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Meshless Local Petrov-Galerkin (MLPG) Approaches for Solving the Weakly-Singular Traction & Displacement Boundary Integral Equations

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Abstract: The general Meshless Local Petrov-Galerkin (MLPG) type weak-forms of the displacement & traction boundary integral equations are presented, for solids undergoing small deformations. These MLPG weak forms provide the most general basis for the numerical solution of the non-hyper-singular displacement and traction BIEs [given in Han, and Atluri (2003)], which are simply derived by using the gradients of the displacements of the fundamental solutions [Okada, Rajiyah, and Atluri (1989a,b)]. By employing the various types of test functions, in the MLPG-type weak-forms of the non-hyper-singular dBIE and tBIE over the local sub-boundary surfaces, several types of MLPG/BIEs are formulated, while also using several types of non-element meshless interpolations for trial functions over the surface of the solid. Three specific types of MLPG/BIEs are formulated in the present study, according to three different types of test functions assumed over a local sub-boundary surface, as: a) the weight function in the MLS, for formulating the MLPG/BIE1; b) a Dirac delta function for formulating the collocation method (MLPG/BIE2); c) the trial function itself, for formulating the MLPG/BIE6. As a special case, the MLPG/BIE6 leads to symmetric systems of equations, and are presented in default in the present study. Numerical examples, presented in the accompanying part II of this paper, show that the present methods are very promising, especially for solving the elastic problems in which the singularities in displacemens, strains, and stresses, are of primary concern.

keyword: Meshless Local Petrov-Galerkin approach (MLPG), Boundary Integral Equations (BIE), Non-Hypersingular dBIE/tBIE, Moving Least Squares (MLS), Radial Basis Functions (RBF), MLPG/BIE.

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1 Introduction

Most problems in mechanics are characterized by partial differential equations, in space and time. The development of approximate methods for the solution of these PDEs has attracted the attention of engineers, physicists and mathematicians for several decades. In the beginning, the finite difference methods were extensively developed to solve these equations. As being derived from the variational principles, or their equivalent weak-forms, the finite element methods have emerged as the most successful methods to solve these partial differential equations, over the past three decades. Recently, the so-called meshless methods of discretization have become very attractive, as they are efficient for solving PDEs by avoiding the tedium of mesh-generation, especially for those problems having complicated geometries, as well as those involving large strains. As a systematic framework for developing various meshless methods, the Meshless Local Petrov-Galerkin (MLPG) approach has been proposed as a fundamentally new concept [Atluri and Zhu (1998); Atluri and Shen (2002a,b)]. The generality of the MLPG approach, based on the symmetric or unsymmetric weak-forms of the PDEs, and using a variety of interpolation methods (trial functions), test functions, and integration schemes with/without background cells, has been widely investigated [Atluri and Shen (2002a,b)]. The many research successes in solving PDEs, demonstrate that the MLPG method is one of the most promising alternative methods for computational mechanics.

On the other hand, the boundary integral equations (BIEs) have been developed for solving PDEs during the past 25 years. They are very efficient in certain applications, in comparison to the domain-solution methods, such as the finite element methods. The BIE methods become even more powerful, when several fast algorithms are combined, such as the panel-clustering method, multi-pole expansions, fast Fourier-transforms, wavelet methods, and so on. It is well known that the BIEs are derived from the unsymmetric- weak-forms of the governing PDEs, by using the fundamental solutions as test functions, and applied to solve linear elastic isotropic solid problems [Okada, Rajiyah, and Atluri (1990)], 3-D dynamic problems [Hatzigeorgiou, and Beskos (2002)], cracked plate problems [Wen, Aliabadi, and Young (2003), El-Zafrany (2001)], acoustic problems [Gaul, Fischer,and Nackenhorst (2003)], and biological systems [Muller-Karger, Gonzalez, Aliabadi and Cerrolaza] (2001)]. However, hyper-singularities are encountered, when the displacement BIEs are directly differentiated to obtain the traction BIEs, usually for solving crack problems. Much has been written about the subject of de-singularization [Richardson and Cruse (1996)]. In contrast, as far back as 1989, Okada, Rajiyah, and Atluri (1989a,b, 1990) have proposed a simple way to *directly derive* the integral equations for gradients of displacements, rather than first derive the dBIE, and then differentiate it, as is most common in the literature. It simplified the derivation processes by taking the gradients of the displacements of the foundation solutions as the test functions, while writing the balance laws in their weak- form. It resulted in "non-hyper-singular" boundary integrals of the gradients of displacements. It has been applied to solve the nonlinear problems successfully. Recently, this concept has been followed and extended for a *directly-derived* traction BIE [Han and Atluri (2003)], which is also "non-hyper-singular" [$1/r^2$], as opposed to being "hyper-singular" [$1/r^3$], as in the most common literature for tBIEs. Han and Atluri (2003) have also proposed a very straight-forward and simple manner to de-singularize the "non-hyper-singular" integrals, in order to render them numerically tractable, with only a weak singularity. This formulation has been successfully applied to solve the crack problems [Han and Atluri (2002)].

In this paper, we reuse the directly-derived non-hyper-singular displacement & traction BIEs as in Han and Atluri (2003), and write *their local weak-forms in the local sub-boundary surfaces, through the MLPG approach* [Atluri and Zhu (1998) and Atluri and Shen (2002a,b)]. From this general MLPG approach, various boundary solution methods can be easily derived by choosing, a) a variety of the meshless interpolation schemes for the trial functions, b) a variety of test functions over the local sub-boundary surfaces, and c) a variety of numerical integration schemes. The typical choices for the meshless interpolations for trial functions include: moving least squares (MLS); radial basis functions (RBF); partition of unity (PU); Shepard function; and so on. The choices for the test functions over each local sub-boundary surface are also many; these may include: the Direct delta function (leading to the "collocation method"); the weight functions in MLS; as well as the nodal trial functions themselves (leading to the "symmetric Galerkin method"). The integration schemes can be based on the collocation points, background cells (as in "element free methods"), or based on nodal influence domains, which lead to truly meshless methods. Such variants have been excellently summarized in Atluri and Shen (2002a,b) in the complementary formulation of the MLPG approach for domain- solutions. In this paper, we focus on developing the general MLPG/BIE approach, and demonstrate its variants as the collocation method (MLPG/BIE1); MLS interpolation with its weight function as the test function (MLPG/BIE2); and general interpolation with the nodal trial function as the nodal test function (MLPG/BIE6), similar to those studied in Atluri and Shen (2002a,b). It should be pointed out that MLPG/BIE methods here are not limited to those variants that are presented here; and many other special suitable combinations are feasible, according to the problems to be solved. Some such special cases have already been demonstrated in prior literature, as: the boundary node method (BNM) [Gowrishankar and Mukherjee (2002)], the hybrid boundary node method (HBNM) [Zhang and Yao (2001)], the boundary cloud method (BCM) [Li and Aluru (2003)] and so on.

The outline of the paper is as follows: Section 2 summarizes the non-hypersingular displacement and traction BIEs [Han and Atluri (2003)]; In Section 3, the local weak- forms of these dBIEs / tBIEs are written, using the general MLPG approach, and are further de-singularized from their non-hyper-singular forms; Section 4 describes several variants of the MLPG/BIE approaches, by using the moving least squares and radial basis functions as the interpolation functions for trial functions. Some conclusions are made in Section 5; by choosing, however, to discuss a variety of numerical results based on the MLPG/BIE approaches, in a companion second part of this paper.

2 Non-Hyper-singular Displacement and Traction BIEs

This section summarizes, for the sake of completeness, the non-hypersingular displacement and traction BIEs for a linear elastic, homogeneous, isotropic solid. They were proposed and discussed in detail in Han and Atluri (2003), by extending the original ideas for non-hyper-singular boundary integral equations for displacements and their gradients given in [Okada, Rajiyah, and Atluri (1989a,b), Okada, and Atluri (1994)].

Consider a linear elastic, homogeneous, isotropic body in a domain Ω, with boundary $\partial\Omega$. The Lame' constants of the linear elastic isotropic body are λ and μ; and the corresponding Young's modulus and Poisson's ratio are E and υ, respectively. We use Cartesian coordinates ξ_i, and the attendant base vectors $\mathbf{e}_i$, to describe the geometry in Ω. The solid is assumed to undergo infinitesimal deformations. The equations of balance of linear and angular momentum can be written as:

$$\nabla\cdot\sigma+\mathbf{f}=\mathbf{0};\quad \sigma=\sigma^{\mathrm{t}};\quad \nabla=\mathbf{e}_i\frac{\partial}{\partial\xi_i} \tag{1}$$

The strain-displacement relations are:

$$\varepsilon=\frac{1}{2}(\nabla\mathbf{u}+\mathbf{u}\nabla) \tag{2}$$

The constitutive relations of an isotropic linear elastic homogeneous solid are:

$$\sigma=\lambda\mathbf{I}\,(\nabla\cdot\mathbf{u})+2\mu\varepsilon \tag{3}$$

It is well known [Fung & Tong (2001)] that the displacement vector, which is a continuous function of ξ, can be derived, in general, from the Galerkin-vector-potential φ such that:

$$\mathbf{u}=\nabla^2\varphi-\frac{1}{2(1-\upsilon)}\nabla(\nabla\cdot\varphi) \tag{4}$$

Using (2), (3) and (4) into (1), it is easily found (in the absence of body force $\mathbf{f}$) that:

$$\nabla\cdot\sigma=\mu\nabla^2\nabla^2\varphi=\mathbf{0}\quad\text{or}\quad \nabla^2\nabla^2\varphi=\mathbf{0} \tag{5}$$

Consider a point unit load applied in an arbitrary direction $\mathbf{e}^p$ at a generic location $\mathbf{x}$ in a linear elastic isotropic homogeneous infinite medium as shown in Fig. 1. It is well-known [Fung and Tong (2001)] that the displacement solution corresponding to this unit point load is given by the Galerkin-vector-displacement-potential:

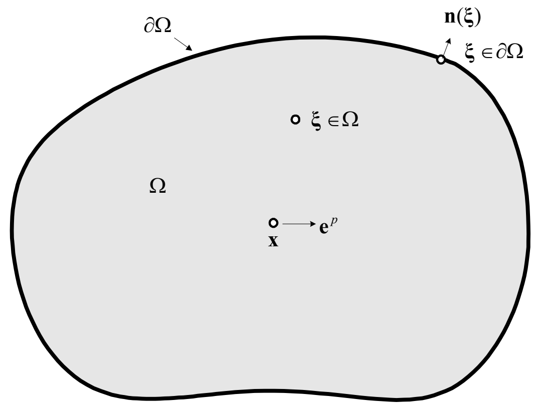

Figure 1 : A solution domain with source point **x** and target point ξ

$$\varphi^{*p}=(1-\upsilon)F^*\mathbf{e}^p \tag{6}$$

where

$$F^*=\frac{r}{8\pi\mu(1-\upsilon)}\quad\text{for 3D problems} \tag{7a}$$

and

$$F^*=\frac{-r^2\ln r}{8\pi\mu(1-\upsilon)}\quad\text{for 2D problems} \tag{7b}$$

where $r=\|\xi-\mathbf{x}\|$

Thus, φ^{*p} in (6) is the solution, in infinite space, to the differential equation (in the coordinates ξ),

$$\mu\nabla^2\nabla^2\varphi^{*p}+\delta(\mathbf{x},\xi)\mathbf{e}^p=\mathbf{0}; \tag{8}$$

$$\nabla\cdot\sigma(\mathbf{x},\xi)+\delta(\mathbf{x},\xi)\mathbf{e}^p=\mathbf{0}$$

or

$$\mu(1-\upsilon)\nabla^2\nabla^2F^*+\delta(\mathbf{x},\xi)=0 \tag{9}$$

The corresponding displacements are derived from the Galerkin-vector-displacement- potential, using (4)a, as:

$$u_i^{*p}(\mathbf{x},\xi)=(1-\upsilon)\delta_{pi}F^*_{,kk}-\frac{1}{2}F^*_{,pi} \tag{10}$$

The gradients of the displacements in (10) are:

$$u^{*p}_{i,j}(\mathbf{x},\xi)=(1-\upsilon)\delta_{pi}F^*_{,kkj}-\frac{1}{2}F^*_{,pij} \quad (11)$$

The corresponding stresses in a linear elastic homogenous isotropic body are given by:

$$\sigma^{*p}_{ij}(\mathbf{x},\xi)\equiv E_{ijkl}u^{*p}_{k,l}=\mu[(1-\upsilon)\delta_{pi}F^*_{,kkj}+\upsilon\delta_{ij}F^*_{,pkk}-F^*_{,pij}] + \mu(1-\upsilon)\delta_{pj}F^*_{,kki} \quad (12)$$

These stresses are seen to satisfy the balance laws:

$$\sigma^{*p}_{ij,i}(\mathbf{x},\xi)=\mu(1-\upsilon)\delta_{pj}F^*_{,kkii}=-\mu\delta_{pj}\delta(\mathbf{x},\xi);\quad \sigma^{*p}_{ij}=\sigma^{*p}_{ji} \quad (13)$$

We define two functions ϕ^{*p}_{ij} and ψ^{*p}_{ij}, as

$$\phi^{*p}_{ij}(\mathbf{x},\xi)\equiv-\mu(1-\upsilon)\delta_{pj}F^*_{,kki} \quad (14a)$$

$$\psi^{*p}_{ij}(\mathbf{x},\xi)\equiv\sigma^{*p}_{ij}(\mathbf{x},\xi)+\phi^{*p}_{ij}(\mathbf{x},\xi) =\mu[(1-\upsilon)\delta_{pi}F^*_{,kkj}+\upsilon\delta_{ij}F^*_{,pkk}-F^*_{,pij}] \quad (14b)$$

Then, from (13) and (14), it can be seen that:

$$\phi^{*p}_{ij,i}(\mathbf{x},\xi)=-\sigma^{*p}_{ij,i}(\mathbf{x},\xi)=-\mu(1-\upsilon)\delta_{pj}F^*_{,kkii} =\mu\delta_{pj}\delta(\mathbf{x},\xi) \quad (15a)$$

$$\psi^{*p}_{ij,i}(\mathbf{x},\xi)=\mathbf{0}\ \ [\text{divergence of } \psi^{*p}(\mathbf{x},\xi)=\mathbf{0}] \quad (15b)$$

Hence, as a divergence free tensor, $\psi^{*p}(\mathbf{x},\xi)$ must be a curl of another divergence free tensor. We choose to rewrite it in term of F^* from Eq. (14)b, as:

$$\psi^{*p}_{ij}(\mathbf{x},\xi)=\mu[(1-\upsilon)\delta_{pi}F^*_{,kkj}+\upsilon\delta_{ij}F^*_{,pkk}-F^*_{,pij}] \equiv e_{ist}G^{*p}_{tj,s} \quad (16)$$

where, by definition,

$$G^{*p}_{ij}(\mathbf{x},\xi)=\mu[(1-\upsilon)e_{ipj}F^*_{,kk}-e_{ikj}F^*_{,pk}] \quad (17)$$

We write the kernel functions for 3D problems, from Eqs.(10), (12), and (14)(a,b), as:

$$u^{*p}_{i}(\mathbf{x},\xi)=\frac{1}{16\pi\mu(1-\upsilon)r}[(3-4\upsilon)\delta_{ip}+r_{,i}r_{,p}] \quad (18)$$

$$G^{*p}_{ij}(\mathbf{x},\xi)=\frac{1}{8\pi(1-\upsilon)r}[(1-2\upsilon)e_{ipj}+e_{ikj}r_{,k}r_{,p}] \quad (19)$$

$$\sigma^{*p}_{ij}(\mathbf{x},\xi)=\frac{1}{8\pi(1-\upsilon)r^2} [(1-2\upsilon)(\delta_{ij}r_{,p}-\delta_{ip}r_{,j}-\delta_{jp}r_{,i})-3r_{,i}r_{,j}r_{,p}] \quad (20)$$

and for 2D plane strain problems, as:

$$u^{*p}_{i}(\mathbf{x},\xi)=\frac{1}{8\pi\mu(1-\upsilon)}[-(3-4\upsilon)\ln r\delta_{ip}+r_{,i}r_{,p}] \quad (21)$$

$$G^{*p}_{ij}(\mathbf{x},\xi)=\frac{1}{4\pi(1-\upsilon)}[-(1-2\upsilon)\ln r\, e_{ipj}+e_{ikj}r_{,k}r_{,p}] \quad (22)$$

$$\sigma^{*p}_{ij}(\mathbf{x},\xi)=\frac{1}{4\pi(1-\upsilon)r} [(1-2\upsilon)(\delta_{ij}r_{,p}-\delta_{ip}r_{,j}-\delta_{jp}r_{,i})-2r_{,i}r_{,j}r_{,p}] \quad (23)$$

Let u_i be the trial functions for displacements, to satisfy Eq. (1), without the body forces f_i (but include them later, when necessary), in terms of u_i, when Eqs. (2) and (3) are used. By taking the fundamental solution $u^{*p}_i(\mathbf{x},\xi)$ as the test functions, and with the consideration of its properties in Eq. (8), the weak-form of the equilibrium Eq. (1) can be written as,

$$u_p(\mathbf{x})=\int_{\partial\Omega}t_j(\xi)u^{*p}_j(\mathbf{x},\xi)\,dS -\int_{\partial\Omega}n_i(\xi)u_j(\xi)\sigma^{*p}_{ij}(\mathbf{x},\xi)\,dS \quad (24)$$

Instead of the ***scalar*** weak form of Eq. (1), as in Eq. (24), we may also write a ***vector*** weak form of Eq. (1), by using the tensor test functions $u^{*p}_{i,j}(\mathbf{x},\xi)$ in Eq. (11) [as originally proposed in Okada, Rajiyah, and Atluri (1988,1989)] as:

$$-u_{p,k}(\mathbf{x})=\int_{\partial\Omega}n_i(\xi)E_{ijmn}u_{m,n}(\xi)u^{*p}_{j,k}(\mathbf{x},\xi)\,dS -\int_{\partial\Omega}n_k(\xi)E_{ijmn}u_{m,n}(\xi)u^{*p}_{j,i}(\mathbf{x},\xi)\,dS +\int_{\partial\Omega}n_n(\xi)E_{ijmn}u_{m,k}(\xi)u^{*p}_{j,i}(\mathbf{x},\xi)\,dS \quad (25)$$

Eqs. (24) and (25) were originally given in [Okada, Rajiyah, and Atluri (1989a,b)], and the notion of using unsymmetric weak-forms of the differential equations, to obtain integral representations for displacements, was presented in [Atluri (1985)]. It should be noted that the integral equations for $u_p(\mathbf{x})$ and $u_{p,k}(\mathbf{x})$ as in Eqs. (24) and (25) are derived independently of each other. On the other hand, if we derive the integral equation for the displacement-gradients, by directly differentiating $u_p(\mathbf{x})$ in Eq. (24), a hyper-singularity is clearly introduced.

Eq. (24) is the original displacement BIE (dBIE) in its strongly-singular form before any regularization. On the other hand, Eq. (25) are the non-hypersingular ("strongly-singular" with "$1/r^2$" type singularities only for 3D problems) integral equations for displacement gradients in a homogeneous linear elastic solid, as originally derived in Okada, Rajiyah, and Atluri (1989a,b). It is but a simple extension to derive a non-hypersingular integral equation for tractions in a linear elastic solid, from Eq. (25),

$$\begin{aligned} -\sigma_{ab}(\mathbf{x}) = -E_{abpk}u_{p,k}(\mathbf{x}) = \int_{\partial\Omega} t_q(\xi)\sigma_{ab}^{*q}(\mathbf{x},\xi)\,dS \\ +\int_{\partial\Omega} D_p u_q(\xi) e_{nlp} E_{abkl}\sigma_{nq}^{*k}(\mathbf{x},\xi)\,dS \\ \equiv \int_{\partial\Omega} t_q(\xi)\sigma_{ab}^{*q}(\mathbf{x},\xi)\,dS + \int_{\partial\Omega} D_p u_q(\xi)\Sigma_{abpq}^{*}(\mathbf{x},\xi)\,dS \end{aligned} \quad (26)$$

where the surface tangential operator D_t is defined as,

$$D_t = n_r e_{rst}\frac{\partial}{\partial\xi_s} \quad (27)$$

and by definition,

$$\Sigma_{ijpq}^{*}(\mathbf{x},\xi) = E_{ijkl}e_{nlp}\sigma_{nq}^{*k}(\mathbf{x},\xi) \quad (28)$$

Contracting Eq. (26) with $n_a(\mathbf{x})$, we have

$$\begin{aligned} -t_b(\mathbf{x}) = n_a(\mathbf{x})\sigma_{ab}^{(}\mathbf{x}) = \int_{\partial\Omega} t_q(\xi) n_a(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\xi)\,dS \\ +\int_{\partial\Omega} D_p u_q(\xi) n_a(\mathbf{x})\Sigma_{abpq}^{*}(\mathbf{x},\xi)\,dS \end{aligned} \quad (29)$$

Eqs. (24) and (29) can be directly used together, for solving mixed boundary value problems, including fracture mechanics. They can also be used to derive the symmetric Galerkin boundary element method (SGBEM) after de-singularization [Han and Atluri (2002)]. In the present paper, *the local weak-forms of the dBIE and tBIE (viz., Eqs 24, and 29) themseleves, are written, by using the most general Meshless Local Petrov- Galerkin (MLPG) approaches [Atluri and Shen (2002 a,b), in the following sections.*

3 The MLPG approach

The MLPG approach was first proposed by Atluri and Zhu (1998) for solving linear potential problems, by using domain discretization techniques. The MLPG approach uses either a local symmetric weak form, or an unsymmetric weak form of the governing equation *over the local sub domain, and such local domains may overlap each other*. The generality of the MLPG, and its variants, are comprehensively investigated in Atluri and Shen (2002a,b). In this section, we apply the general MLPG approach to solving the displacement and traction BIEs, as in their non-hyper-singluar strong forms, given in Eqs. (24) and (29). To make the current formulation fully general, we focus on the derivation of the weak-forms themselves, by ignoring for the moment, the details of the interpolation (trial) functions, the test functions, and the integration schemes; while leaving these details to be discussed in the next section.

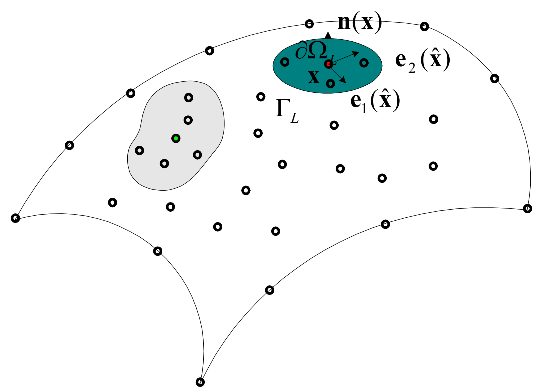

Figure 2 : a local sub boundary around point **x**

Consider a local sub-boundary surface $\partial\Omega_L$, *with its boundary contour* Γ_L, as *a part of the whole boundary-surface, as shown in Fig. 2, for a 3-D solid*. Eqs. (24) and (29) may be satisfied in weak-*forms* over the sub-boundary surface $\partial\Omega_L$, by using a Local Petrov-Galerkin scheme, as:

$$\begin{aligned} \int_{\partial\Omega_L} w_p(\mathbf{x})u_p(\mathbf{x})dS_x = \int_{\partial\Omega_L} w_p(\mathbf{x})dS_x \int_{\partial\Omega} t_j(\xi)u_j^{*p}(\mathbf{x},\xi)\,dS \\ -\int_{\partial\Omega_L} w_p(\mathbf{x})dS_x \int_{\partial\Omega} n_i(\xi)u_j(\xi)\sigma_{ij}^{*p}(\mathbf{x},\xi)\,dS \end{aligned} \quad (30a)$$

$$-\int_{\partial\Omega_L} w_b(\mathbf{x})t_b(\mathbf{x})dS_x = \int_{\partial\Omega_L} w_b(\mathbf{x})dS_x \int_{\partial\Omega} t_q(\xi)n_a(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\xi)\,dS_\xi + \int_{\partial\Omega_L} w_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\xi)n_a(\mathbf{x})\Sigma_{abpq}^{*}(\mathbf{x},\xi)\,dS_\xi \tag{30b}$$

where $w_b(\mathbf{x})$ is a test function. *If $w_b(\mathbf{x})$ is chosen as a Dirac delta function, i.e. $w_b(\mathbf{x}) = \delta(\mathbf{x},\mathbf{x}_m)$ at $\partial\Omega_L$, we obtain the standard "collocation" method for displacement and traction BIEs, at the collocation point $\mathbf{x}_m$.* Some basic identities of the fundamental solution in Eq. (6) may be derived from its weak-form as [Han and Atluri (2003)],

$$\int_\Omega [\nabla\cdot\boldsymbol{\sigma}^{*p}(\mathbf{x},\xi)+\delta(\mathbf{x},\xi)\mathbf{e}^p]\cdot\mathbf{u}(\mathbf{x})d\Omega = \mathbf{0} \tag{31a}$$

$$\int_{\partial\Omega} n_p(\xi)\sigma_{pq}(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\xi)dS + \int_{\partial\Omega} D_p u_q(\mathbf{x})\Sigma_{abpq}^{*}(\mathbf{x},\xi)\,dS + \sigma_{ab}(\mathbf{x}) = 0 \tag{31b}$$

One may obtain the fully desingularized dBIE and tBIE for the standard "collocation" methods, by making the use of these identities, as,

$$0 = \int_{\partial\Omega} t_j(\xi)u_j^{*p}(\mathbf{x},\xi)\,dS - \int_{\partial\Omega} n_i(\xi)[u_j(\xi)-u_j(\mathbf{x})]\sigma_{ij}^{*p}(\mathbf{x},\xi)\,dS \tag{32a}$$

$$0 = \int_{\partial\Omega} [t_q(\xi)-n_p(\xi)\sigma_{pq}(\mathbf{x})]n_a(\mathbf{x})\sigma_{ab}^{*q}(\mathbf{x},\xi)\,dS + \int_{\partial\Omega} [D_p u_q(\xi)-(D_p u_q)(\mathbf{x})]n_a(\mathbf{x})\Sigma_{abpq}^{*}(\mathbf{x},\xi)\,dS \tag{32b}$$

If $w_b(\mathbf{x})$ is chosen such that it is continuous over the local sub boundary-surface $\partial\Omega_L$ and zero at the contour Γ_L, one may apply Stokes' theorem to Eq. (30), and re-write it as:

$$\frac{1}{2}\int_{\partial\Omega_L} w_p(\mathbf{x})u_p(\mathbf{x})dS_x = \int_{\partial\Omega_L} w_p(\mathbf{x})dS_x \int_{\partial\Omega} t_j(\xi)u_j^{*p}(\mathbf{x},\xi)\,dS_\xi + \int_{\partial\Omega_L} w_p(\mathbf{x})dS_x \int_{\partial\Omega} D_i(\xi)u_j(\xi)G_{ij}^{*p}(\mathbf{x},\xi)\,dS_\xi + \int_{\partial\Omega_L} w_p(\mathbf{x})dS_x \int_{\partial\Omega}^{CPV} n_i(\xi)u_j(\xi)\phi_{ij}^{*p}(\mathbf{x},\xi)\,dS_\xi \tag{33a}$$

$$-\frac{1}{2}\int_{\partial\Omega_L} t_b(\mathbf{x})w_b(\mathbf{x})dS_x = \int_{\partial\Omega_L} D_a w_b(\mathbf{x})dS_x \int_{\partial\Omega} t_q(\xi)G_{ab}^{*q}(\mathbf{x},\xi)\,dS_\xi - \int_{\partial\Omega} t_q(\xi)\,dS_\xi \int_{\partial\Omega_L}^{CPV} n_a(\mathbf{x})w_b(\mathbf{x})\phi_{ab}^{*q}(\mathbf{x},\xi)dS_x + \int_{\partial\Omega_L} D_a w_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\xi)H_{abpq}^{*}(\mathbf{x},\xi)\,dS_\xi \tag{33b}$$

where G_{ab}^{*q} is defined in Eq.(17); ϕ_{ab}^{*q} is defined in Eq. (14)a. After some algebra, similar to the one used for obtaining the G_{ab}^{*q} term [σ_{ab}^{*q} in Eq. (14)], the quantity H_{abpq}^{*} is defined for Σ_{abpq}^{*}, as detailed in Han and Atluri (2003), as,

$$H_{ijpq}^{*}(\mathbf{x},\xi) = \mu^2[-\delta_{ij}F_{,pq}+2\delta_{ip}F_{,jq} + 2\delta_{jq}F_{,ip}-\delta_{pq}F_{,ij}-2\delta_{ip}\delta_{jq}F_{,bb} + 2\upsilon\delta_{iq}\delta_{jp}F_{,bb}+(1-\upsilon)\delta_{ij}\delta_{pq}F_{,bb})] \tag{34}$$

With Eqs. (7), we can write H_{ijpq}^{*} for 3D problems as:

$$H_{ijpq}^{*}(\mathbf{x},\xi) = \frac{\mu}{8\pi(1-\upsilon)r}[4\upsilon\delta_{iq}\delta_{jp}-\delta_{ip}\delta_{jq}-2\upsilon\delta_{ij}\delta_{pq} + \delta_{ij}r_{,p}r_{,q}+\delta_{pq}r_{,i}r_{,j}-2\delta_{ip}r_{,j}r_{,q}-\delta_{jq}r_{,i}r_{,p}] \tag{35a}$$

and for 2D plain strain problems as:

$$H_{ijpq}^{*}(\mathbf{x},\xi) = \frac{\mu}{4\pi(1-\upsilon)}[-4\upsilon\ln r\delta_{iq}\delta_{jp}+\ln r\delta_{ip}\delta_{jq}+2\upsilon\ln r\delta_{ij}\delta_{pq} + \delta_{ij}r_{,p}r_{,q}+\delta_{pq}r_{,i}r_{,j}-2\delta_{ip}r_{,j}r_{,q}-\delta_{jq}r_{,i}r_{,p}] \tag{35b}$$

If the test function $w_b(\mathbf{x})$ is chosen to be identical to a function that is energy-conjugate to u_p (for dBIE) and t_b (for tBIE), namely, the nodal trial function $\hat{t}_p(\mathbf{x})$ and $\hat{u}_b(\mathbf{x})$, respectively, we obtain the local weak-forms of the symmetric Galerkin dBIE and tBIE, as:

$$\frac{1}{2}\int_{\partial\Omega_L} \hat{t}_p(\mathbf{x})u_p(\mathbf{x})dS_x = \int_{\partial\Omega_L} \hat{t}_p(\mathbf{x})dS_x \int_{\partial\Omega} t_j(\xi)u_j^{*p}(\mathbf{x},\xi)\,dS_\xi + \int_{\partial\Omega_L} \hat{t}_p(\mathbf{x})dS_x \int_{\partial\Omega} D_i(\xi)u_j(\xi)G_{ij}^{*p}(\mathbf{x},\xi)\,dS_\xi + \int_{\partial\Omega_L} \hat{t}_p(\mathbf{x})dS_x \int_{\partial\Omega}^{CPV} n_i(\xi)u_j(\xi)\phi_{ij}^{*p}(\mathbf{x},\xi)\,dS_\xi \tag{36a}$$

$$
\begin{aligned}
&-\frac{1}{2}\int_{\partial\Omega_L} t_b(\mathbf{x})\hat{u}_b(\mathbf{x})dS_x \\
&\quad=\int_{\partial\Omega_L} D_a\hat{u}_b(\mathbf{x})dS_x\int_{\partial\Omega} t_q(\xi)G^{*q}_{ab}(\mathbf{x},\xi)\, dS_\xi \\
&\qquad-\int_{\partial\Omega} t_q(\xi)\, dS_\xi \int^{CPV}_{\partial\Omega_L} n_a(\mathbf{x})\hat{u}_b(\mathbf{x})\phi^{*q}_{ab}(\mathbf{x},\xi)dS_x \\
&\qquad+\int_{\partial\Omega_L} D_a\hat{u}_b(\mathbf{x})dS_x\int_{\partial\Omega} D_p u_q(\xi)H^{*}_{abpq}(\mathbf{x},\xi)\, dS_\xi
\end{aligned}
\tag{36b}
$$

4 Variants of the MLPG/BIE: Several Types of Interpolation (Trial) and Test functions, and Integration schemes

The general MLPG/BIE approaches are given through Eqs. (32), (33) and (36). For a so-called meshless implementation, we need to introduce a meshless interpolation scheme to approximate the trial functions over the surface of a three-dimensional body. It is very common to adopt the moving least squares (MLS) interpolation scheme for interpolating the trial functions over a 3-D surface, as it has been done successfully in meshless domain methods [Atluri and Zhu (1998); Atluri and Shen (2002 a,b)]. Unfortunately, the moment matrix in the MLS interpolation sometimes becomes singular, when it is applied to three-dimensional surface approximation, if Cartesian coordinates are used. An alternative choice is to use the curvilinear coordinates [Chati, Mukherjee and Paulino (2001)]. Also, a varying polynomial basis has been used, instead of the complete basis [Li and Aluru (2003)]. However, these authors reported that worse results are sometimes obtained. As another interpolation scheme, radial basis functions (RBF) are becoming more and more attractive, in meshless methods. RBF was first introduced to BEM by Nardini and Brebbia (1982) for the domain interpolation for the Dual Reciprocity Method and used for the direct solution [Tsai, Young and Cheng (2002)]. Recently, RBFs are also used for the boundary value interpolations due to their simplicity. However, they are not as easy to pass the patch test as in other interpolation methods, because of their lack of completeness. A modified RBF is proposed by introducing some polynomial basis for solving such problems. In this section, we summarize the MLS and RBF approaches and their applications in MLPG/BIEs.

Consider a sub-boundary surface $\partial\Omega_x$, the neighborhood of a point $\mathbf{x}$, which is local in the global boundary $\partial\Omega$. To approximate the distribution of function u in $\partial\Omega_x$, over a number of scattered points $\{\mathbf{x}_I\}$, $(I = 1,2,...,n)$, the moving least squares approximate $u(\mathbf{x})$ of u, $\forall\mathbf{x} \in \partial\Omega_x$, can be defined by

$$u(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})\mathbf{a}(\mathbf{x})\ \forall\mathbf{x} \in \partial\Omega_x \tag{37}$$

where $\mathbf{P}^T(\mathbf{x}) = [P_1(\mathbf{x}), P_2(\mathbf{x}), \ldots, P_m(\mathbf{x})]$ is a monomial basis of order m; and $\mathbf{a}(\mathbf{x})$ is a vector containing coefficients, which are functions of the Cartersian coordinates $[x_1,x_2,x_3]$ or the surface curvilinear coordinates $[s_1,s_2]$, depending on the monomial basis. They are determined by minimizing a weighted discrete L_2 norm, defined, as:

$$J(\mathbf{x}) = \sum_{i=1}^{m} w_i(\mathbf{x})[\mathbf{P}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) - \hat{u}_i]^2 \tag{38}$$

where $w_i(\mathbf{x})$ are weight functions and $\hat{u}_i$are the fictitious nodal values.

The stationarity of in Eq. (38) with respect to $\mathbf{a}(\mathbf{x})$ leads to following linear relation between $\mathbf{a}(\mathbf{x})$ and $\hat{\mathbf{u}}$,

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\hat{\mathbf{u}} \tag{39}$$

where matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are defined by

$$A(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})\mathbf{w}(\mathbf{x})\mathbf{P}(\mathbf{x}) \quad \mathbf{B}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})\mathbf{w}(\mathbf{x}) \tag{40}$$

The MLS approximation is well defined only when the matrix A in Eq. (40) is non-singular. It needs to be reconditioned if the monomial basis is defined in the Cartersian coordinates for 3D surface approximation.

Substituting Eq. (39) into Eq. (37), one obtains

$$u(\mathbf{x}) = \mathbf{\Phi}^T(\mathbf{x})\hat{\mathbf{u}} \quad \forall\mathbf{x} \in \partial\Omega_x \tag{41}$$

where $\mathbf{\Phi}(\mathbf{x})$ is usually called the shape function of the MLS approximation, defined as,

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \tag{42}$$

The weight function in Eq. (38) defines the range of influence of node I. Normally it has a compact support. The possible choices are the Gaussian and spline weight functions with compact supports, as given in Eqs. (43) and (44), respectively.

$$w(\mathbf{x}) = \begin{cases} \dfrac{\exp[-(\frac{d_I}{r_I})^{2k}] - \exp[-(\frac{r_I}{c_I})^{2k}]}{1-\exp[-(\frac{r_I}{c_I})^{2k}]} & 0 \le d_I \le r_I \\ 0 & d_I > r_I \end{cases} \tag{43}$$

$$w(\mathbf{x})=\begin{cases}1-6(\frac{d_I}{r_I})^2+8(\frac{d_I}{r_I})^3-3(\frac{d_I}{r_I})^4 & 0\le d_I\le r_I\\ 0 & d_I>r_I\end{cases} \quad (44)$$

where $d_I = |\mathbf{x}-\mathbf{x}_I|$ is the distance or the length of the geodesic on boundary between $\mathbf{x}$ and $\mathbf{x}_I$, depending to the basis functions; r_I and c_I are constants and define the shape and the range of influence of node I.

It should be pointed out that the shape functions given in Eq. (42) are based on the fictitious nodal values. This introduces an additional complication, since all the nodal values in BIEs are the direct boundary values, a situation which is totally different from the domain meshless methods. As a practical way, a conversion matrix may be used to map the fictitious values to true values.

As an alternative meshless interpolation, RBFs may interpolate the distribution of function u in $\partial\Omega_x$, over a number of scattered points $\{\mathbf{x}_I\}$, $(I = 1,2,...,n)$, as,

$$u(\mathbf{x}) = \mathbf{R}^T(\mathbf{x})\mathbf{a}\ \forall\mathbf{x}\in\partial\Omega_x \quad (45)$$

where $\mathbf{R}(\mathbf{x})$ is a set of radial basis functions centered around the scattered points, and $\mathbf{a}$ is a constant vector containing the coefficients. To improve the completeness of the interpolation functions, monomial basis functions $\mathbf{P}(\mathbf{x})$, as used in MLS, are introduced in Eq. (45) as the "augmented" functions. One may rewrite the modified RBF interpolation, from Eq. (45), as,

$$u(\mathbf{x}) = \mathbf{R}^T(\mathbf{x})\mathbf{a}+\mathbf{P}^T(\mathbf{x})\mathbf{b} \quad \forall\mathbf{x}\in\partial\Omega_x \quad (46)$$

where $\mathbf{b}$ is a constant vector containing the coefficients, instead of a function vector as in MLS. It should be pointed out that these "augmented" functions may also introduce the non-unique interpolation if $\mathbf{P}(\mathbf{x})$ is defined on the Cartersian coordinates.

To determine the coefficients $\mathbf{a}$ and $\mathbf{b}$, one may enforce the interpolation to satisfy the given values at the scattered points as,

$$u(\mathbf{x}_I) = \mathbf{R}^T(\mathbf{x}_I)\mathbf{a}+\mathbf{P}^T(\mathbf{x}_I)\mathbf{b} \quad (47a)$$

and

$$\sum_{I=1}^{n} P_j(\mathbf{x}_I)a_I \quad j = 1,2,\dots,m \quad (47b)$$

By substituting the solution of Eq. (47) into Eq. (46), one obtains

$$u(\mathbf{x}) = \mathbf{\Phi}^T(\mathbf{x})\mathbf{u} \quad \forall\mathbf{x}\in\partial\Omega_x \quad (48)$$

where $\mathbf{\Phi}(\mathbf{x})$ is the shape function of the RBF approximation. It has a very important feature for imposing the given boundary values, that it has the delta function properties, i. e., $\Phi_J(\mathbf{x}_I) = \delta_{IJ}$.

Although there is no limit on the choice of the radial basis functions in Eq.(46), a practical one may be the positive definiteness, in order to ensure the unique solution of Eq. (47). In addition, it is better to have their compact support, which reduces the range of the influence of a node. Thus, banded equations are obtained in Eq. (47), which makes it more efficient to determine the constant coefficients in Eq. (46), especially when data sets are larger. As another advantage, it may allow the integration to be performed based on the nodal influence domains, instead on the background cells, which makes the resultant MLPG/BIE method to be a truly meshless method.

Some compactly supported positive definite radial basis functions with C^2 continuities are listed from [Wendland (1995)] as,

$$R(\mathbf{x}) = \begin{cases}(1-d_I/r_I)^3(3d_I/r_I+1) & 0\le d_I\le r_I\\ 0 & d_I>r_I\end{cases} \quad (49)$$

$$R(\mathbf{x}) = \begin{cases}(1-d_I/r_I)^4(4d_I/r_I+1) & 0\le d_I\le r_I\\ 0 & d_I>r_I\end{cases} \quad (50)$$

Other choices of RBFs are the Gaussians and splines with compact supports, as the weight functions used in MLS.

Once the interpolation functions are determined over the whole global boundary, the MLPG/BIE in its collocation format, in Eq. (32), can be directly used to develop the meshless BIEs. As a good example of this particular approach, the boundary node method (BNM) [Chati, Mukherjee, and Paulino (2001)] has been developed by using MLS approximation, *in which the hyper-singular BIEs were used and the back-ground cells were used to perform the integrals over the whole boundary*.

Table 1 : Variants of MLPG/BIE

MLPG/BIE weak-form	Interpolation	Test function	Integration
MLPG/BIE1 : The fully desingularized weak forms, in Eq. (33)	A) Moving Least Square (MLS) with various weight function	Piece-wise continuous function	a) element mesh
MLPG/BIE2 : The standard collocation methods, in Eq. (32)	B) Radial basis functions with polynomials	Dirac delta function	b) background cell
MLPG/BIE6 : The fully desingularized weak forms, in Eq. (36)	C) any other interpolation for scattered points	Nodal shape function	c) nodal influence domain

Furthermore, various procedures for discretization may also be applied for evaluating the boundary integrals. In the traditional BEMs, the boundary integration is performed over "elements". Without loss of generality, one may write it in natural coordinates η, as,

$$\int_{\partial\Omega} f(\mathbf{x},\xi)\,dS = \sum_{i=1}^{NE} \int_{Elem_i} f(\mathbf{x},\eta)\,J(\eta)d\eta \tag{51}$$

where NE is the number of the elements and J is the determinant of Jacobi matrix.

Considering a meshless approximation, one may define the non-overlapping cells for the whole boundary as,

$$\partial\Omega = \bigcup_{i=1}^{NC} Cell_i \quad \text{and } Cell_i \cap Cell_j = 0 \text{ for } \forall i \neq j \tag{52}$$

where NC is the number of cells. Thus, one may obtain the boundary integrals, in the meshless form, as,

$$\int_{\partial\Omega} f(\mathbf{x},\xi)\,dS = \sum_{i=1}^{NC} \int_{Cell_i} f(\mathbf{x},\mathbf{s})\,d\mathbf{s} \tag{53}$$

in which the curvilinear coordinates $\mathbf{s}$ are used.

In addition, if the variables are interpolated from the nodal ones as in Eqs. (41) and (48) and the range of influence of each node is compact, one set of overlapped nodal domains may be defined as,

$$\partial\Omega = \bigcup_{i=1}^{NN} DOI_i \quad \text{and } DOI_i \cap DOI_j \neq 0 \text{ for } \exists i \neq j \tag{54}$$

where NN is the number of nodes and DOI_i is the influence domain of node i. *The boundary integrals can be re-written in the truly meshless form, as,*

$$\int_{\partial\Omega} f(\mathbf{x},\xi)\,dS = \sum_{i=1}^{NN} \int_{DOI_i} f(\mathbf{x},\mathbf{s})\,d\mathbf{s} \tag{55}$$

This integration scheme has been applied to BNM for solving potential problems [Gowrishankar and Mukherjee (2002)].

Besides the traditional collocation methods, the weakly-singular BIEs in Eq. (33) can also be easily used for developing the meshless BIEs in their numerically tractable weak-forms. All piece-wise continuous functions can be used here as the test functions. For example, all the weight functions in MLS approximation, and the radial basis functions discussed above, are suitable for such purposes. *In addition, one may also directly use the nodal shape function as the nodal test functions as in Eqs. (41) and (48), which leads to the symmetric BIEs in Eq. (36). In summary, the several variants, of the general MLPG/BIE approach, are defined through a suitable combination of the trial-function interpolation scheme, the choice for the test functions, and the choices for the integration scheme, as listed in table 1.*

5 Closure

In this paper, we presented the "Meshless Local Petrov-Galerkin BIE Methods" (MLPG/BIE), by using the concept of the general meshless local Petrov-Galerkin (MLPG) approach developed in Atluri et al [1998, 2002a,b]. The several variants of the MLPG/BIE solution methods are also formulated, in terms of the varieties of the interpolation schemes for trial functions, the test functions, and the integration schemes. With the use of a nodal influence domain, truly meshless BIEs are also presented. Although the formulations in the paper are presented for elastostatic problems, they can be easily developed for solving other partial differential equations, including those in potential problems, fluid mechanics, acoustics, electromagnetic problems, etc.

Numerical results, using several variants of the MLPG/BIEs as developed here, are presented in a companion paper.

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