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The Identification of Elastic Moduli of a Stratified Layer Through Localized Surface Probes, with Biomedical Applications

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Abstract: We discuss the inverse problem of the recovery of the distribution of the elastic moduli of a stratified layer, based on measurements of the surface displacement under localized surface loads. A general parametric solution and a numerical procedure for computing the parameters are presented. Examples of numerical results are given. The problem and its solution are related to the monitoring of elastic properties of living tissues.

1 Introduction

Computing the mechanical properties of a medium based on its response to external mechanical action takes new impetus in conjunction with the problem of imaging of the elastic properties of living tissues [Gao et al (1996)]. As is well known, the mechanical properties of a tissue are an effective indicator of its functional state, its age-related changes, muscle training, state of skin, amount of subcutaneous fat, etc. Significant changes in the elasticity of soft tissue are usually related to pathological processes. Therefore, the elastic moduli of the tissue can be used, in addition to other widely accepted clinical factors, as an aid to physicians in the diagnosis process.

Now-a-days, in spite of many new imaging modalities, palpation is still widely used as a self-screening procedure for the detection of hard masses in a human body. Its efficiency, however, is limited by its high subjectivity. The goal of elasticity imaging in general is to develop a kind of surrogate, objective remote palpation [Gao et al (1996)]. Ultrasound, NMR or X-rays are often used to this purpose. However, these methods are comparatively expensive, and not always safe for the patient (namely, X-rays) [see, for instance, Liu and Ferrari (2003)]. Therefore, the development of *in vivo* techniques that are simple to use, inexpensive and are not armful to the patient is an important issue. In particular, the goal of a number of efforts aim at developing a pre-screening self-palpation device, able to estimate the state of surface living tissue on the base of its mechanical properties [see, for example, Sarvazyan and Ponomarev (1987); Timanin et al (1997)]. The approaches and devices proposed are based on a non-invasive estimate of the elasticity and the viscosity of the tissue by touching it with a steady action or using a vibrating piston. Such methods and devices are safe and do not use any form of hazardous radiation, they are easy-to-use, simple and inexpensive. However, the hard mathematical and numerical methods must be used on this way to convert the results of experimental measurements into mechanical properties of the tissue. An accurate quantitative indentification of these properties *in vivo* using experimental data measured by any proposed method could permit to detect a tissue abnormalities and to control state of tissue during training, in sport medicine, in traumatology, when using cosmetic creams, processing of restoration, etc. Unfortunately, greatly simplified approaches only were proposed in static [Sarvazyan and Ponomarev (1987)] and dynamic [Timanin (1989); Timanin et al (1997)] cases to estimate unknown mechanical moduli of the object under investigation.

Some aspects of the problem have been considered by the authors in previous publications [Goldstein and Skovoroda (1989); Skovoroda (1989); Skovoroda (1996); Skovoroda and Aglimov (1997); Skovoroda and Aglimov (1998)]. This paper summarizes their mathematical approach and the numerical method they propose for reconstructing the elastic moduli applicable to skin and subcutaneous tissues in static case. A layered medium and continuously inhomogeneous medium under a local surface loading are considered. A general theory and some computational algorithms are presented. Numerical experiments are performed to test a possibility of recovering the elastic properties through the inhomogeneous layer on the base of displacement measurements.

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It may also be possible to solve inverse problems for layered media such as the present one, using displacement and fraction boundary-integral-equation methods [Kubo (1993); Han and Atluri (2003)].

2 Piecewise-uniform Stratified Elastic Layer

Let us first consider a stratified compressible elastic layer under a symmetric surface loading. Put the origin of a cylindrical coordinate system (r,θ,z) at the center of the loaded region of the surface $z=0$, with the z axis directed inward the layered medium.

The stress tensor σ_{ij} at the surface $z=0$ has to satisfy the boundary conditions, which we assume to be

$$\sigma_{rz}(r,0)=0, \quad \sigma_{zz}(r,0)=-P(r). \tag{1}$$

Due to the symmetry of the loading, we get the displacement vector in the form $\mathbf{U}=(U,0,W)$, with $U=U(r,z)$ and $W=W(r,z)$. This leads to the following form of the strain tensor:

$$\varepsilon_{rr}=\frac{\partial U}{\partial r},\ \varepsilon_{\theta\theta}=\frac{U}{r},\ \varepsilon_{zz}=\frac{\partial W}{\partial z},\varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial r}\right),$$
$$\sigma_{r\theta}=\sigma_{z\theta}=\varepsilon_{r\theta}=\varepsilon_{z\theta}=0.$$

If the thickness H of the layer system is finite, we assume that the displacement vanishes at the remote boundary $z=H$, i.e.

$$U(r,H)=0,\ W(r,H)=0. \tag{2}$$

If the layer is a half-space $(H=\infty)$, we assume that the displacement and the stress vanish for $z\to\infty$.

We also assume that displacement and stress are continuous across boundaries between layers $z=h_i$:

$$U^i=U^{i+1},\ W^i=W^{i+1},$$
$$\sigma^i_{rz}=\sigma^{i+1}_{rz},\ \sigma^i_{zz}=\sigma^{i+1}_{zz},\ \ i=1,\ldots,N-1, \tag{3}$$

where N is the number of layers.

The bulk balance equation in this case takes the form [Ilyushin (1978); Novatski (1975); Rabotnov (1979)]

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{(\sigma_{rr}-\sigma_{\theta\theta})}{r}=0,$$
$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=0. \tag{4}$$

To close the problem, we assume the medium to be linearly elastic and isotropic

$$\sigma_{ij}=2\mu\varepsilon_{ij}+\lambda e\delta_{ij}, \tag{5}$$

where λ and μ are the Lame coefficients, δ_{ij} is the Kronecker delta symbol and $e=\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz}$.

When the Lame coefficients are constant within each layer, the general solution of the problem within the *i*-th layer is given by [Nikishin and Shapiro (1970)] and presented in Appendix 1.

Equations (A1.1) provide the general solution of the direct elastic problem for a symmetrically loaded layered medium with known mechanical properties. The solution for each particular case can be obtained combining (A1.1) for *i=1,...,N*, with the continuity conditions (3) and the appropriate boundary conditions.

In this paper we focus on the inverse problem: the mechanical properties of the layers are unknown and need to be found on the base of a small quantity of experimental data on displacements.

In general such a computation implies an iterative procedure during which the model parameters are adjusted in search of the best fit between experimental measurements and theoretical predictions. It is important to note that this procedure converges to the unique solution only if the measured date depend monotonically on the model parameters. It should be mentioned that in practice the convergence speed is still crucial.

To illustrate the numerical aspects of the inverse problem, i.e. the problem of the quantitative reconstruction of elastic properties of a layered medium using limited experimental information, we consider two cases: a decompressive probe, and an indenter-like probe.

2.1 Decompressive probe

The decompressive probe proposed in [Sarvazyan and Ponomarev (1987)] is a thin-walled cylindrical jar of thin walls under which it is possible to create a negative normal pressure at the surface of a human body. The magnitude of the normal suction P can be precisely controlled inside the probe. Boundary conditions (1) can be specified as follows

$$\sigma_{rz}(r,0)=0; \quad \sigma_{zz}(r,0)=\begin{cases} P,\ r<a \\ -Q,\ r\in[a,b] \\ 0,\ r>b \end{cases} \tag{6}$$

where a and b are the internal and external radii of the probe, and Q is the normal pressure at the contact between the probe wall and the skin. If the probe is thin-walled, i.e. if parameter $d = 1 - a/b$ is small, we can admit that Q = *const*. In this case the balance of forces acting on the probe equilibrium leads to

$$Q = Pa^2/(b^2 - a^2) = P(1-d)^2/d(2-d). \tag{7}$$

Thanks to linearity, the solution of the direct problem with boundary conditions (6) has the form $F = F_1 + F_2$ where F_1 and F_2 are the solutions of the two simpler problems with boundary conditions

$$\sigma_{rz}(r,0) = 0, \quad \sigma_{zz}(r,0) = \begin{cases} Q+P, \ r < a \\ 0, \ r > a \end{cases} \tag{6a}$$

and

$$\sigma_{rz}(r,0) = 0, \quad \sigma_{zz}(r,0) = \begin{cases} -Q, \ r < b \\ 0, \ r > b \end{cases} \tag{6b}$$

respectively. Taking into account the general form of the solution (A1.1), we use the Hankel transformation for boundary conditions (6a) and (6b). The general form of Hankel's transformation for a step function is:

$$P(r) = \begin{cases} P^0, \ r < r^0 \\ 0, \ r > r^0 \end{cases} = r^0 P^0 \int_0^\infty J_1(\alpha r^0) J_0(\alpha r) d\alpha \tag{8}$$

If the problem under a general step function $\sigma_{zz}(r,0)$ given by Eq. (8) being solved, then one can easily find the solution with boundary conditions (6a) and (6b) by specifying the values P^0 and r^0.

2.1.1 *Homogeneous semi-infinite body*

The general solution has the simplest form in case of homogeneous semi-infinite body ($N = 1$, $H = \infty$). By incorporating Eqs (1), (A1.1) and (8) we obtain

$$A_1 = B_1 = 0, A_2 = 2\nu p/\alpha, \quad B_2 = p$$

where $p = r^0 P^0 J_1(\alpha r^0)/2\mu\alpha^2$.

Therefore, the appropriate solution $F\left\langle P^0, r^0 \right\rangle = \{U, W, \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}\}$ has the following form

$$U(r,z) = \frac{r^0 P^0}{2\mu} \int_0^\infty \alpha^{-1} e^{-\alpha z} J_1(\alpha r^0) J_1(\alpha r)(1 - 2\nu - \alpha z) d\alpha$$

$$W(r,z) = -\frac{r^0 P^0}{2\mu} \int_0^\infty \alpha^{-1} e^{-\alpha z} J_1(\alpha r^0) J_0(\alpha r)[2(1-\nu) + \alpha z] d\alpha$$

$$\sigma_{rr}(r,z) = r^0 P^0 \int_0^\infty e^{-\alpha z} J_1(\alpha r^0)[J_0(\alpha r)[(1-\alpha z) - \frac{J_1(\alpha r)}{\alpha r}(1 - 2\nu - \alpha z)] d\alpha$$

$$\sigma_{\theta\theta}(r,z) = r^0 P^0 \int_0^\infty e^{-\alpha z} J_1(\alpha r^0)[2\nu J_0(\alpha r) + \frac{J_1(\alpha r)}{\alpha r}(1 - 2\nu - \alpha z)] d\alpha$$

$$\sigma_{zz}(r,z) = r^0 P^0 \int_0^\infty e^{-\alpha z} J_1(\alpha r^0) J_0(\alpha r)(1 + \alpha z) d\alpha$$

$$\sigma_{rz}(r,z) = r^0 P^0 \int_0^\infty e^{-\alpha z} J_1(\alpha r^0) J_0(\alpha r) \alpha z d\alpha \tag{9}$$

As has been said, the general solution F for the boundary conditions given by Eqs (6) is a sum of two particular solutions, at particular boundary conditions (6a) and (6b) such that

$$F = F\left\langle -Q, b \right\rangle + F\left\langle Q+P, a \right\rangle \tag{10}$$

For instance, the vertical displacement of the layer surface equals

$$W(r,0) = 2(1-\nu^2)\frac{1}{E} \int_0^\infty \alpha^{-1} J_0(\alpha r)[bQ J_1(\alpha b) - a(Q+P) J_1(\alpha a)] d\alpha \tag{11}$$

where $E = 2\mu(1+\nu)$ is the Young modulus of the medium.

To simplify formulas we posite $d = 1 - a/b$ and

$$x = r/b, \quad \beta = \alpha b \tag{12}$$

With these notations taking into account (7) expression (11) takes the form

$$W(r,0) = 2(1-\nu^2)\frac{bP(1-d)}{Ed(2-d)}\int_0^\infty \beta^{-1}J_0(\beta x)[(1-d)J_1(\beta) - J_1(\beta(1-d))]d\beta,$$

It is known that [Korn and Korn (1968)]

$$J_1(\beta(1-d)) = J_1(\beta) + d[J_1(\beta) - \beta J_0(\beta)] + O(d^2)$$

for small d's we could obtain

$$W(r,0) = -(1-\nu^2)b\frac{P}{E}\int_0^\infty \beta^{-1}J_0(\beta x)[2J_1(\beta) - \beta J_0(\beta)]d\beta + O(d) \quad (13)$$

and, letting $d \to 0$:

$$W(r,0) = -(1-\nu^2)b\frac{P}{E}\overline{\Phi}_1(x), \quad \overline{\Phi}_1(x) = \int_0^\infty \beta^{-1}J_0(\beta x)[2J_1(\beta) - \beta J_0(\beta)]d\beta. \quad (14)$$

As is well known, soft tissues are nearly incompressible, i.e. their Poisson's ratio ν is very close to its limit value $\nu^0 = 0.5$. This simplifies the relation between the Young modulus E and shear modulus $\mu = E/2(1+\nu) \approx E/3$. Parameters E and μ play the major role in diagnostics, and their quantitative estimation is the ultimate goal of tissue elasticity identification [Gao et al (1996)].

The unknown Young's modulus can be estimated from Eq. (14) as follows

$$E = -(1-\nu^2)bP\frac{\overline{\Phi}_1(x)}{W(r,0)} \approx -\frac{3}{4}bP\frac{\overline{\Phi}_1(x)}{W(r,0)}$$

Hence, the elastic moduli of an elastic half-space can be recovered if the vertical displacement at a single point of the surface available. Moreover, there is no need to measure the absolute value of the vertical displacement. If only the relative displacement $W(r_1,0) - W(r_2,0)$ is measured, the Young modulus may be estimated by the formula

$$E = -(1-\nu^2)bP\frac{\overline{\Phi}_1(x_1) - \overline{\Phi}_1(x_2)}{W(r_1,0) - W(r_2,0)}.$$

Note that this formula is more sensitive to noise since noisy experimental displacement are subtracted in the denominator.

2.1.2 *Homogeneous layer of finite thickness*

If a homogeneous layer of finite thickness H is considered, equations (1), (2), (A1.1) and (8) lead to

$$\alpha A_1 = -p\frac{[2(n-\overline{\beta})^2 + (1+2\overline{\beta})(1+n) + k_2^2(1-n)(1+2n)]}{\Delta},$$
$$\alpha A_2 = p\frac{[2(n+\overline{\beta})^2 + (1-2\overline{\beta})(1+n) + k_1^2(1-n)(1+2n)]}{\Delta},$$
$$B_1 = p\frac{[(1+2\overline{\beta}) + k_2^2(1+2n)]}{\Delta},$$
$$B_2 = p\frac{[(1-2\overline{\beta}) + k_1^2(1+2n)]}{\Delta}. \quad (15)$$

where

$$n = 1-2\nu,\ \overline{\beta} = \alpha H,\ k_1 = \exp(\overline{\beta}),\ k_2 = \exp(-\overline{\beta}),$$
$$\Delta = (1+2n)(k_1^2 + k_2^2) + (1+2n)^2 + (1+4\overline{\beta}^2). \quad (16)$$

Consequently, the solution $F\left\langle P^0, r^0\right\rangle$ for $z = 0$ takes the form

$$U(r,0) = \frac{r^0P^0}{2\mu}\int_0^\infty \frac{J_1(\alpha r^0)}{\alpha\Delta}J_1(\alpha r)[(k_1^2+k_2^2)(1+2n)n + 2n - 4(n^2+\overline{\beta}^2)]d\alpha$$

$$W(r,0) = \frac{r^0P^0(1+n)}{2\mu}\int_0^\infty \frac{J_1(\alpha r^0)}{\alpha\Delta}J_0(\alpha r)[4\overline{\beta} - (k_1^2 - k_2^2)(1+2n)]d\alpha$$

$$\sigma_{rr}(r,0) = r^0P^0\int_0^\infty \frac{J_1(\alpha r^0)}{\Delta}\{J_0(\alpha r)[(k_1^2+k_2^2)(1+2n) + 2(1-2n) - 4(n^2+\overline{\beta}^2)] - J_1(\alpha r)[(k_1^2+k_2^2)(1+2n)n - 2n - 4(n^2+\overline{\beta}^2)]/\alpha r\}d\alpha$$

$$\sigma_{\theta\theta}(r,0) = r^0P^0\int_0^\infty \frac{J_1(\alpha r^0)}{\Delta}\{(1-n)J_0(\alpha r)[(k_1^2+k_2^2)(1+2n)+2] + J_1(\alpha r)[(k_1^2+k_2^2)(1+2n)n - 2n - 4(n^2+\overline{\beta}^2)]/\alpha r\}d\alpha \quad (17)$$

and the solution of the general problem is determined by Eqs (10) and (17) by specifying P^0 and r^0 according to

Eqs (6a) and (6b). In particular, for the vertical displacement at the surface we obtain

$$W(r,0)=2(1-\nu^2)\frac{1}{E}\int_0^\infty \alpha^{-1}\Phi(\overline{\beta})J_0(\alpha r)$$
$$[bQJ_1(\alpha b)-a(Q+P)J_1(\alpha a)]d\alpha \qquad (18)$$

where $\Phi(\overline{\beta})=[m(k_1^2-k_2^2)-4\overline{\beta}]/[m^2+m(k_1^2+k_2^2)+(1+4\overline{\beta}^2)]$, $m=3-4\nu$. (Note that Eq. (18) reduces to Eq. (11) at $H\to\infty$).

Taking into account (7) and using notations already given in Eqs (12), (16) and $B=b/H$

Eq. (18) may be rewritten in the form

$$W(r,0)=2(1-\nu^2)\frac{bP(1-d)}{Ed(2-d)}\int_0^\infty \beta^{-1}\Phi(\beta/B)J_0(\beta x)$$
$$[(1-d)J_1(\beta)-J_1(\beta(1-d))]d\beta$$

Then, for small values of d, one obtains

$$W(r,0)=-(1-\nu^2)b\frac{P}{E}\int_0^\infty \beta^{-1}\Phi(\beta/B)J_0(\beta x)$$
$$[2J_1(\beta)-\beta J_0(\beta)]d\beta+O(d)$$

which parallels Eq. (13).

However, differently from the half-space, we have now two unknowns: the Young modulus E and the thickness of the layer H. To solve the inverse problem we consider the relative displacements between two points on the surface:

$$D_i=W(r_0,0)-W(r_i,0)=$$
$$-(1-\nu^2)b\frac{P}{E}[\Phi_1(B,x_0)-\Phi_1(B,x_i)],\quad i=1,2. \qquad (19)$$

Note that ratio Φ_2 defined below does not depend on E:

$$\Phi_2=D_1/D_2=\frac{\int_0^\infty \Phi(\beta/B)f(\beta,x_0,x_1)d\beta}{\int_0^\infty \Phi(\beta/B)f(\beta,x_0,x_2)d\beta},$$

where $f(\beta,x_0,x_i)=\beta^{-1}[2J_1(\beta)-\beta J_0(\beta)][J_0(\beta x_0)-\beta J_0(\beta x_i)]$, $i=1,2$. By incorporating this property and solving the nonlinear equation $\Phi_2(B)=D_1^{\exp}/D_2^{\exp}$, where the upper index "exp" denotes a datum, we can find the value $B=B^*$ and, therefore, the unknown thickness of the layer $H=b/B^*$. The unknown Young's modulus $E$of a homogeneous layer can be calculated from anyone of the two equations (19) using the previously computed value B^*, as follows:

$$E=-(1-\nu^2)bP\frac{\Phi_{1R}(B^*,x_0,x_i)}{D_i^{\exp}}$$
$$\approx-\frac{3}{4}bP\frac{\Phi_{1R}(B^*,x_0,x_i)}{D_i^{\exp}},$$

where $\Phi_{1R}(B^*,x_0,x_i)=\Phi_1(B^*,x_0)-\Phi_1(B^*,x_i)$

2.1.3 Stratified layer

As has been said, in the case of a stratified layer $4N$ unknowns A_j^i and B_j^i, *i=1,...,N* , $j=1,2$ need be determined as functions of α, in order to solve the direct elastic problem. These unknowns can be evaluated using the set of solutions (A1.1), continuity conditions (3) and boundary conditions (2) and (6). More precisely, to obtain the solution $F\left\langle P^0,r^0\right\rangle$ you need satisfy the boundary conditions

$$\alpha A_1^1+\alpha A_2^1+(1-n_1)B_1^1-(1-n_1)B_2^1=0,$$
$$-\alpha A_1^1+\alpha A_2^1+n_1B_1^1+n_1B_2^1=p, \qquad (20)$$

$$\alpha k_1^NA_1^N-\alpha k_2^NA_2^N+(1+\beta_N)k_1^NB_1^N+(1-\beta_N)k_2^NB_2^N=0,$$
$$\alpha k_1^NA_1^N+\alpha k_2^NA_2^N-(2n_N-\beta_N)k_1^NB_1^N$$
$$+(2n_N+\beta_N)k_2^NB_2^N=0, \qquad (21)$$

and the continuity conditions (3) for $i=2,\dots,N$

$$-\alpha k_1^iA_1^{i-1}-\alpha k_2^iA_2^{i-1}+(2n_{i-1}-\beta_i)k_1^iB_1^{i-1}$$
$$-(2n_{i-1}+\beta_i)k_2^iB_2^{i-1}+\alpha k_1^iA_1^i+\alpha k_2^iA_2^i$$
$$-(2n_i-\beta_i)k_1^iB_1^i+(2n_i+\beta_i)k_2^iB_2^i=0, \qquad (22)$$

$$\alpha k_1^iA_1^{i-1}-\alpha k_2^iA_2^{i-1}+(1+\beta_i)k_1^iB_1^{i-1}+(1-\beta_i)k_2^iB_2^{i-1}$$
$$-\alpha k_1^iA_1^i+\alpha k_2^iA_2^i-(1+\beta_i)k_1^iB_1^i-(1-\beta_i)k_2^iB_2^i=0, \qquad (23)$$

$$-\alpha k_1^iA_1^{i-1}+\alpha k_2^iA_2^{i-1}+(n_{i-1}-\beta_i)k_1^iB_1^{i-1}$$
$$+(n_{i-1}+\beta_i)k_2^iB_2^{i-1}]+m_i[\alpha k_1^iA_1^i$$
$$-\alpha k_2^iA_2^i-(n_i-\beta_i)k_1^iB_1^i-(n_i+\beta_i)k_2^iB_2^i]=0, \qquad (24)$$

$$[\alpha k_1^iA_1^{i-1}+\alpha k_2^iA_2^{i-1}+(1-n_{i-1}+\beta_i)k_1^iB_1^{i-1}$$
$$-(1-n_{i-1}-\beta_i)k_2^iB_2^{i-1}]-m_i[\alpha k_1^iA_1^i+\alpha k_2^iA_2^i$$
$$+(1-n_i+\beta_i)k_1^iB_1^i-(1-n_i-\beta_i)k_2^iB_2^i]=0. \qquad (25)$$

Here

$$n_i = 1-2\nu_i,\ \beta_i = \alpha h_i,\ k_1^i = \exp(\beta_i),\ k_2^i = \exp(-\beta_i),$$
$$p = r^0 P^0 J_1(\alpha r^0)/2\mu_1\alpha^2,\ m_i = \mu_i/\mu_{i-1}.$$

The algorithm to solve the system of 4N linear equations (20)-(25) is presented in Appendix 2.

Using the linearity property (Eq. (10)), the expression

$$W(r,0) = \frac{r^0P^0}{2\mu_1}\int_0^\infty \alpha^{-1}J_1(\alpha r^0)J_0(\alpha r)[2n_1(B_1^1-B_2^1) - \alpha(A_1^1+A_2^1)]d\alpha,$$

and specifications of P^0 and r^0 according to Eqs (6a) and (6b) we find the vertical displacement of the layer surface, $z=0$, as follows

$$W(r,0) = \frac{1}{2\mu_1}\int_0^\infty \alpha^{-1}J_0(\alpha r)[2n_1(B_1^1-B_2^1)-\alpha(A_1^1+A_2^1)] [a(Q+P)J_1(\alpha a)-bQJ_1(\alpha b)]d\alpha.$$

Again, using the notations

$$x = r/b,\quad \beta = \alpha b,\quad d = 1-a/b,$$
$$\psi(\alpha) = 2n_1(B_1^1-B_2^1)-\alpha(A_1^1+A_2^1),$$

and taking into account (7) we obtain

$$W(r,0) = \frac{bP(1-d)}{2\mu_1(2-d)}\int_0^\infty \beta^{-1}\psi(\beta/b)J_0(\beta x)\{J_1[\beta(1-d)] - (1-d)J_1(b)\}d\beta.$$

For $d\to 0$

$$W(r,0) = \frac{bP(1+\nu_1)}{2E_1}\Phi_1^*(x),$$
$$\Phi_1^*(x) = \int_0^\infty \beta^{-1}\psi(\beta/b)J_0(\beta x)[2J_1(\beta)-\beta J_0(\beta)]d\beta \qquad (26)$$

Note that the limit property $\psi(\alpha)\to -2(1-\nu_1)$ at $a\to\infty$, drastically simplifies the computation of the integrals in Eq. (26).

All formulas presented above are easily adapted to the case of a stratified half-space by letting $h_N = H\to\infty$, as well as for an incompressible medium, letting $\nu_i\to 0.5$.

The inverse problem in the case of a layered medium is more complicated. In principle, the elastic moduli E_i and the thicknesses h_i of all layers are to be recovered. In order to find these $K\le 2N$ unknowns for each given N we need at least K equations

$$W(r_k,0) = W^{\exp}(r_k),\quad k=1,\dots,K^*,\quad K^*\ge K. \qquad (27)$$

where the left-hand parts are computed numerically as presented above, and the right-hand parts are measured experimentally.

If $K^* = K$ we can try to solve Eqs (27) as a system of nonlinear equations for K unknown parameters of the model using any appropriate numerical procedure. When $K^* > K$ the system (27) is overdetermined, and a least square approach, for example, could be used instead.

Some examples for $K^* = K$ are given later on. In these examples a three-layered medium is considered, for which the experimental information on displacement is replaced by exact solutions of the appropriate direct elastic problem. Newton's iterative procedure is used to solve the system of nonlinear equations (27). A description of these numerical experiments is given in Table 1. The relative boundary positions are denoted as $H_i = h_i/b$ and values $\nu_i = 0.45$ are used. The results of using the Newton iterative procedure as a functions of the iteration index are given in Figs 1-5.

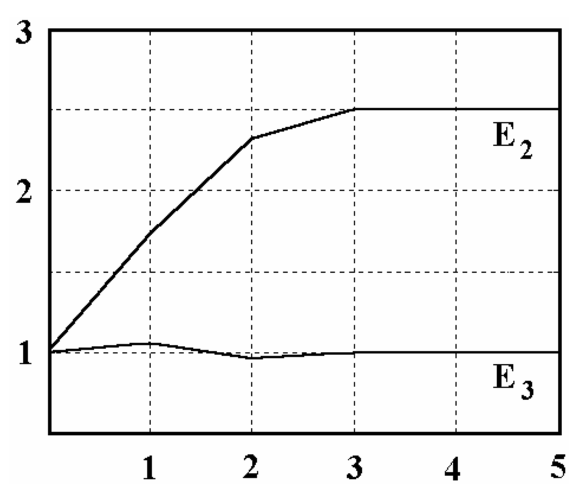


Figure 1 : Young moduli of the second and third layers E_2, and E_3 as a function of the iteration index

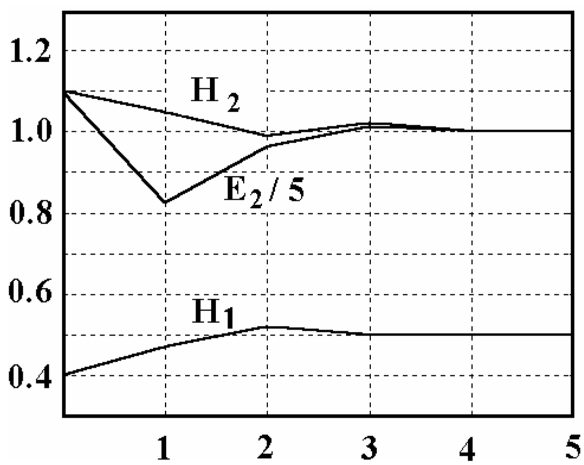


Figure 2 : Young modulus and boundary locations of the internal layer as a function of the iteration index

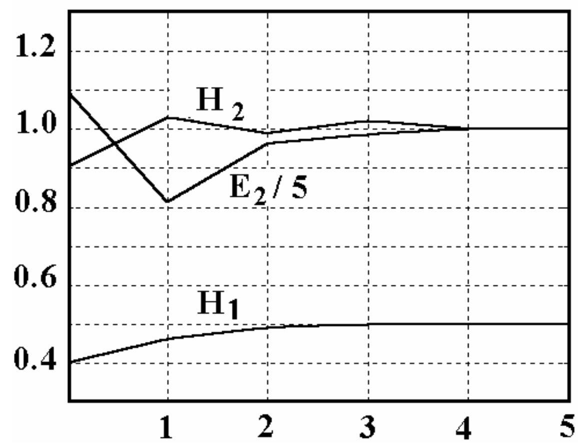


Figure 3 : Young modulus and boundary locations of the internal layer as a function of the iteration index

The first example (Fig.1) assumes that the thickness of the strata are known, while their mechanical properties are to be recovered. The following two examples (Figs 2, 3) are devoted to the problem of strata detection. The elastic properties and the thickness of the strata are recovered from different starting points. The fourth example (Fig.4) addresses the problem of the detection of a stratum whose thickness is known, but exact location and elastic modulus are to be found. Also the modulus of the deeper stratum is unknown in this case. The last example (Fig.5) is remarkable. It presents the results obtained by assuming that there are four strata when identifying the mechanical properties of a three-layered tissue. As can be seen from Fig. 5, the initially incorrect number of strata is corrected by the iterative procedure that attributes the same modulus to the second and third strata (E_2=E_3=2.5).

Note, that the described procedure demonstrates a high speed of convergence: no more that 4 Newton's iterations are needed to approach the exact solution of the inverse problem to within a relative error $\approx 10^{-3}$ in all examples considered.

2.2 *Indenter-like probe*

This section considers an indenter-like probe as an alternative to the previously considered decompressive probe. This problem is akin to the problem considered above:

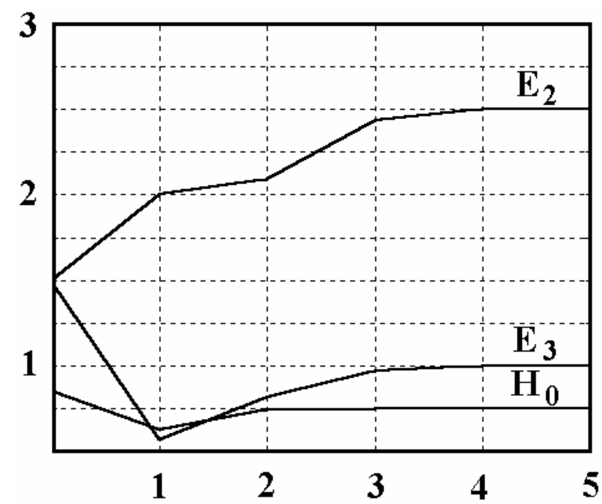


Figure 4 : Young moduli of the second and third layers E_2 and E_3 and the location of the center of the second layer as functions of the iteration index

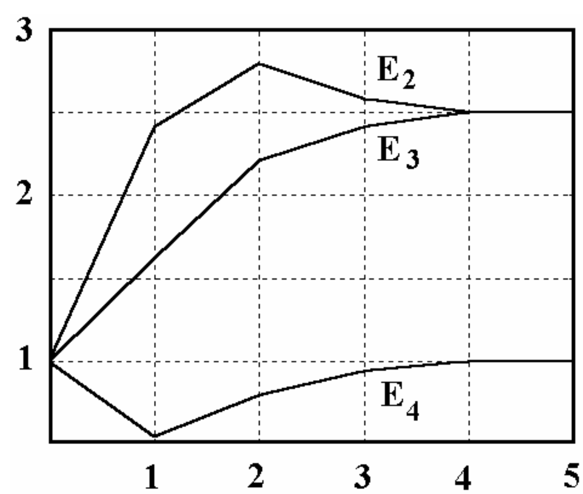


Figure 5 : Young moduli of four-layer model of the the three-layered medium as a function of iteration index

Table 1 :

Fig	Given parameters	Unknown parameters	Starting values of the unknown parameters	Exact Solution
1	E_1=1, H_1=0.5, H_2=1, H_3=H=2 W^{exp}(0.7), W^{exp}(0.9)	E_2, E_3	E_2=E_3=1	E_2=2.5, E_3=1
2	E_1=E_3=2, H_3=H=2, W^{exp}(0.2), W^{exp}(0.5), W^{exp}(0.7)	H_1, H_2, E_2	H_1=0.4, H_2=1.1 E_2=5.5	H_1=0.5, H_2=1 E_2=5
3	E_1=E_3=2, H_3=H=2, W^{exp}(0.2), W^{exp}(0.5), W^{exp}(0.7)	H_1, H_2, E_2	H_1=0.4, H_2=0.9 E_2=5.5	H_1=0.5, H_2=1 E_2=5
4	E_1=E_3=1, H_3=H=2, H_2-H_1=0.5 W^{exp}(0.2), W^{exp}(0.5), W^{exp}(0.7)	H_0=(H_1+H_2)/2 E_2, E_3	H_0=0.85 E_2=E_3=1.5	H_0=0.75 E_2=E_3=1
5	E_1=1, H_1=0.5, H_2=0.75, H_3=1, H_4=H=2 W^{exp}(0.2), W^{exp}(0.5), W^{exp}(0.7)	E_2, E_3, E_4	E_2=E_3=E_4=1	E_2=E_3=2.5 E_4=1

boundary conditions (6) have to be replaced by

$$\sigma_{rz}(r,0)=0,\ 2\pi\int_0^b r\sigma_{zz}(r,0)dr=-P,$$
$$W(r,0)=W_0,\ r<b;\ \sigma_{zz}(r,0)=0,\ r>b, \tag{28}$$

where b is the external radius of the cylindrical indenter and P is the force it exerts. If indenter is not glued to the skin these conditions need to be coupled with the unilateral constraint $\sigma_{zz}(r,0)<0$.

It is well known [Novatski (1975); Rabotnov (1979)] that the stress distribution under an indenter with a sharp edge is highly nonuniform and increases in magnitude near the edge. Taking into account this phenomenon, to design an effective numerical procedure to solve the direct elastic problem we use a nonuniform grid $[a_i,b_i]$, $i=1,..,M$, where $a_1=0$, $a_i=b_{i-1}$, $b_M=b$, $b_{i-1}-a_{i-1}>b_i-a_i$. Thus, the loaded circle of radius b is subdivided into a finite set of rings of the internal and external radii a_i and b_i, respectively. Assuming that the normal pressure p_i is constant on each ring $[a_i,b_i]$ we find these M unknowns p_i using the condition $W(r,0)=W_0=const$, $r<b$. In all the examples we present, a Chebishev grid with M =8 was used.

We construct the solution F^i of the direct problem for each ring $[a_i,b_i]$ with the boundary conditions

$$\sigma_{rz}=0,\quad \sigma_{zz}^i=\{-p_i,r\in[a_i,b_i];\ 0,r\notin[a_i,b_i]\} \tag{29}$$

as the sum of two solutions, corresponding respectively to the simplified boundary conditions

$$\sigma_{rz}=0,\quad \sigma_{zz}^i=\{-p_i,r<b_i;\ 0,r>b_i\} \tag{30}$$

and

$$\sigma_{rz}=0,\quad \sigma_{zz}^i=\{p_i,r<a_i;\ 0,r>a_i\} \tag{31}$$

which we determine using Hankel's transformation (Eq. (8)) for step functions.

2.3 *Homogeneous half-space*

If a homogeneous half-space is considered ($H\to\infty$) the solution (9) holds for each boundary condition (30) or (31), and the solution for boundary conditions (29) takes the form

$$F^i=F\langle -p_i,b_i\rangle+F\langle p_i,a_i\rangle.$$

The vertical displacement at the surface is (cfr. Eq. (11))

$$W_i(r,0)= \tag{32}$$
$$2(1-\nu^2)\frac{p_i}{E}\int_0^\infty \alpha^{-1}J_0(\alpha r)[b_iJ_1(\alpha b_i)-a_iJ_1(\alpha a_i)]d\alpha$$

Using the notations

$$x=r/b,\quad \beta=\alpha b,\quad c_{0i}=b_i/b,$$
$$c_{1i}=a_i/b,\quad \lambda_i=2bp_i(1-\nu^2)/EW_0$$

we rewrite Eq. (32) as follows

$$W_i(x,0) = W_0\lambda_i \int_0^\infty \beta^{-1} J_0(\beta x)[c_{0i}J_1(\beta c_{0i}) - c_{1i}J_1(\beta c_{1i})]d\beta,$$

Therefore, the general solution of the direct elastic problem in this case takes the form

$$W(x,0) = W_0 \sum_{i=1}^{N} \lambda_i \int_0^\infty \beta^{-1} J_0(\beta x)[c_{0i}J_1(\beta c_{0i}) - c_{1i}J_1(\beta c_{1i})]d\beta \quad (33)$$

The unknowns $\lambda_i,\ i = 1,\ldots,M$ can be determined solving the system of linear equations

$$W(x_j,0) = W_0,\ \ j = 1,\ldots,M^*,\ \ x_j < 1,\ \ M^* \geq M. \quad (34)$$

The second condition (28) couples the total indenting force P and the vertical displacement of the indenter W_0:

$$P = \frac{\pi b W_0 E}{2(1-\nu^2)} \sum_{i=1}^{N} \lambda_i (c_{0i}^2 - c_{1i}^2) \quad (35)$$

In all the examples we present, we take $M^* = M$ and $r_j = (a_j + b_j)/2$. If one takes

$M^* > M$, the system (34) is overdetermined and a least-square method should be used to estimate the M unknowns $\lambda_i,\ i = 1,\ldots,M$.

The unknown Young's modulus of a half-space can be determined in two different ways. If the total force P and the vertical displacement of the indenter W_0 are both known, formula (35) can be used. If not, the experimentally measured displacement of any particular point $x > 1$ and formula (33) should be used instead.

2.3.1 Homogeneous layer of finite thickness

If a homogeneous layer of finite thickness H is considered, the basic solution $F\left\langle P^0, r^0\right\rangle$ in the form (17) holds and the general solution of the forward elastic problem takes the form

$$W(x,0) = W_0 \sum_{i=1}^{N} \lambda_i \int_0^\infty \beta^{-1} \Phi(\beta/B) J_0(\beta x) [c_{0i}J_1(\beta c_{0i}) - c_{1i}J_1(\beta c_{1i})]d\beta. \quad (36)$$

Note, that Eq. (36) reduces to Eq. (33) for $H \to \infty$.

Again, the unknowns $\lambda_i,\ i = 1,\ldots,M$ can be determined solving the system of linear equations (34), and the two unknowns E and Hcan be determined from Eqs (35) and (27), where the left-hand terms are the theoretically predicted and the right-hand terms the experimentally measured values of the vertical displacement of the surface outside the indenter $r_k > b$.

2.3.2 Stratified medium

In the case of a stratified medium these conditions serve also to estimate the unknown parameters. The basic solution $F\left\langle P^0, r^0\right\rangle$ and computational technique (20)-(25) and (A2.1)-(A2.6) remain valid in this case. The general solution of the direct elastic problem takes the form

$$W(x,0) = W_0 \sum_{i=1}^{N} \lambda_i \int_0^\infty \beta^{-1} \Psi(\beta/b) J_0(\beta x) [c_{0i}J_1(\beta c_{0i}) - c_{1i}J_1(\beta c_{1i})]d\beta \quad (37)$$

where

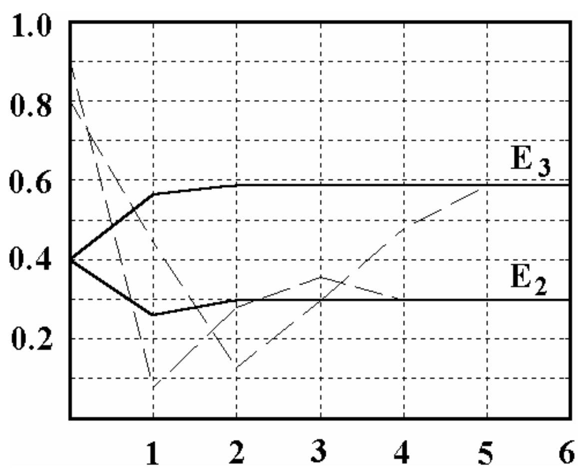
$$\Psi(\alpha) = [2n_1(B_1^1 - B_2^1) - \alpha(A_1^1 + A_2^1)]/2(1-\nu_1).$$

Note that in principle one could use in Eq. (27) also values of the measured under the surface displacement. The vertical displacements of an internal point, including the vertical displacements at internal boundaries can be used as well. Internal vertical displacements have the same form given by Eq. (37) where term Ψ is a function of α and z. Note also that all derived formulas can be easily adopted to the case of a half-space by letting $h_N = H \to \infty$, or for a incompressible medium, letting $\nu_i \to 0.5$.

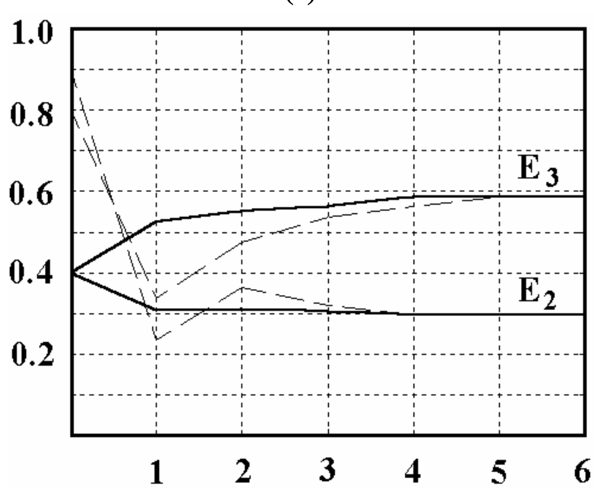
Table 2 describes five numerical experiments for a three-layered medium. The values of E_2 and E_3 are recovered relatively to the elastic modulus of the first (top) layer (we assume $E_1 = 1$). The case when the identification of the elastic properties is based on vertical displacements measured on the surface $W_0(r)$ and Eq. (27) is considered in Fig. 6.a. At variance with this, the vertical displacements at the first internal boundary $W_1(r)$ are used in Figs 6.b and 6.c. The last example (Fig. 6.c) and two preceding ones (Figs. 6.a and 6.b) correspond to the same three-layered medium flipped upside down. The value $\nu_i = 0.49$ is assumed in all strata.

Table 2 :

Fig	Given parameters	Unknown parameters	Starting values of the unknown parameters	An exact of the unknown
6.a solid line	H_1=1, H_2=2, H_3=H=3.5 $W_0^{exp}(1.5)$, $W_0^{exp}(2)$	E_2, E_3	E_2=E_3=0.4	E_2=0.296 E_3=0.587
6.a dashed line	H_1=1, H_2=2, H_3=H=3.5 $W_0^{exp}(1.5)$, $W_0^{exp}(2)$	E_2, E_3	E_2=0.9, E_3=0.8	E_2=0.296 E_3=0.587
6.b solid line	H_1=1, H_2=2, H_3=H=3.5 $W_1^{exp}(0.1)$, $W_1^{exp}(1.6)$	E_2, E_3	E_2=E_3=0.4	E_2=0.296 E_3=0.587
6.b dashed line	H_1=1, H_2=2, H_3=H=3.5 $W_1^{exp}(0.1)$, $W_1^{exp}(1.6)$	E_2, E_3	E_2=0.9, E_3=0.8	E_2=0.296 E_3=0.587
6.c	H_1=1.5, H_2=2.5, H_3=H=3.5 $W_1^{exp}(0.1)$, $W_1^{exp}(1.6)$	E_2, E_3	E_2=0.9, E_3=0.8	E_2=0.504 E_3=1.704

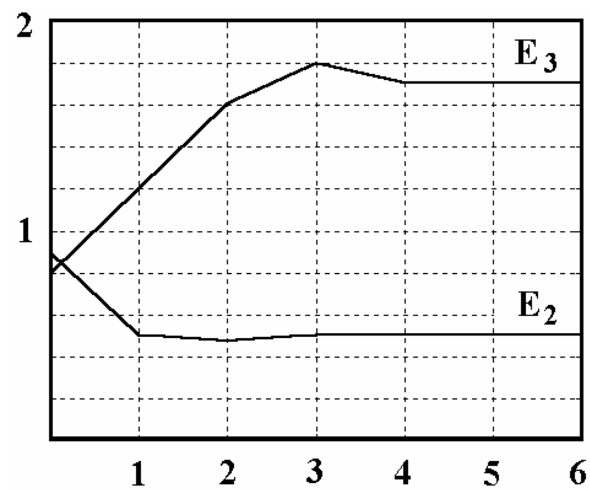


(a)



(b)

Figure 6 : Relative Young moduli of the second and third layers E_2 and E_3 as functions of the iteration index



(c)

Figure 6 : (continued) Relative Young moduli of the second and third layers E_2 and E_3 as functions of the iteration index

Figs 6.a - 6.c present the results given by our iterative procedure as a function of the number of iterations. As in the case of the decompressive probe, no more than 4-5 Newton's iterations are needed to approach the exact solution of the inverse problem to within a relative error $\approx 10^{-3}$ in each case considered.

3 Smoothly Stratified Elastic Strate

The same model of a stratified medium can be applied in two different ways. On one hand, this model could be applied with a small number of internal strata to characterize some piecewise averaged elasticity distribution. Namely, a three-layered model seems advantageous when average elastic properties of the tissue, subcutaneous fat and muscle are of interest. On the other hand, if a more detailed analysis is needed, the layered model could be applied with a large number N of strata. In general, the layered model with large N can be used as a discrete approximation to a smoothly stratified elastic layer. In this case, the distribution of elastic moduli is approximated by a stepwise constant distribution. The formulas we presented constitute a specific finite discretization of the equilibrium equations, which satisfy exactly the original problem within each thin approximately homogeneous layer. Being coupled with an effective numerical procedure presented in Appendix 2 the layered model constitutes a robust approach to the detailed quantitative estimation of elastic properties of the stratified medium even for a large number of unknowns (i.e., for a large number of internal layers).

The simplified approach based on a reduced set of parameters can be used to characterize the elasticity distribution even if for a smoothly stratified layer. In fact, any appropriate class of continuous functions can be used to represent the elasticity distribution within the layer, instead of the stepwise constant distribution considered so far. We give a demonstration of this approach for the case of an incompressible medium.

In this section, the Young (or the shear) modulus of the medium is assumed to depend smoothly on z, and the general Hankel transformation of displacement components is used:

$$U(r,z) = \int_0^\infty \alpha \hat{U}(\alpha,z) J_1(\alpha r)\, d\alpha,$$

$$W(r,z) = \int_0^\infty \alpha \hat{W}(\alpha,z) J_0(\alpha r)\, d\alpha. \tag{38}$$

Substituting (38) into the incompressibility condition

$$e = \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z} = 0$$

leads to

$$\hat{U} = -\hat{W}'/\alpha \tag{39}$$

while the stress-strain relation for an incompressible medium takes the form [3]

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + q\delta_{ij}, \tag{40}$$

where the constituitively undetermined internal pressure $q = (\sigma_{rr} + \sigma_{\varphi\varphi} + \sigma_{zz})/3$ is the limit value of the product λe in (5) when $\nu \to 0.5$. The notation $f' = \partial f/\partial z$ is used in Eq. (39) and later on.

Using the Hankel transformation of q in the form

$$q(r,z) = \int_0^\infty \alpha \hat{q}(\alpha,z) J_1(\alpha r)\, d\alpha,$$

Eqs (38), (39), (40), (4) and eliminating $\hat{q}$ we obtain the equation coupling $\hat{W}$ and E

$$W'''' + 2\kappa_1 W''' + (\kappa_2 - 2\beta)W'' - 2\beta\kappa_1 W' + \beta(\kappa_2 + \beta)W = 0, \tag{41}$$

where $\beta = \alpha^2$, $\kappa_1 = E'/E, \kappa_2 = E''/E$.

In this case, the stress-strain relation (40) leads to

$$\sigma_{zz}(r,z) = \frac{E}{3}\int_0^\infty J_0(\alpha r)\{4\alpha W' - [\kappa_1(\hat{W}'' + \beta\hat{W}) + (\hat{W}''' + \beta\hat{W}')]/\alpha\}\, d\alpha,$$

$$\sigma_{rz}(r,z) = -\frac{E}{3}\int_0^\infty J_1(\alpha r)(\hat{W}'' + \beta\hat{W})\, d\alpha.$$

Using the Hankel transformation of the vertical displacement and of the normal pressure at the surface

$$W_0 r) = \int_0^\infty \alpha \hat{W}_0(\alpha) J_0(\alpha r)\, d\alpha$$

$$P(r) = \int_0^\infty \alpha \hat{P}(\alpha) J_0(\alpha r)\, d\alpha$$

and bearing in mind conditions (1) and (2), we obtain the boundary conditions for Eq. (41)

$$\hat{W}'(\alpha,H) = \hat{W}(\alpha,H) = 0, \tag{42}$$

$$\hat{W}''(\alpha,0) + \beta\hat{W}(\alpha,0) = 0, \tag{43}$$

$$\hat{W}'''(\alpha,0) - 3\beta\hat{W}'(\alpha,0) = 3\beta\hat{P}(\alpha)/E(0). \tag{44}$$

If $W_0(r)$ is known, then we also have

$$\hat{W}(\alpha,0) = \hat{W}_0(\alpha). \tag{45}$$

Note that the boundary-value problem (41)-(45) with known $\hat{W}_0(\alpha)$ and $\hat{P}(\alpha)$ (i.e., with known $W_0(r)$ and $P(r)$) is overdetermined if $E(z)$ is given. Conditions (42)-(44) are sufficient to obtain the solution $\hat{W}_0(\alpha,z)$. The vertical displacement at any point including the layer's surface can be reconstructed by (38). Using the solution $W(r,0)$ of the direct elastic problem the inverse problem (i.e. the problem of recovering $E(z)$) can be formulated on the base of Eqs (27). The unknown function $E(z)$ can be approximated in any appropriate class of smooth functions determined by a reduced set of parameters. It should be pointed out that a precise specification of the local surface loading is needed to compute in this way the left-hand parts of Eqs (27). Note also that the number K^* of Eqs (27) should be no less than the number of these unknown parameters.

If both functions $W_0(r)$ and $P(r)$ are known for all $r \geq 0$, there is no need to consider any particular specification of the local surface loading. Let us rewrite Eqs (41)-(45) using the notation

$$V(\alpha,z) = \hat{W}(\alpha,z)/\hat{W}_0(\alpha).$$

$$V'''' + 2\kappa_1 V''' + (\kappa_2 - 2\beta)V'' - 2\beta\kappa_1 V' + \beta(\kappa_2+\beta)V = 0, \tag{46}$$

$$V'(\alpha,H) = V(\alpha,H) = 0, \tag{47}$$

$$V''(\alpha,0) = -\beta, \tag{48}$$

$$V(\alpha,0) = 1, \tag{49}$$

$$V'''(\alpha,0) - 3\beta V'(\alpha,0) = 3\beta G(\alpha)/E(0), \quad G(\alpha) = \hat{P}(\alpha)/\hat{W}_0(\alpha). \tag{50}$$

Formulas (46)-(50) show that in this case the inverse problem can be formulated in a different way, independently of the surface loading. Indeed, if the function $E(z)$ was known, the solution $V(\alpha,z)$ of the problem given by Eqs (46)-(49) could be obtained and, therefore, the function $G(\alpha)$ computed through the first equation in (3). Hence, the ratio $G(\alpha) = \hat{P}(\alpha)/\hat{W}_0(\alpha)$ depends only on the function $E(z)$; therefore, it is the same for all surface loadings. On the other hand, the function $G(\alpha)$ can be obtained according to the last equation in (3) using the known functions $\hat{W}_0(\alpha)$ and $\hat{P}(\alpha)$ (i.e., with known $W_0(r)$ and $P(r)$).

Bearing this in mind, we can formulate the inverse problem as a minimization problem for the error function

$$g(\alpha) = |G(\alpha) - G^*(\alpha)|,$$

where $G(\alpha)$ is the theoretical prediction corresponding to the trial distribution $E(z)$, and $G^*(\alpha)$ is computed from the experimentally measured $W_0(r)$ and $P(r)$, which reflect the real distribution $E^*(z)$.

Some numerical experiments were performed to demonstrate the robustness of this approach to detect tissue abnormality [Skovoroda (1996)]. It has been shown that the recovering of elastic properties within the layer is possible even when the abnormality is superimposed to a physiological inhomogeneity of the layer occurs and the function $G(\alpha) = \hat{P}(\alpha)/\hat{W}_0(\alpha)$ is noisy. Here we present some basic aspects of this approach.

In the examples we give here, the function $G^*(\alpha)$ is replaced by the exact solution of the direct elastic problem based on the given distribution $E^*(z)$, similarly to what has been done in Sects 2.1 and 2.2.

Let us assume the function $E^*(z)$ as follows

$$E^*(z) = E(0)\{1 + e_0(z) + \sum_{i=1}^{N} A_i^* e_i^*(z)\}, \tag{51}$$

where the term $[1 + e_0(z),]$ represents the physiological inhomogeneity of the layer and the terms

$$A_i^* e_i^*(z) = \exp\left[-t_i^* |z - z_i^0 * |^{n_i}\right], \quad i = 1,\ldots,N,$$

represent the abnormal inclusions. In all the examples considered, we use the direct and back sweep procedure [Samarski and Nikolaev (1978)] to obtain the numerical solution of the boundary value problem (46)-(49), with $n_i = 3$.

Figs 7-9 illustrate the relation between the function $g(\alpha)$ and the parameters of inclusion. The functions $g(\alpha)$ given in Fig. 7 correspond to the case when $e_0(z) = 0$, the single hard inclusion with the parameters $A_1^* = 1$, $t_1^* = 1200$ is located at $z_1^0 * = 0.2$, 0.3, 0.4, 0.5, 0.6, respectively and $G(\alpha)$ for a homogeneous layer is computed. The same case with $z_1^0 * = 0.4$ is illustrated in Fig. 8. Different inclusions characterised by $A_1^* =$ -0.75, -0.5, -0.25, 0.5, 1, 1.5 are considered. Finally, the functions $g(\alpha)$ for $A_1^* = 1$, $z_1^0 * = 0.4$, t_1^* = 250, 500, 1000 and 1500 are given in Fig. 9. These figures clearly show that the function $g(\alpha)$ depends monotonically on the parameters of the inclusion (position, size and shape).

Figures 10 and 11 illustrate the recovery of a distribution of elastic modulus when the space

$$E(z) = E(0)\{1 + \sum_{j=1}^{M} A_j^* e_j^*(z)\}$$

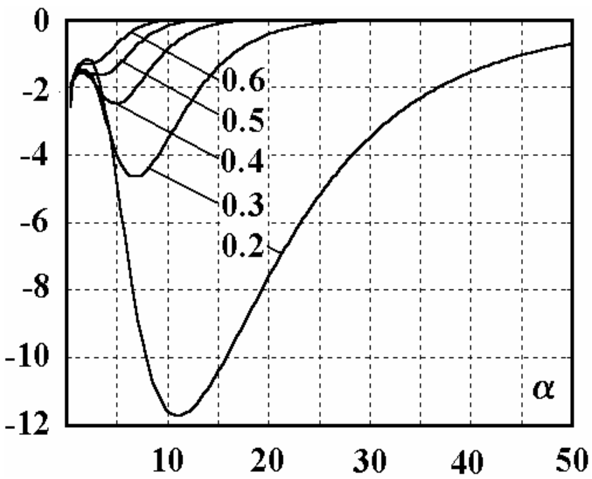


Figure 7 : Error function $g(\alpha)$, computed for the different position of inclusion

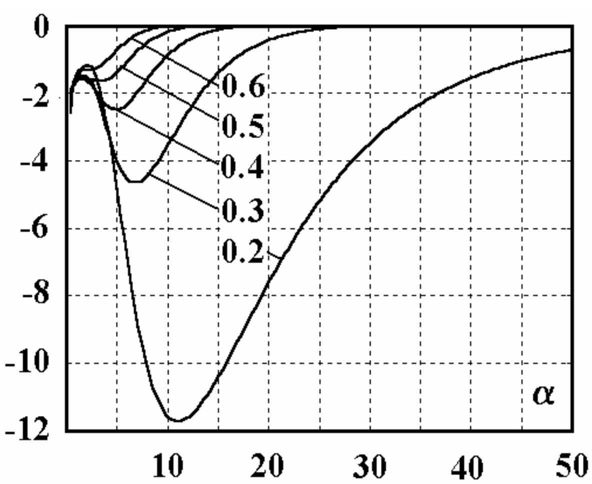


Figure 8 : Error function $g(\alpha)$, computed for the different magnitudes of inclusion

with

$e_j(z) = \exp[-t_j(z-z_j^0)^2], j = 1,\ldots,M,$

is used to approximate the unknown distribution $E^*(z)$. Namely, M = 9, $z_j^0 = jh$, $t_j = 4\ln 2/h^2$, $h = 0.1$ are used. The set of unknown parameters in this case is $\{A_j\}, j = 1,\ldots,M$, and these parameters are estimated by minimizing the total error

$$D = \int_0^\infty |G(\alpha) - G^*(\alpha)|\, d\alpha. \qquad (52)$$

Note, that $G(\alpha) \to const$ when $\alpha \to \infty$. Hence, the error

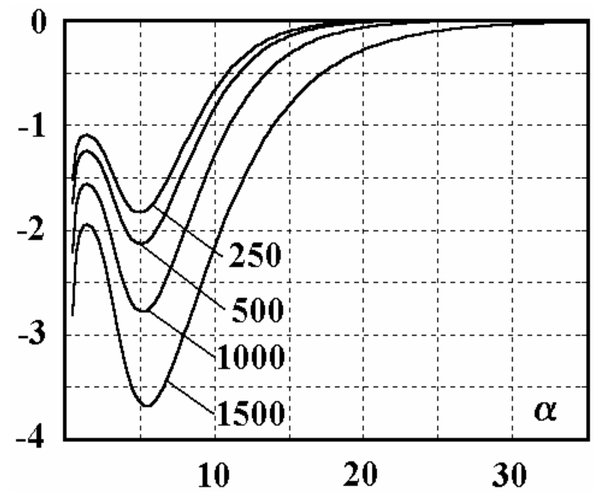


Figure 9 : Error function $g(\alpha)$, computed for the different shape of inclusion

minimization procedure can be drastically simplified by evaluating the integral in Eq. (52) on a bounded interval. Note that a deep inclusion correspond to a function $g(\alpha)$ much smaller than that corresponding to an inclusion located close to the surface. This phenomenon is clearly shown by Fig. 7. It suggests that weighted error functions could be more advantageous as compared with Eq. (52).

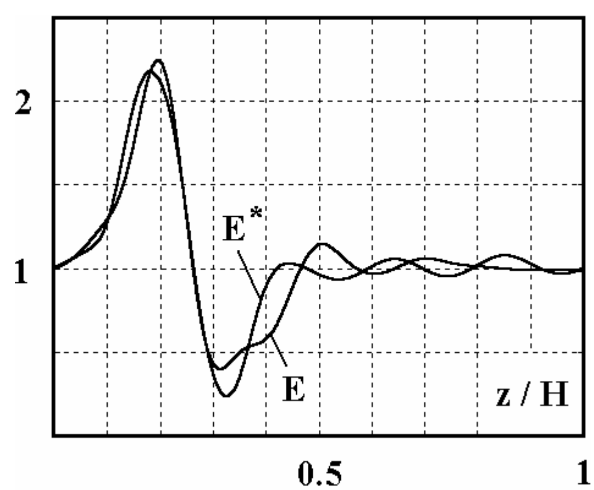


Figure 10 : The actual and reconstructed distribution $E(z)$

The iterative procedure of minimization is based on a gradient method, and starts with a trial solution $\{A_j*, t_j*, z_j^0*\}$, $j = 1,..,M$, *j=1,...,M*. The step sizes of

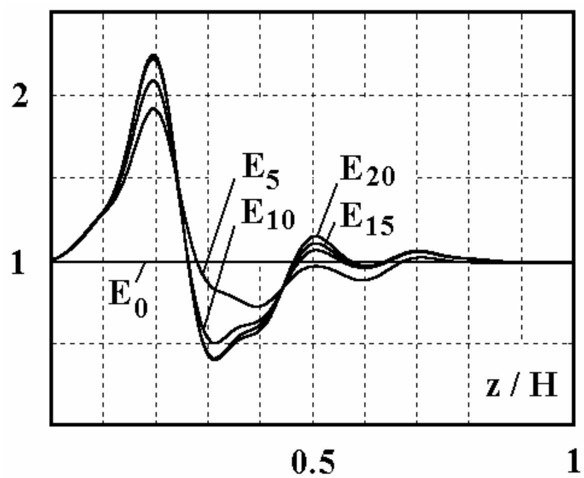


Figure 11 : Distribution $E(z)$ reconstructed using different number of iterations

the gradient method for each particular unknown are chosen on the base on three estimates of *D*, then the minimum of *D* is locally predicted using a second order polynomial approximation (under the restriction of a decreasing error). Then, a global linear predictor is used to upgrade all unknowns simultaneously. This damps out oscillations in convergence. If the total error remains nearly constant, then the step size is reselected separately using the local quadratic predictor as described above.

The term $1+e_0(z)$, in (51) in these examples is simulated by weighted first ten terms of Fourier expension with random coefficients.

The actual distribution $E^*(z)$ and the recovered distribution $E(z)$ are compared in Fig. 10. A few stages of the iterative procedure are represented in Fig. 11, to illustrate how the iterative process converges.

4 Conclusion

When you press on the surface of a human body with a mechanical probe, a specific mechanical reaction is determined by the elastic properties of the underlying tissues. The problem is to recover these properties using limited information on surface displacement. As we show in this paper, a number of mathematical approaches can be followed to formalize this problem, which lead to successful numerical techniques.

This paper presents the mathematical and numerical aspects of the problem when all discussed approaches are tested on the results of numerical experiments instead of real experimental data. This is done in order to highlight the basic features of our approach. The approaches discussed in sections 2 and 3 have been successfully used by Skovoroda and Aglimov (1997); Skovoroda and Aglimov (1998) to reconstruct the mechanical properties of a layered visco-elastic medium on the base of experimental measures of surface impedance effected with an harmonically vibrating cylindrical indenter [Timonim (1989)]. The quality of the results was encouraging. In particular, it was found that the differences between measured and predicted surface impedance components were not greater than the experimental noise.

The results from soft tissues coupled with the results of a comparison of experimentally measured and theoretically predicted displacement and strain fields from inhomogeneous tissue – like platforms [Skovoroda (1994)] lead to conclusion, that small displacement and strain of the object are sufficient for mechanical properties reconstruction and restricted by signal – to – noise ratio of experimental data only.

As is demonstrated here, a sound mathematical modeling gives a robust procedure for a quantitative estimation of elastic tissue parameters even with limited experimental information available.

The analytical-numerical method we developed seems promising for solving the identification problem of mechanical properties of stratified structural elements.

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Appendix 1. The solution of forward elastic problem for homogeneous layer

When the Lame coefficients are constant within the layer, i.e. when the layer is homogeneous, the solution of the direct elastic problem is [5]

$$
\begin{aligned}
U^i(r,z) &= \int_0^\infty \alpha J_1(\alpha r)(B^i + \alpha S^i + \alpha z R^i)\, d\alpha,\\
W^i(r,z) &= \int_0^\infty \alpha J_0(\alpha r)[2(1-2\nu_i)R^i - \alpha A^i - \alpha z B^i]\, d\alpha,\\
\sigma^i_{rr}(r,z) &= 2\mu_i \int_0^\infty \alpha^2 \{J_0(\alpha r)[(1+2\nu_i)B^i + \alpha S^i + \alpha z R^i]\\
&\quad - \frac{J_1(\alpha r)}{\alpha r}(B^i + \alpha S^i + \alpha z R^i)\}\, d\alpha,\\
\sigma^i_{\theta\theta}(r,z) &= 2\mu_i \int_0^\infty \alpha^2 \{2\nu_i J_0(\alpha r) B^i\\
&\quad + \frac{J_1(\alpha r)}{\alpha r}(B^i + \alpha S^i + \alpha z R^i)\}\, d\alpha,\\
\sigma^i_{zz}(r,z) &= 2\mu_i \int_0^\infty \alpha^2 J_0(\alpha r)[(1-2\nu_i)B^i - \alpha S^i - \alpha z R^i]\, d\alpha,\\
\sigma^i_{rz}(r,z) &= 2\mu_i \int_0^\infty \alpha^2 J_1(\alpha r)(2\nu_i R^i + \alpha A^i + \alpha z B^i)\, d\alpha,
\end{aligned}
\tag{A1.1}
$$

where $\nu = \gamma/2(\lambda+\mu)$ is Poisson's ratio, $J_{0,1}$ are Bessel's functions of the first kind of order 0,1 [4], and

$$
B = B_1 e^{\alpha z} + B_2 e^{-\alpha z}, \quad R = B_1 e^{\alpha z} - B_2 e^{-\alpha z},
$$
$$
A = A_1 e^{\alpha z} + A_2 e^{-\alpha z}, \quad S = A_1 e^{\alpha z} - A_2 e^{-\alpha z}.
$$

Note that the $4N$ parameters A^i_j and B^i_j, $j = 1, 2$, $i=1,\dots,N$ depend on α.

Appendix 2. The algorithm to solve the system of 4N linear equations, arising in the case of layered medium

To solve the system of $4N$ linear equations (20)-(25) a procedure based on direct and back sweeps [8] was advocated to be advantageous. Taking into account [8] we formalize the procedure as follows

$$A_j^i = \Phi_{j1}^i A_2^N + \Phi_{j2}^i B_2^N,$$
$$B_j^i = \Psi_{j1}^i A_2^N + \Psi_{j2}^i B_2^N, \ i = 1,\ldots,N, \ j = 1,2 \qquad \text{(A2.1)}$$

This procedure is simple, extremely fast and does not require any general $4N$x$4N$ matrix inversion. According to Eqs (A2.1), all unknowns are linear combinations of two of them. The coefficients Φ_{jk}^i and $\Psi_{jk}^i$of Eqs (A2.1) are specified below and, therefore, the general $4N$x$4N$ matrix inversion problem is reduced to 2x2 case.

To arrange this procedure several steps are needed. First, we immediately obtain for $i = N$ and $j = 2$

$$\Phi_{21}^N = 1, \Phi_{22}^N = 0, \Psi_{21}^N = 0, \Psi_{22}^N = 1. \qquad \text{(A2.2)}$$

Second, we could obtain from Eqs (21)

$$\Phi_{11}^N = -K(1+2\beta_N - 2n_N)/(1+2n_N),$$
$$\Phi_{12}^N = -K(\beta_N^2 + 4n_N)/\alpha(1+2n_N),$$
$$\Psi_{11}^N = 2\alpha K/(1+2n_N),$$
$$\Psi_{12}^N = -K(1-2\beta_N - 2n_N)/(1+2n_N), \qquad \text{(A2.3)}$$

where $K = (k_2^N)^2$. Third, Eqs (22)-(25) yield

$$A_1^{i-1} = \Big\{[m_i(1+2n_i)+1]A_1^i +$$
$$(m_i-1)[n_i(1+2\beta_i-2n_i)B_1^i/\alpha -$$
$$((1+2\beta_i-2n_i)A_2^i + (2\beta_i^2-2n_i^2+3n_i)B_2^i/\alpha)(k_2^i)^2]\Big\}$$
$$\div 4(1-\nu_{i-1}),$$
$$B_1^{i-1} = \Big\{(m_i+2n_i+1)B_1^i +$$
$$(m_i-1)[2\alpha A_2^i - (1-2\beta_i-2n_i)B_2^i)(k_2^i)^2]\Big\}/4(1-\nu_{i-1}),$$
$$A_2^{i-1} = \{[m_i(1+2n_i)+1]A_2^i -$$
$$(m_i-1)[n_i(1-2\beta_i-2n_i)B_2^i/\alpha +$$
$$((1-2\beta_i-2n_i)A_1^i - (2\beta_i^2-2n_i^2+3n_i)B_1^i/\alpha)(k_1^i)^2]\}$$
$$\div 4(1-\nu_{i-1}),$$
$$B_2^{i-1} = \{(m_i+2n_i+1)B_2^i -$$
$$(m_i-1)\ [2\alpha A_1^i + (1+2\beta_i-2n_i)B_1^i)(k_1^i)^2]\}/4(1-\nu_{i-1}) \qquad \text{(A2.4)}$$

Therefore, using (A2.1) and (A2.4) we conclude that

$$\Phi_{1j}^{i-1} = \Big\{[m_i(1+2n_i)+1]\Phi_{1j}^i +$$
$$(m_i-1)[n_i(1+2\beta_i-2n_i)\Psi_{1j}^i/\alpha -$$
$$((1+2\beta_i-2n_i)\Phi_{2j}^i + (2\beta_i^2-2n_i^2+3n_i)\Psi_{2j}^i/\alpha)(k_2^i)^2]\Big\}$$
$$\div 4(1-\nu_{i-1}),$$
$$\Psi_{1j}^{i-1} = \Big\{(m_i+2n_i+1)\Psi_{1j}^i +$$
$$(m_i-1)[2\alpha\Phi_{2j}^i - (1-2\beta_i-2n_i)\Psi_{2j}^i)(k_2^i)^2]\Big\}/4(1-\nu_{i-1}),$$
$$\Phi_{2j}^{i-1} = \Big\{[m_i(1+2n_i)+1]\Phi_{2j}^i -$$
$$(m_i-1)[n_i(1-2\beta_i-2n_i)\Psi_{2j}^i/\alpha +$$
$$((1-2\beta_i-2n_i)\Phi_{1j}^i - (2\beta_i^2-2n_i^2+3n_i)\Psi_{1j}^i/\alpha)(k_1^i)^2]\Big\}$$
$$\div 4(1-\nu_{i-1}),$$
$$\Psi_{2j}^{i-1} = \Big\{(m_i+2n_i+1)\Psi_{2j}^i -$$
$$(m_i-1)[2\alpha\Phi_{1j}^i + (1+2\beta_i-2n_i)\Psi_{1j}^i)(k_1^i)^2]\Big\}/4(1-\nu_{i-1}).$$
$$j = 1,2 \qquad \text{(A2.5)}$$

Finally, we obtain from Eqs (20) using Eqs (A2.1) for $i = 1$

$$A_2^N = -ps_{12}/\Delta, \quad B_2^N = ps_{11}/\Delta, \qquad \text{(A2.6)}$$

where

$$\Delta = s_{11}s_{22} - s_{21}s_{12},$$
$$s_{1j} = \alpha(\Phi_{1j}^1 + \Phi_{2j}^1) + (1-n_1)(\Psi_{1j}^1 - \Psi_{2j}^1),$$
$$s_{2j} = \alpha(\Phi_{2j}^1 - \Phi_{1j}^1) + n_1(\Psi_{1j}^1 + \Psi_{2j}^1),$$
$$j = 1,2.$$

Hence, the matrix inversion procedure is completed.