 CMES, vol.5, no.1, pp.35-43, 2004

Micropolar Theory and Its Applications to Mesoscopic and Microscopic Problems

Youping Chen[1], James D Lee[2] and Azim Eskandarian[1]

Abstract: This paper addresses the need of theories and simulations for material body of mesoscopic and microscopic sizes. An overview of polar theories is presented. The micropolar theory proposed by Eringen is introduced and compared with other polar theories. Constitutive equations of micropolar thermo-visco-elastic solid are derived. Finite element analyses have been performed for a few sample problems with wide range of length scales. Based on the discussion, comparison and computer simulations, the unique feature and applicability of micropolar theory are demonstrated.

keyword: continuum, micropolar, finite element, length scale.

1 Introduction

The entire physical science is based on two fundamental physical models: (1) discrete atomic models, (2) continuum field theories. Those two models provide foundations for all physical theories and the success of both models has been demonstrated and tested throughout the history of science in explaining and predicting diverse physical phenomena. There is no doubt that quantum mechanics, molecular dynamics and lattice dynamics are fundamental to science. On the other hand, the resort to the continuum model is often justified on the basis of expediency and practicality.

In the enormous gap between two physical models resides a very rich physical world that is extremely relevant to science and technology. Hence, significant efforts have been made to try to reach this region from either the atomic or the continuum end. Molecular dynamics simulation with several hundreds to several millions of atoms has become an extremely popular tool to explore the nanotechnology and to reveal the hidden mechanisms and correlations underlying the macroscopic behavior. It is estimated that one billion atoms can be included in a calculation with the current most advanced parallel computing technology; this would allow a specimen size up to four microns on the time scale of picoseconds. Obviously, this is still too far to reach a real time simulation for a specimen of finite size.

The approach to this intermediate region from the continuum end begins with polar theories. Those theories supply just the necessary apparatus to treat various problems relevant to materials with microstructure and micromotion. The aim of this paper is to illustrate why polar theories are complimentary, and when used properly they bear fruits that cannot be harvested by means of either atomic model or classic continuum model alone.

2 An Overview of Polar Theories

The beginning of the rational theories of polar continua goes back to E. and F. Cosserat (1909) in 1909, who, in a remarkable memoir, developed a theory of elasticity with the concept of "triedre" by means of a variational principle. The Cosserats obtained equations for the balance of momenta in the dynamic case. But they did not give a specific microinertia, nor did a conservation law for the microinertia tensor, which are crucial to the construction of constitutive equations and dynamical problems. The rigid directors used by the Cosserats and later by others to represent rigid rotations did not have metrical significance. Consequently, difficulties are encountered in the treatment of material symmetry regulations in constitutive equations. Gunther (1958) and Schaefer (1967) recapitulated the Cosserats elasticity and remarked on its connection to dislocations. In that theory, the anti-symmetric part of the stress tensor is redundant and it remains undetermined.

Extensive developments, however, have taken place dur-

[1] Center for Intelligent System Research,
School of Engineering and Applied Science,
The George Washington University,
Ashburn, Virginia 20147, USA
[2] Department of Mechanical and Aerospace Engineering,
School of Engineering and Applied Science,
The George Washington University,
Washington, DC 2005

ing the past decades. Among the various contributors to this field it is worthwhile to mention Truesdell and Toupin (1960)], Mindlin and Tiersten (1962), Toupin (1962), and Eringen (1962). These early theories are mostly known as the "constrained" continuum theories. Later Mindlin (1964) constructed a theory of elasticity with microstructure, Green and Rivilin (1964) a general multipolar theory. These theories, as well as "constrained" continuum theories, have close contact with Eringen's polar (micromorphic and micropolar) theory (Eringen and Suhubi (1964), Eringen (1965, 1967), Lee (1971), Lee and Eringen (1971a, 1971b, 1974)). In this polar theory, the material body is envisioned as a collection of a large number of deformable particles (point particles), each was endowed with three directors. A material point is then equipped with the independent degrees of freedom for rigid rotations (micropolar) or for both stretches and rotations (micromorphic), in addition to the classical translation degrees of freedom of the center. A new conservation law of microinertia, which was missing from all other polar theories, was derived in both the micropolar and micromorphic theories. The evolution of the inertia tensor with motion determines the anisotropic character of the body at any time. Moreover, the phase transitions are the results of the change of the microinertia tensor with temperature. Without the balance law of the microinertia tensor, basic field equations are incomplete and the evolution of the constitution of the body with motion cannot be determined.

As the recent application of metals and polymers at the micron scale are multiplying, several plasticity theories for application in micron region have been proposed. The theory invoking strengthening by the Laplacians of effective strain has proved effective in dealing with plastic instabilities at dislocation patterning or shear localization (Aifantis (1984), Zbib and Afantis (1989), Polizzotto and Borino (1998), Shizawa and Zbib (1999)). A variety of size dependent phenomena, including torsion of thin wires, growth of microvoids, indentation tests, et al., have been interpreted in terms of stretching gradient effect or rotation gradient effect or both by a strain gradient plasticity (Fleck and Hutchinson (1997)). This theory involves all components of the strain gradient tensor and work conjugate higher-order stresses in the form of couple stresses and double stresses, and falls within Toupin's (1962) couple stress theory. Shizawa and Zbib (1999) developed a gradient theory for thermo-elastic-plasticity, in which the concept of dislocation density tensor was introduced and thermodynamically consistent constitutive equations for plastic stretch, plastic spin and back stress were derived. For numerical implementation of strain gradient theories it is worthwhile to mention the recent works by Tenek and Aifantis (2002), Chen, Huang and Wittmann (2002), and Tang, Shen and Atluri (2003).

The basic objective of those recently developed gradient theories is the extension of continuum mechanics to the micron range for applications to phenomena involving size effect. They are close to polar theory. However, the essential difference between Eringen's polar theory and those newly developed strain gradient theories is obvious: the micromotion in Eringen's theory is not treated as the gradient of the macro-strain.

2.1 The Framework of Polar Theories (micromorphic and micropolar)

Eringen's microcontinuum field theories constitute extensions of the classical field theories to microscopic space and time scales. In terms of a physical picture, a material body is envisioned as a continuous collection of a large number of deformable particles whose motions can be expressed as

$$x_k = x_k(X,t) \quad , \tag{1}$$

$$\xi_k = \chi_{kK}(X,t)\Xi_K \tag{2}$$

It is seen that the macromotion, eqn. (1), accounts for the motion of the centroid of the particle while the micromotion, eqn. (2), specifies the changing of orientation and the deformation of the inner structures of the particle. The inverse motions can be written as

$$X_K = X_K(\mathbf{x},t) \ , \ \Xi_K = \overline{\chi}_{Kk}(\mathbf{x},t)\xi_k, \tag{3}$$

with

$$\chi_{kK}\,\overline{\chi}_{Kl} = \delta_{kl} \ , \ \overline{\chi}_{Kk}\,\chi_{kL} = \delta_{KL} \quad . \tag{4}$$

The balance laws of micromorphic continuum were obtained by Eringen and Suhubi (1964) and Eringen (1964,1992) as follows:

$$\dot{\rho} + \rho v_{k,k} = 0 \quad , \tag{5}$$

$$\frac{d}{dt} i_{kl} = i_{km}\omega_{lm} + i_{lm}\omega_{km} \quad , \tag{6}$$

$$t_{kl,k} + \rho f_l = \rho \dot{v}_l \quad , \tag{7}$$

$$m_{klm,k} + t_{ml} - s_{ml} + \rho l_{lm} = \sigma_{lm} \quad , \tag{8}$$

$$\rho \dot{e} = t_{kl} v_{l,k} + (s_{kl} - t_{kl})\omega_{lk} + m_{klm}\omega_{lm,k} - q_{k,k} + \rho h \quad , \tag{9}$$

$$- \rho(\dot{\psi} + \eta\dot{\theta}) + t_{kl} v_{l,k} + (s_{kl} - t_{kl})\omega_{lk} + m_{klm}\omega_{lm,k} - q_k \theta_{,k}/\theta \geq 0 \quad , \tag{10}$$

where the Helmholtz free energy is defined as $\psi \equiv e - \eta\theta$; and ρ, ***i, f, l, v,*** θ, ***q***, η, e are mass density, microinertia, body force, body moment, velocity, absolute temperature, heat flux, entropy, internal energy, respectively; ***t, s, m*** may be named as the stress, micro-stress average, and moment stress, respectively. The microgyation tensor $\boldsymbol{\omega}$ and the spin inertia $\boldsymbol{\sigma}$ are defined as

$$\omega_{kl} \equiv \dot{\chi}_{kK} \overline{\chi}_{Kl} \quad , \tag{11}$$

$$\sigma_{kl} = i_{ml}(\dot{\omega}_{km} + \omega_{kn}\omega_{nm}) \quad . \tag{12}$$

Notice that, from eqn. (6), the microinertia tensor ***i*** is not constant; together with the microgyation tensor and its time rate, it forms the spin inertia $\boldsymbol{\sigma}$ that plays a vital role in the dynamic equations for the balance of moment of momentum, eqn. (8). It should also be pointed out that in micromorphic (micropolar) theory the micromotion (microrotation) is independent of the macromotion. Without this property, one cannot address the change of orientations of long chain molecules in liquid crystals (Lee (1971), Lee and Eringen (1971a, 1971b, 1974).

2.2 Comparison with other Polar Theories

Actually, Eringen's polar theory can be reduced to Mindlin's microstructure theory (1964) based on *small strain* and *slow motion assumptions*. Identifying the micro-displacement u'_j in Mindlin's theory as $\xi_j - \delta_{jK}\Xi_K$ in Eringen's theory, then, by approximating the inverse micro-motion $\overline{\chi}_{Kk}$ (eqn. (3)) to be the shifter δ_{Kk} based upon *small strain assumption*, one has

$$u'_j \approx (\chi_{jK} - \delta_{jK})\delta_{Kk}\xi_k = (\chi_{jk} - \delta_{jk})\xi_k \equiv \psi_{kj}\xi_k \; , \tag{13}$$

$$1/3\,\rho' d^2_{kl}(\text{Mindlin's theory}) \Leftrightarrow \rho i_{kl}(\text{Eringen's theory}) . \tag{14}$$

However, in Mindlin's theory, $\rho' d^2_{kl}$ are treated as constants, and, as a consequence, the spin inertia σ_{kl} in eqn. (12) is reduced to $\rho' d^2_{mk}\ddot{\psi}_{ml}/3$ (Mindlin's eqn. (4.1-2), based on the *slow motion assumption*. Further, there is a special case in Mindlin's theory, referred as micro-homogeneous assumption: *the distinction between micro- and macro- motions is removed* by making the constraint $\psi_{ij} = u_{j,i}$. Then the so-called relative stress σ_{ij} in Mindlin's theory becomes indeterminate (cf. eqn. (3.4-2)). To eliminate the indeterminate relative stress from the balance laws and to *neglect the body couple and inertia terms*, Mindlin's eqn. (4.1-1) and (4.1-2) can be combined as

$$-\mu_{ijk,ij} + \tau_{ik,i} + f_k = 0 \quad , \tag{15}$$

which can be seen in many strain gradient theories (Fleck and Hutchinson (1997)). Now it is obvious that the micromorphic theory and its special case, micropolar theory, stem from more elaborate considerations of microstructures and micro-motions.

3 Micropolar Theory

Micropolar theory is a special case of micromorphic theory with the assumption that the micromotion is a microrotation only, i.e.,

$$\overline{\chi}_{Kk} = \chi_{kK} \quad , \tag{16}$$

which implies

$$\omega_{kl} = \dot{\chi}_{kK}\chi_{lK} = -\chi_{kK}\dot{\chi}_{lK} = -\omega_{lk} \quad . \tag{17}$$

The anti-symmetry of the microgyration tensor leads to the definition of micro-angular-velocity

$$\omega_k = -\varepsilon_{kij}\omega_{ij}/2 \quad , \tag{18}$$

where $\boldsymbol{\varepsilon}$ is the permutation tensor. Also, eqn. (4) and eqn. (16) leads to the Rodrignes formula

$$\chi_{KL} = \cos\phi\,\delta_{KL} - \sin\phi\,\varepsilon_{KLM} n_M + (1 - \cos\phi) n_K n_L \quad , \tag{19}$$

where $\chi_{KL} \equiv \delta_{kK}\chi_{kL}$, δ_{kK} is the shifter tensor, ϕ_K is the microrotation vector, $\phi \equiv \sqrt{\phi_K\phi_K}$ and $n_K \equiv \phi_K/\phi$. For small rotation, eqn. (19) is reduced to

$$\chi_{KL} \cong \delta_{KL} - \varepsilon_{KLM}\phi_M \quad . \tag{20}$$

The balance laws for microinertia, moment of momentum, and energy may be rewritten as

$$\frac{d j_{kl}}{dt} + (\varepsilon_{kmn} j_{lm} + \varepsilon_{lmn} j_{km})\omega_n = 0 \quad , \tag{21}$$

$$m_{kl,k}+\varepsilon_{lmn}t_{mn}+\rho l_l=\rho\sigma_l \quad , \tag{22}$$

$$\rho\dot{e}=t_{kl}(v_{l,k}+\varepsilon_{lkm}\omega_m)+m_{kl}\omega_{l,k}-q_{k,k}+\rho h\,, \tag{23}$$

where

$$j_{kl}\equiv i_{mm}\delta_{kl}-i_{kl} \quad , \tag{24}$$

$$l_k\equiv-\varepsilon_{kij}\,l_{ij} \quad , \tag{25}$$

$$\sigma_k\equiv-\varepsilon_{kij}\,\sigma_{ij}=j_{kl}\dot{\omega}_l+\varepsilon_{kmn}j_{nl}\omega_m\omega_l \quad , \tag{26}$$

$$m_{kl}\equiv-\varepsilon_{lij}\,m_{kij} \quad . \tag{27}$$

Define the generalized Lagrangian strain tensors and the generalized Piola-Kirchhoff Stress tensors as follows:

$$\begin{aligned}E_{KL}&\equiv x_{k,K}\chi_{kL}-\delta_{KL}\\ \Gamma_{KL}&\equiv-\varepsilon_{LMN}\chi_{kK}\chi_{kM,N}/2 \quad ,\end{aligned} \tag{28}$$

$$\begin{aligned}T_{KL}&\equiv\tfrac{\rho^o}{\rho}t_{kl}X_{K,k}\chi_{lL}\\ M_{KL}&\equiv\tfrac{\rho^o}{\rho}m_{kl}X_{K,k}\chi_{lL} \quad ,\end{aligned} \tag{29}$$

then the Clausius-Duhem inequality, eqn. (10), can be rewritten as

$$-\rho^o(\dot{\psi}+\eta\dot{\theta})+T_{KL}\dot{E}_{KL}+M_{KL}\dot{\Gamma}_{KL}-Q_K\theta_{,K}/\theta\geq0, \tag{30}$$

where $Q_K\equiv\frac{\rho^o}{\rho}q_kX_{K,k}$.

To derive the constitutive equations for micropolar thermo-visco-elasticsolid, following the axiom of equipresence, let ψ, η, $\mathbf{Q}$, $\mathbf{T}$, and $\mathbf{M}$ be functions of θ, $\dot{\theta}$, $\nabla\theta$, $\mathbf{E}$, $\dot{\mathbf{E}}$, $\mathbf{\Gamma}$, and $\dot{\mathbf{\Gamma}}$. Then the Clausius-Duhem inequality leads to

$$\begin{aligned}&-\rho^o\{\frac{\partial\psi}{\partial\theta}\dot{\theta}+\frac{\partial\psi}{\partial\dot{\theta}}\ddot{\theta}+\frac{\partial\psi}{\partial\nabla\theta}\cdot\nabla\dot{\theta}+\frac{\partial\psi}{\partial\mathbf{E}}:\dot{\mathbf{E}}\\ &+\frac{\partial\psi}{\partial\dot{\mathbf{E}}}:\ddot{\mathbf{E}}+\frac{\partial\psi}{\partial\mathbf{\Gamma}}:\dot{\mathbf{\Gamma}}+\frac{\partial\psi}{\partial\dot{\mathbf{\Gamma}}}:\ddot{\mathbf{\Gamma}}+\eta\dot{\theta}\} \quad .\\ &+\mathbf{T}:\dot{\mathbf{E}}+\mathbf{M}:\dot{\mathbf{\Gamma}}-\frac{\mathbf{Q}\cdot\nabla\theta}{\theta}\geq0\end{aligned} \tag{31}$$

Because this inequality is linear in $\ddot{\theta}$, $\nabla\dot{\theta}$, $\ddot{\mathbf{E}}$, $\ddot{\mathbf{\Gamma}}$, it can only be valid if

$$\psi=\psi(\theta,\,\mathbf{E},\,\mathbf{\Gamma}) \quad , \tag{32}$$

$$\begin{aligned}&-\rho^o(\frac{\partial\psi}{\partial\theta}+\eta)\dot{\theta}+(\mathbf{T}-\rho^o\frac{\partial\psi}{\partial\mathbf{E}}):\dot{\mathbf{E}}\\ &+(\mathbf{M}-\rho^o\frac{\partial\psi}{\partial\mathbf{\Gamma}}):\dot{\mathbf{\Gamma}}-\frac{\mathbf{Q}\cdot\nabla\theta}{\theta}\geq0 \quad .\end{aligned} \tag{33}$$

We may now decompose the entropy and the generalized Piola-Kirchhoff Stresses into two parts: the elastic (reversible) part and the dissipative (irreversible) part as follows.

$$\eta=\eta^e+\eta^d\equiv-\frac{\partial\psi}{\partial\theta}+\eta^d(\theta,\dot{\theta},\nabla\theta,\mathbf{E},\dot{\mathbf{E}},\mathbf{\Gamma},\dot{\mathbf{\Gamma}}), \tag{34}$$

$$\mathbf{T}=\mathbf{T}^e+\mathbf{T}^d\equiv\rho^o\frac{\partial\psi}{\partial\mathbf{E}}+\mathbf{T}^d(\theta,\dot{\theta},\nabla\theta,\mathbf{E},\dot{\mathbf{E}},\mathbf{\Gamma},\dot{\mathbf{\Gamma}}). \tag{35}$$

$$\mathbf{M}=\mathbf{M}^e+\mathbf{M}^d\equiv\rho^o\frac{\partial\psi}{\partial\mathbf{\Gamma}}+\mathbf{M}^d(\theta,\dot{\theta},\nabla\theta,\mathbf{E},\dot{\mathbf{E}},\mathbf{\Gamma},\dot{\mathbf{\Gamma}})\ , \tag{36}$$

and then

$$-\rho^o\eta^d\dot{\theta}+\mathbf{T}^d:\dot{\mathbf{E}}+\mathbf{M}^d:\dot{\mathbf{\Gamma}}-\mathbf{Q}\cdot\nabla\theta/\theta\geq0 \quad . \tag{37}$$

It is seen that $\{\eta^d,\dot{\theta}\}$, $\{\mathbf{Q},\nabla\theta\}$, $\{\mathbf{T}^d,\dot{\mathbf{E}}\}$, and $\{\mathbf{M}^d,\dot{\mathbf{\Gamma}}\}$ are four pairs of thermodynamic conjugates in micropolar theory. From now on we assume that the material considered is orthotropic. Also, it is noticed that, under the Lagrangian coordinate transformation

$$X_K^*=S_{KL}X_L \quad , \tag{38}$$

$\mathbf{M}$ and $\mathbf{\Gamma}$are transformed as

$$\begin{aligned}M_{KL}^*&=\det(\mathbf{S})S_{KM}S_{LN}M_{MN}\\ \Gamma_{KL}^*&=\det(\mathbf{S})S_{KM}S_{LN}\Gamma_{MN} \quad .\end{aligned} \tag{39}$$

To derive linear constitutive equations, let ψ , η^d , $\mathbf{T}^d$, $\mathbf{M}^d$, $\mathbf{Q}$ be expanded as polynomials of their arguments as follows.

$$\begin{aligned}\rho^o\psi&=\Sigma^o-\rho^o\eta^oT-\frac{\gamma T^2}{2T^o}-\alpha_{KL}TE_{KL}\\ &+\tfrac{1}{2}A_{KLMN}E_{KL}E_{MN}+\tfrac{1}{2}B_{KLMN}\Gamma_{KL}\Gamma_{MN} \quad ,\end{aligned} \tag{40}$$

$$\eta^d=-(a^1\dot{T}+a^3_{KL}\dot{E}_{KL})/\rho^oT^o, \tag{41}$$

$$Q_K=-b^2_{KL}T_{,L}-b^4_{KLM}\dot{\Gamma}_{LM} \quad , \tag{42}$$

$$T^d_{KL}=c^1_{KL}\dot{T}+c^3_{KLMN}\dot{E}_{MN} \quad , \tag{43}$$

$$M^d_{KL}=d^2_{KLM}T_{,M}+d^4_{KLMN}\dot{\Gamma}_{MN} \quad , \tag{44}$$

where $T\equiv\theta-T^o$, $T^o>0$, $|T|<T^o$; T^o is the reference temperature and T is the temperature variation. The elastic parts of the entropy and the stresses are obtained as

$$\eta^e=\eta^o+\frac{\gamma T}{\rho^oT^o}+\frac{\alpha_{KL}E_{KL}}{\rho^o} \quad , \tag{45}$$

$$T^e_{KL} = -\alpha_{KL}T + A_{KLMN}E_{MN} \quad , \tag{46}$$

$$M^e_{KL} = B_{KLMN}\Gamma_{MN} \quad . \tag{47}$$

The Clausius-Duhem inequality now requires that, for all $\dot{T}$, $\nabla\theta$, $\dot{\mathbf{E}}$ and $\dot{\mathbf{\Gamma}}$,

$$\begin{aligned} &\frac{a^1}{T^o}(\dot{T})^2 + c^3_{KLMN}\dot{E}_{KL}\dot{E}_{MN} + d^4_{KLMN}\dot{\Gamma}_{KL}\dot{\Gamma}_{MN} \\ &+ \frac{b^2_{KL}}{\theta}T_{,K}T_{,L} + (\frac{a^3_{KL}}{T^o} + c^1_{KL})\dot{T}\dot{E}_{KL} \\ &+ (d^2_{KLM} + \frac{b^4_{MKL}}{\theta})\dot{\Gamma}_{KL}T_{,M} \geq 0 \quad . \end{aligned} \tag{48}$$

4 Finite Element Formulation

The law of conservation of energy can now be expressed as

$$\begin{aligned} &(T+T^o)\{\frac{\gamma}{T^o}\dot{T} + \alpha_{KL}\dot{E}_{KL} \\ &-(a^1\ddot{T} + a^3_{KL}\ddot{E}_{KL})/T^o\} + Q_{K,K} \\ &= \Phi + \rho^o h \quad , \end{aligned} \tag{49}$$

where

$$\begin{aligned} \Phi \equiv\ &\frac{a^1}{T^o}(\dot{T})^2 + c^3_{KLMN}\dot{E}_{KL}\dot{E}_{MN} \\ &+ d^4_{KLMN}\dot{\Gamma}_{KL}\dot{\Gamma}_{MN} + (\frac{a^3_{KL}}{T^o} + c^1_{KL})\dot{T}\dot{E}_{KL} \\ &+ d^2_{KLM}\dot{\Gamma}_{KL}T_{,M} \quad . \end{aligned} \tag{50}$$

If the temperature variation is small, i.e., $|T| \ll T^o$, then eqn. (49) is further reduced to

$$\begin{aligned} &-a^1\ddot{T} - a^3_{KL}\ddot{E}_{KL} + \gamma\dot{T} + T^o\alpha_{KL}\dot{E}_{KL} + Q_{K,K} \\ &= \Phi + \rho^o h \quad . \end{aligned} \tag{51}$$

Multiply eqn. (51) by δT, integrate over the volume, use the Green-Gauss theorem, and then one obtains

$$\begin{aligned} &\int_V \{-a^1\ddot{T} - a^3_{KL}\ddot{E}_{KL} + \gamma\dot{T} + T^o\alpha_{KL}\dot{E}_{KL}\}\delta T\, dV \\ &\qquad - \int_V Q_K \delta T_{,K} dV \\ &= \int_V (\Phi + \rho^o h)\delta T\, dV - \int_{S_q} q^*\delta T\, dS \quad , \end{aligned} \tag{52}$$

where $Q_K N_K$ is specified to be q^* on S_q; N_K is the outward normal of the surface in question.

From eqn. (7) and eqn. (22),

$$\begin{aligned} &\int_V \{[t_{kl,k} + \rho(f_l - \dot{v}_l)]\delta v_l \\ &+ [m_{kl,k} + \varepsilon_{lmn}t_{mn} + \rho(l_l - \sigma_l)]\delta\omega_l\}dV \\ &= 0 \quad , \end{aligned} \tag{53}$$

and it leads to

$$\begin{aligned} &\int_V \{\rho^o(\dot{v}_M\delta v_M + \sigma_M\delta\omega_M) + T_{KL}\delta\dot{E}_{KL} + M_{KL}\delta\dot{\Gamma}_{KL}\}dV \\ &= \int_V \rho^o(f_M\delta v_M + l_M\delta\omega_M)dV + \int_{S_t} t^*_M \delta v_M dS + \\ &\quad \int_{S_m} m^*_M \delta\omega_M dS \quad , \end{aligned} \tag{54}$$

where $T_{KL}\chi_{ML}N_K$ is specified to be t^*_M on S_t; $M_{KL}\chi_{ML}N_K$ is specified to be m^*_M on S_m; $\{v_M, \omega_M, f_M, \sigma_M\} = \delta_{mM}\{v_m, \omega_m, f_m, \sigma_m\}$.

In finite element formulation, the temperature T, displacements U_K, microrotation ϕ_K at a generic point within an element can be linked to the corresponding nodal values through the shape functions, i.e.,

$$\begin{aligned} T &= N_\alpha T_\alpha \\ U_K &= N_{K\alpha}U_\alpha \quad . \\ \phi_K &= N_{K\alpha}\phi_\alpha \end{aligned} \tag{55}$$

The temperature gradient and the strains can be expressed as

$$\begin{aligned} &T_{,K} = N_{\alpha,K}T_\alpha \stackrel{\Delta}{=} C_{K\alpha}T_\alpha \\ &E_{KL} \cong U_{L,K} + \varepsilon_{LKM}\phi_M \\ &\stackrel{\Delta}{=} B_{LK\alpha}U_\alpha + \varepsilon_{LKM}N_{M\alpha}\phi_\alpha \quad . \\ &\Gamma_{KL} \cong \phi_{L,K} \stackrel{\Delta}{=} B_{LK\alpha}\phi_\alpha \end{aligned} \tag{56}$$

After lengthy but straightforward derivation, the following dynamic finite element equations are obtained

$$\begin{aligned} &\begin{vmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} \\ \mathbf{M}^{31} & \mathbf{M}^{32} & \mathbf{M}^{33} \end{vmatrix} \begin{vmatrix} \ddot{\mathbf{u}} \\ \ddot{\phi} \\ \ddot{\mathbf{T}} \end{vmatrix} + \begin{vmatrix} \mathbf{C}^{11} & \mathbf{C}^{12} & \mathbf{C}^{13} \\ \mathbf{C}^{21} & \mathbf{C}^{22} & \mathbf{C}^{23} \\ \mathbf{C}^{31} & \mathbf{C}^{32} & \mathbf{C}^{33} \end{vmatrix} \begin{vmatrix} \dot{\mathbf{u}} \\ \dot{\phi} \\ \dot{\mathbf{T}} \end{vmatrix} \\ &+ \begin{vmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} \\ 0 & 0 & \mathbf{K}^{33} \end{vmatrix} \begin{vmatrix} \mathbf{u} \\ \phi \\ \mathbf{T} \end{vmatrix} = \begin{vmatrix} \mathbf{F} \\ \mathbf{L} \\ \mathbf{Q} \end{vmatrix} \quad , \end{aligned} \tag{57}$$

where

$$M^{11}_{\alpha\beta} = \int \rho^o N_{K\alpha} N_{K\beta} dV = M^{11}_{\beta\alpha} \quad , \tag{58}$$

$$M^{22}_{\alpha\beta} = \int \rho^o j_{KL} N_{K\alpha} N_{K\beta} dV = M^{22}_{\beta\alpha} \, , \tag{59}$$

$$M^{33}_{\alpha\beta} = \int -a^1 N_\alpha N_\beta dV = M^{33}_{\beta\alpha} \quad , \tag{60}$$

$$M^{31}_{\alpha\beta} = \int -a^3_{KL} B_{LK\beta} N_\alpha dV \quad , \tag{61}$$

$$M^{32}_{\alpha\beta} = \int -a^3_{KL} \varepsilon_{LKP} N_{P\beta} N_\alpha dV, \tag{62}$$

$$K^{11}_{\alpha\beta} = \int A_{KLMN} B_{LK\alpha} B_{NM\beta} dV = K^{11}_{\beta\alpha} \quad , \tag{63}$$

$$K^{22}_{\alpha\beta} = \int (B_{KLMN} B_{LK\alpha} B_{NM\beta} + A_{KLMN} \varepsilon_{LKP} \varepsilon_{NMQ} N_{P\alpha} N_{Q\beta}) dV = K^{22}_{\beta\alpha} \, , \tag{64}$$

$$K^{33}_{\alpha\beta} = \int b^2_{KL} C_{L\beta} C_{K\alpha} dV = K^{33}_{\beta\alpha} \quad , \tag{65}$$

$$K^{12}_{\alpha\beta} = \int A_{KLMN} \varepsilon_{NMP} N_{P\beta} B_{LK\alpha} dV = K^{21}_{\beta\alpha} \quad , \tag{66}$$

$$K^{21}_{\alpha\beta} = \int A_{KLMN} B_{NM\beta} \varepsilon_{LKP} N_{P\alpha} dV = K^{12}_{\beta\alpha}, \tag{67}$$

$$K^{13}_{\alpha\beta} = \int -\alpha_{KL} N_\beta B_{LK\alpha} dV \quad , \tag{68}$$

$$K^{23}_{\alpha\beta} = \int (-\alpha_{KL} \varepsilon_{LKP} N_{P\alpha} N_\beta + d^2_{KLM} C_{M\beta} B_{LK\alpha}) dV \quad , \tag{69}$$

$$C^{11}_{\alpha\beta} = \int c^3_{KLMN} B_{LK\alpha} B_{NM\beta} dV = C^{11}_{\alpha\beta} \quad , \tag{70}$$

$$C^{22}_{\alpha\beta} = \int (d^4_{KLMN} B_{NM\beta} B_{LK\alpha} + c^3_{KLMN} \varepsilon_{LKP} \varepsilon_{NMQ} N_{P\alpha} N_{Q\beta}) dV = C^{22}_{\beta\alpha} \quad , \tag{71}$$

$$C^{33}_{\alpha\beta} = \int \gamma N_\alpha N_\beta dV = C^{33}_{\beta\alpha} \quad , \tag{72}$$

$$C^{12}_{\alpha\beta} = \int c^3_{KLMN} B_{LK\alpha} \varepsilon_{NMP} N_{P\beta} dV = C^{21}_{\beta\alpha} \quad , \tag{73}$$

$$C^{21}_{\alpha\beta} = \int c^3_{KLMN} B_{NM\beta} \varepsilon_{LKP} N_{P\alpha} dV = C^{12}_{\beta\alpha} \quad , \tag{74}$$

$$C^{13}_{\alpha\beta} = \int c^1_{KL} B_{LK\alpha} N_\beta dV \quad , \tag{75}$$

$$C^{31}_{\alpha\beta} = \int T^o \alpha_{KL} B_{LK\beta} N_\alpha dV \quad , \tag{76}$$

$$C^{23}_{\alpha\beta} = \int c^1_{KL} \varepsilon_{LKP} N_{P\alpha} N_\beta dV \quad , \tag{77}$$

$$C^{32}_{\alpha\beta} = \int (T^o \alpha_{KL} \varepsilon_{LKP} N_{P\beta} N_\alpha + b^4_{KLM} C_{K\alpha} B_{ML\beta}) dV \quad , \tag{78}$$

$$\mathbf{F}_\alpha = \int_V \rho^o f_K N_{K\alpha} dV + \int_{S_t} T^*_K N_{K\alpha} dS, \tag{79}$$

$$\mathbf{L}_\alpha = \int_V \rho^o l_K N_{K\alpha} dV + \int_{S_m} M^*_K N_{K\alpha} dS \quad , \tag{80}$$

$$\mathbf{Q}_\alpha = \int_V (\rho^o h + \Phi) N_\alpha dV - \int_{S_q} q^* N_\alpha dS \quad , \tag{81}$$

and, in eqn. (57), $\mathbf{u}$, ϕ, $\mathbf{T}$ are the displacements, microrotations, and temperatures at the nodes. We emphasize that the dissipations Φ, defined in eqn. (50), which are nonlinear terms due to the thermodynamics second law, are included in this formulation. Also, it is noticed that the inclusion of $\ddot{\mathbf{T}}$ in eqn. (57) is the consequence of the temperature-rate dependency incorporated in the constitutive theory. On the other hand, if temperature-rate dependency is excluded, then

$$a^1 = a^3_{KL} = c^1_{KL} = 0, \tag{82}$$

and eqn. (57) is reduced to

$$\begin{vmatrix} \mathbf{M}^{11} & 0 & 0 \\ 0 & \mathbf{M}^{22} & 0 \\ 0 & 0 & \mathbf{C}^{33} \end{vmatrix} \begin{vmatrix} \ddot{\mathbf{u}} \\ \ddot{\phi} \\ \dot{\mathbf{T}} \end{vmatrix} + \begin{vmatrix} \mathbf{C}^{11} & \mathbf{C}^{12} \\ \mathbf{C}^{12} & \mathbf{C}^{22} \\ \mathbf{C}^{13} & \mathbf{C}^{23} \end{vmatrix} \begin{vmatrix} \dot{\mathbf{u}} \\ \dot{\phi} \end{vmatrix} + \begin{vmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}^{33} \end{vmatrix} \begin{vmatrix} \mathbf{u} \\ \phi \\ \mathbf{T} \end{vmatrix} = \begin{vmatrix} \mathbf{F} \\ \mathbf{L} \\ \mathbf{Q} \end{vmatrix} \quad . \tag{83}$$

If the viscous effect is further excluded, then eqn. (83) can be rewritten as

$$\mathbf{C}^{33} \dot{\mathbf{T}} + \mathbf{K}^{33} \mathbf{T} = \mathbf{Q} \quad , \tag{84}$$

$$\begin{vmatrix} \mathbf{M}^{11} & 0 \\ 0 & \mathbf{M}^{22} \end{vmatrix} \begin{vmatrix} \ddot{\mathbf{u}} \\ \ddot{\phi} \end{vmatrix} + \begin{vmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{vmatrix} \begin{vmatrix} \mathbf{u} \\ \phi \end{vmatrix} = \begin{vmatrix} \mathbf{F} - \mathbf{K}^{13} \mathbf{T} \\ \mathbf{L} - \mathbf{K}^{23} \mathbf{T} \end{vmatrix}, \tag{85}$$

and **Q** in eqn. (84) is reduced to

$$\mathbf{Q}_\alpha = \int_V \rho^o h N_\alpha \, dV - \int_{S_q} q^* N_\alpha \, dS \quad , \qquad (86)$$

which means there is no dissipation; the temperature field is not influenced by the mechanical variables; the temperature, which causes thermal stresses, will influence the mechanical field as forcing terms.

A finite element computer program, named POLAR, was developed based on the general constitutive equations of micropolar thermo-visco-elastic solid.

5 Numerical Results

Case 1: crack problem

For illustrative purpose, consider a specimen of rectangular shape be subjected to an applied normal stress ($t_{yy} = \bar{t}$), perpendicular to a centered line crack along the x-axis, at the boundary, as shown in Fig.1. This mode I fracture problem is analyzed by using POLAR. The normalized stress, $t_{yy}/\bar{t}$, at the center of the element closest to the crack tip are calculated. Specimens of different length scales, keeping the ratios of the length-width-thickness, the finite element mesh, the material properties and the applied stress unchanged, are analyzed and the results are shown in Fig. 2. Because of the stress singularity at the crack tip, the stresses at the center of the element closest to the crack tip will increase if the mesh near the crack tip is refined. However, in this work, we keep the finite element mesh unchanged and we investigate the polar effect as the length scale changes. It is clearly seen that the classical elasticity predicts that the normalized stress of specimens at different length scales (such as meter, micron, and nanometer) remains to be constant. On the other hand, the micropolar theory clearly indicates the existence of the size effect as the stress at the crack tip decreases as the length scale of the specimen decreases; when the length goes up to larger scales, the effect of polarity would diminish. This result is in qualitative consistent with the experimental observations on cellular material (Lakes (1995) and molecular dynamics simulations of alloy system (Dang and Grujicic (1996), deCelis, Argon and Yip (1983)).

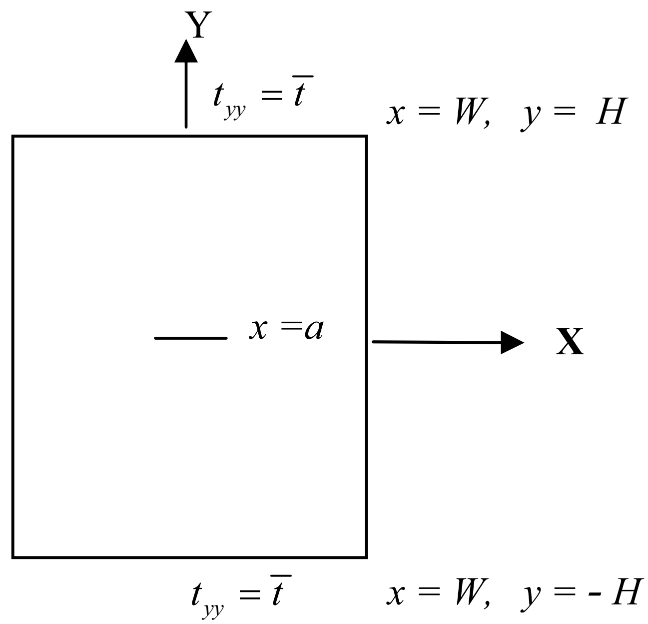

Figure 1 : Center-cracked specimen subjected to mode-I tensile stress (W/H/a =1/1.5/0.415)

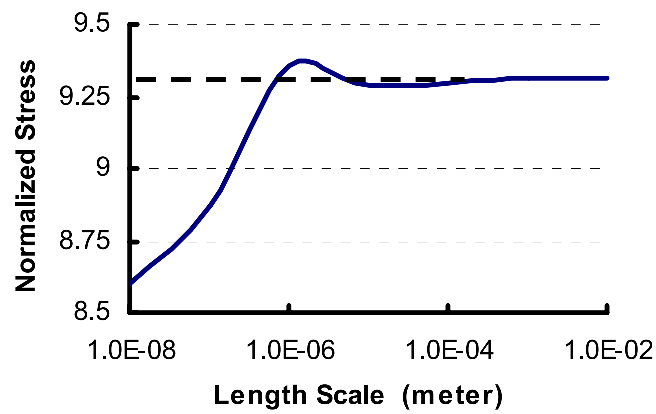

Figure 2 : Effect of Length Scale on Stress Near the Crack Tip Broke line: classical elasticity, solid line: micropolar elasticity

Case 2: natural frequency of a beam

Next, POLAR is used to find the natural frequencies of a cantilever beam (length/thickness = 10/1) with different length scales. The classical elasticity predicts that the product of the natural frequency and the length scale, ωl, remains to be constant. However, as can be seen in Fig.3, the dimensionless quantity, $\omega l/v$, called normalized natural frequency, (v is the speed of longitudinal wave), changes as the length scale changes. It indicates that as the length scale decreases, the material become more rigid. This is also in good qualitative agreement with molecular dynamics simulation results of polycrystalline (Shibutani, Vitek and Bassani (1997), and experimental observations on cellular and fibrous materials (Lakes (1995)).

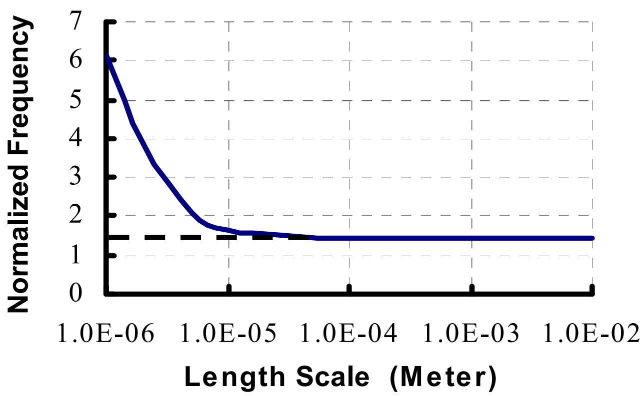

Figure 3 : Natural Frequencies of Cantilever Beam with Different Length Scales Broke line: classical elasticity, solid line: micropolar elasticity

6 Discussions

The numerical examples have shown that, for a micropolar material such as molecular crystal or granular material, the material may become more rigid as the length scale decreases, which is consistent with atomic solutions. It is worthwhile to notice that this scaling effect is different with that of nonlocal material since the later becomes less stiff as the length scale decrease (Chen, Lee and Eskandarian (2002)). Also, it is found that the stress at the crack tip decreases as the size decreases, and is much smaller than that predicated by classical continuum mechanics. This is consistent with a well-known observation "the smaller, the stronger", which implies that the yield strength reduces as the length scale increases, and also, the real stresses may decrease as the length scale decreases.

When the size of a specimen is reduced to small length scales, individual subbody response to the external disturbance may become too significant to be ignored. Micropolar theory stems from the considerations of microstructure and micromotion. Its promise in applications to granular type materials and small size problems is illustrated in the current work.

Acknowledgement: The support to this work by National Science Foundation under Award Number CMS-0115868 is gratefully acknowledged.

References

Aifantis, E. C. (1984): On the microstructural origin of certain inelastic models. *Trans. ASME J. Eng. Mater. Technol.*, 106, pp.326-330

Böhm, H. J.; Han, W.; Eckschlager, A. (2004): Multi-inclusion Unit Cell Studies of Reinforcement Stresses and Particle Failure in Discontinuously Reinforced Ductile Matrix Composites, *CMES: Computer Modeling in Engineering & Sciences*, Volume 5, No. 1, pp. 5-20.

Chen, J.; Huang Y.; Wittmann F. H. (2002): Computational Simulations of Micro-Indentation Tests Using Gradient Plasticity. *CMES: Computer Modeling in Engineering & Sciences*, 3, pp. 743-754

Chen, Y.; Lee J. D.; Eskandarian A. (2002): Dynamic meshless method applied to nonlocal cracked problems. *Theoretical and Applied Fracture Mechanics*, 38, pp.293-300.

Cosserat, E. F. (1909): Theorie des corps deformable, Hermann, Paris.

Dang, P.; Grujicic, M. (1996): The effect of crack-tip material evolution on fracture toughness – an atomistic simulation study of the Ti-V alloy system, *Acta mater*, 45, pp75-87.

deCelis, B.; Argon, A. S.; Yip, S. (1983): Molecular dynamics simulation of crack tip processes in alpha-iron and copper. *J. Appl. Phys.*, 54, pp.4864-4878.

Eringen, A. C. (1962): Nonlinear theory of continuous media. McGraw-Hill, New York.

Mindlin, R. D. (1964): Microstructure in linear elasticity. *Arch. Rat. Mech. Anal.*, 16, pp.51-78.

Eringen, A. C. (1964): Simple micro-fluids. *Int. J. Eng. Sci.*, 2, pp205-217.

Eringen, A. C.; Suhubi, E. S. (1964): Nonlinear theory of simple micro-elastic solids - I. *Int. J. Engng Sci.*, 2, pp.189-203

Eringen, A. C. (1965): Theory of micropolar continua, in *Developments in Mechanics.* (T. C. Huang and M. W. Johnson, Jr., eds), 3. Wiley, New York.

Eringen, A. C. (1967): Mechanics of micromorphic materials, *Proc. Int. Congr. Appl. Mech.* 11^{th}. Springerverlag, Berlin.

Fleck, N. A.; Hutchinson, J. W. (1997): Strain Gradient plasticity. *Advances in Applied Mechanics*, 13, pp.295-361.

Green, A. E.; Rivlin, R. S. (1964): Multipolar continuum Mechanics. *Arch. Rat. Mech. Anal*, 17, pp.113.

Gunther, W. (1958): Zur Static und Kinematik des Cosseratschen Kontinuums, *Abh. Braunschweig. Wiss. Ges*, 10, pp195.

Kwon, Y. W.; Roach, K. (2004): Unit-cell Model of 2/2-twill Woven Fabric Composites for Multi-scale Analysis, *CMES: Computer Modeling in Engineering & Sciences*, Volume 5, No. 1, pp. 63-72.

Lakes, R. (1995): Experimental methods for study of Cosserat elastic solids and other generalized elastic continua, in Continuum Models for Materials with Microstructure (edited by Muhlhaus), John Wiley $ Sons Ltd.

Lee, J. D. (1971): Mechanics of Liquid Crystals, Ph.D. Dissertation, Department of Mechanical and Aerospace Engineering, Princeton University.

Lee, J. D.; Eringen, A. C. (1971a): Wave Propagation in Nematic Liquid Crystals. *J. Chem. Phys.*, 54, pp.5027-5034.

Lee, J. D.; Eringen, A. C. (1971b): Alignment of Nematic Liquid Crystals. *J. Chem. Phys.*, 55, pp. 4504-4508.

Lee, J. D.; Eringen, A. C. (1974): Relations of Two continuum Theories of Liquid Crystals.*Ordered Fluids and Liquid Crystals*, American Chemical Society, pp.315-330.

Mindlin, R. D.; Tiersten, H. F. (1962): Effects of couple stresses in linear elasticity, *Arch. Rat. Mech. Anal*, 11, pp.415-448.

Mindlin, R. D. (1964): Microstructure in linear elasticity. *Arch. Rat. Mech. Anal*, 16, pp.51-78.

Polizzotto, C.; Borino, G. (1998): A thermodynamics-based formulation of gradient-dependent plasticity. *Eur. J. Mech. A/Solids.*, 17, pp.741-761

Shizawa, K.; Zbib, H. M. (1999): A thermodynamical theory of gradient elastoplasticity with dislocation density tensor. I: Fundamentals. *Int. J. of Plasticity.* , 15, pp.899-938.

Schaefer (1967): Das Cosserat-Kontinuum. *Z. Angew. Math, Mech.*, 47, pp.34.

Shibutani, Y.; Vitek, V.; Bassani, J. L. (1997): Non-local properties of inhomogeneous structures by linking approach of generalized continuum to atomistic model. *Int. J. Mech. Sci.*, 40, pp.129-137.

Tang, Z.; Shen, S.; Atluri, S. N. (2003): Analysis of Materials with Strain-Gradient Effects: A Meshless Local Petrov-Galerkin(MLPG) Approach, with Nodal Displacements only. *CMES: Computer Modeling in Engineering & Sciences.*, 4, pp. 177-196

Tenek, L. T; Aifantis, E. C. (2002): A Two-dimensional Finite Element Implementation of a Special Form of Gradient Elasticity. *CMES: Computer Modeling in Engineering & Sciences*, 3, pp. 731-742

Toupin, R. A. (1962): Elastic materials with couple-stresses. *Arch. Rational Mech. Anal.*, 11, pp.385-414

Truesdel, C.; Toupin, R. A. (1960). The classic field theories, in *Handbuch der Physik (S. Flugge, ed.)*. III/1, Springer, Berlin.

Zbib, H. M.; Aifantis, E. C. (1989) A gradient-dependence flow theory of plasticity: application to metal and soil instability. *Appl. Mech. Rev.*, 42, pp.295-304