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On the application of MQ-RBF to the valuation of derivative securities

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Abstract: The general intractability of derivative security pricing models to numerical techniques arguably remains one of the preeminant problems of mathematical finance. In particular, the valuations of such models may be represented as solutions of variational inequalities of evolutionary type typically characterized by their high number of degrees of freedom, unbounded domains, and asymptotic behavior. We consider the application of Multi-Quadratic Radial Basis Functions (MQ-RBF) to the problem of option pricing.

1 Introduction

We consider here the valuation of American options written on several risky assets. Such an option represents a contract to buy or sell a prescribed asset for a predetermined amount (the exercise price) up until a specified time in the future (the expiration date). The purpose of option pricing is to determine the "fair" value of the option and its (expected) optimal exercise time. Options which may only be exercised at the expiration date are known as European. As represented by [Black and Scholes (1973)] and [Merton (1973)], the value of an option formally equates to the value of a portfolio consisting of a position in a safe asset, typically a money-market account, and the risky assets on which the option is written. The determination of the investment distribution which eliminates arbitrage opportunities dictates the value of the portfolio and consequently the option. An introduction to the theory of option pricing may be found in [Hull (1993)], [Wilmott et al. (1993)], [Duffie (1996)].

As analytic solutions do not generally exist for optimal stopping problems, numerical methods are necessitated. In terms of numerically approximating the value function of an option's contract, we remark that dynamic programming based algorithms have typically been employed, particularly in the single state space case when explicit time differencing may be employed (so-called lattice methods) without any great computational cost (cf. [Geske and Shastri (1985)]). When the time horizon is known to be realized (e.g European options), Monte Carlo and Quasi Monte Carlo methods have been shown to be effective in higher dimensions (cf. [Niederreiter (1992)], [Boyle (1977)]). Recently, attempts at combining these two approaches have led to dynamic programming simulation based techniques (cf. [Broadie and Glasserman (1997)], [Longstaff and Schwartz (2001)]). These appear attractive when the number of assets is "large" and the possibility of early exercise times is limited to a "small" set.

The fundamental issue in applying field equation methods for computing option prices is the development of computationally viable techniques. To this end, one poses restrictions of the original problem on a sequence of exhausting approximating domains such that boundary conditions are inferred from the payoff. The application of Multi-Quadratic Radial Basis Functions (MQ-RBF) to option pricing has been investigated by [Hon (2002)] and [Marcozzi et al. (2001)], [Choi and Marcozzi (2001)], [Marcozzi, Choi and Chen (1999)]. A general framework for option valuation utilizing finite elements may be found in the papers by [Marcozzi (2001)], [Marcozzi (to appear)]. Finite difference applications may be found in [Lamberton and Lapeyre (1996)], [Zhang (1997)], [Wilmott et al. (1993)]. Our objective is to compare the relative efficiencies of the these approaches for prototypical problems arising in financial engineering.

The outline of this paper is as follows; in section 2 we consider the mathematical formulation of the option pricing problem. In particular, the valuation of an option contract is a stochastic control problem whose value function satisfies a variational inequality of evolutionary type. In section 3, we consider the discretization of the variational inequality by Multi Quadratic - Radial Basis Functions (MQ-RBF), a collocation algorithm. As an application, in section 4 we develop the method for the case of an option on a single risky asset. Numerical results are presented in section 5 for a variety of option pricing prob-

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lems from mathematical finance.

2 Variational characterization

In this section, we review the mathematical foundations of option valuation. In section 2.1, the options contract is defined as the expected value of a stochastic control problem, evaluated under an appropriate measure. This value function in turn may be characterized as the solution to a variational inequality of evolution type. We consider temporal semi-discretization of the variational characterization in section 2.2.

2.1 Definition of the economy

The economy that we consider consists of m "risky" assets, modeled by diffusion, and one "risk-free" asset, a money-market account, featuring a risk-free rate of return r. The actual contract of the option is represented by a function ψ, the "obstacle." More specifically, the economy maintains m independent exogenous sources of uncertainty represented by the Brownian motion $\{B_t\}_{t\geq 0}$ defined on the probability space $(\Omega, F, \mathbb{P})$, where $F = \{F_t\}_{t\geq 0}$ denotes the standard augmentation of the natural filtration. We suppose that the state variable $X_t \in \mathbb{R}^n$ evolves for $s > t$ according to the Itô process

$$dX_s = b(X_s,s)\,ds + \sigma(X_s,s)\,dB_s \tag{1a}$$

such that

$$X_t = x\,, \tag{1b}$$

where x is the initial state. For example, in valuing a "basket" of stocks, we consider for convenience that $X_t = (X_1, X_2, \ldots, X_n)$ such that $X_i = \ln S_i$, where S_i is the price of the i^{th} stock.

Heuristically, option valuation can be viewed as a decision process. Relative to the process (1), there exists two possibilities at time $t > 0$: exit the process or continue on. These two decisions are not equally favorable but depend on efficiencies represented by an actualization of the final payout. The stopping strategy which provides the best result "in the long run" is defined as the value of the option. That is, associated with the flow (1), the decision variable θ, and the obstacle ψ, we define the (expected) *cost* function

$$J_t^x(\theta) = \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^{\theta} r(X_s,s)\,ds\right)\psi(X_\theta,\theta)\right] \tag{2a}$$

and the valuation (optimal stopping) problem

$$u(x,t) = \sup_{\theta\in T_{[t,T]}} J_t^x(\theta)\,, \tag{2b}$$

where $T < \infty$ represents the *expiration* date of the option and $T_{[t,T]}$ the set of stopping times in $[t,T]$.

In order to characterize the *value* function u, we introduce the weighted Sobolev space $W^{m,p,\mu}$, where m is a non-negative integer, $1 \leq p \leq \infty$, and $0 < \mu < \infty$. Let $W^{m,p,\mu}$ denote the space of functions $u \in L^p(\mathbb{R}^n, e^{-\mu|x|}\,dx)$ whose weak derivatives of all orders $\leq m$ belong to $L^p(\mathbb{R}^n, e^{-\mu|x|}\,dx)$. We equip $W^{m,p,\mu}$ with the norm

$$\|u\|_{m,p,\mu} = \left\{\sum_{k\leq m}\int_{\mathbb{R}^n} |D^k u(x)|^p \cdot e^{-\mu|x|}\,dx\right\}^{1/p}.$$

If X has norm $\|\cdot\|_X$, the space $L^p([0,T];X)$ consists of the set of measurable functions $g:[0,T]\to X$ such that $\int_{[0,T]} \|g(t)\|_X^p\,dt < \infty$.

In order to obtain a well-defined valuation for the contract ψ relative to the process (1), we make the following assumptions:

(A) The *drift* vector $b:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}^n$ is C^1 and has bounded derivatives.

(B) The *dispersion* matrix $\sigma:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}^{n\times m}$ is $C^{2,1}$, bounded and has bounded derivatives.

(C) The coercivity condition holds; that is, there exists a constant $\eta > 0$ such that

$$\xi^T a\,\xi \geq \eta\,\|\xi\|^2\,,$$

for all $\xi\in\mathbb{R}^m$ and each $(x,t)\in\mathbb{R}^n\times(0,\infty)$. The $(n\times n)$-matrix $a(x,t) := \sigma(x,t)\,\sigma(x,t)^T$ is known as the *diffusion* matrix.

(D) The *obstacle* $\psi\in L^p([0,T];W^{2,p,\mu})$, where $p > (n/2)+1$, is non-negative and bounded.

We note that the boundedness imposed in (D) is a formal restriction which will be removed subsequently; it requires, in effect, that $\psi(x,\cdot) = O\left(e^{\varepsilon|x|^2}\right)$, for some $\varepsilon > 0$. Assumptions (A) - (C) ensure that the functions b and σ satisfy the global Lipschitz condition

$$\|b(x,t)-b(y,t)\| + \|\sigma(x,t)-\sigma(y,t)\| \leq c\,\|x-y\|\,,$$

for every $0 \le t < \infty$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, for some constant $c > 0$. This suffices to ensure that there exists a unique t-continuous strong solution of (1) (e.g. [Karatzas and Shreve (1988)]).

The option valuation problem is to determine $u \in L^2([0,T];H^1_{loc}) \cap L^p([0,T];W^{2,p,\mu})$, $\partial u/\partial t \in L^2([0.T];L^2_{loc}) \cap L^p([0,T];W^{0,p,\mu})$ such that

$$\frac{\partial u}{\partial t} - A\,u - r\,u \ge 0\,; \quad u \ge \psi\,, \tag{3a}$$

$$\left\{\frac{\partial u}{\partial t} - A\,u - r\,u\right\} \cdot \{u - \psi\} = 0\,, \tag{3b}$$

where

$$u(x,T) = \psi(x,T)\,, \tag{3c}$$

for all $x \in \mathbb{R}^n$. Here, A is the characteristic operator associated with (1) given by

$$A\,f := -\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial f}{\partial x_j}\right) + \sum_j a_j \frac{\partial f}{\partial x_j}\,, \tag{4a}$$

where

$$a_j = \sum_i \frac{\partial a_{ij}}{\partial x_i} - b_j\,;\ j = 1, \ldots n\,. \tag{4b}$$

Supposing (A) - (D), that r satisfies (A), and that $\partial\psi/\partial t - A\psi - r\,\psi \in L^p([0,T];W^{0,p,\mu})$, there exist a unique solution to (4) represented by (3), for all p sufficiently large and μ sufficiently small (cf. [Bensoussan and Lions (1982)], [Lamberton and Lapeyre (1996)]). In particular, it follows from the Sobolev embedding theorem that $u(\cdot,t) \in C^0$ for all $t \in [0,T)$. For $n = 1$, the solution u may be seen to be infinitely smooth on the continuation region and C^1 globally (cf. [Van Moerbeke (1976)]).

We remark that in the case of European valuations, the variational inequality (3) reduces to the parabolic equation $\partial u/\partial t - A\,u - r\,u = 0$ subject to the terminal condition (3c). Moreover, it is possible to effect the change of variables $\tau = T - t$, in which case (3) becomes an *initial* (as opposed to a "terminal") value problem in τ.

Mathematically, the assumptions (A) - (D) ensure the well-posedness of the variational characterisation of the option value. In a financial context, they ensure that the option may be valued in the economy; that is, that the m-risky and one safe asset "span" the portfolio.

2.2 Approximation by exhausting domains

As a practical consideration, the value function u may be realized as the pointwise limit of a convergent sequence of functions which satisfy approximating variational inequalities on a sequence of exhausting domains. That is, let $\{\Omega_k\}$ denote an increasing sequence of bounded open domains for which $\cup\Omega_k = \mathbb{R}^n$ and $\Psi \in L^p([0,T];W^{2,p,\mu})$ a regularization of ψ such that $\widetilde{\psi} := \psi - \Psi \to 0$ as $\|x\| \to \infty$. Of course, one may consider as a special case that $\Psi = \psi$, but this is not required and consequently $\widetilde{\psi}$ does not necessarily vanish on $\partial\Omega_k$. We consider $\widetilde{u}_k := u_k - \Psi$, where u_k approximates $u - \psi$ on Ω_k and $\widetilde{u}_k|_{\partial\Omega_k} \equiv \widetilde{\psi}|_{\partial\Omega_k}$, by construction. In particular, there exists a unique $\widetilde{u}_k \in L^2([0,T];H^1_{\widetilde{\psi}}(\Omega_k) \cap H^2(\Omega_k))$, $\partial\widetilde{u}_k/\partial t \in L^2([0,T];L^2(\Omega_k))$ such that

$$\frac{\partial\widetilde{u}_k}{\partial t} - A\,\widetilde{u}_k - r\,\widetilde{u}_k \ge \widetilde{f}\ \ a.e.\ \text{on}\ \Omega_k \times [0,T)\,;$$
$$\widetilde{u}_k \ge \widetilde{\psi}\ \text{on}\ \overline{\Omega}_k \times [0,T]\,, \tag{5a}$$

$$\left\{\frac{\partial\widetilde{u}_k}{\partial t} - A\,\widetilde{u}_k - r\,\widetilde{u}_k - \widetilde{f}\right\} \cdot \{\widetilde{u}_k - \widetilde{\psi}\}$$
$$= 0\ \ a.e.\ \text{ on }\ \Omega_k \times [0,T)\,, \tag{5b}$$

where $\widetilde{f} = -\partial\widetilde{\psi}/\partial t + A\,\widetilde{\psi} + r\,\widetilde{\psi}$ and

$$\widetilde{u}_k(x,T) = \widetilde{\psi}(x,T)\,, \tag{5c}$$

for all $x \in \Omega_k$, where $H^1_{\widetilde{\psi}}(\Omega_k) := \{v \in H^1(\Omega_k) \mid v = \widetilde{\psi}$ on $\partial\Omega_k\}$. It follows then for any compact set $\overline{G} \subset \mathbb{R}^n$ that

$$\max_{t\in[0,T]} \|u(x,t) - u_k(x,t)\|_{L^\infty(\overline{G})} \to 0\ \text{ as }\ k \to \infty \tag{6}$$

(cf. [Bensoussan and Lions (1982)], [Lamberton and Lapeyre (1996)]). The result (6) is a statement that the asymptotic behavior (that is, $u_k|_{\partial\Omega_k}$) cannot "appreciably" effect the solution in any fixed bounded region within a finite interval of time. In effect, the estimate (6) indicates that *any* well-posed problem on Ω_k suffices as an approximation on an exhasuting sequence.

Clearly, any convergence realized beyond the estimate (6) will rely on the quality of the boundary conditions imposed upon the frontier of Ω_k. To this end, we consider two types of boundary conditions consistent with (5);

(i) *Artificial boundary condition*. We impose the constraint as the Dirichlet data $\widetilde{u}_k = 0$ on $\partial\Omega_k$.

(ii) *Natural (no) boundary condition*. Formally, we suppose $\partial\widetilde{u}_k/\partial t - A\widetilde{u}_k - r\widetilde{u}_k - \widetilde{f} \to 0$ as $x \in \Omega_k \to x_0 \in \partial\Omega_k$; differentiation being understood in a one-sided sense.

We remark that *no* explicit boundary information is transmitted by condition (ii), which represents an attempt to minimize the approximation error introduced by the artificial boundary conditions in (i). The motivation for (ii) is to attain similar accuracies as (i) utilizing computational domains of smaller diameter. We remark, however, that the well-posedness of (5) and the estimate (6) are not demonstrated in this case.

In order to obtain an implicit semi-discretization of (5), we let $\Delta t = T/M$, for some positive integer M, $t_m = m\cdot\Delta t$, and define the sequence $\{\widetilde{u}_k^m\}$ by recurrence starting with

$$\widetilde{u}_k^M = \widetilde{\psi}, \tag{7a}$$

where, for $m = 1,\dots,M-1$,

$$\frac{1}{\Delta t}\widetilde{u}_k^m - A\widetilde{u}_k^m - a_0\widetilde{u}_k^m \geq \widetilde{f} - \frac{1}{\Delta t}\widetilde{u}_k^{m+1} \;\; a.e. \text{ on } \Omega_k\,; \qquad \widetilde{u}_k \geq \widetilde{\psi} \text{ on } \overline{\Omega}_k \tag{7b}$$

$$\left\{\frac{1}{\Delta t}\widetilde{u}_k^m - A\widetilde{u}_k^m - a_0\widetilde{u}_k^m - \widetilde{f} + \frac{1}{\Delta t}\widetilde{u}_k^{m+1}\right\}\cdot\{\widetilde{u}_k - \widetilde{\psi}\} = 0 \; a.e. \text{ on } \Omega_k. \tag{7c}$$

That is, we utilize a backwards Euler temporal discretization. For each $1 \leq m < M$, we note that the relation (7) is a stationary variational inequality which is known to be uniquely solvable for case (i) boundary conditions. Moreover, by allowing piecewise continuation on the interval $[m\Delta t,(m+1)\Delta t)$ and imposing the constraint at the boundary, it follows that $\widetilde{u}_k^m$ converges weakly to $\widetilde{u}_k$ (cf. [Bensoussan and Lions (1982)], [Glowinski, Lions and Tremolieres (1981)]). We consider the space discretization of (7) by MQ-RBF in the next section.

3 MQ-RBF discretization on bounded domains

Multi-Quadratic Radial Basis Functions (MQ-RBF) constitute a collocation method for the discretization of differential operators in space (cf. [Kansa (1990)]). Proceeding formally, we suppose a given mesh $\{x_1,x_2,\dots,x_N\} \subset \Omega_h$ and the so-called MQ-RBF approximation

$$\widetilde{u}_h(x,t) = \sum_{j=1}^{N} \alpha_j(t)\cdot\phi(r_j(x))\,, \tag{8a}$$

such that

$$\phi(r_j(x)) = \left[r_j^2 + \widetilde{c}^{\,2}\right]^{1/2} \quad ; \quad r_j(x) = \|x - x_j\|\,, \tag{8b}$$

where $\widetilde{c}$ is a constant known as the *shape* parameter. For a uniform mesh with spacing $h = \|x_i - x_{i-1}\|$, we suppose $\widetilde{c} = 4h$; although $\widetilde{c}$ may be optimized for particular applications (cf. [Kansa (1990)], [Carlson and Foley (1991)]). Substituting (8) into (7), we obtain

$$\Phi\,\alpha^{(M)} = \widetilde{\psi}(x,T)\,, \tag{9a}$$

and define sequentially for $m = M, M-1,\dots,1$;

$$w^{(m)} := \left[\frac{1}{\Delta\tau}\Phi - L\right]\alpha^{(m)} - \frac{1}{\Delta\tau}\Phi\alpha^{(m-1)} := A\alpha^{(m)} - a\,, \tag{9b}$$

and

$$z^{(m)} := \Phi\alpha^{(m)} - b\,, \tag{9c}$$

where $\alpha^{(m)}$, resp. $b := \widetilde{\psi}^{(m)}$, is an element in $\mathbb{R}^n$ such that $\alpha_j^{(m)} = \alpha_j(T - m\cdot\Delta t)$ and $\widetilde{\psi}_j^{(m)} = \widetilde{\psi}(\,\cdot\,, m\cdot\Delta t)$. That is, the exponent m is the time stepping parameter. The matrix $\Phi = [\phi(r_{ij})]$ is symmetric positive definite and so invertible (e.g. [Golub and Van Loan (1996)]) and the operator L is linear and strongly coercive. We define for each m the discrete linear complimentary problem (q,M) as follows; find $w^{(m)}, z^{(m)}$ satisfying

$$[w^{(m)}]^t\cdot z^{(m)} = 0 \quad : \quad w^{(m)} \geq 0 \quad : \quad z^{(m)} \geq 0, \tag{10a}$$

where

$$w^{(m)} = Mz^{(m)} + q\,, \tag{10b}$$

such that

$$M = A\Phi^{-1} = \frac{1}{\Delta\tau}I - L\Phi^{-1} \text{ and } q = A\Phi^{-1}b - a\,.$$

For each $m = M, M-1,\dots,1$, there exists a unique solution to (q,M), for all Δt sufficiently small (cf. [Lemke (1965)], [Cottle, Pang and Stone (1992)]). Consequently, at each step, the MQ-RBF discretization (10) is uniquely solvable for $\alpha^{(m)}$.

Relative to the implementation of (10), we note:

(a) Asymptotic performance may *only* be realized on the approximating region $\overline{G}$ of estimate (6) and not over the entire computational domain Ω_k.

(b) Decreasing time and mesh sizing Δt and h without increasing $\mathrm{diam}\{\Omega_h\}$ will *not* result in convergence.

We remark that a so-called Bermuda approximation to the American formulation would be to solve (9a) for $\alpha^{(M)}$, time step back Δt solving $w^{(M-1)} A\alpha^{(M-1)} - a = 0$ for $\alpha^{(M-1)}$, and then applying the constraint $z^{(M-1)} \geq 0$. This procedure may then be continued inductively for $m = M-2, M-3, \ldots, 1$. It subsequently follows from Lemke's algorithm [Lemke (1965)] that the so-called Bermuda approximation represents a "first" complementary approximation to the American option valuation. We note also that upon solving $w \equiv 0$ we obtain the European valuation.

4 Put option on a single stock.

We consider here the explicit MQ-RBF construction in the case of an option written on a single stock. To this end, we suppose that $X_t = \ln S_t$, where S_t denotes the stock price, satisfies the stochastic differential equation

$$dX_t = \left(r - \sigma^2/2\right) dt + \sigma\, dB_t\,,$$

such that

$$X_0 = x\,,$$

where $t > 0$ represents time, r the risk-free rate of return (typically that of a money market account), σ the so-called volatility of the asset, and dB_t the Brownian motion with respect to the risk-neutral measure.

We further distinguish two types of options; calls and puts. A call option gives the holder the right to buy the asset under contract at the agreed upon exercise price, while a put option provides the holder with the opportunity to sell the asset at the fixed price. As may be expected, the pay-off properties of these two options are opposite of one another. Mathematically, an American option is a nonnegative, adapted process $\{g(t)\}_{0\leq t\leq T}$, where $g(t)$ is the payoff of the claim if exercised at time t such that $g(t) = (E - S_t)^+ := \max\{E - S_t, 0\}$ for a put option and $g(t) = (S_t - E)^+$ for a call option. Here, E denotes the exercise price. Henceforth, we shall consider only put options as these alone admit early exercise opportunities when the option is vanilla (that is, pays no dividends). Also, without loss of generality, we may suppose $E = 1$. As per (2), the fair (arbitrage-free) price associated with an American put is given by

$$P(x,t) = \sup_{\theta\in T_{[t,T]}} \mathbb{E}\left[e^{-r(\theta-t)}\left(1 - e^{X_\theta}\right)^+ | F_t\right],$$

(cf. [Merton (1973)]). "Arbitrage-free" refers to the apparent fact that a profit cannot be realized by exercising the option without incurring some associated market risk. Formally, the put price $P(x,t)$ satisfies the variational inequality for all $x \in \mathbb{R}$ and $t \in (0,T)$;

$$\min\left\{ -\frac{\partial P}{\partial t}(x,t) - \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial x^2}(x,t) - (r - \sigma^2/2)\frac{\partial P}{\partial x}(x,t) + rP(x,t)\,,\; P(x,t) - (1 - e^x)^+ \right\} = 0, \tag{11a}$$

subject to the terminal condition

$$P(x,T) = (1-x)^+\,, \tag{11b}$$

for all $x \in \mathbb{R}$. In particular, we nondimensionalize (11) by letting

$$t = T - \frac{\tau}{\frac{1}{2}\sigma^2} \quad : \quad P(x,\tau) = e^{-\frac{1}{2}(\kappa-1)x - \frac{1}{4}(\kappa+1)^2\tau} p(x,\tau),$$

where $\kappa = r/\frac{1}{2}\sigma^2$. The nondimensional put value is then seen to satisfy the parabolic variational inequality

$$\min\left\{ -\frac{\partial p}{\partial \tau}(x,\tau) + \frac{\partial^2 p}{\partial x^2}(x,\tau)\,,\; p(x,\tau) - e^{\frac{1}{4}(\kappa+1)^2\tau}\left[e^{\frac{1}{2}(\kappa-1)x} - e^{\frac{1}{2}(\kappa+1)x}\right]^+ \right\} = 0, \tag{12a}$$

for all $(x,\tau) \in \mathbb{R}\times\mathbb{R}_+$ such that

$$p(x,0) = \left[e^{\frac{1}{2}(\kappa-1)x} - e^{\frac{1}{2}(\kappa+1)x}\right]^+, \tag{12b}$$

for all $x \in \mathbb{R}$. That is,

$$-\frac{\partial p(x,\tau)}{\partial \tau} + \frac{\partial^2 p(x,\tau)}{\partial x^2} \geq 0 \quad ; \quad p(x,\tau) - e^{\kappa_1\tau}[e^{\kappa_2 x} - e^{\kappa_3 x}]^+ \geq 0\,, \tag{13a}$$

$$\left\{\frac{\partial p}{\partial \tau}(x,\tau) - \frac{\partial^2 p}{\partial x^2}(x,\tau)\right\} \cdot \left\{p(x,\tau) - e^{\kappa_1\tau}[e^{\kappa_2 x} - e^{\kappa_3 x}]^+\right\} = 0\,, \tag{13b}$$

for all $(x,\tau) \in \mathbb{R}\times\mathbb{R}_+$ such that

$$p(x,0) = \left[e^{\kappa_2 x} - e^{\kappa_3 x}\right]^+ \quad ; \quad x \in \mathbb{R}\,, \tag{13c}$$

where

$$\kappa_1 = \frac{1}{4}(\kappa+1)^2, \quad \kappa_2 = \frac{1}{2}(\kappa-1), \quad \kappa_3 = \frac{1}{2}(\kappa+1).$$

We wish now to discretize (13) through the use of Multi-Quadratic Radial Basis Functions relative to a given mesh $\{x_1, x_2, \ldots, x_N\}$. To this end, we suppose

$$p(x,\tau) = \sum_{j=1}^{N} \alpha_j(\tau) \cdot \phi(r_j(x)), \tag{14a}$$

such that

$$\phi(r_j(x)) = \left[r_j^2 + \widetilde{c}^2\right]^{\frac{1}{2}} \quad ; \quad r_j(x) = |x - x_j|, \tag{14b}$$

where $\widetilde{c}$ is the shape parameter. Substituting (14) into (13), it follows for all $\tau \in \mathbb{R}_+$ and $i = 1,2,\ldots,N$;

$$\left\{\sum_{j=1}^{N} \frac{d\alpha_j}{d\tau}(\tau)\,\phi(r_{ij}) - \sum_{j=1}^{N} \alpha_j(\tau)\left[\frac{1}{\phi(r_{ij})} - \frac{(x_i - x_j)}{\phi^3(r_{ij})}\right]\right\} \tag{15a}$$

$$\times \left\{\sum_{j=1}^{N} \alpha_j(\tau)\phi(r_{ij}) - e^{\kappa_1 \tau}\left[e^{\kappa_2 x_i} - e^{\kappa_3 x_i}\right]^+\right\} = 0,$$

such that

$$\sum_{j=1}^{N} \frac{d\alpha_j}{d\tau}(\tau)\,\phi(r_{ij}) - \sum_{j=1}^{N} \alpha_j(\tau)\left[\frac{1}{\phi(r_{ij})} - \frac{(x_i - x_j)}{\phi^3(r_{ij})}\right] \geq 0, \tag{15b}$$

$$\sum_{j=1}^{N} \alpha_j(\tau)\phi(r_{ij}) - e^{\kappa_1 \tau}\left[e^{\kappa_2 x_i} - e^{\kappa_3 x_i}\right]^+ \geq 0, \tag{15c}$$

and

$$\sum_{j=1}^{N} \alpha_j(0)\phi(r_{ij}) = \left[e^{\kappa_2 x} - e^{\kappa_3 x}\right]^+. \tag{15d}$$

for all $x \in \mathbb{R}$, where $r_{ij} = r_i(x_j)$. Discetizing (15) now uniformly in time, we suppose

$$\frac{d\alpha_j}{d\tau}(\tau) \approx \frac{\alpha_j^{(m)} - \alpha_j^{(m-1)}}{\Delta\tau},$$

for $\Delta\tau > 0$ and $m = 1,2,\ldots,M$, where

$$\alpha_j^{(m)} = \alpha_j(m \cdot \Delta\tau)$$

and $M \cdot \Delta\tau = \frac{1}{2}\sigma^2 T$. Let

$$w_i := \sum_{j=1}^{N} \frac{1}{\Delta\tau}\phi(r_{ij})\,\alpha_j^{(m)} - \sum_{j=1}^{N}\left[\frac{1}{\phi(r_{ij})} - \frac{(x_i - x_j)}{\phi^3(r_{ij})}\right]\alpha_j^{(m)} - \sum_{j=1}^{N} \frac{1}{\Delta\tau}\phi(r_{ij})\,\alpha_j^{(m-1)}, \tag{16}$$

or

$$w := \left[\frac{1}{\Delta\tau}\Phi - L\right]\alpha^{(m)} - \frac{1}{\Delta\tau}\Phi\alpha^{(m-1)} := A\alpha^{(m)} - a, \tag{17a}$$

and

$$z_i := \sum_{j=1}^{N} \phi(r_{ij})\,\alpha_j^{(m)} - e^{\kappa_1(m\cdot\Delta\tau)}\left[e^{\kappa_2 x_i} - e^{\kappa_3 x_i}\right]^+,$$

or

$$z := \Phi\alpha^{(m)} - b, \tag{17b}$$

where $i = 1,2,\ldots,N$. We may now define the discrete linear complimentary problem (q,M) as follows; find w,z satisfying

$$w^t \cdot z = 0 \quad ; \quad w \geq 0 \quad ; \quad z \geq 0, \tag{18a}$$

where

$$w = Mz + q \tag{18b}$$

such that

$$M = A\Phi^{-1} = \frac{1}{\Delta\tau}I - L\Phi^{-1}$$

and

$$q = A\Phi^{-1}b - a.$$

It then follows that there exists a unique solution to (q,M), for all $\Delta\tau$ sufficiently small, and consequently the RBF discretization (17) is uniquely solvable. It follows readily that the European put satisfies the relation

$$w = 0$$

subject to the initial condition (15d).

5 Numerical experiments

In order to verify the convergence and efficiencies of the MQ-RBF discretization, we consider the valuation of a variety of options from mathematical finance. To this end, we value European and American style options written on a single asset and two noncorrelated assets, as well as an option on a foreign currency in a stochastic interest rate economy. In section 5.1, we present case related details, benchmarking the MQ-RBF discretization to the finite element (FEM) implementation of [Marcozzi (2001)]. In section 5.2, we discuss general conclusions.

In the case of European options considered below, comparisons will be made to the analytic solution whenever possible. For American options, or when analytic solutions are not available for the European case, computed finite element solutions anticipated to be at least two orders of magnitude more accurate than the approximations are utilized as benchmarks.

Although the order of the MQ-RBF discretization in space is theoretically unknown, our results appear to indicate second-order convergence for solutions which are globally C^1. To this end, we consider uniform spactial mesh sizes $h = \Delta x_i$ and fixed $\Delta t/h^2 = \beta$, where we choose for computational purposes $\beta = 1$, a sufficiently small constant, unless otherwise noted. Note that the choice of β effects the relative performance (accuracy) of the method. As such, we expect to achieve asymptotic convergence as $\Delta t \to 0$, $h \to 0$, and the radius of the computational domain grows without bound; that is, $R_c \to \infty$ (cf. Remarks (a) and (b) of section 3).

The approximation and computational domains were constructed as n-squares with (maximum norm) radius, R_a and R_c, respectively. The center of the approximation domain G and the computational domain Ω_k were aligned "at the money" (detailed in each example). In particular, the radius $R_a = 0.1$ was utilized for all computations. Results are reported in the (discrete) maximum norm where we considered only those nodal values within the approximation region $\overline{G}$.

5.1 Case studies

The following formulations are provided in the so-called risk-neutral measure, which provides for complete markets and unique option prices. We remark that the process (1) is fully specified by its drift b and volatility σ whereas the options themselves are uniquely determined by their payoff attributes represented by the obstacle ψ and duration T in (2).

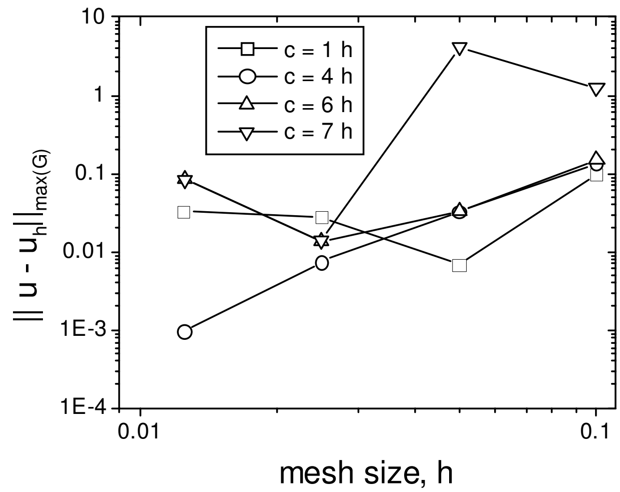


Figure 1 : Effect of shape parameter on convergence.

(*I*) *European put option on a single risky asset.*

We consider $X_t = \ln S_t$, where $S_t > 0$ denotes the assets price at time $t > 0$. In the so-called risk neutral valuation, the drift $b = r - (1/2)\sigma^2$. The payoff is defined such that

$$\psi(x,T) = \max\left\{E - e^x, 0\right\},$$

where E denotes the exercise price of the option. In case (i) applications, the artificial boundary conditions are specified by the constraint. As indicated in [Marcozzi (2001)], any boundary conditions for which the problem is well-posed suffice. In particular, we utilize Dirichlet data obtained from the depreciated payoff

$$\psi(x,t) = \max\left\{E \cdot e^{-r(T-t)} - e^x, 0\right\},$$

for the European option. In the single asset case, the European put price is given by the celebrated Black-Scholes formula. The stochastic process (1) was defined such that $\sigma = 0.3$, $r = 0.1$, $T = 1.0$, while for the payoff we assumed $E = 100$. The center of the computational grid (and of G) was taken to be $\ln(E)$.

The effect of mesh and computational domain size for the one-asset European put was considered in [Marcozzi, Choi and Chen (1999)]. We simply note here that the results are similar to those of example (II) below; relying on the refinement of the mesh size and enlargement

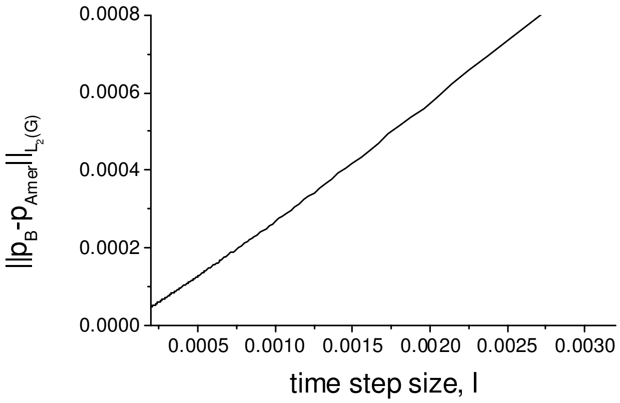


Figure 2 : Bermudian approximation to the American put option.

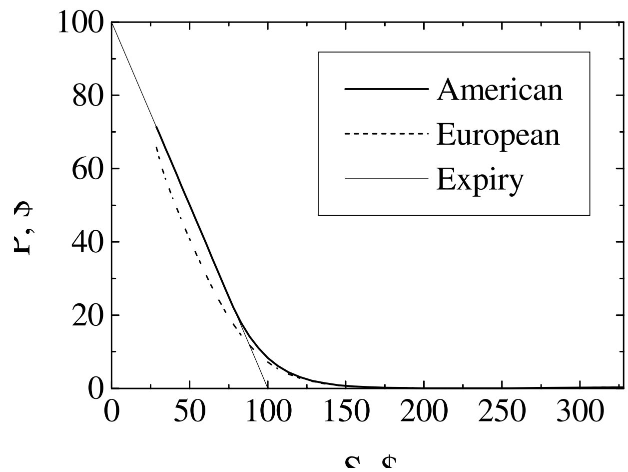


Figure 3 : Put option values.

of the computational domain to obtain convergence. Figure 1 demonstrates the convergence of the MQ-RBF approximation for the European case and case (ii) boundary conditions as a function of the mesh size and shape parameter for $R_c = 1.0$. Results were similar for the case (i) artificial boundary condition application. In particular, we note the sensitivity of the results on the choice of the shape parameter. For $\widetilde{c} = 1$ or h^2, we found that the matrix Φ was nearly singular. This may portend the utilization of numerical accuracies greater than the double precision employed here. In Figure 2, we consider the effect of valuing an American option by its so-called Bermudian approximation. As expected, the approximation improves as mesh size decreases. Figure 3 plots the MQ-RBF valuations for the American and European options as a function of stock price.

(*II*) *Put option on the geometric average of two assets*.

Generalizing the single asset case, each component of the diffision represents the logorithm of a stocks price; that is, $X_i = \ln S_i$, for $i = 1,2$. We again suppose that the drift components $b_j = r - (1/2)\,\sigma_j^2\,(j = 1,2)$ in the risk neutral economy. The payoff for a put option on the geometric average of two assets is defined to be

$$\psi(x,T) = \max\left\{E - (e^{x_1} \cdot e^{x_2})^{1/2}, 0\right\},$$

where we again suppose the constraint for a European option to be the depreciated payoff

$$\psi(x,t) = \max\left\{E \cdot e^{-r(T-t)} - (e^{x_1} \cdot e^{x_2})^{1/2}, 0\right\}.$$

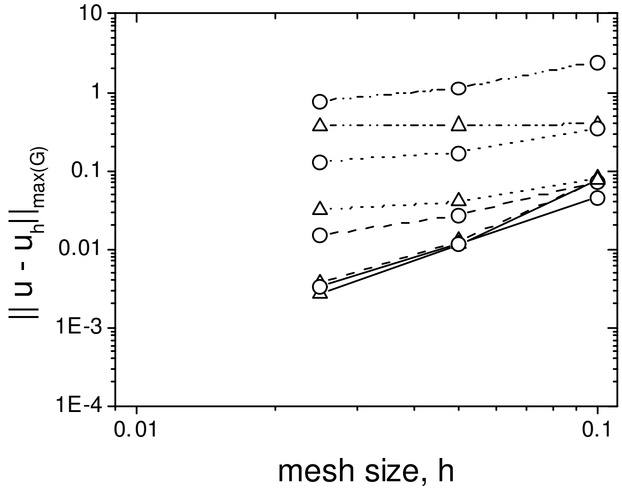
For the corresponding American option the obstacle is given by

$$\psi(x,t) = \max\left\{E - (e^{x_1} \cdot e^{x_2})^{1/2}, 0\right\}.$$

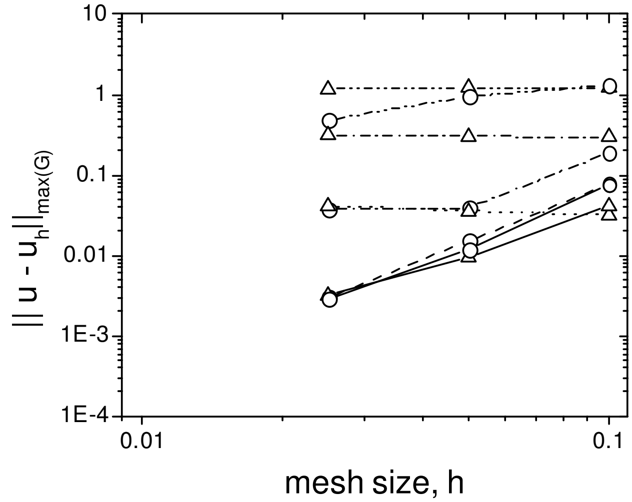
Here E denotes the exercise price of the option. Unlike the single asset case, there does not exist an analytic solution with which to compare the quality of the approximation. However, by introducing the process $Y_t = X_1 + X_2$ such that $\psi(y) = \max\{E - e^{y/2}, 0\}$, we are able to value these options by numerically approximating Y_t, which has only a single source of uncertainty (that is, by solving the single state variable variational inequality for Y_t). The FEM solution for Y_t (with $h = 0.001$) was utilized to benchmark the general MQ-RBF and FEM approximations.

We suppose $\sigma_{ii} = 0.3$, $\sigma_{ij} = 0.0$, and $r = 0.1$ to define the diffusion (1) and $E = 100$ and $T = 1.0$ to define the option. The center of the computational grid (and of G) was taken to be $(\ln(E), \ln(E))$. For the MQ-RBF computations, we consider the so-called Bermudian approximation to the American option case.

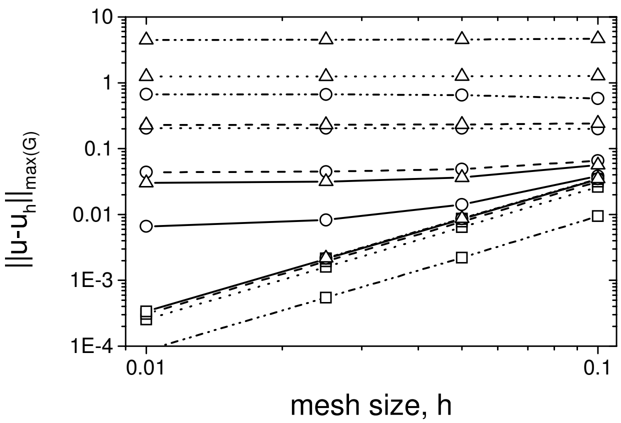
Figure 4 presents convergence results for the European an American cases, both for the MQ-RBF and FEM spatial discretizations. In particular, we note the need to both refine the mesh sizing while simultaneously enlarging the radius of the computational domain. In general, the case (ii) formulation tends to achieve its asymptotic convergence rate for smaller computational domains than the case (i) boundary conditions. As with the single-asset case, Figure 4 suggests that the MQ-RBF discretization



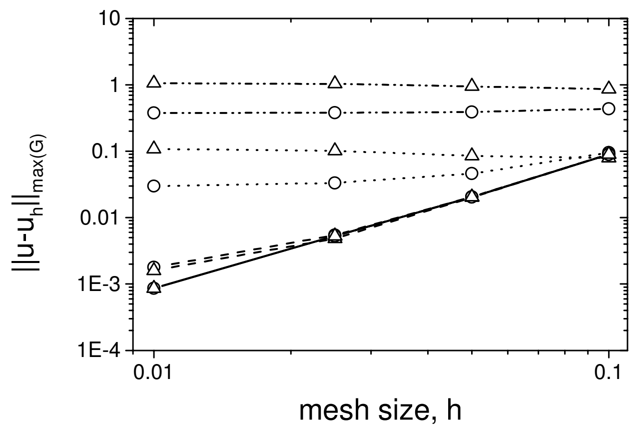
a.) European put with MQ-RBF



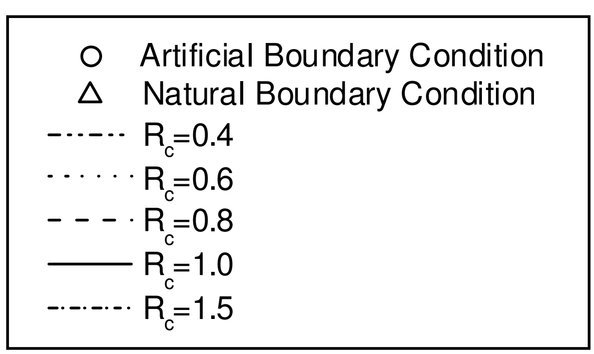
b.) American put with MQ-RBF



c.) European put with FEM



d.) American put with FEM



e.) Key

Figure 4 : Put options on the geometric average of two assets.

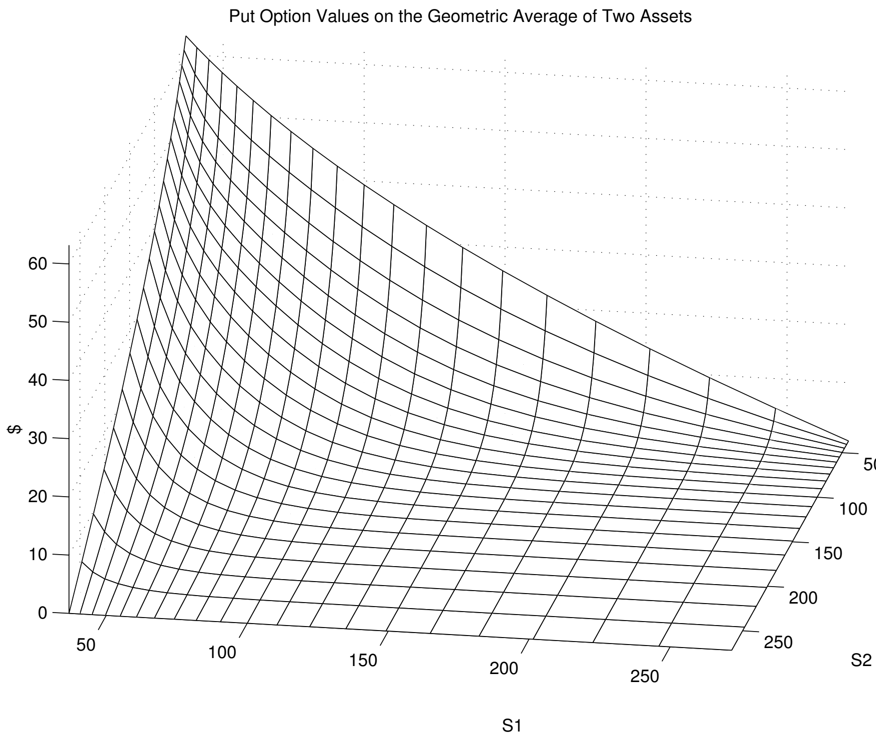


Figure 5 : American put option on the geometric average of two assets.

is second-order in space. Figure 5 is a plot of the value function in terms of asset prices.

(*III*) *Option on a foreign currency*.

We consider the valuation of options written on a foreign currency when interest rates are stochastic and the diffusion matrix representing the global economy is coercive. In this case, an American call (put) option gives the holder the right to buy (sell) a fixed amount of a foreign currency at a predeterminded price at any time until a fixed expiration date. As usual, the corresponding European option can be exercised only at the expiration date. In particular, an option on a foreign currency represents an option on a foreign treasury bond. When denominated in domestic currency through the spot exchange rate, such instruments may be considered exotic domestic securities. Models encompassing options on a foreign currency are quite general and include such securities as corporate bonds and domestic treasuries as special cases.

We denote the (stochastic) instantaneous risk-free rate of return at time t within the domestic economy by $r = X_1$, the foreign economy by X_2, and the logarithm of the cross-currency exchange rate by X_3. It follows then that the payoff of a put on a foreign currency is given by

$$\psi(x,T) = \max\{E - e^{x_3}, 0\},$$

such that in the European case the discounted constraint is specified such that

$$\psi(x,t) = \begin{cases} \max\{E - e^{x_3}, 0\}, & ifX_1 < 0 \\ \max\{E \cdot e^{-x_1(T-t)} - e^{x_3}, 0\}, & ifX_1 \geq 0, \end{cases}$$

and by

$$\psi(x,t) = \max\{E - e^{x_3}, 0\},$$

in the American case, where E is the exercise price of the option. The specific model utilized is introduced in [Choi and Marcozzi (2001)]. The general model for American options on a foreign currency under stochastic interests rates was first developed in Amin and Jarrow [Amin and Jarrow (1991)], which includes an analytic solution applicable to the European case specified here. In particular, we suppose the following dispersion matrix;

$$\sigma = \begin{bmatrix} 0.05 & 0.005 & 0.005 \\ 0.005 & 0.05 & 0.005 \\ 0.03 & 0.03 & 0.3 \end{bmatrix}$$

in addition to taking

$$b_1 = 0.005\,, \;\; b_2 = 0.005 - \sum_{i=1}^{3} \sigma_{2i}\sigma_{3i}\,,$$

$$b_3 = x_1 - x_2 - \sum_{i=1}^{3} \sigma_{3i}^2/2\,,$$

an exercise price of $E = 1$, and a duration of $T = 1$.

For a maximum norm computational domain radius of 2.5, a uniform spacial mesh size of 0.09, a time step of 1/360, and for a duration of 1 year, we obtained a European valuation "at the money" of $u_E = 0.1143$ at (0.05, 0.05, 0.0) utilizing the MQ-RBF discretization and case (ii) boundary conditions. This compares to the exact value of $u_{exact} = 0.1144$. For a MQ-RBF Bermudian approximation to the American option valuation, we obtained $u_A = 0.1171$, which compares to $u_{FE} = 0.1172$ obtained utilizing finite elements.

5.2 Further discussion

With respect to other field equation methods, the advantage of MQ-RBF is its simplicity. As a collocation technique, coding is elementary. As such, MQ-RBF discretizations appear to hold promise for the complex free-boundary geometries of mathematical finance problems for which the state space dimension may readily exceed three. We note that MQ-RBF was able to obtain accuracies comparable to a linear peicewise (second-order spatial) finite element implementation. For variational inequalities on unbounded domains (or degenerate operators), convergence was only achieved upon allowing the temporal and spatial mesh sizes to vanish as the size of the computational domain increases without bound. We also note that this convergence may only be obtained on the compact approximation region and not, in general, over the entirety of the computational domain. We remark that this asymptotic convergence has only been realized in the case of *uniform* meshes, suggesting the need for domain decomposition algorithms when local mesh refinement is optimal.

Additionally, MQ-RBF discretizations appear to be more stable than their FEM counterpart with respect to the application of the natural boundary conditions. This may be a consequence of the global footprint of the MQ-RBF method as opposed to the compact support of the basis functions utilized in FEM. As the so-called natural boundary condition potentially represents increased computational efficiencies when applied to the problems of mathematical finance, this may prove significant for MQ-RBF applications when the number of independent assets is "large."

The difficulties with MQ-RBF pertain to the resultant full matrix Φ, which leads to very large condition numbers, and a lack of theoretical justification of the method. To combat these difficulties, domain decomposition methods have been employed (cf. [Marcozzi, Choi and Chen (1999)]) as well as so-called compactly supported radial basis functions (cf. [Schaback and Wendland (1999)]). Likewise, the issue of the shape parameter remains unresolved in MQ-RBF, although we suspect that this dependence is a consequence of the full coefficient matrix and the accuracy of the discrete solver, as opposed to any particular choice of $\tilde{c}$.

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