

Non-Hyper-Singular Boundary Integral Equations for Acoustic Problems, Implemented by the Collocation-Based Boundary Element Method

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Abstract: The weak-form of Helmholtz differential equation, in conjunction with vector test-functions (which are gradients of the fundamental solutions to the Helmholtz differential equation in free space) is utilized as the basis in order to directly derive non-hyper-singular boundary integral equations for the velocity potential, as well as its gradients. Thereby, the presently proposed boundary integral equations, for the gradients of the acoustic velocity potential, involve only $O(r^{-2})$ singularities at the surface of a 3-D body. Several basic identities governing the fundamental solution to the Helmholtz differential equation for velocity potential, are also derived for the further desingularization of the strongly singular integral equations for the potential and its gradients to be only weakly-singular. These weakly-singular equations are denoted as R- ϕ -BIE, and R- q -BIE, respectively. [i.e., containing singularities of $O(r^{-1})$ only at the boundary] Collocation-based boundary-element numerical approaches [denoted as BEM-R- ϕ -BIE, and BEM-R- q -BIE] are implemented to solve these R- ϕ -BIE, and R- q -BIE. The lower computational costs of BEM-R- ϕ -BIE and BEM-R- q -BIE, as compared to the previously published Symmetric Galerkin BEM based solutions of R- ϕ -BIE [Qian, Han and Atluri (2004)] are demonstrated through examples involving acoustic radiation as well as scattering from 3-D bodies.

keyword: Boundary integral equations, hyper-singularity, collocation

1 Introduction

It is well known that the solutions of the conventional boundary integral equations are nonunique at the fictitious eigenfrequencies for exterior acoustic or elastic wave problems in the frequency domain. The fictitious eigenfrequencies have no physical meaning for the ex-

terior problems, and are the manifestations of the drawback of the mathematical formulation of the conventional boundary element method. To circumvent this, hypersingular boundary integral equations (HBIEs), which are the derivatives of the conventional boundary integral equations, have become a useful alternative approach. Liu & Chen (1999) and Burton & Miller (1971) linearly combine the surface Helmholtz integral equation for the potential, and the integral equation for the normal derivative of potential at the surface (HBIEs), to circumvent the problem of nonuniqueness at characteristic frequencies. Their method is very effective, and is labeled as CHIE (Composite Helmholtz Integral Equation), or CONDOR (Composite Outward Normal Derivative Overlap Relation) by Reut [Reut (1985)]. However, the smoothness requirement, which implies that the derivatives of the density function must be Holder continuous, is a serious theoretical issue associated with the HBIE formulation [Liu and Chen (1999)]. Therefore, only boundary elements with C^1 continuity near each node, such as the Overhauser and Hermite elements, can be applied in the implementation of HBIEs [Liu and Rizzo (1992)]. This requirement hinders the applications of HBIEs, because of the complexity of C^1 elements. Another key issue for hypersingular boundary integral equations, is the application of regularization techniques, which are commonly employed to improve the approach by reducing the problem to the one involving $O(r^{-1})$ singular integrals near the point of singularity. Chien, Rajiyah, and Atluri (1990) employed some known identities of the fundamental solution from the associated interior Laplace problem, to regularize the hypersingular integrals. This concept has been applied by many successive researchers: The regularized normal derivative equation in Wu, Seybert, and Wan (1991) is sought to be converged in the Cauchy principal value sense, rather than in the finite-part sense, and the computation of tangential derivatives is required everywhere on the boundary. Recently, Yan, Hung, and Zheng (2003) employed

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a discretized operator matrix for improving the intensive computation of double surface integrals, by replacing the evaluation of double surface integral with the evaluation of two discretized operator matrices. In general, most of the *regularization techniques* published so far in literature for evaluating the hyper-singular integrals in the acoustic BIEs, arise from certain identities associated with the fundamental solution to the Laplace equations.

However, in the present paper, without directly differentiating the derivatives of the conventional boundary integral equation for the potential, which will result in the hyper-singular integrals, novel non-hyper-singular boundary integral equations are derived directly, for the gradients of the velocity potential. The acoustic potential gradients are related to the sound velocity in their physical meaning. The basic idea of using the gradients of the fundamental solution to the Helmholtz differential equation, as *vector test-functions* to write the weak-form of the original Helmholtz differential equation, and thereby directly derive a *non-hyper-singular boundary integral equations for velocity potential gradients*, has its origins in [Okada, Rajiyah, and Atluri (1989a, b), Okada and Atluri (1994)], who use the displacement and velocity gradients to directly establish the displacement and displacement gradient boundary integral equations in elastic/plastic solid problems, as well as traction boundary integral equations [Han and Atluri (2003a, b)], which are very simple to be implemented numerically. The current method can be shown to be fundamentally different from the regularized normal derivative equation by Wu *et al.* [Wu, Seybert, and Wan (1991)], who used the tangential derivatives to reduce the singularity.

The boundary integral equations for the potential [labeled here as ϕ -BIE], and its gradient [labeled here as q -BIE], which are newly presented in the present paper, are only strongly singular [$O(r^{-2})$]. These *strongly singular* ϕ -BIE, and q -BIE are further regularized to *only weakly singular* [$O(r^{-1})$] types, which are labeled here as R- ϕ -BIE, R- q -BIE, respectively. This is achieved by using certain basic identities of the fundamental solution of the Helmholtz differential equation for potential. These basic identities, in their most general form, are also newly derived in this paper. These basic identities are derived from the most general scalar and vector weak-forms of the Helmholtz differential equation for potential, governing the fundamental solution itself. The boundary element methods [BEM] derived by simply collocating

the R- ϕ -BIE, and R- q -BIE, as developed in the present paper, are referred to as the BEM-R- ϕ -BIE, and BEM-R- q -BIE, respectively. In addition, general Petrov-Galerkin based methods can be formulated as shown in [Qian, Han and Atluri (2004)] to solve the R- ϕ -BIE, and R- q -BIE, in their weak senses. With this general Petrov-Galerkin formulation, one can easily derive different methods, such as the (Symmetric Galerkin Boundary Element Method) SGBEM-R- $[\phi&q]$ -BIE, by using various alternative functions as the test function. In the present paper, the BEM-R- ϕ -BIE and BEM-R- q -BIE methods employ the Dirac Delta function as the test function in the general Petrov-Galerkin approach. In the present BEM-R- ϕ -BIE and BEM-R- q -BIE, C^0 continuity of ϕ as well as q over the boundary elements is sufficient for numerical implementation. There are several reasons for implementing BEM-R- ϕ -BIE and BEM-R- q -BIE method here. First of all, the BEM-R- $[\phi&q]$ -BIE, SGBEM-R- $[\phi&q]$ -BIE, and the MLPG approaches [Atluri and Zhu (1998)], [Atluri and Shen (2002a, b)], will result in a family of methods with the general Petrov-Galerkin formulation as the basis. Numerical examples demonstrate that the BEM-R- $[\phi&q]$ -BIE are far more efficient than the SGBEM-R- $[\phi&q]$ -BIE, and its implementation is easier as well.

Meshless methods for solving BIEs have been developed through the meshless local Petrov-Galerkin (MLPG) approaches [Han, and Atluri (2003b)]. The meshless method, as an alternative numerical approach to eliminate the well known drawbacks in the finite element and boundary element methods, has attracted much attention recently [Atluri, Han, and Shen (2003); Atluri, and Shen (2002a, b)]. More importantly, by using the moving least square approximation scheme or radial basis function as the trial functions, the MLPG method make it possible to overcome the curse, in which the element size in the discretization model, either in the finite element method or the traditional boundary element method, must be less than the wavelength of the acoustic wave. These MLPG methods wherein the ϕ and q are approximated by meshless interpolations in the surface are also subjects of our studies.

2 The non-hyper-singular boundary integral equations in acoustics

The Helmholtz differential equation governing the acoustic velocity potential ϕ for time-harmonic $e^{i\omega t}$ waves, can

be written as:

$$\nabla^2 \phi + k^2 \phi = 0 \quad (1)$$

where i is the imaginary unit, ω is the angular frequency of the acoustic wave, and $k = \omega/c$ is the wave number. At any field point ξ , the velocity potential of Eq. 1, due to a point sound source at \mathbf{x} (Fig. 1), is well known as the free-space Green's function $\phi^*(\mathbf{x}, \xi)$, which is listed here for 2- and 3-D problems, respectively, as:

$$\phi^*(\mathbf{x}, \xi) = \frac{i}{4} H_0^{(1)}(kr) \text{ in 2D} \quad (2a)$$

where $H_0^{(1)}(kr)$ denotes the Hankel function of the first kind, and $r = |\mathbf{x} - \xi|$;

$$\phi^*(\mathbf{x}, \xi) = \frac{1}{4\pi r} e^{-ikr} \text{ in 3D} \quad (2b)$$

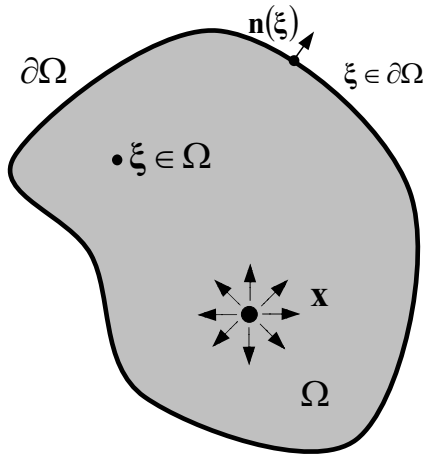


Figure 1 : A solution domain with source point \mathbf{x}

Therefore, the so-called fundamental solution of the Helmholtz equation is governed by the wave equation,

$$\phi_{,ii}^*(\mathbf{x}, \xi) + k^2 \phi^*(\mathbf{x}, \xi) + \delta(\mathbf{x}, \xi) = 0 \quad (3)$$

2.1 Boundary integral equations for the velocity potential ϕ -BIE

By using $\bar{\phi}$ as the test function to enforce the Helmholtz equation, Eq. 1 in terms of the trial function ϕ , in a weak-sense, the weak form of Helmholtz equation can be written as,

$$\int_{\Omega} (\nabla^2 \phi + k^2 \phi) \bar{\phi} d\Omega = 0 \quad (4)$$

A “symmetric weak-form” is obtained by applying the divergence theorem once in Eq. 4,

$$\int_{\partial\Omega} n_i \phi_{,i} \bar{\phi} dS - \int_{\Omega} \phi_{,i} \bar{\phi}_{,i} d\Omega + \int_{\Omega} k^2 \phi \bar{\phi} d\Omega = 0 \quad (5)$$

where, both the trial function ϕ , as well as the test functions $\bar{\phi}$ are only required to be first-order differentiable. By applying the divergence theorem twice in Eq. 4, we obtain the “unsymmetric weak-form”,

$$\int_{\partial\Omega} n_i \phi_{,i} \bar{\phi} dS - \int_{\partial\Omega} n_i \phi \bar{\phi}_{,i} dS + \int_{\Omega} \phi (\bar{\phi}_{,ii} + k^2 \bar{\phi}) d\Omega = 0 \quad (6)$$

Now, the test functions $\bar{\phi}$ are required to be second-order differentiable, while ϕ is not required to be differentiable [Han and Atluri (2003a)] in Ω .

By using the fundamental solution $\phi^*(\mathbf{x}, \xi)$ as the test function $\bar{\phi}$ in Eq. 6, and with the property from Eq. 3, we obtain the integral equation for ϕ :

$$\begin{aligned} \phi(\mathbf{x}) &= \int_{\partial\Omega} q(\xi) \phi^*(\mathbf{x}, \xi) dS - \int_{\partial\Omega} \phi(\xi) \Theta^*(\mathbf{x}, \xi) dS \quad \mathbf{x} \in \Omega \end{aligned} \quad (7)$$

where, by definition,

$$q(\xi) = \frac{\partial \phi(\xi)}{\partial n_{\xi}} = n_k(\xi) \phi_{,k}(\xi) \quad \xi \in \partial\Omega$$

and the kernel function $\Theta^*(\mathbf{x}, \xi)$ is defined as,

$$\Theta^*(\mathbf{x}, \xi) = \frac{\partial \phi^*(\mathbf{x}, \xi)}{\partial n_{\xi}} = n_k(\xi) \phi_{,k}^*(\mathbf{x}, \xi) \quad \xi \in \partial\Omega$$

Eq. 7 is the conventional BIE for ϕ , which is widely used in literature, and is hereafter referred to as the ϕ -BIE. One can use Green's second identity directly to obtain Eq. 7, however, to keep the consistency of the presented paper, the notion of a weak-form is used here.

The nonuniqueness of the Helmholtz integral equation, Eq. 7, is well known; it possesses nontrivial solutions at some characteristic frequencies [Chien, Rajiyah, and Atluri (1990)]. Many researchers have investigated and expended substantial efforts in solving this problem of nonuniqueness. Burton & Miller stated that the nonuniqueness can be restored by combining a second integral equation. If we differentiate Eq. 7 directly with respect to \mathbf{x}_k , we obtain the second integral equation for the potential gradients $\phi_{,k}(\mathbf{x})$. One term in this equation is hyper-singular, since $\frac{\partial \Theta^*(\mathbf{x}, \xi)}{\partial x_k}$ is of order $O(r^{-3})$ for a 3D problem. A wide body of literature is devoted to deal with the hyper-singularity in this equation.

2.2 Presently proposed Non-Hyper-Singular q-BIE

The novel method in this paper starts from writing a vector weak-form [as opposed to a scalar weak-form] of the governing equation Eq. 4 by using the vector test function $\bar{\phi}_{,k}$, as in [Okada, Rajiyah, and Atluri (1989a, 1989b)]:

$$\int_{\Omega} (\phi_{,ii} + k^2\phi) \bar{\phi}_{,k} d\Omega = 0 \quad \text{for } k=1,2,3 \quad (8)$$

After applying the divergence theorem three times in Eq. 8, and by using the gradients of the fundamental solution, viz., $\phi_{,k}^*(\mathbf{x}, \boldsymbol{\xi})$, as the test functions, and using the identity from Eq. 3, we obtain

$$\begin{aligned} -\phi_{,k}(\mathbf{x}) &= \int_{\partial\Omega} q(\boldsymbol{\xi}) \phi_{,k}^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &+ \int_{\partial\Omega} D_t \phi(\boldsymbol{\xi}) e_{ikt} \phi_{,i}^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &+ \int_{\partial\Omega} k^2 n_k(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}) \phi^*(\mathbf{x}, \boldsymbol{\xi}) dS \end{aligned} \quad (9)$$

we have used the definition of the surface tangential operator as

$$D_t = n_r e_{rst} \frac{\partial}{\partial \xi_s}$$

in which e_{rst} is the permutation symbol.

We refer to Eq. 9, hereafter, as the presently proposed non-hyper-singular q-BIE. The ϕ -BIE [Eq. 7], and q-BIE [Eq. 9], are derived independently of each other. The most interesting feature of the “directly derived” integral equations Eq. 9, for $\phi_{,k}(\mathbf{x})$, is that they are non-hyper-singular; viz, the highest order singularity in the kernels appearing in Eq. 9 is only $O(r^{-2})$ in a 3D problem.

3 Further regularization of ϕ -BIE and q-BIE [i.e. reducing them to be of only (r^{-1}) singularity]

3.1 Basic properties of the fundamental solution ϕ^*

Before we proceed with the further regularization of the strongly-singular ϕ -BIE and q-BIE, certain fundamental properties of ϕ^* are derived, so that they will enable us to derive simple, straightforward and elegant weakly-singular R-[ϕ &q]-BIE.

We write the weak form of Eq. 3 governing the fundamental solution over the domain, using an arbitrary function $\phi(\mathbf{x})$ in Ω as the test function; and thus we obtain a

basic identity of the fundamental solution $\phi^*(\mathbf{x}, \boldsymbol{\xi})$ to be:

$$\begin{aligned} &\int_{\partial\Omega} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) dS \\ &+ \int_{\Omega} k^2 \phi^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) d\Omega + \phi(\mathbf{x}) = 0 \quad \text{at } \mathbf{x} \in \Omega \end{aligned} \quad (10)$$

When the point \mathbf{x} approaches a smooth boundary, i.e., $\mathbf{x} \rightarrow \partial\Omega$, the first term in Eq. 10 can be written as

$$\begin{aligned} \lim_{\mathbf{x} \rightarrow \partial\Omega} \int_{\partial\Omega} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) dS &= \int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) dS \\ &- \frac{1}{2} \phi(\mathbf{x}) \end{aligned} \quad (11)$$

where CPV denotes a Cauchy Principal Value integral. The physical meaning of Eq. 11 can be understood by rewriting Eq. 10 as:

$$\begin{aligned} &\int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) dS \\ &+ \int_{\Omega} k^2 \phi^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) d\Omega + \frac{1}{2} \phi(\mathbf{x}) = 0 \quad \text{at } \mathbf{x} \in \partial\Omega \end{aligned} \quad (12)$$

Eq. 12 means that only a half of the sound source $\phi(\mathbf{x})$ at point \mathbf{x} is applied to the domain Ω , when the point \mathbf{x} approaches a smooth boundary, $\mathbf{x} \rightarrow \partial\Omega$. We may write Eq. 12 in a more general form as:

$$\begin{aligned} &\int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) dS + \int_{\Omega} k^2 \phi^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) d\Omega \\ &+ \varepsilon \phi(\mathbf{x}) = 0 \end{aligned} \quad (13)$$

where

$$\varepsilon = \begin{cases} 0 & \text{for } x \text{ is interior to } \partial\Omega \\ 1 & \text{for } x \text{ is exterior to } \partial\Omega \\ 1/2 & \text{for } x \in \text{smooth } \partial\Omega \\ \theta/4\pi & \text{for } x \in \partial\Omega \text{ (a sharp corner with angle } \theta) \end{cases}$$

On the other hand, we may also consider the vector test functions $\bar{\phi}_{,k}$ to have constant values, as, $\bar{\phi}_{,k}(\boldsymbol{\xi}) = \phi_{,k}(\mathbf{x})$. Then by applying the divergence theorem, and with the basic identities of fundamental solution (Eq. 10 and Eq. 12), we can obtain:

$$\begin{aligned} &\int_{\partial\Omega} n_i(\boldsymbol{\xi}) \psi_i(\mathbf{x}) \phi_{,k}^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &+ \int_{\partial\Omega} e_{ikt} D_t \phi(\mathbf{x}) \phi_{,i}^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &- \int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) \phi_{,k}(\mathbf{x}) dS + \frac{1}{2} \phi_{,k}(\mathbf{x}) = 0 \quad \mathbf{x} \in \partial\Omega \end{aligned} \quad (14)$$

in which, we define the local coordinates, for numerical implementation purposes, at point \mathbf{x} on the boundary $\partial\Omega$, as shown in Fig. 2. We then have $\psi(\mathbf{x})$ in terms of $\phi_{,i}(\mathbf{x})$ on the boundary, as:

$$\begin{cases} \psi_3(\mathbf{x}) = q(\mathbf{x}) \\ \psi_1(\mathbf{x}) = t_i(\mathbf{x}) \phi_{,i}(\mathbf{x}) \\ \psi_2(\mathbf{x}) = s_i(\mathbf{x}) \phi_{,i}(\mathbf{x}) \end{cases} \quad (15)$$

Therefore, the vector $\psi(\mathbf{x})$ in local coordinates comes from two physical terms: the gradient $q(\mathbf{x})$ is along the vector in the outward normal direction, and the other two gradients of the potential $\phi(\mathbf{x})$ are along the vectors in the other two tangential directions.

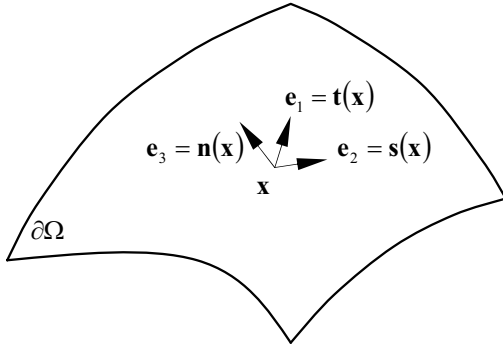


Figure 2 : The local coordinates at a boundary point \mathbf{x}

Generally, by using other carefully chosen weak forms of Eq. 3, any number of “properties” of the fundamental solution can be derived [Han and Atluri (2003a)]. Now, we can use properties 13 and 15 to derive simple, straightforward and elegant further regularizations of the strongly-singular BIEs for ϕ , and $\phi_{,k}$.

3.2 Regularization of ϕ -BIE, Eq. 7

Eq. 7 can be implemented numerically without any difficulty, if it is restricted only for boundary points, ie., $\mathbf{x} \in \partial\Omega$, because $n_k(\xi) \phi_{,k}^*(\mathbf{x}, \xi)$ contains only the weak singularity $[O(r^{-1})]$. Most researchers have implemented the ϕ -BIE based on this equation and solved the boundary problems. On the other hand, when one consider a domain point which is approaching the boundary, one may encounter the higher order singularity $[O(r^{-2})]$ with Eq. 7. We suggest here ways to remedy this.

Subtracting Eq. 10 from Eq. 7, we obtain:

$$\begin{aligned} & \int_{\partial\Omega} q(\xi) \phi^*(\mathbf{x}, \xi) dS \\ & - \int_{\partial\Omega} [\phi(\xi) - \phi(\mathbf{x})] \Theta^*(\mathbf{x}, \xi) dS \\ & + \int_{\Omega} k^2 \phi^*(\mathbf{x}, \xi) \phi(\mathbf{x}) d\Omega = 0 \end{aligned} \quad (16)$$

With Eq. 12, Eq. 16 is applicable at point \mathbf{x} on the boundary $\partial\Omega$, as:

$$\begin{aligned} & \int_{\partial\Omega} q(\xi) \phi^*(\mathbf{x}, \xi) dS \\ & - \int_{\partial\Omega} [\phi(\xi) - \phi(\mathbf{x})] \Theta^*(\mathbf{x}, \xi) dS \\ & = \int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \xi) \phi(\mathbf{x}) dS + \frac{1}{2} \phi(\mathbf{x}) \quad \mathbf{x} \in \partial\Omega \end{aligned} \quad (17)$$

in which $\phi(\xi) - \phi(\mathbf{x})$ becomes $O(r)$ when $\xi \rightarrow \mathbf{x}$, and at the same time Eq. 17 becomes weakly-singular $[O(r^{-1})]$. A reference node on the boundary may be used for a point close to the boundary, for regularization purpose. [Han and Atluri (2003a)]. Therefore, one can evaluate all the integrals in Eq. 17 numerically, for both the boundary points and the points close to the boundary. We refer to Eq. 17 as the regularized ϕ -BIE or “R- ϕ -BIE”, which involves singularities of order $O(r^{-1})$ only.

If $\partial\Omega$ has corners, ϕ may be expected to have a variation of $r^{+\lambda}$ ($\lambda < 1$) near the corners. In such cases, $\phi(\xi) - \phi(\mathbf{x})$ may become $O(r^{\lambda-1})$ when $\xi \rightarrow \mathbf{x}$, and thus, in a theoretical sense, Eq. 17 is no longer weakly singular. However, in a numerical solution of R- ϕ -BIE Eq. 17 directly, through a collocation process, to derive a ϕ Boundary Element Method (BEM-R- ϕ -BIE), we envision using only C^0 polynomial interpolations of ϕ and q . Thus, in the numerical implementation of the BEM-R- ϕ -BIE by a collocation process, we encounter only weakly singular integrals. By using C^0 elements and employing an adaptive boundary-element refinement strategy near corners at the boundary, one may extract the value of ($\lambda < 1$) in the asymptotic solution for ϕ near such a corner [Qian, Han and Atluri (2004)].

Also, a Petrov-Galerkin scheme can be used to write the weak-form for Eq. 17. By choosing the test function to be identical to a function which is energy-conjugate to $\phi(\mathbf{x})$, viz. the trial function, we obtain the symmetric Galerkin ϕ -BIE form (hereafter referred to as the SGBEM-R- ϕ -BIE). This novel formulation for a

Symmetric Galerkin Boundary Element Method for the weakly singular BEM-R- ϕ -BIE is presented elsewhere [Qian, Han and Atluri (2004)].

3.3 Regularization of q -BIE, Eq. 9

By subtracting Eq. 9 from Eq. 14, and contracting with $n_k(\mathbf{x})$ on both sides, we can obtain the fully regularized form of Eq. 9 as,

$$\begin{aligned} -\frac{1}{2}q(\mathbf{x}) &= \int_{\partial\Omega} [q(\boldsymbol{\xi}) - n_i(\boldsymbol{\xi})\psi_i(\mathbf{x})] \hat{\Theta}^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &+ \int_{\partial\Omega} k^2 n_k(\mathbf{x}) n_k(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}) \phi^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &+ \int_{\partial\Omega} [D_t \phi(\boldsymbol{\xi}) - (D_t \phi)(\mathbf{x})] n_k(\mathbf{x}) e_{ikt} \phi_{,i}^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &+ \int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) q(\mathbf{x}) dS \end{aligned} \quad (18)$$

where the kernel function $\hat{\Theta}^*(\mathbf{x}, \boldsymbol{\xi})$ is defined as,

$$\hat{\Theta}^*(\mathbf{x}, \boldsymbol{\xi}) = -\frac{\partial \phi^*(\mathbf{x}, \boldsymbol{\xi})}{\partial n_x} = n_k(\mathbf{x}) \frac{\partial \phi^*(\mathbf{x}, \boldsymbol{\xi})}{\partial \xi_k} \quad \boldsymbol{\xi} \in \partial\Omega$$

We label Eq. 18 as the regularized q -BIE, or “R- q -BIE”. Suppose $\partial\Omega$ is smooth; then $[q(\boldsymbol{\xi}) - n_i(\boldsymbol{\xi})\psi_i(\mathbf{x})]$ and $[D_t \phi(\boldsymbol{\xi}) - (D_t \phi)(\mathbf{x})]$ become $O(r)$ when $\boldsymbol{\xi} \rightarrow \mathbf{x}$, and Eq. 18 becomes weakly singular [$O(r^{-1})$] on a 3D problem. Thus, one can evaluate all the integrals in Eq. 18 numerically, and applicable to any point \mathbf{x} on the boundary $\partial\Omega$. On the other hand, if $\partial\Omega$ has corners, $[q(\boldsymbol{\xi}) - n_i(\boldsymbol{\xi})\psi_i(\mathbf{x})]$ and $[D_t \phi(\boldsymbol{\xi}) - (D_t \phi)(\mathbf{x})]$ become $O(r^{\lambda-1})$ when $\boldsymbol{\xi} \rightarrow \mathbf{x}$, and thus, in a theoretical sense, Eq. 18 is no longer weakly singular. However, in a numerical implementation of the R- q -BIE, viz. Eq. 18, directly, through a collocation process, to derive a q Boundary Element Method (BEM-R- q -BIE), we envision using only C^0 polynomial interpolations of ϕ and q . Thus, in the numerical implementation of the BEM- q -BIE by a collocation of Eq. 18, we encounter only weakly singular integrals.

Similarly, a Petrov-Galerkin scheme can be used to write the weak-form for Eq. 18. By choosing the test function to be identical to a function which is energy-conjugate to $q(\mathbf{x})$, viz. the trial function $\hat{\phi}(\mathbf{x})$, we obtain the symmetric Galerkin q -BIE form (herein referred to as the SGBEM-R- q -BIE). This novel formulation for a SGBEM is presented elsewhere. [Qian, Han and Atluri (2004)]

Now, the ϕ -BIE and the q -BIE have been fully desingularized [i.e., they contain singularities of order $O(r^{-1})$ only

in 3-D problems] simply, and elegantly in the present work.

4 Numerical results

In the implementation process, the BEM-R- ϕ -BIE and BEM-R- q -BIE are combined as in Eq. 19, and the coefficient β is chosen to be a small complex number.

$$\begin{aligned} &\int_{\partial\Omega} q(\boldsymbol{\xi}) \phi^*(\mathbf{x}, \boldsymbol{\xi}) dS - \int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) dS \\ &- \frac{1}{2}\phi(\mathbf{x}) - \int_{\partial\Omega} [\phi(\boldsymbol{\xi}) - \phi(\mathbf{x})] \Theta^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &+ \beta \left\{ \frac{1}{2}q(\mathbf{x}) + \int_{\partial\Omega} [q(\boldsymbol{\xi}) - n_i(\boldsymbol{\xi})\psi_i(\mathbf{x})] \hat{\Theta}^*(\mathbf{x}, \boldsymbol{\xi}) dS \right. \\ &+ \int_{\partial\Omega} k^2 n_k(\mathbf{x}) n_k(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}) \phi^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &+ \int_{\partial\Omega} [D_t \phi(\boldsymbol{\xi}) - (D_t \phi)(\mathbf{x})] n_k(\mathbf{x}) e_{ikt} \phi_{,i}^*(\mathbf{x}, \boldsymbol{\xi}) dS \\ &\left. + \int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) q(\mathbf{x}) dS \right\} = 0 \end{aligned} \quad (19)$$

In order to check the accuracy and efficiency of the proposed method, two different representative acoustic problems are considered: (1) the pulsating sphere problem; and (2) acoustic scattering from a rigid sphere.

4.1 Pulsating sphere

The field radiated from a pulsating sphere into the infinite homogeneous medium is chosen as an example for the exterior problem. The analytical solution for the acoustic pressure for a sphere of radius a , pulsating with uniform radial velocity v_a , is given by [Chien, Rajiyah, and Atluri (1990)]

$$\frac{p(r)}{z_0 v_a} = \frac{a}{r} \frac{ika}{1 + ika} e^{-ik(r-a)} \quad (20)$$

where z_0 is the characteristic impedance, $p(r)$ is the acoustic pressure at distance r , and k is the wave number. For the purpose of comparison, BEM-R- $[\phi \& q]$ -BIE and SGBEM-R- $[\phi \& q]$ -BIE, the whole sphere is considered for modeling: a 24 element model and a 64 element model, as shown in Fig. 3. The models are discretized by using 8-node isoparametric quadrilateral elements. The evaluation of all integrals of kernels is performed by using 3x3 standard Gaussian quadrature.

In Fig. 4 and Fig. 5, the real and imaginary parts of dimensionless surface acoustic pressures are plotted with

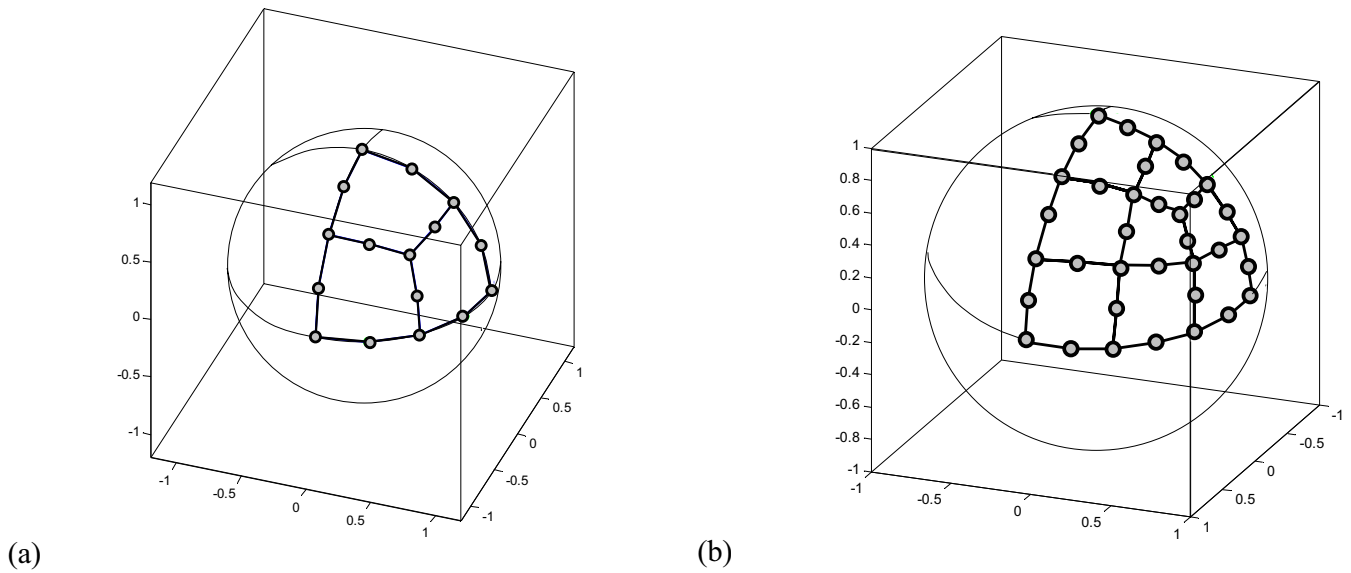


Figure 3 : Surface discretization with quadrilateral elements (a) 24 element model; (b) 64 element model

respect to the reduced frequency ka . Fig. 4 presents the numerical solutions with 24 elements, while results with 64 elements are plotted in Fig. 5. The present results are seen to converge to the analytical solution, with a mesh refinement. It is obvious that the conventional BIE method fails to provide unique solutions near $k = \pi, 2\pi \dots$, which is also demonstrated in many earlier works, such as [Yan, Hung, and Zheng (2003)]. The present BEM-R- $[\phi \& q]$ -BIE solutions and exact solution have a good agreement between with ka up to 8.0. The accuracy of BEM-R- $[\phi \& q]$ -BIE is lower than that of the SGBEM in the coarse model (24 element) as shown in Fig. 4; however, by refining the mesh size, an acceptable result can be obtained easily with a comparatively low computation cost as shown in Fig. 4, by using BEM-R- $[\phi \& q]$ -BIE. The other method of increasing the accuracy of BEM-R- $[\phi \& q]$ -BIE is to use higher order Gaussian Quadrature. The computational costs shown in Tab. 1 are based on the MATLAB code running on the desktop with 1.5 GHz Intel Pentium IV CPU, and 512MB Memory. The SGBEM-R- $[\phi \& q]$ -BIE is much slower, because of the double integral evaluation for every element, and the SGBEM's code for evaluating weakly singular integrals is more complicated than in BEM [Qian, Han and Atluri (2004)]. On the contrary, the BEM-R- $[\phi \& q]$ -BIE only encounters single integral evaluation for every element, and therefore, it is faster than the SGBEM.

Table 1 : The comparison of the computational costs between BEM-R- $[\phi \& q]$ -BIE and SGBEM-R- $[\phi \& q]$ -BIE

	24 Element Model	64 Element Model
BEM-R- $[\phi \& q]$ -BIE	60 s	378 s
SGBEM-R- $[\phi \& q]$ -BIE	284 s	1346 s

4.2 Scattering from a rigid sphere

To render the present method to be applicable to scattering problems, only a small change is necessary to Eq. 17, as:

$$\begin{aligned}
 \frac{1}{2}\phi(\mathbf{x}) - \phi^i(\mathbf{x}) &= \int_{\partial\Omega} q(\boldsymbol{\xi}) \phi^*(\mathbf{x}, \boldsymbol{\xi}) dS \\
 &- \int_{\partial\Omega} [\phi(\boldsymbol{\xi}) - \phi(\mathbf{x})] \Theta^*(\mathbf{x}, \boldsymbol{\xi}) dS \\
 &- \int_{\partial\Omega}^{CPV} \Theta^*(\mathbf{x}, \boldsymbol{\xi}) \phi(\mathbf{x}) dS \quad (21)
 \end{aligned}$$

where $\phi^i(\mathbf{x})$ is the incident acoustic potential; and also a similar change to Eq. 18 is needed. In this example, the acoustic scattering of plane incident waves, with a unit amplitude (e^{-ikx}), from a rigid sphere is considered. Thereby, $\partial\phi/\partial n = 0$ on the surface of the sphere. The magnitudes of the ratio of $\phi^s(\mathbf{x})$ to $\phi^i(\mathbf{x})$, where $\phi(\mathbf{x}) = \phi^s(\mathbf{x}) + \phi^i(\mathbf{x})$, at $r = 5a$ are plotted against the

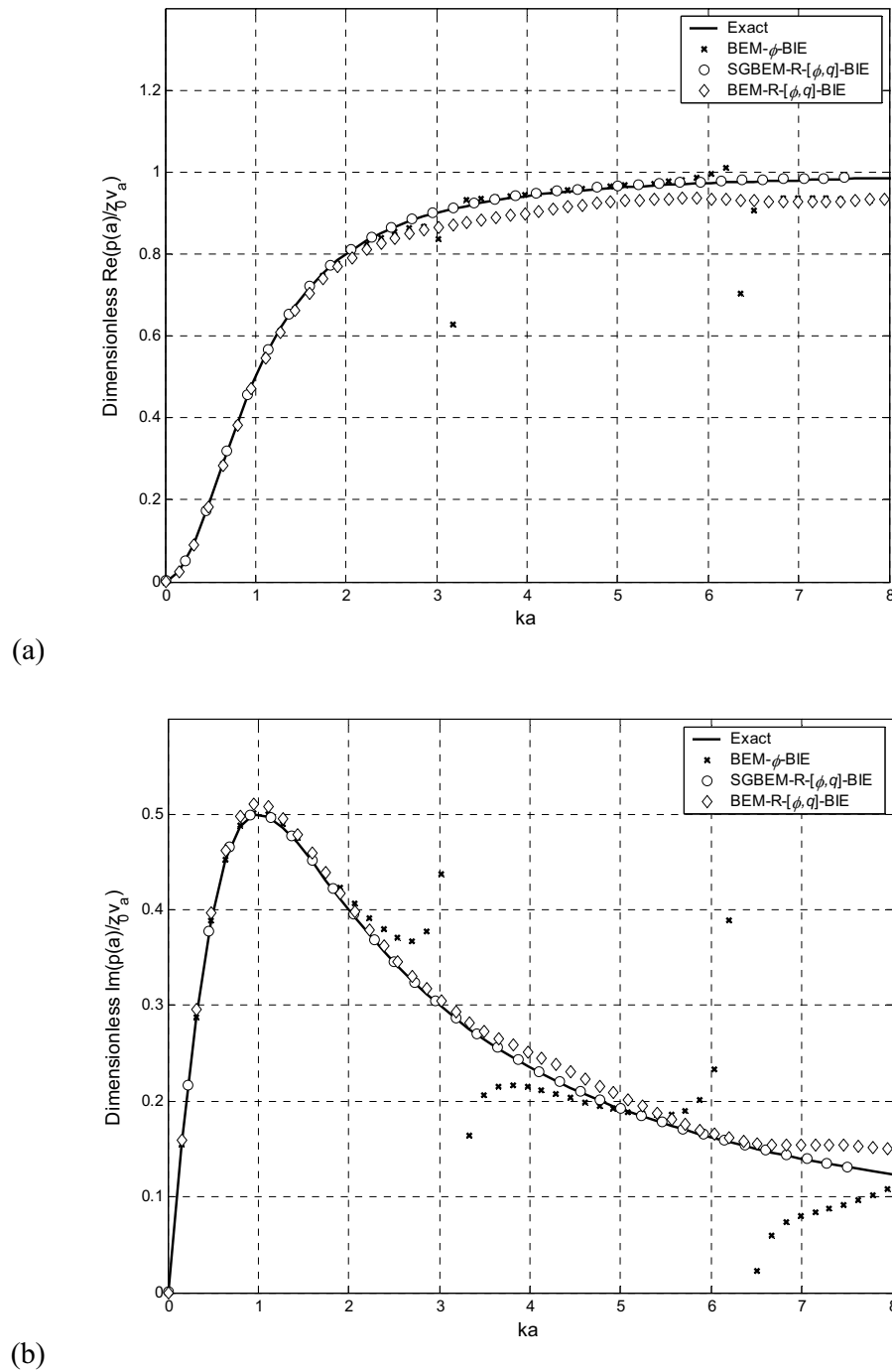


Figure 4 : Dimensionless surface acoustic pressure of a pulsating (24 elements): (a) real part; (b) imaginary part

angle, for one of the fictitious frequencies of the exterior problem, as $ka = \pi$. Four discretization models are used here, in which N is the total number of element on the whole sphere, and the analytical solution is computed by the equation in [Morse, Ingard (1968)].

Fig. 6 is a comparison of the analytical solution and four element models, in which only regular 8 node quadrilateral elements are used. The solutions show that the method converges, as the number of elements increases, and the BEM-R-[ϕ & q]-BIE solutions have a fairly good

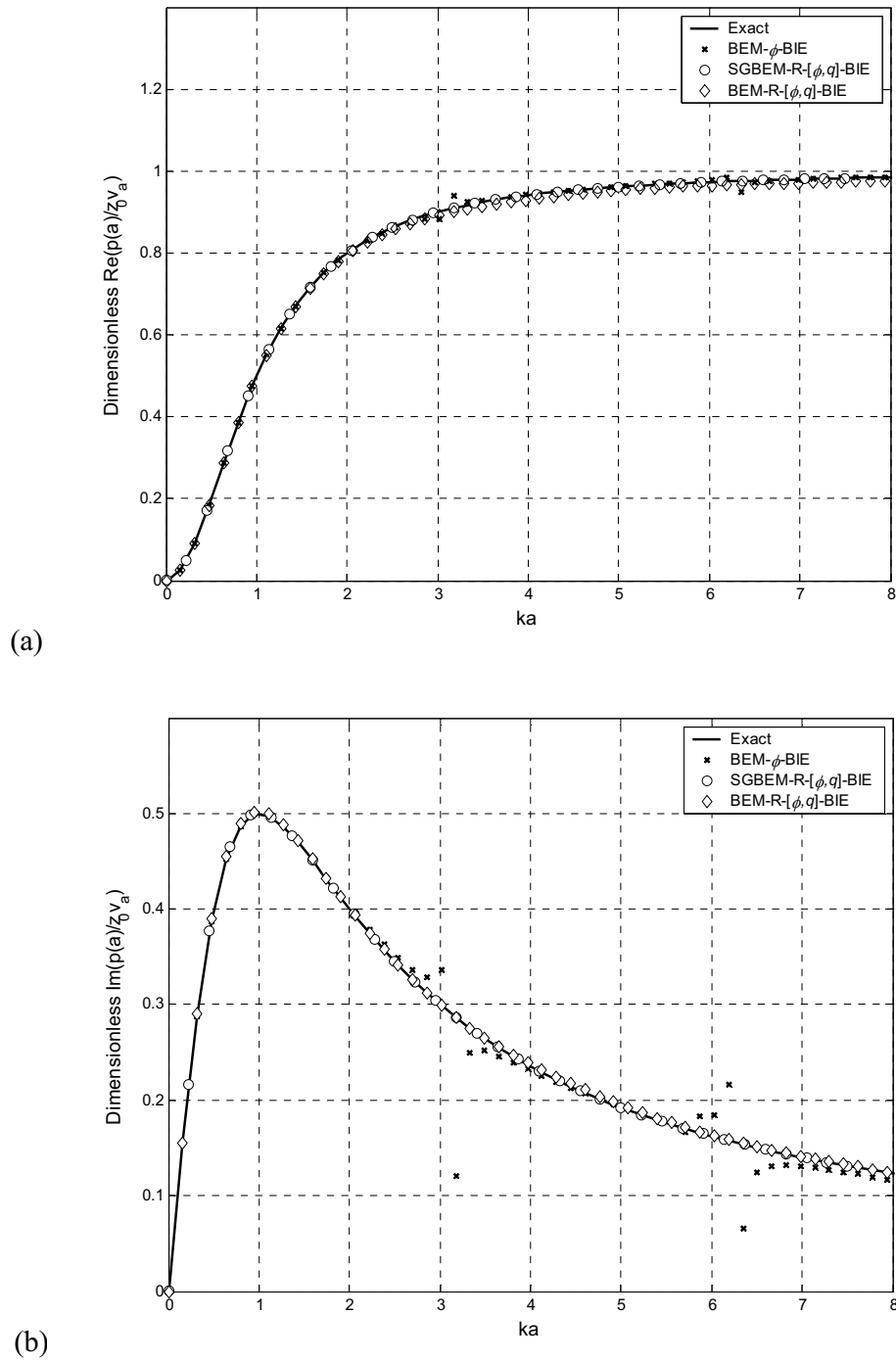


Figure 5 : Dimensionless surface acoustic pressure of a pulsating (64 elements): (a) real part; (b) imaginary part

agreement with the analytical solution, with the use of a relatively small number of elements. Moreover, the C^0 elements have been demonstrated by this example to give fairly good results for most values of θ except near the forward scattering direction.

5 Conclusion

The weak-form of Helmholtz differential equation with vector test-functions (which are spatial gradients of the free-space fundamental solutions) is employed, as the basis in the present paper in order to directly derive non-

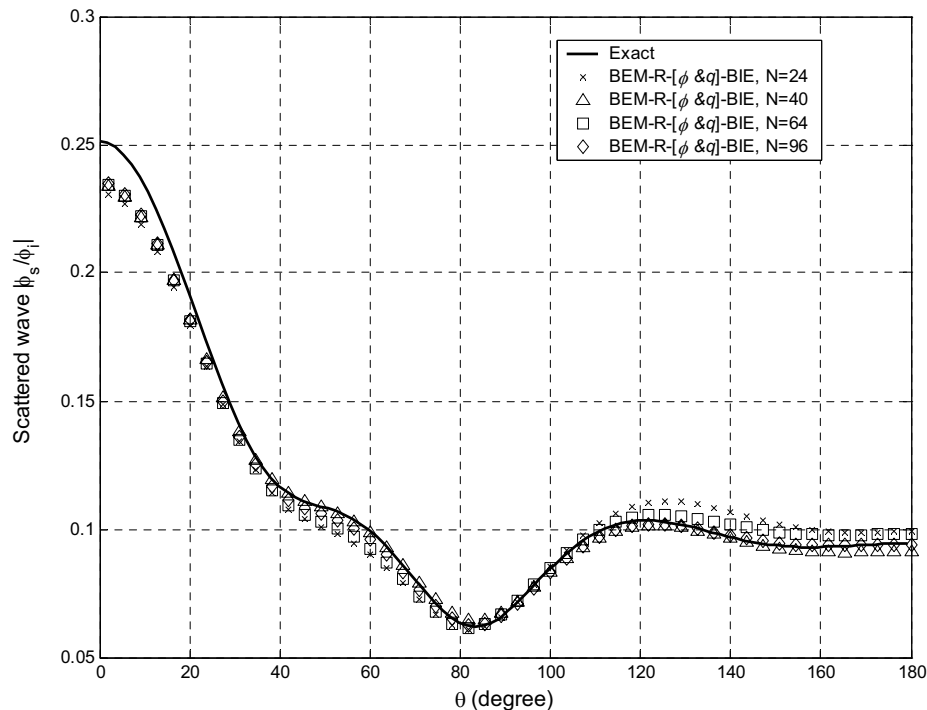


Figure 6 : Scattering from the rigid sphere at $r = 5a$, $ka = \pi$; with BEM-R- $[\phi \& q]$ -BIE

hyper-singular boundary integral equations. Thereby, the difficulties with hyper-singular integrals, involved in the composite Helmholtz integral equations presented by Burton and Miller [Burton and Miller (1971)], can be overcome. Further desingularization of the strongly singular integrals to the order of $O(r^{-1})$ is made possible with the use of certain basic identities of the fundamental solution of the Helmholtz differential equation for potential. Therefore, the formulation of the novel weakly singular $[O(r^{-1})]$ boundary integral equations is based on the weak-form of Helmholtz differential equation only. These new weakly-singular integral equations designated as R- ϕ -BIE and R- q -BIE, respectively, are solved by using direct collocations. The attendant boundary element methods are desingularized as BEM-R- $[\phi \& q]$ -BIE in this paper. The weak singularities make the present approach highly accurate and efficient in the numerical implementation, and the non-uniqueness problem is resolved as well. Moreover, there is no requirement of smoothness of the chosen trial functions for ϕ and q , and C^0 continuity is sufficient for numerical implementation.

Fig. 7 compares the element size and wavelength for achieving accuracies with under 5% and 10% error, in pulsating sphere problem by using BEM-R- ϕ -BIE, and

BEM-R- q -BIE. The error is measured in the magnitude of the acoustic pressure at $r = 5a$. For the boundary element method, and the finite element method, it is well known that the mesh size has to be less than the wave length of the acoustic wave, in order to obtain an acceptable solution. This is shown in Fig. 7. Further effort will be made in extending the present approach, using the Meshless Local Petrov Galerkin approach, to develop MLPG-R- ϕ -BIE, and MLPG-R- q -BIE, respectively. These MLPG methods are expected to be a good way to improve the mesh size requirement in the numerical methods for solving the present R- $[\phi \& q]$ -BIE.

An alternate approach to cope with very high frequency acoustic radiation and scattering problems is to use the method of asymptotics pioneered by Ufimtsev [2003] in electromagnetics, and is being developed in the context of asymptotics by Ufimtsev and Atluri [2004a, 2004b].

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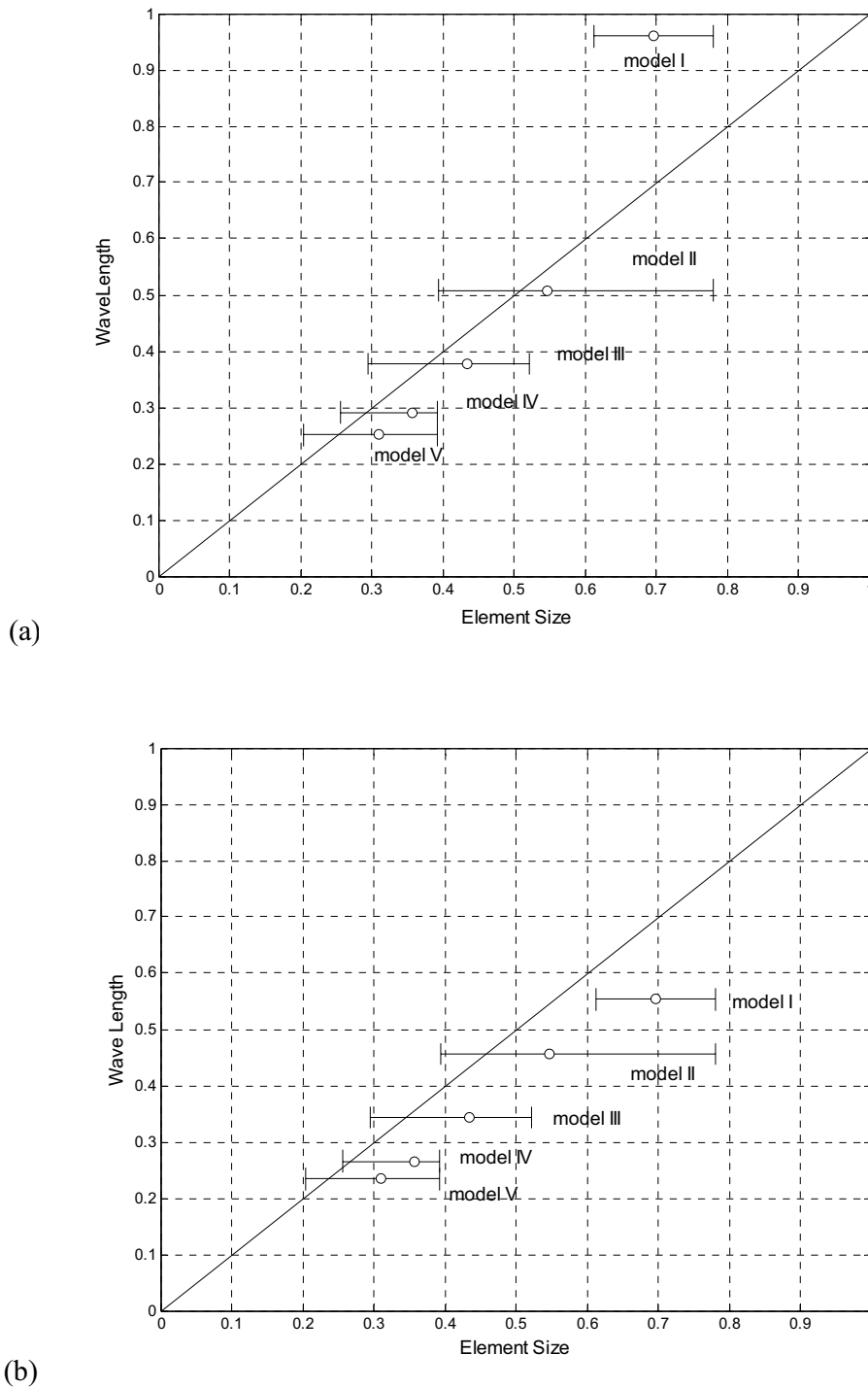


Figure 7 : Relationship between wave length and element size in pulsating sphere case (a) 5% error; (b) 10% error. model I: N=24; II: N=40; III: N=64; IV: N=96; V: N=128.

References

Atluri, S. N. (1985): Computational solid mechanics (finite elements and boundary elements) present status and

future directions, *The Chinese Journal of Mechanics*.

Atluri, S. N.; Han, Z. D.; Shen, S. (2003): Meshless Local Petrov-Galerkin (MLPG) approaches for weakly-

- singular traction & displacement boundary integral equations, *CMES: Computer Modeling in Engineering & Sciences* 4(5): 507-516.
- Atluri, S. N.; Shen, S.** (2002a): *The meshless local Petrov-Galerkin (MLPG) method*. Tech. Science Press, 440 pages.
- Atluri, S. N.; Shen, S.** (2002b): The meshless local Petrov-Galerkin (MLPG) method: A simple & less-costly alternative to the finite element and boundary element method. *CMES: Computer Modeling in Engineering & Sciences* 3(1): 11-52.
- Atluri, S. N.; Zhu, T.** (1998): A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. *Computational Mechanics* 22: 117-127.
- Burton, A.J., Miller, G.F.** (1971): The application of the integral equation method to the numerical solution of some exterior boundary value problems. *Proc. R. Soc. London Ser. A* 323: 201-210.
- Chien, C.C.; Rajiyah, H.; Atluri, S.N.** (1990): An effective method for solving the hypersingular integral equations in 3-D acoustics. *J. Acoust. Soc. Am.* 88(2): 918-937.
- Han, Z.D.; Atluri, S. N.** (2002): SGBEM (for Cracked Local Subdomain) – FEM (for uncracked global Structure) Alternating Method for Analyzing 3D Surface Cracks and Their Fatigue-Growth. *CMES: Computer Modeling in Engineering & Sciences* 3(6): 699-716.
- Han, Z.D.; Atluri, S.N.** (2003a): On simple formulations of weakly-singular tBIE&dBIE, and Petrov-Galerkin approaches, *CMES: Computer Modeling in Engineering & Sciences* 4(1): 5-20.
- Han, Z.D.; Atluri, S. N.** (2003b): Truly Meshless Local Petrov-Galerkin (MLPG) solutions of traction & displacement BIEs. *CMES: Computer Modeling in Engineering & Sciences* 4(6): 665-678.
- Liu, Y.J.; Chen, S.H.** (1999): A new form of the hypersingular boundary integral equation for 3-D acoustics and its implementation with C^0 boundary elements. *Computer Methods in Applied Mechanics and Engineering* 173: 375-386.
- Liu, Y.J.; Rizzo, F.J** (1992): A weakly-singular form of the hypersingular boundary integral equation applied to 3-D acoustic wave problems. *Computer Methods in Applied Mechanics and Engineering* 96: 271-287.
- Morse, P.M.; Ingard, K.U.** (1968): *Theoretical Acoustics*. McGraw-Hill Book, New York, 927 pages
- Okada, H.; Atluri, S. N.** (1994): Recent developments in the field-boundary element method for finite/small strain elastoplasticity, *Int. J. Solids Struct.* 31(12-13): 1737-1775.
- Okada, H.; Rajiyah, H.; Atluri, S. N.** (1989a): A Novel Displacement Gradient Boundary Element Method for Elastic Stress Analysis with High Accuracy, *J. Applied Mech.* April: 1-9.
- Okada, H.; Rajiyah, H.; Atluri, S. N.** (1989b): Non-hypersingular integral representations for velocity (displacement) gradients in elastic/plastic solids (small or finite deformations), *Comp. Mech.* 4: 165-175.
- Okada, H.; Rajiyah, H.; Atluri, S. N.** (1990): A full tangent stiffness field-boundary-element formulation for geometric and material non-linear problems of solid mechanics, *Int. J. Numer. Meth. Eng.* 29(1): 15-35.
- Qian, Z.Y.; Han, Z.D.; Atluri, S.N.** (2004): Directly Derived Non-Hyper-Singular Boundary Integral Equations for Acoustic Problems, and Their Solution through Petrov-Galerkin Schemes. *CMES: Computer Modeling in Engineering & Sciences* 5(6): 541-562.
- Reut, Z.** (1985): On the pboundary integral methods for the exterior acoustic problem. *J. Sound and Vib.* 103: 297-298.
- Ufimtsev, P.Ya** (2003) *Theory of Edge Diffraction in Electromagnetics*, Tech Science Press, Eucino, CA
- Ufimtsev, P.Ya; Atluri, S.N.** (2004a) An analysis of the classical Sommerfeld and Pauli asymptotics in the wedge diffraction problem, (submitted)
- Ufimtsev, P.Ya; Atluri, S.N.** (2004b) Classical computational methods in the theory of acoustics. (to appear)
- Wu, T.W.; Seybert, A.F.; Wan, G.C.** (1991): On the numerical implementation of a Cauchy principal value integral to insure a unique solution for acoustic radiation and scattering. *J. Acoust. Soc. Am.* 90: 554-560.
- Yan, Z.Y.; Hung, K.C.; Zheng, H.** (2003): Solving the hypersingular boundary integral equation in three-dimensional acoustics using a regularization relationship, *J. Acoust. Soc. Am.* 113(5): 2674-2683.