. Nour-Omid and Rankin (1991) separate the rigid body motion from the total one and also derive an unsymmetric tangent stiffness matrix for rods and shells including large rotations. A symmetric tangent stiffness for finitely stretched and rotated 3-D beams is presented by Iura and Atluri (1989), in which an equilibrium state and a conservative system are assumed. Sansour and Bufler (1992) suggest that if we define the configuration space by using a rotation vector instead of a rotation tensor, we can derive a symmetric tangent operator. In addition, Bufler (1993) shows that the tangent stiffness of a conservative system is always symmetric, even if the rotation field is involved. However, in Bufler (1993), the Gâteaux derivative with respect to the rotation tensor is defined in a similar manner to the derivative on a vector space. In Simo (1992) it is proposed to replace the conventional definition of the second Gâteaux derivative by the covariant derivative, which is defined by the Levi-Civita connection and which is symmetric even in a non-equilibrium state. Aforementioned discussions are summarized by Makowski and Stumpf (1995), in which the question whether the tangent operator is symmetric or not is controversially discussed.

In the latest research works, Gotou, Kuwataka and Iwakuma (2003) derive an unsymmetric tangent stiffness matrix in which rotational degrees of freedom are expressed by Eulerian angles. On the other hand, Ijima, Obiya, Iguchi and Goto (2003) present a symmetric formulation for space frames by introducing element end coordinates. The authors (2003) also present a symmetric variational formulation for 3-dimensinal Timoshenko's beam and discuss the accuracy of the co-rotational formulation. As for shell analyses, Basar and Kintzel (2003) derive a symmetric tangent equation, in which an assumption is introduced for the variation and the linearization of rotation fields.

In this paper, the symmetry of the tangent operator is reconsidered in the context of the conventional Gâteaux derivative, while Simo (1992) and Makowski and Stumpf (1995) discuss the problem in terms of the differential manifold and the Lie algebra. Especially, we deal with the problem in the context of the material (Lagrangean) and the spatial (Eulerian) descriptions. As a result, a

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symmetric tangent operator with the rotation field can be derived as far as the Lagrangean description is adopted and a conservative system is assumed. It is also shown that the symmetry of the tangent operator holds even at a non-critical (non-equilibrium) point.

The remainder of this paper is as follows. Section 2 deals with preliminaries. In Section 3, it is symbolically shown that the second Gâteaux derivative of the total potential energy becomes symmetric in the Lagrangean description. In Section 4, the application of the present concept to the shell problem is presented, in which the functional established by the authors is employed [Suetake, Iura, and Atluri (1999); Atluri, Iura, and Suetake (2001); Atluri, Iura and Vasudevan (2001)]. Concluding remarks is given in Section 5.

## 2 Preliminaries

Throughout this paper, we indicate vectors with an under bar like $\underline{\boldsymbol{v}}$ and tensors with an under tilde like $\underset{\sim}{\boldsymbol{T}}$.

Let $\underline{\boldsymbol{X}}$ be a point in the reference configuration $B$. On the other hand, let $\underline{\boldsymbol{x}}$ be a point in the current configuration $b$. If we introduce convected coordinates $\xi^i$, base vectors at points $\underline{\boldsymbol{X}}$ and $\underline{\boldsymbol{x}}$ are defined as follows:

$$\underline{\boldsymbol{G}}_i = \frac{\partial \underline{\boldsymbol{X}}}{\partial \xi^i} \quad , \quad \underline{\boldsymbol{g}}_i = \frac{\partial \underline{\boldsymbol{x}}}{\partial \xi^i}. \tag{1}$$

We refer to $\underline{\boldsymbol{G}}_i$ in the material (Lagrangean) description and $\underline{\boldsymbol{g}}_i$ in the spatial (Eulerian) description.

We define a differential operator $\underline{\boldsymbol{\nabla}}$ [see, for example, Ogden (1984)]:

$$\underline{\boldsymbol{\nabla}} = \underline{\boldsymbol{g}}^i \frac{\partial}{\partial \xi^i}. \tag{2}$$

By using the operator $\underline{\boldsymbol{\nabla}}$, the gradients of a scalar and a vector fields, $\phi$ and $\underline{\boldsymbol{f}}$, are defined as follows:

$$\underline{\boldsymbol{\nabla}}\phi = \frac{\partial \phi}{\partial \xi^i}\underline{\boldsymbol{g}}^i, \tag{3}$$

$$\underline{\boldsymbol{\nabla}} \otimes \underline{\boldsymbol{f}} = \underline{\boldsymbol{g}}^i \otimes \underline{\boldsymbol{f}}_{,i} \quad , \quad \underline{\boldsymbol{f}} \otimes \underline{\boldsymbol{\nabla}} = \underline{\boldsymbol{f}}_{,i} \otimes \underline{\boldsymbol{g}}^i. \tag{4}$$

where $(\ )_{,i}$ denotes a partial derivative with respect to $\xi^i$. Therefore, the gradient of the vector field $\underline{\boldsymbol{\nabla}}\phi$ is given by

$$\underline{\boldsymbol{\nabla}} \otimes (\underline{\boldsymbol{\nabla}}\phi) = (\underline{\boldsymbol{\nabla}}\phi) \otimes \underline{\boldsymbol{\nabla}} = \phi|_{ij}\, \underline{\boldsymbol{g}}^i \otimes \underline{\boldsymbol{g}}^j, \tag{5}$$

where $\phi|_{ij}$ is the second covariant derivative of a scalar field $\phi$. Note that Eq. (5) is a symmetric.

For shell problems, we define several quantities. Let $\underline{\boldsymbol{X}}_0$ be a generic point on the undeformed shell reference surface $S_0$. The covariant base vectors at $\underline{\boldsymbol{X}}_0$ on $S_0$ are defined by

$$\underline{\boldsymbol{A}}_\alpha = \frac{\partial \underline{\boldsymbol{X}}_0}{\partial \xi^\alpha} \quad , \qquad \underline{\boldsymbol{A}}_3 = \frac{1}{2\sqrt{A}} e^{\alpha\beta} \underline{\boldsymbol{A}}_\alpha \times \underline{\boldsymbol{A}}_\beta, \tag{6}$$

where Greek indices take 1 or 2, $e^{\alpha\beta}$ is a permutation symbol, and $A = \det(A_{\alpha\beta}) = \det(\underline{\boldsymbol{A}}_\alpha \cdot \underline{\boldsymbol{A}}_\beta)$. By the definition as Eq. (6), the base vector $\underline{\boldsymbol{A}}_3$ becomes a unit normal to $S_0$ and the covariant base vectors in the undeformed shell domain can be written as

$$\underline{\boldsymbol{G}}_\alpha = \underline{\boldsymbol{A}}_\alpha + \xi^3 \underline{\boldsymbol{A}}_{3,\alpha} \quad , \qquad \underline{\boldsymbol{G}}_3 = \underline{\boldsymbol{A}}_3. \tag{7}$$

The contravariant base vectors $\underline{\boldsymbol{A}}^i$ and $\underline{\boldsymbol{G}}^i$ are defined by $\underline{\boldsymbol{A}}^i \cdot \underline{\boldsymbol{A}}_j = \underline{\boldsymbol{G}}^i \cdot \underline{\boldsymbol{G}}_j = \delta^i_j$ where $\delta^i_j$ is the Kronecker's delta.

Let the vectors $\underline{\boldsymbol{X}}_0$ and $\underline{\boldsymbol{A}}_3$ be mapped into $\underline{\boldsymbol{x}}_0$ and $\underline{\boldsymbol{a}}_3$. The base vectors at $\underline{\boldsymbol{x}}_0$ are defined by

$$\underline{\boldsymbol{a}}_\alpha = \underline{\boldsymbol{x}}_{0,\alpha} = \underline{\boldsymbol{A}}_\alpha + \underline{\boldsymbol{u}}_{0,\alpha}, \tag{8}$$

where $\underline{\boldsymbol{u}}_0$ is a displacement vector at $\underline{\boldsymbol{X}}_0$. If we assume that a straight fiber normal to $S_0$ is mapped into another straight fiber after deformation, the base vector $\underline{\boldsymbol{g}}_i$ in the deformed shell domain can be expressed as

$$\underline{\boldsymbol{g}}_\alpha = \underline{\boldsymbol{a}}_\alpha + \xi^3 \underline{\boldsymbol{a}}_{3,\alpha} \quad , \quad \underline{\boldsymbol{g}}_3 = \underline{\boldsymbol{a}}_3. \tag{9}$$

At this stage, we introduce an arbitrary rigid rotation $\underset{\sim}{\boldsymbol{R}}$ and alternative strain measures $\underline{\boldsymbol{c}}_i$ and $\underline{\boldsymbol{b}}_\alpha$ as follows:

$$\underline{\boldsymbol{c}}_i = \underset{\sim}{\boldsymbol{R}}^T \cdot \underline{\boldsymbol{a}}_i \quad , \quad \underline{\boldsymbol{b}}_\alpha = \underset{\sim}{\boldsymbol{R}}^T \cdot \underline{\boldsymbol{a}}_{3,\alpha}. \tag{10}$$

We also define a stress-resultant vector and a stress-couple vector for shells:

$$\underline{\boldsymbol{N}}^i = \int_{\xi^3} \underline{\boldsymbol{t}}^i \mu_0 d\xi^3, \tag{11}$$

$$\underline{\boldsymbol{H}}^\alpha = \int_{\xi^3} \xi^3 \underline{\boldsymbol{t}}^\alpha \mu_0 d\xi^3. \tag{12}$$

where $\underline{\boldsymbol{t}}^i$ is a stress vector defined by the first Piola-Kirchhoff stress tensor as $\underline{\boldsymbol{t}}^i = \underline{\boldsymbol{G}}^i \cdot \underset{\sim}{\boldsymbol{t}}$ and $\mu_0$ is defined by using the curvature tensor $B^\alpha{}_\beta$ and the Gaussian curvature $B$ of the undeformed shell reference surface $S_0$ as $\mu_0 = 1 - B^\rho{}_\rho \xi^3 + B(\xi^3)^2$. We also introduce alternative stress-resultant and stress-couple vectors as

$$\underline{\hat{\boldsymbol{N}}}^i = \underline{\boldsymbol{N}}^i \cdot \underset{\sim}{\boldsymbol{R}} \quad \text{or} \quad \underline{\boldsymbol{N}}^i = \underset{\sim}{\boldsymbol{R}} \cdot \underline{\hat{\boldsymbol{N}}}^i, \tag{13}$$

$$\underline{\hat{\boldsymbol{H}}}^i = \underline{\boldsymbol{H}}^i \cdot \underset{\sim}{\boldsymbol{R}} \quad \text{or} \quad \underline{\boldsymbol{H}}^i = \underset{\sim}{\boldsymbol{R}} \cdot \underline{\hat{\boldsymbol{H}}}^i. \tag{14}$$

We should note that these stress measures are conjugate with the strain measures defined by Eq. (10).

In the latter section, a functional for shells will be expressed by the above alternative stress and strain measures.

## 3 Gâteaux Derivative of Functional

We discuss here the Gâteaux derivatives of a potential energy $\Pi = \Pi(\underline{\boldsymbol{x}})$, which always exists for a conservative system. The argument of the potential $\Pi$, namely, $\underline{\boldsymbol{x}}$ is expressed by $\underline{\boldsymbol{G}}_i$ in the Lagrangean description, while by $\underline{\boldsymbol{g}}_i$ in the Eulerian description.

We deal with two mappings: one is represented by a vector field (displacement) $\underline{\boldsymbol{u}}$ and the other by a rotation field $\underset{\sim}{\boldsymbol{R}}$. In the following subsections, both mappings are discussed separately.

### 3.1 Vector Field

The mapping form $\underline{\boldsymbol{X}}$ to $\underline{\boldsymbol{x}}$ shall be represented by a vector field $\underline{\boldsymbol{u}}$ such that $\underline{\boldsymbol{x}} = \underline{\boldsymbol{X}} + \underline{\boldsymbol{u}}$. We introduce a variation of $\underline{\boldsymbol{u}}$,that is, $\delta\underline{\boldsymbol{u}} = \varepsilon\underline{\boldsymbol{h}}$, where $\varepsilon$ is an infinitesimal parameter and $\underline{\boldsymbol{h}}$ is an arbitrary vector. Since $\underline{\boldsymbol{u}}$ belongs to an Euclidean vector space, an infinitesimal transformation $\underline{\tilde{\boldsymbol{x}}}$ is simply given by

$$\underline{\tilde{\boldsymbol{x}}} = \underline{\boldsymbol{x}} + \varepsilon\underline{\boldsymbol{h}}. \tag{15}$$

Note that the variation vector $\underline{\boldsymbol{h}}$ does not depend on the current configuration $\underline{\boldsymbol{x}}$. Therefore, the first Gâteaux derivative of $\Pi$ is written as

$$\begin{aligned} d\Pi(\underline{\boldsymbol{x}};\underline{\boldsymbol{h}}) &= \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \tilde{\Pi}(\underline{\tilde{\boldsymbol{x}}}) \\ &= \left(\frac{\partial\tilde{\Pi}}{\partial\xi^i}\underline{\boldsymbol{g}}^i \cdot \frac{\partial\underline{\tilde{\boldsymbol{x}}}}{\partial\varepsilon}\right)\Big|_{\varepsilon=0} = \underline{\nabla}\Pi \cdot \underline{\boldsymbol{h}} \end{aligned} \tag{16}$$

where $\underline{\nabla}\Pi = \underline{\boldsymbol{0}}$ corresponds to a critical point, that is, an equilibrium equation.

Since we adopt $\underline{\boldsymbol{g}}_i$ as base vectors, Eq. (16) is the Eulerian description. However, if we express the base vectors $\underline{\boldsymbol{g}}_i$ by $\underline{\boldsymbol{G}}_i$ and $\underline{\boldsymbol{u}} = u^i\underline{\boldsymbol{G}}_i$, Eq. (16) can be regarded as the Lagrangean description. Consequently, Eq. (16) can be interpreted as both the Eulerian and the Lagrangean descriptions. For simplicity, we make the expression of the base vectors remain $\underline{\boldsymbol{g}}_i$. The definition of the differential operator $\underline{\boldsymbol{\nabla}}$ also remains Eq. (2).

In the similar manner to Eq. (16), we can derive the second Gâteaux derivative of $\Pi$ for an infinitesimal transformation $\underline{\boldsymbol{x}}^* = \underline{\boldsymbol{x}} + \tau\underline{\boldsymbol{v}}$:

$$\begin{aligned} d^2\Pi(\underline{\boldsymbol{x}};\underline{\boldsymbol{h}},\underline{\boldsymbol{v}}) &= d[\underline{\nabla}\Pi \cdot \underline{\boldsymbol{h}}](\underline{\boldsymbol{x}};\underline{\boldsymbol{v}}) \\ &= \frac{\partial}{\partial\tau}\Big|_{\tau=0} \{\underline{\nabla}\Pi^*(\underline{\boldsymbol{x}}^*) \cdot \underline{\boldsymbol{h}}\} \\ &= \left\{\frac{\partial\underline{\boldsymbol{x}}^*}{\partial\tau} \cdot (\underline{\nabla} \otimes \underline{\nabla}\Pi^*)\right\}\Big|_{\tau=0} \cdot \underline{\boldsymbol{h}} = \underline{\boldsymbol{v}} \cdot (\underline{\nabla} \otimes \underline{\nabla}\Pi) \cdot \underline{\boldsymbol{h}}. \end{aligned} \tag{17}$$

where we consider that the arbitrary vector $\underline{\boldsymbol{h}}$ does not depend on $\underline{\boldsymbol{x}}$.

Apparently, the right hand side of Eq. (17) is commutable with respect to $\underline{\boldsymbol{h}}$ and $\underline{\boldsymbol{v}}$. Therefore, we can conclude that the second Gâteaux derivative of $\Pi$ is a symmetric bilinear form in both the Lagrangean and the Eulerian descriptions. In addition, the above discussion does not depend on whether or not the current configuration point $\underline{\boldsymbol{x}}$ is a critical one (an equilibrium point).

### 3.2 Rotation Field

In this subsection, our discussion is restricted to the rigid body mapping form $\underline{\boldsymbol{X}}$ to $\underline{\boldsymbol{x}}$ such that $\underline{\boldsymbol{x}} = \underset{\sim}{\boldsymbol{R}} \cdot \underline{\boldsymbol{X}}$ in order to clarify the relationship between the rotation field and the Gâteaux derivative. Since the space of the rotation field $\underset{\sim}{\boldsymbol{R}}$ is $SO(3)$, not a vector space, the second Gâteaux derivative of the potential $\Pi$ cannot be simply defined as Eq. (17). Such concept is similar to the aforementioned literatures.

In the beginning, we introduce an infinitesimal rotation $\underset{\sim}{\boldsymbol{Q}}$ in order to define an infinitesimal transformation $\tilde{\underline{\boldsymbol{x}}}$ such that

$$\tilde{\underline{\boldsymbol{x}}} = \underset{\sim}{\boldsymbol{Q}} \cdot \underset{\sim}{\boldsymbol{R}} \cdot \underline{\boldsymbol{X}} = \underset{\sim}{\boldsymbol{Q}} \cdot \underline{\boldsymbol{x}}. \tag{18}$$

In the Eulerian description, we can represent the infinitesimal rotation $\underset{\sim}{\boldsymbol{Q}}$ by

$$\underset{\sim}{\boldsymbol{Q}} = \underset{\sim}{\boldsymbol{I}} + \varepsilon \underline{\boldsymbol{\phi}_\delta} \times \underset{\sim}{\boldsymbol{I}}, \tag{19}$$

where $\varepsilon$ is an infinitesimal parameter again, $\underline{\boldsymbol{\phi}_\delta}$ is an arbitrary axial vector, and $\underset{\sim}{\boldsymbol{I}}$ is the identity tensor. The second term in Eq. (19) corresponds to the skew-symmetric tensor $\delta\underset{\sim}{\boldsymbol{R}} \cdot \underset{\sim}{\boldsymbol{R}}^T$ and means the Eulerian variation of rotation as stated by Atluri and Cazzani (1995).

Substitution of Eq. (19) into Eq. (18) leads to the following expression:

$$\tilde{\underline{\boldsymbol{x}}} = (\underset{\sim}{\boldsymbol{I}} + \varepsilon \underline{\boldsymbol{\phi}_\delta} \times \underset{\sim}{\boldsymbol{I}}) \cdot \underline{\boldsymbol{x}} = \underline{\boldsymbol{x}} + \varepsilon \underline{\boldsymbol{\eta}} \;\; ; \;\; \underline{\boldsymbol{\eta}} = \underline{\boldsymbol{\phi}_\delta} \times \underline{\boldsymbol{x}}. \tag{20}$$

Since the variation with the rotation field has a vector space structure locally, the first Gâteaux derivative of $\Pi$ can be defined in the same way as Eq. (16):

$$d\Pi(\underline{\boldsymbol{x}};\underline{\boldsymbol{\eta}}) = \underline{\nabla}\Pi \cdot \underline{\boldsymbol{\eta}}. \tag{21}$$

The second Gâteaux derivative of $\Pi$, however, cannot be defined as Eq. (17), since the variation vector $\underline{\boldsymbol{\eta}} = \underline{\boldsymbol{\phi}_\delta} \times \underline{\boldsymbol{x}}$ apparently depends on the current configuration $\underline{\boldsymbol{x}}$. For an infinitesimal transformation $\underline{\boldsymbol{x}}^* = \underline{\boldsymbol{x}} + \tau \underline{\boldsymbol{w}}$, in which $\underline{\boldsymbol{w}}$ also involves the rotation field such that $\underline{\boldsymbol{w}} = \underline{\boldsymbol{\phi}_\delta}^* \times \underline{\boldsymbol{x}}$, the second Gâteaux derivative of $\Pi$ is defined as follows:

$$\begin{aligned} d^2\Pi(\underline{\boldsymbol{x}};\underline{\boldsymbol{\eta}},\underline{\boldsymbol{w}}) &= d[\underline{\nabla}\Pi \cdot \underline{\boldsymbol{\eta}}](\underline{\boldsymbol{x}};\underline{\boldsymbol{w}}) \\ &= \frac{\partial}{\partial\tau}\Big|_{\tau=0} \{\underline{\nabla}\Pi^*(\underline{\boldsymbol{x}}^*) \cdot \underline{\boldsymbol{\eta}}^*\} \\ &= \{\tfrac{\partial \underline{\boldsymbol{x}}^*}{\partial\tau} \cdot (\underline{\nabla} \otimes \underline{\nabla}\Pi^*) \cdot \underline{\boldsymbol{\eta}}^* \\ &\qquad + \underline{\nabla}\Pi^*(\underline{\boldsymbol{x}}^*) \cdot \tfrac{\partial}{\partial\tau}(\underline{\boldsymbol{\phi}_\delta} \times \underline{\boldsymbol{x}}^*)\}\Big|_{\tau=0} \\ &= \underline{\boldsymbol{w}} \cdot (\underline{\nabla} \otimes \underline{\nabla}\Pi) \cdot \underline{\boldsymbol{\eta}} + \underline{\nabla}\Pi \cdot (\underline{\boldsymbol{\phi}_\delta} \times \underline{\boldsymbol{w}}). \end{aligned} \tag{22}$$

Equation (22) is not commutable with respect to $\underline{\boldsymbol{\eta}}$ and $\underline{\boldsymbol{w}}$. As a result, the second Gâteaux derivative of $\Pi$ is not a symmetric bilinear form in the Eulerian description unless the current configuration point is critical ($\underline{\nabla}\Pi = \underset{\sim}{\boldsymbol{O}}$).

On the other hand, the situation is changed in the Lagrangean description. As stated by Atluri and Cazzani (1995), the variation of rotation $\delta\underset{\sim}{\boldsymbol{R}} \cdot \underset{\sim}{\boldsymbol{R}}^T$ is represented by a rotation vector $\underline{\boldsymbol{\omega}}$ in the Lagrangean description. Since the rotation tensor $\underset{\sim}{\boldsymbol{R}}$ is expressed by the rotation vector $\underline{\omega}$ as

$$\underset{\sim}{\boldsymbol{R}} = \cos\theta \underset{\sim}{\boldsymbol{I}} + \frac{\sin\theta}{\theta} \underline{\boldsymbol{\omega}} \times \underset{\sim}{\boldsymbol{I}} + \frac{1-\cos\theta}{\theta^2} \underline{\boldsymbol{\omega}} \otimes \underline{\boldsymbol{\omega}}, \tag{23}$$

the skew-symmetric tensor $\delta\underset{\sim}{\boldsymbol{R}} \cdot \underset{\sim}{\boldsymbol{R}}^T$ is given by

$$\delta\underset{\sim}{\boldsymbol{R}} \cdot \underset{\sim}{\boldsymbol{R}}^T = (\underset{\sim}{\boldsymbol{T}} \cdot \delta\underline{\boldsymbol{\omega}}) \times \underset{\sim}{\boldsymbol{I}}, \tag{24}$$

where

$$\begin{aligned} \underset{\sim}{\boldsymbol{T}} = \tfrac{\sin\theta}{\theta} \underset{\sim}{\boldsymbol{I}} + \tfrac{1-\cos\theta}{\theta^2} \underline{\boldsymbol{\omega}} \times \underset{\sim}{\boldsymbol{I}} \\ + \tfrac{1}{\theta^2}(1 - \tfrac{\sin\theta}{\theta}) \underline{\boldsymbol{\omega}} \otimes \underline{\boldsymbol{\omega}} \end{aligned} , \tag{25}$$

[See Atluri and Cazzani (1995)]. In Eqs. (23) to (25), the rotation vector $\underline{\boldsymbol{\omega}}$ is written as $\underline{\boldsymbol{\omega}} = \theta \underline{\boldsymbol{e}}$ where $\theta$ is a rotation angle and $\underline{\boldsymbol{e}}$ is a unit rotation axis.

If we replace the variation of the rotation vector $\delta\underline{\boldsymbol{\omega}}$ with $\varepsilon\hat{\underline{\boldsymbol{\omega}}}$, instead of Eq. (19), the infinitesimal rotation $\underset{\sim}{\boldsymbol{Q}}$ can be written by

$$\underset{\sim}{\boldsymbol{Q}} = \underset{\sim}{\boldsymbol{I}} + \delta\underset{\sim}{\boldsymbol{R}} \cdot \underset{\sim}{\boldsymbol{R}}^T = \underset{\sim}{\boldsymbol{I}} + \varepsilon(\underset{\sim}{\boldsymbol{T}} \cdot \hat{\underline{\boldsymbol{\omega}}}) \times \underset{\sim}{\boldsymbol{I}}, \tag{26}$$

where $\hat{\underline{\boldsymbol{\omega}}}$ is an arbitrary vector. Substitution of Eq. (26) into Eq. (18) leads to the following infinitesimal transformation:

$$\tilde{\underline{x}} = \{\underset{\sim}{I} + \varepsilon(\underset{\sim}{T}\cdot\hat{\underline{\omega}})\times\underset{\sim}{I}\}\cdot\underline{x} = \underline{x} + \varepsilon\breve{\underline{\eta}};\ \breve{\underline{\eta}} = (\underset{\sim}{T}\cdot\hat{\underline{\omega}})\times\underline{x} \quad . \tag{27}$$

At the same time, we obtain an infinitesimal transformation for the rotation vector as $\tilde{\underline{\omega}} = \underline{\omega} + \varepsilon\hat{\underline{\omega}}$.

The first Gâteaux derivative of the potential $\Pi$ has the same form as Eq. (21):

$$d\Pi(\underline{x};\breve{\underline{\eta}}) = \underline{\nabla}\Pi\cdot\breve{\underline{\eta}}. \tag{28}$$

The second Gâteaux derivative of $\Pi$ also can be defined as Eq. (22) for an infinitesimal transformation $\underline{x}^* = \underline{x} + \tau\breve{\underline{w}}$:

$$\begin{aligned} d^2\Pi(\underline{x};\breve{\underline{\eta}},\breve{\underline{w}}) &= d[\underline{\nabla}\Pi\cdot\breve{\underline{\eta}}](\underline{x};\breve{\underline{w}}) \\ &= \frac{\partial}{\partial\tau}\bigg|_{\tau=0}\{\underline{\nabla}\Pi^*(\underline{x}^*)\cdot\breve{\underline{\eta}}^*\} \\ &= [\frac{\partial\underline{x}^*}{\partial\tau}\cdot(\underline{\nabla}\otimes\underline{\nabla}\Pi^*)\cdot\{(\underset{\sim}{T}^*\cdot\hat{\underline{\omega}})\times\underline{x}^*\} \\ &\quad +\underline{\nabla}\Pi^*(\underline{x}^*)\cdot\frac{\partial}{\partial\tau}\{(\underset{\sim}{T}^*\cdot\hat{\underline{\omega}})\times\underline{x}^*\}]\bigg|_{\tau=0} \\ &= \breve{\underline{w}}\cdot(\underline{\nabla}\otimes\underline{\nabla}\Pi)\cdot\breve{\underline{\eta}} + \underline{\nabla}\Pi\cdot\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\{(\underset{\sim}{T}^*\cdot\hat{\underline{\omega}})\times\underline{x}^*\}. \end{aligned} \tag{29}$$

In Eq. (29), both $\underset{\sim}{T}^*$ and $\underline{x}^*$ correspond to an infinitesimal transformation for the rotation as $\underline{\omega}^* = \underline{\omega} + \tau\dot{\underline{\omega}}$. Especially, likewise Eq. (27), the vector $\breve{\underline{w}}$ is given by $\breve{\underline{w}} = (\underset{\sim}{T}\cdot\dot{\underline{\omega}})\times\underline{x}$.

The first term of the right hand side of Eq. (29) is commutable with respect to $\breve{\underline{\eta}}$ and $\breve{\underline{w}}$ and therefore symmetric bilinear form. The second term, however, does not seem to be symmetric. On the other hand, since the rotation vector $\underline{\omega}$ makes a vector space, it is expected that the second Gâteaux derivative also becomes symmetric as suggested by Sansour and Bufler (1992). In what follows, we examine whether the second term of Eq. (29) is symmetric or not.

The second term of the right hand side of Eq. (29) is rewritten as

$$\begin{aligned} &\underline{\nabla}\Pi\cdot\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\{(\underset{\sim}{T}^*\cdot\hat{\underline{\omega}})\times\underline{x}^*\} \\ &= \underline{\nabla}\Pi\cdot\{(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\underset{\sim}{T}^*\cdot\hat{\underline{\omega}})\times\underline{x} \\ &\qquad +(\underset{\sim}{T}\cdot\hat{\underline{\omega}})\times(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\underline{x}^*)\} \\ &= \hat{\underline{\omega}}\cdot[(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\underset{\sim}{T}^*)^T\cdot(\underline{x}\times\underline{\nabla}\Pi) \\ &\qquad +\underset{\sim}{T}^T\cdot\{(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\underline{x}^*)\times\underline{\nabla}\Pi\}]. \end{aligned} \tag{30}$$

where we consider tensor formulae $\underline{c}\cdot\underline{a}\times\underline{b} = \underline{a}\cdot\underline{b}\times\underline{c}$ and $\underset{\sim}{A}\cdot\underline{a} = \underline{a}\cdot\underset{\sim}{A}^T$.

Since $\underset{\sim}{T}^*$ is evaluated for $\underline{\omega}^* = \underline{\omega} + \tau\dot{\underline{\omega}}$, instead of $\underline{\omega}$, in Eq. (25) and $\underline{x}^* = \underline{x} + \tau\breve{\underline{w}}$, we can straightforwardly derive the following expressions:

$$\begin{aligned} \frac{\partial}{\partial\tau}\bigg|_{\tau=0}\underset{\sim}{T}^* &= \frac{1}{\theta^2}(\underline{\omega}\cdot\dot{\underline{\omega}})(\underset{\sim}{R} - \underset{\sim}{T}) \\ &\quad + \frac{1-\cos\theta}{\theta^2}\{\dot{\underline{\omega}} - \frac{1}{\theta^2}(\underline{\omega}\cdot\dot{\underline{\omega}})\underline{\omega}\}\times\underset{\sim}{I} \\ &+\frac{1}{\theta^2}(1 - \frac{\sin\theta}{\theta})\{\dot{\underline{\omega}}\otimes\underline{\omega} + \underline{\omega}\otimes\dot{\underline{\omega}} - \frac{2}{\theta^2}(\underline{\omega}\cdot\dot{\underline{\omega}})\underline{\omega}\otimes\underline{\omega}\} \end{aligned} \tag{31}$$

and

$$\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\underline{x}^* = \breve{\underline{w}} = (\underset{\sim}{T}\cdot\dot{\underline{\omega}})\times\underline{x}. \tag{32}$$

Substituting Eqs. (31) and (32) into the square bracket in Eq. (30), after careful and straightforward calculation, we can finally obtain the following formula:

$$\begin{aligned} &(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\underset{\sim}{T}^*)^T.\{(\underset{\sim}{R}\cdot\underline{X})\times\underline{\nabla}\Pi\} \\ &+\underset{\sim}{T}^T\cdot\{(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\underset{\sim}{R}^*\cdot\underline{X})\times\underline{\nabla}\Pi\} = \underset{\sim}{\Gamma}(\underline{X},\underline{\nabla}\Pi).\dot{\underline{\omega}}, \end{aligned} \tag{33}$$

where we consider $\underline{x} = \underset{\sim}{R}\cdot\underline{X}$ and the tensor $\underset{\sim}{\Gamma}(\underline{a},\underline{b})$ is given by

$$\begin{aligned} \underset{\sim}{\Gamma}(\underline{a},\underline{b}) &= \frac{1}{\theta}[C_1\underset{\sim}{I} + \frac{1-\cos\theta}{\theta}(\underline{a}\otimes\underline{b} + \underline{b}\otimes\underline{a}) \\ &+\frac{1}{\theta}(\cos\theta - \frac{\sin\theta}{\theta})\{(\underline{a}\times\underline{b})\otimes\underline{\omega} + \underline{\omega}\otimes(\underline{a}\times\underline{b})\} \\ &\quad +\frac{1}{\theta^2}\{\sin\theta - \frac{2(1-\cos\theta)}{\theta}\}\{(\underline{\omega}\cdot\underline{b})(\underline{a}\otimes\underline{\omega} \\ &\qquad +\underline{\omega}\otimes\underline{a}) + (\underline{\omega}\cdot\underline{a})(\underline{b}\otimes\underline{\omega} + \underline{\omega}\otimes\underline{b})\} \\ &\qquad\quad +\frac{1}{\theta^2}C_2\underline{\omega}\otimes\underline{\omega}], \end{aligned} \tag{34}$$

where

$$C_1 = -\sin\theta(\underline{\boldsymbol{a}}\cdot\underline{\boldsymbol{b}}) + \frac{1}{\theta}(\cos\theta - \frac{\sin\theta}{\theta})(\underline{\boldsymbol{\omega}}\cdot\underline{\boldsymbol{a}}\times\underline{\boldsymbol{b}}) + \frac{1}{\theta^2}\{\sin\theta - \frac{2(1-\cos\theta)}{\theta}\}(\underline{\boldsymbol{\omega}}\cdot\underline{\boldsymbol{b}})(\underline{\boldsymbol{\omega}}\cdot\underline{\boldsymbol{b}}) \tag{35}$$

and

$$C_2 = (\sin\theta - \theta\cos\theta)(\underline{\boldsymbol{a}}\cdot\underline{\boldsymbol{b}}) - \{\sin\theta + \frac{3}{\theta}(\cos\theta - \frac{\sin\theta}{\theta})\}(\underline{\boldsymbol{\omega}}\cdot\underline{\boldsymbol{a}}\times\underline{\boldsymbol{b}}) + \frac{1}{\theta}\{\sin\theta - \frac{5\sin\theta}{\theta} + \frac{8(1-\cos\theta)}{\theta^2}\}(\underline{\boldsymbol{\omega}}\cdot\underline{\boldsymbol{b}})(\underline{\boldsymbol{\omega}}\cdot\underline{\boldsymbol{b}}). \tag{36}$$

Note that $\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\boldsymbol{a}},\underline{\boldsymbol{b}})$ is a symmetric tensor. Therefore, the second term of Eq. (29) is commutable with $\hat{\underline{\boldsymbol{\omega}}}$ and $\dot{\underline{\boldsymbol{\omega}}}$. The formula (33) and the tensor $\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\boldsymbol{a}},\underline{\boldsymbol{b}})$ play a key role for the derivation of the symmetric tangent operator in shell problems.

Consequently, the second Gâteaux derivative, Eq. (29), can be rewritten as

$$d^2\Pi(\underline{\boldsymbol{x}};\breve{\underline{\boldsymbol{\eta}}},\breve{\underline{\boldsymbol{w}}}) = \{(\underset{\sim}{\boldsymbol{T}}\cdot\hat{\underline{\boldsymbol{\omega}}})\times\underline{\boldsymbol{x}}\}\cdot(\underline{\nabla}\otimes\underline{\nabla}\Pi)\cdot\{(\underset{\sim}{\boldsymbol{T}}\cdot\dot{\underline{\boldsymbol{\omega}}})\times\underline{\boldsymbol{x}}\} + \hat{\underline{\boldsymbol{\omega}}}.\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\boldsymbol{X}},\underline{\nabla}\Pi).\dot{\underline{\boldsymbol{\omega}}}. \tag{37}$$

which is also commutable with $\hat{\underline{\boldsymbol{\omega}}}$ and $\dot{\underline{\boldsymbol{\omega}}}$. The symmetry of Eq. (37) also does not depend on whether or not the current configuration is critical.

## 4 Variational Formulation for Shells

In the previous section, it is confirmed that the second Gâteaux derivative of the total potential energy becomes symmetric in the Lagrangean description even when involving the rotation field. This section deals with the application of the above discussion to shell problems. As an example of the nonlinear solid mechanics, we consider a finitely deformed thick shell model that incorporates finite rotations.

### 4.1 Mixed Variational Principle

A mixed type variational principle for shells employed here has been already established by Suetake, Iura, and Atluri (1999). The variational principle involves the rotation tensor $\underset{\sim}{\boldsymbol{R}}$ as an independent variable. The functional for the mixed variational principle is written by using the stress and the strain measures defined in Section 2 as follows:

$$F(\underline{\boldsymbol{u}}_0,\boldsymbol{\omega},\underline{\boldsymbol{c}}_3,\hat{\underline{\boldsymbol{N}}}^\alpha,\hat{\underline{\boldsymbol{H}}}^\alpha,\hat{\underline{\boldsymbol{N}}}^3) = \int_{S_0}[-W_C(\hat{\underline{\boldsymbol{N}}}^\alpha,\hat{\underline{\boldsymbol{H}}}^\alpha,\hat{\underline{\boldsymbol{N}}}^3) + \underset{\sim}{\boldsymbol{R}}\cdot\hat{\underline{\boldsymbol{N}}}^\alpha\cdot(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha}) + \hat{\underline{\boldsymbol{H}}}^\alpha\cdot\underline{\boldsymbol{c}}_{3,\alpha} + \hat{\underline{\boldsymbol{N}}}^3\cdot\underline{\boldsymbol{c}}_3 - \overline{\underline{\boldsymbol{p}}}\cdot\underline{\boldsymbol{u}}_0 - \overline{\underline{\boldsymbol{m}}}\cdot(\underset{\sim}{\boldsymbol{R}}\cdot\underline{\boldsymbol{c}}_3-\underline{\boldsymbol{A}}_3)]\sqrt{A}d\xi^1 d\xi^2, \tag{38}$$

where $W_C$ is a complementary energy density, $\overline{\underline{\boldsymbol{p}}}$ is an external force, is $\overline{\underline{\boldsymbol{m}}}$ an external moment, and the rotation tensor $\underset{\sim}{\boldsymbol{R}}$ is represented by the rotation vector $\underline{\boldsymbol{\omega}}$ by Eq. (23). In this section we also assume a conservative system.

For simplicity, we omit several terms that concern boundary conditions and therefore only the principal part of the functional is presented in Eq. (38). In addition, for the application to numerical formulation, the rotation field $\underset{\sim}{\boldsymbol{R}}$ or $\underline{\boldsymbol{\omega}}$ is assumed to be constant within a sub-region, i.e., within a finite element.

### 4.2 Gâteaux Derivatives

In this subsection, we calculate the first and the second Gâteaux derivatives of the functional $F$. Here, we express the first Gâteaux derivative simply as an operator $\delta$ and the second one as $\Delta$.

From Eq. (38), we obtain the following expression as the first Gâteaux derivative of $F$:

$$\delta F = \int_{S_0}[-\delta\hat{\underline{\boldsymbol{N}}}^\alpha\cdot\frac{\partial W_C}{\partial\hat{\underline{\boldsymbol{N}}}^\alpha} - \delta\hat{\underline{\boldsymbol{H}}}^\alpha\cdot\frac{\partial W_C}{\partial\hat{\underline{\boldsymbol{H}}}^\alpha} - \delta\hat{\underline{\boldsymbol{N}}}^3\cdot\frac{\partial W_C}{\partial\hat{\underline{\boldsymbol{N}}}^3} + \delta\underset{\sim}{\boldsymbol{R}}\cdot\hat{\underline{\boldsymbol{N}}}^\alpha\cdot(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha}) + \delta\hat{\underline{\boldsymbol{N}}}^\alpha\cdot\underset{\sim}{\boldsymbol{R}}^T\cdot(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha}) + \delta\underline{\boldsymbol{u}}_{0,\alpha}\cdot\underset{\sim}{\boldsymbol{R}}\cdot\hat{\underline{\boldsymbol{N}}}^\alpha + \delta\hat{\underline{\boldsymbol{H}}}^\alpha\cdot\underline{\boldsymbol{c}}_{3,\alpha} + \delta\underline{\boldsymbol{c}}_{3,\alpha}\cdot\hat{\underline{\boldsymbol{H}}}^\alpha + \delta\hat{\underline{\boldsymbol{N}}}^3\cdot\underline{\boldsymbol{c}}_3 + \delta\underline{\boldsymbol{c}}_3\cdot\hat{\underline{\boldsymbol{N}}}^3 - \delta\underline{\boldsymbol{u}}_0\cdot\overline{\underline{\boldsymbol{p}}} - (\delta\underset{\sim}{\boldsymbol{R}}\cdot\underline{\boldsymbol{c}}_3 + \delta\underline{\boldsymbol{c}}_3\cdot\underset{\sim}{\boldsymbol{R}}^T)\cdot\overline{\underline{\boldsymbol{m}}}]\sqrt{A}d\xi^1 d\xi^2. \tag{39}$$

For linear elastic material, the complementary energy density $W_C$ is given by a bilinear form of the stress measures, for example,

$$W_C(\hat{\underline{\boldsymbol{N}}}^\alpha,\hat{\underline{\boldsymbol{H}}}^\alpha,\hat{\underline{\boldsymbol{N}}}^3) = \frac{1}{2}(\hat{\underline{\boldsymbol{N}}}^\alpha\cdot\underset{\sim\alpha\beta}{\boldsymbol{C}^{(s)}}\cdot\hat{\underline{\boldsymbol{N}}}^\beta + \hat{\underline{\boldsymbol{H}}}^\alpha\cdot\underset{\sim\alpha\beta}{\boldsymbol{C}^{(b)}}\cdot\hat{\underline{\boldsymbol{H}}}^\beta + \hat{\underline{\boldsymbol{N}}}^3\cdot\underset{\sim}{\boldsymbol{C}^{(d)}}\cdot\hat{\underline{\boldsymbol{N}}}^3), \tag{40}$$

where $\underset{\sim\alpha\beta}{\boldsymbol{C}^{(s)}}$, $\underset{\sim\alpha\beta}{\boldsymbol{C}^{(b)}}$, and $\underset{\sim}{\boldsymbol{C}^{(d)}}$ are constant symmetric compliance tensors.

In what follows, we discuss only the Lagrangean description. Therefore, it is assumed that the variation of rotation, $\delta\underset{\sim}{\boldsymbol{R}}\cdot\underset{\sim}{\boldsymbol{R}}^T$, can be expressed as Eq. (24). Considering Eq. (40) and the vector formula $\underline{\boldsymbol{a}}\times\underline{\boldsymbol{b}}\cdot\underline{\boldsymbol{c}}=\underline{\boldsymbol{a}}\cdot\underline{\boldsymbol{b}}\times\underline{\boldsymbol{c}}$ again, we obtain the following expression from Eq. (39):

$$\begin{aligned}
\delta F = \int_{S_0} [&\delta\underline{\boldsymbol{u}}_{0,\alpha}\cdot\underset{\sim}{\boldsymbol{R}}\cdot\underline{\hat{\boldsymbol{N}}}^{\alpha}\\
&\underline{\underline{+\delta\underline{\boldsymbol{\omega}}\cdot\underset{\sim}{\boldsymbol{T}}^T\cdot\{(\underset{\sim}{\boldsymbol{R}}\cdot\underline{\hat{\boldsymbol{N}}}^{\alpha})\times(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha})-(\underset{\sim}{\boldsymbol{R}}\cdot\underline{\boldsymbol{c}}_3)\times\underline{\overline{\boldsymbol{m}}}\}}}\\
&+\delta\underline{\boldsymbol{c}}_{3,\alpha}\cdot\underline{\hat{\boldsymbol{H}}}^{\alpha}+\delta\underline{\boldsymbol{c}}_3\cdot(\underline{\hat{\boldsymbol{N}}}^{3}-\underset{\sim}{\boldsymbol{R}}^T\cdot\underline{\overline{\boldsymbol{m}}})\\
&+\delta\underline{\hat{\boldsymbol{N}}}^{\alpha}\cdot\{\underset{\sim}{\boldsymbol{R}}^T\cdot(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha})-\underset{\sim\alpha\beta}{\boldsymbol{C}^{(s)}}\cdot\underline{\hat{\boldsymbol{N}}}^{\beta}\}\\
&+\delta\underline{\hat{\boldsymbol{H}}}^{\alpha}\cdot(\underline{\boldsymbol{c}}_{3,\alpha}-\underset{\sim\alpha\beta}{\boldsymbol{C}^{(b)}}\cdot\underline{\hat{\boldsymbol{H}}}^{\beta})\\
&+\delta\underline{\hat{\boldsymbol{N}}}^{3}\cdot(\underline{\boldsymbol{c}}_3-\underset{\sim}{\boldsymbol{C}}^{(d)}\cdot\underline{\hat{\boldsymbol{N}}}^{3})-\delta\underline{\boldsymbol{u}}_0\cdot\underline{\overline{\boldsymbol{p}}}]\sqrt{A}d\xi^1d\xi^2.
\end{aligned}\tag{41}$$

Next we calculate the second Gâteaux derivative of $F$. In the calculation of the second Gâteaux derivative, we must pay attention to the underlined terms in Eq. (41), because it includes $\Delta\underset{\sim}{\boldsymbol{T}}$. The Gâteaux derivative of the underlined terms in Eq. (41) involves the following part:

$$\begin{aligned}
&\Delta\underset{\sim}{\boldsymbol{T}}^T\cdot\{(\underset{\sim}{\boldsymbol{R}}\cdot\underline{\hat{\boldsymbol{N}}}^{\alpha})\times(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha})\}\\
&\qquad+\underset{\sim}{\boldsymbol{T}}^T\cdot\{(\Delta\underset{\sim}{\boldsymbol{R}}\cdot\underline{\hat{\boldsymbol{N}}}^{\alpha})\times(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha})\}\\
&-\Delta\underset{\sim}{\boldsymbol{T}}^T\cdot\{(\underset{\sim}{\boldsymbol{R}}\cdot\underline{\boldsymbol{c}}_3)\times\underline{\overline{\boldsymbol{m}}}\}-\underset{\sim}{\boldsymbol{T}}^T\cdot\{(\Delta\underset{\sim}{\boldsymbol{R}}\cdot\underline{\boldsymbol{c}}_3)\times\underline{\overline{\boldsymbol{m}}}\}
\end{aligned}\tag{42}$$

In view of the formula (33), it is easily understood that the expression (42) is rewritten by using the tensor $\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\boldsymbol{a}},\underline{\boldsymbol{b}})$ as:

$$\{\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\hat{\boldsymbol{N}}}^{\alpha},\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha})-\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\boldsymbol{c}}_3,\underline{\overline{\boldsymbol{m}}})\}\cdot\Delta\underline{\boldsymbol{\omega}}\tag{43}$$

Finally, we obtain the second Gâteaux derivative of $F$ as follows:

$$\begin{aligned}
&\Delta\delta F\\
&=\int_{S_0}[\delta\underline{\boldsymbol{u}}_{0,\alpha}\cdot\{-(\underset{\sim}{\boldsymbol{R}}\cdot\underline{\hat{\boldsymbol{N}}}^{\alpha})\times\underset{\sim}{\boldsymbol{T}}\cdot\Delta\underline{\boldsymbol{\omega}}+\underset{\sim}{\boldsymbol{R}}\cdot\Delta\underline{\hat{\boldsymbol{N}}}^{\alpha}\}\\
&+\delta\underline{\boldsymbol{\omega}}\cdot\underset{\sim}{\boldsymbol{T}}^T\cdot(\underset{\sim}{\boldsymbol{R}}\cdot\underline{\hat{\boldsymbol{N}}}^{\alpha})\times\Delta\underline{\boldsymbol{u}}_{0,\alpha}+\delta\underline{\hat{\boldsymbol{N}}}^{\alpha}\cdot\underset{\sim}{\boldsymbol{R}}^T\cdot\Delta\underline{\boldsymbol{u}}_{0,\alpha}\\
&\quad+\delta\underline{\boldsymbol{\omega}}\cdot\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\hat{\boldsymbol{N}}}^{\alpha},\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha})-\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\boldsymbol{c}}_3,\underline{\overline{\boldsymbol{m}}})\}\cdot\Delta\underline{\boldsymbol{\omega}}\\
&\quad+\delta\underline{\boldsymbol{\omega}}\cdot\underset{\sim}{\boldsymbol{T}}^T\cdot\{\underline{\overline{\boldsymbol{m}}}\times\underset{\sim}{\boldsymbol{R}}\cdot\Delta\underline{\boldsymbol{c}}_3-(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha})\times\underset{\sim}{\boldsymbol{R}}\cdot\Delta\underline{\hat{\boldsymbol{N}}}^{\alpha}\}\\
&\quad-\delta\underline{\boldsymbol{c}}_3\cdot\underset{\sim}{\boldsymbol{R}}^T\cdot\underline{\overline{\boldsymbol{m}}}\times\underset{\sim}{\boldsymbol{T}}\cdot\Delta\underline{\boldsymbol{\omega}}\\
&\qquad\qquad+\delta\underline{\hat{\boldsymbol{N}}}^{\alpha}\cdot\underset{\sim}{\boldsymbol{R}}^T\cdot(\underline{\boldsymbol{A}}_\alpha+\underline{\boldsymbol{u}}_{0,\alpha})\times\underset{\sim}{\boldsymbol{T}}\cdot\Delta\underline{\boldsymbol{\omega}}\\
&\quad+\delta\underline{\boldsymbol{c}}_{3,\alpha}\cdot\Delta\underline{\hat{\boldsymbol{H}}}^{\alpha}+\delta\underline{\hat{\boldsymbol{H}}}^{\alpha}\cdot\Delta\underline{\boldsymbol{c}}_{3,\alpha}\\
&\qquad+\delta\underline{\boldsymbol{c}}_3\cdot\Delta\underline{\hat{\boldsymbol{N}}}^{3}+\delta\underline{\hat{\boldsymbol{N}}}^{3}\cdot\Delta\underline{\boldsymbol{c}}_3-\delta\underline{\hat{\boldsymbol{N}}}^{\alpha}\cdot\underset{\sim\alpha\beta}{\boldsymbol{C}^{(s)}}\cdot\underline{\hat{\boldsymbol{N}}}^{\beta}\\
&-\delta\underline{\hat{\boldsymbol{H}}}^{\alpha}\cdot\underset{\sim\alpha\beta}{\boldsymbol{C}^{(b)}}\cdot\underline{\hat{\boldsymbol{H}}}^{\beta}-\delta\underline{\hat{\boldsymbol{N}}}^{3}\cdot\underset{\sim}{\boldsymbol{C}}^{(d)}\cdot\underline{\hat{\boldsymbol{N}}}^{3})]\sqrt{A}d\xi^1d\xi^2.
\end{aligned}\tag{44}$$

where we consider $\Delta\underset{\sim}{\boldsymbol{T}}^T\cdot\underset{\sim}{\boldsymbol{R}}=-(\underset{\sim}{\boldsymbol{T}}^T\cdot\Delta\underline{\boldsymbol{\omega}})\times\underset{\sim}{\boldsymbol{I}}$ when $\Delta\underset{\sim}{\boldsymbol{R}}\cdot\underset{\sim}{\boldsymbol{R}}^T=(\underset{\sim}{\boldsymbol{T}}\cdot\Delta\underline{\boldsymbol{\omega}})\times\underset{\sim}{\boldsymbol{I}}$ and a vector product between a vector and a second-order tensor as $\underline{\boldsymbol{a}}\times\underset{\sim}{\boldsymbol{A}}$ simply means $\underline{\boldsymbol{a}}\times\underset{\sim}{\boldsymbol{I}})\cdot\underset{\sim}{\boldsymbol{A}}$. In Eq. (44), we present only the part concerning the tangent operator, that is, we omit the part concerning the external force.

It should be noted that since the tensor $\underset{\sim}{\boldsymbol{\Gamma}}(\underline{\boldsymbol{a}},\underline{\boldsymbol{b}})$ is symmetric, Eq. (44) is apparently commutable with respect to $(\delta\underline{\boldsymbol{u}}_0,\delta\underline{\boldsymbol{\omega}},\delta\underline{\boldsymbol{c}}_3,\delta\underline{\hat{\boldsymbol{N}}}^{\alpha},\delta\underline{\hat{\boldsymbol{H}}}^{\alpha},\delta\underline{\hat{\boldsymbol{N}}}^{3})$ and $(\Delta\underline{\boldsymbol{u}}_0,\Delta\underline{\boldsymbol{\omega}},\Delta\underline{\boldsymbol{c}}_3,\Delta\underline{\hat{\boldsymbol{N}}}^{\alpha},\Delta\underline{\hat{\boldsymbol{H}}}^{\alpha},\Delta\underline{\hat{\boldsymbol{N}}}^{3})$. Thus, we can conclude that the tangent operator for shells with the finite rotation field becomes symmetric in the Lagrangean description even at non-equilibrium point.

## 5 Concluding Remarks

We discuss the symmetry of the tangent operator for nonlinear mechanical systems that are conservative. In particular, we deal with the problem in both the Lagrangean and the Eulerian descriptions.

If independent variables are only vector fields, the tangent operator, that is, the second Gâteaux derivative of the functional becomes symmetric in both descriptions. In addition, the symmetry of the tangent operator does not depend on whether or not the current configuration point is at an equilibrium state.

On the other hand, when the functional for the mechanical system incorporates the finite rotation field, we cannot simply reveal the symmetry problem because of the special orthogonality of the rotation group. In the Eulerian description the tangent operator does not become symmetric unless the current state satisfies the equilibrium condition. In the Lagrangean description, however, the tangent operator always becomes symmetric even at a non-critical point.

As an example, we adopt the finitely deformed thick shell problem that involves the rotation field. We evaluate the second Gâteaux derivative of the well-established mixed type functional and confirm that the obtained tangent operator has a symmetric bilinear form.

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