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A Comparative Investigation of Different Homogenization Methods for Prediction of the Macroscopic Properties of Composites

Qing-Sheng Yang[1,2], **Wilfried Becker**[3]

Abstract: The present paper focuses on the comparative investigation of different homogenization methods for fiber composites, void solids and rigid inclusion media. The effective properties of multi-phase media are calculated by three methods, i.e. direct average method of stress and strain, direct average method of strain energy and two-scale expansion method. A comprehensive comparison, in principle and numerically, of these methods is emphasized. It is obvious that the two direct average methods are identical in principle and therefore they give the same numerical results. It is shown that the two-scale expansion method is the same as the direct average concept of field quantities in principle but is expressed by different mathematical form so that it gives very close predictions with that of direct average methods.

keyword: composites; homogenization; effective properties; micromechanics

1 Introduction

A composite body is an assembly of periodic microscopic unit cells or representative volume elements (RVE), as shown in Fig.1. In a composite, all of global characteristics are the same in any RVE, irrespective of its position. Therefore, the analysis for the entire composite body can be replaced by an analysis for a RVE (Hashin, 1983; Mura, 1987).

Effective or macroscopic properties of the composites depend on the geometric and physical properties of the phases. The effective properties can be found by a homogenization approach, based on the microscopic fields and local properties of the heterogeneous media. There are many different homogenization approaches. For example, the direct average method of local field quantities, equivalent inclusion method, bounding method and two-scale expansion method are used to evaluate the effective properties of the composites and give a reasonable prediction within allowable precision. The direct average method is based on the surface or volume average of the field quantities like stresses, strains and strain energy density on the microscopic level and then the effective properties can be found on the macroscopic level, according to the relations between the macroscopic stresses and strains. The direct average of the local field quantities can be performed by the concept of RVE and a numerical method, FEM or BEM, for instance. One of the advantages of applying the direct average method is that arbitrary geometries and properties of components can be easily treated.

The equivalent inclusion method is based on the Eshelby's eigenstrain solution on the single inclusion embedded into an infinite matrix (Eshelby, 1957). This method need not use the average of field quantities for analysis of the effective properties of composites. But the effective properties can be derived in terms of the volume fraction, geometries of the inclusion and the properties of the components. Self-consistent scheme (Hill, 1965; Budiansky, 1965; Taya and Chou, 1981), generalized self-consistent scheme(Christensen and Lo, 1979), differential method(Norries, 1985) and Mori-Tanaka method (Mori and Tanaka, 1973; Weng, 1984; Benveniste, 1987) have been developed from this approach and widely used in evaluation of the elastic properties of various composite materials. However, arbitrary microstructural geometry that is frequently encountered in actual materials cannot be deterministically treated with these models. Fortunately, the situation is changing. A numerical procedure that combines the self-consistent scheme with finite element method has been developed for prediction of the effective properties of composites with arbitrary shaped inclusions (Yang, Tang and Chen, 1994).

[1] Department of Engineering Mechanics, Beijing University of Technology, Beijing 100022, China
[2] Corresponding author: Tel: +86-10-67392760, Fax: +86-10-67391617, Email: qsyang@bjut.edu.cn
[3] Institute of Mechanics, Darmstadt University of Technology, D-64289, Germany

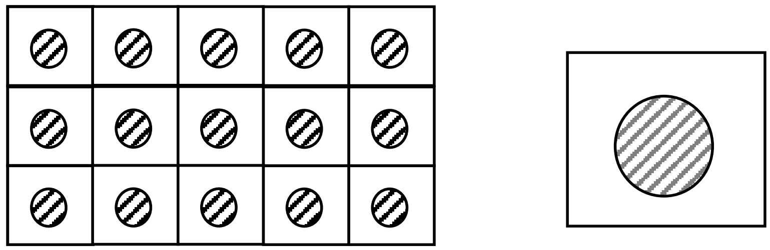


Figure 1 : Periodic microstructure and RVE

Alternatively to direct average and equivalent inclusion methods, the variational method is a unique one that can give the upper and lower bounds of the elastic moduli of composites (Hashin and Shtrikman, 1962). This method has already improved the earlier results of approximate bounds by Voigt (1889) and Reuss (1929).

A relatively new approach, named two-scale expansion method, for homogenization of microstructure consists in the mathematical homogenization based on a two-scale expansion of the displacement field, which originated for analyzing physical systems containing two or more length scales (Benssonusan et al, 1978; Sanchez-Palencia, 1980). It is suitable for multi-phase materials in which the natural scales are the microscopic heterogeneity scale and macroscopic structure scale. In this method, the displacement field is expanded in micro-scale and macro-scale, then following the same approach as direct average method, the effective stiffness coefficients can be found. Several homogenization methods have been compiled in the book by Nemat-Nasser and Hori (1993). A critical review of different homogenization methods and applications in cellular sandwich structures can be found in recent article by Hohe and Becker (2002). The micromechanical theory has been extended into the investigation of flow in pulmonary tissue (Zhong et al 2002). The homogenization of the particulate composites using BEM technology was proposed (Okada et al 2004; Yang and Qin 2004).

Although these homogenization approaches have been widely applied to composites, cracked solids and other multiphase media, the comparative research on different homogenizations is very limited and the critical precision analysis and comparison are lacking so far. Some consistencies or inconsistencies in principle and in quantity exist among these homogenization methods. The present paper focuses on a comprehensive comparison and tends to give a critical review for the different homogenization concepts. Especially, the principles and numerical results of the direct average methods and two-scale expansion method are analyzed.

The article is organized as follows: In the next section, the definition of the effective properties and the direct average methods are reviewed. Then the consistency between the volume average and surface average procedures is identified. In section 3, the two-scale expansion method is outlined. Here the numerical procedure in conjunction with finite element method is emphasized. In section 4, the approximate estimation of the effective properties of composites based on two-scale expansion method is presented. The periodic boundary conditions of RVE are described in section 5. The numerical comparisons of the different homogenization concepts are performed for transversely isotropic composites, void solids and rigid inclusion media in section 6 and then the conclusions of the research are given in final section.

2 Direct average method of field quantities

In the direct average method, the average values of the microscopic field quantities like the stresses, strains or strain energy densities, are calculated by a volume or surface average procedure in a domain, then the effective properties of the composites are predicted according to the relations of the macroscopic stresses, strains and strain energy density.

The volume average of local or microscopic stresses σ_{ij} and strains ε_{ij} can be defined by

$$\overline{\sigma}_{ij} = \frac{1}{V}\int_{\Omega}\sigma_{ij}\mathrm{d}\Omega, \quad \overline{\varepsilon}_{ij} = \frac{1}{V}\int_{\Omega}\varepsilon_{ij}\mathrm{d}\Omega \tag{1}$$

where the superscript bars denote the volume average values of the quantities, e.g. macroscopic or effective quantities. The Ω denotes a domain over which the homogenization is performed and V is its volume. Generally, the domain can be represented by a RVE for the composite with a periodic microstructure. For elastic body, the volume average of the strain energy density can be expressed by

$$\begin{aligned}\overline{w} &= \frac{1}{V}\int_{\Omega} w\mathrm{d}\Omega = \frac{1}{V}\int_{\Omega}\frac{1}{2}\sigma_{ij}\varepsilon_{ij}\mathrm{d}\Omega \\ &= \frac{1}{V}\int_{\Omega}\frac{1}{2}D_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\mathrm{d}\Omega \\ &= \frac{1}{V}\int_{\Omega}\frac{1}{2}C_{ijkl}\sigma_{ij}\sigma_{kl}\mathrm{d}\Omega \end{aligned} \tag{2}$$

where $\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = w$ is strain energy density, D_{ijkl} are local stiffness coefficients and $C_{ijkl}(\mathbf{C} = \mathbf{D}^{-1})$ are local compliance coefficients which are different from phase to phase.

The effective properties which are represented by effective stiffness $\overline{D}_{ijkl}$ or compliance $\overline{C}_{ijkl}$ of the composites can be defined by the elastic relation between the average stresses and strains

$$\overline{\sigma}_{ij} = \overline{D}_{ijkl}\overline{\varepsilon}_{kl}, \quad \overline{\varepsilon}_{ij} = \overline{C}_{ijkl}\overline{\sigma}_{kl} \tag{3}$$

or by equivalence of the strain energy

$$\frac{1}{2}\overline{\sigma}_{ij}\overline{\varepsilon}_{ij} = \frac{1}{V}\int_{\Omega}\frac{1}{2}\sigma_{ij}\varepsilon_{ij}\mathrm{d}\Omega \tag{4}$$

This relation is referred to as the Hill's principle (Hill, 1963; Kroner, 1972; Hazanov 1998).

The linear dependence of the average stresses and strains for elastic body leads to

$$\overline{D}_{ijkl} = \frac{\partial^2\overline{w}}{\partial\overline{\varepsilon}_{ij}\partial\overline{\varepsilon}_{kl}} \tag{5}$$

An explicit form of the effective stiffness coefficients for 3D elastic deformation has been obtained by Hohe and Becker (2001), that is

$$\overline{D}_{ijkl} = \begin{cases} 2\overline{w}(\varepsilon_{ij})\frac{1}{\overline{\varepsilon}_{ij}^2} & i=j,k=l,i=k \\ \overline{w}(\varepsilon_{ij})\frac{1}{2\overline{\varepsilon}_{ij}^2} & i\neq j,k\neq l,i=k,j=l \\ [\overline{w}(\varepsilon_{ij},\varepsilon_{kl}) - \overline{w}(\varepsilon_{ij}) - \overline{w}(\varepsilon_{kl})]\frac{1}{\overline{\varepsilon}_{ij}\overline{\varepsilon}_{kl}} & \\ i=j,k=l,i\neq k & (no\,sum\,for\,i,j) \\ [\overline{w}(\varepsilon_{ij},\varepsilon_{kl}) - \overline{w}(\varepsilon_{ij}) - \overline{w}(\varepsilon_{kl})]\frac{1}{4\overline{\varepsilon}_{ij}\overline{\varepsilon}_{kl}} & \\ i\neq j,k\neq l,(i\neq k\ or\ j\neq l) & \\ [\overline{w}(\varepsilon_{ij},\varepsilon_{kl}) - \overline{w}(\varepsilon_{ij}) - \overline{w}(\varepsilon_{kl})]\frac{1}{2\overline{\varepsilon}_{ij}\overline{\varepsilon}_{kl}} & \\ i=j,k\neq l & \end{cases} \tag{6}$$

where $\overline{w}(\varepsilon_{ij},\varepsilon_{kl})$ denotes the strain energy density for a reference strain state in which only ε_{ij} and ε_{kl} have non-zero values.

The effective quantities of the stresses, strains and strain energy density can be calculated by a surface average procedure for corresponding boundary values. For strains $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, applying the divergence theorem in the second equation of Eq.(1) yields

$$\overline{\varepsilon}_{ij} = \frac{1}{V}\int_{\Omega}\varepsilon_{ij}\mathrm{d}\Omega = \frac{1}{V}\int_{\Gamma}\frac{1}{2}(u_i n_j + u_j n_i)\mathrm{d}\Gamma \tag{7}$$

where Γ is boundary of the domain Ω and n_i is the outward normal vector on the boundary Γ.

The surface average of the stresses can be carried out by partial integration of the first equation of Eq.(1), that is

$$\overline{\sigma}_{ij} = \frac{1}{V}\int_{\Omega}\sigma_{ij}\mathrm{d}\Omega = \frac{1}{V}\int_{\Gamma}\frac{1}{2}(T_i x_j + T_j x_i)\mathrm{d}\Gamma \tag{8}$$

where T_i is the traction vector on the surface of Ω. This implies that the relations $T_i = \sigma_{ij}n_j$ hold. It is shown from Eq.(8) that the average stresses can be calculated by the process of volume average of the stresses in a domain or by the process of surface average of the corresponding tractions on the boundary.

Let us consider, for illustration, that the RVE is a brick shaped domain Ω, as shown in Fig.2, which has been used in most research works. The surface averages of the tractions can be expressed by the following form

$$\overline{\sigma}_{11} = \frac{1}{b}\int_{BC}\sigma_{11}\mathrm{d}\Gamma, \quad \overline{\sigma}_{22} = \frac{1}{a}\int_{DC}\sigma_{22}\mathrm{d}\Gamma \tag{a}$$

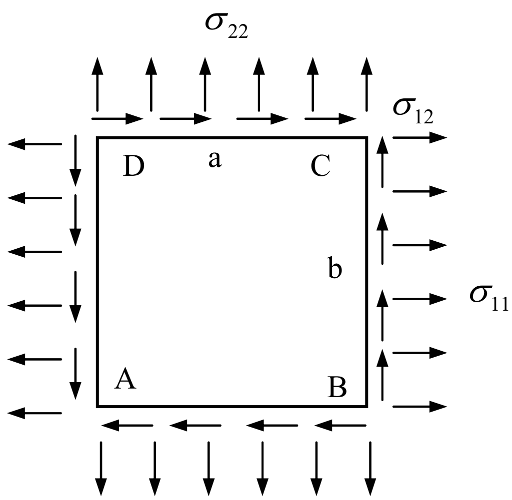


Figure 2 : Traction condition of RVE

$$\overline{\sigma}_{12} = \overline{\sigma}_{21} = \frac{1}{b}\int_{BC}\sigma_{12}\mathrm{d}\Gamma = \frac{1}{a}\int_{DC}\sigma_{21}\mathrm{d}\Gamma \tag{b}$$

The average value of the strain energy density can be expressed also by the boundary values according to the work-energy principle

$$\overline{w} = \frac{1}{V}\int_{\Omega}\frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}\mathrm{d}\Omega = \frac{1}{V}\int_{\Gamma}T_i u_i \mathrm{d}\Gamma \tag{9}$$

In fact, using the Green's theorem, one can easily prove Eq.(9) mathematically.

As is well known, the average of stresses, strains and strain energy density can be calculated by either volume or surface average processes. Once the two of the three quantities are found, then the effective properties of the composites can be predicted according to Eq.(3) or (4).

It is worthwhile to note that the macroscopic stresses, strains and strain energy density can be expressed by the phase volume fractions. For a n-phase composite, the stresses, strains and strain energy density can be written as

$$\overline{\sigma}_{ij} = \sum_{i=1}^{n} v_i \overline{\sigma}_{ij}^{(i)}, \quad \overline{\varepsilon}_{ij} = \sum_{i=1}^{n} v_i \overline{\varepsilon}_{ij}^{(i)}, \quad \overline{w} = \sum_{i=1}^{n} v_i \overline{w}^{(i)} \tag{10}$$

where superscript (i) corresponds with ith phase of the composite and the quantities v_i are referred to volume fractions of the corresponding phases.

Remark 1: *The volume or surface average method of micro-stresses, strains or strain energy density is most direct and basic approach to establish the connection between the micro- and macro-scale quantities. The definition of the effective properties is determinative and explicit.*

3 Two-scale expansion method

In this section, we introduce two coordinate systems: global coordinate x and local coordinate y. The global coordinate x is related to the local coordinate y as

$$y = x/\varepsilon \tag{11}$$

where ε is a very small positive number denoting the ratio between the dimension of a RVE and a structure body. When subjected to structural level loads and displacements, the resulting evolutionary variables, e.g. deformation and stresses, vary from point to point at the macroscopic scale x. Furthermore, a high level of heterogeneity in the microstructure causes a rapid variation of these variables in a small neighborhood ε of the macroscopic point x. In present homogenization theory, a periodic repetition of the microstructure about a macroscopic point x has been assumed, from which the field functions depend periodically on $y = x/\varepsilon$. This characteristic is often termed as Y- periodicity, where Y corresponds to a RVE (Peng and Cao, 2002; Ghosh, Lee and Moorthy, 1996)

The displacement field can be asymptotically expanded as

$$u_i = u_i^{\varepsilon}(x) = u_i^0(x,y) + \varepsilon u_i^1(x,y) + \varepsilon^2 u_i^2(x,y) + \cdots \tag{12}$$

The superscript ε denotes association of the function with the two length scales.

It is noted that

$$\frac{\partial F^{\varepsilon}(x,y)}{\partial x_i} = \frac{\partial F(x,y)}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial F(x,y)}{\partial y_i} \tag{13}$$

where F is a general function. For the strain tensor ε_{ij}, one has

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$
$$= \frac{1}{\varepsilon}\varepsilon_{ij}^{-1}(x,y) + \varepsilon_{ij}^0(x,y) + \varepsilon\varepsilon_{ij}^1(x,y) + \cdots \tag{14}$$

where

$$\varepsilon_{ij}^{-1}(x,y) = \frac{1}{2}\left(\frac{\partial u_i^0}{\partial y_j} + \frac{\partial u_j^0}{\partial y_i}\right) \tag{15a}$$

$$\varepsilon_{ij}^{0}(x,y)=\frac{1}{2}\left(\frac{\partial u_i^0}{\partial x_j}+\frac{\partial u_j^0}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial u_i^1}{\partial y_j}+\frac{\partial u_j^1}{\partial y_i}\right) \tag{15b}$$

$$\varepsilon_{ij}^{1}(x,y)=\frac{1}{2}\left(\frac{\partial u_i^1}{\partial x_j}+\frac{\partial u_j^1}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial u_i^2}{\partial y_j}+\frac{\partial u_j^2}{\partial y_i}\right) \tag{15c}$$

The elastic coefficients D_{ijkl} are periodic functions of x and depend on ε,

$$D_{ijkl}^{\varepsilon}=D_{ijkl}(x/\varepsilon)=D_{ijkl}(y) \tag{16}$$

Thus the stresses can be expressed by

$$\begin{aligned}\sigma_{ij}^{\varepsilon}&\\ &=D_{ijkl}^{\varepsilon}\varepsilon_{kl}\\ &=\frac{1}{\varepsilon}D_{ijkl}^{\varepsilon}\varepsilon_{kl}^{-1}(x,y)+D_{ijkl}^{\varepsilon}\varepsilon_{kl}^{0}(x,y)+\varepsilon D_{ijkl}^{\varepsilon}\varepsilon_{kl}^{1}(x,y)+\cdots\\ &=\frac{1}{\varepsilon}\sigma_{ij}^{-1}(x,y)+\sigma_{ij}^{0}(x,y)+\varepsilon\sigma_{ij}^{1}(x,y)+\cdots\end{aligned} \tag{17}$$

The elastic stress-strain relations are

$$\sigma_{ij}^{n}(x,y)=D_{ijkl}^{\varepsilon}\varepsilon_{kl}^{n}(x,y),\quad n=-1,0,1 \tag{18}$$

From Eqs.(15) and (18), the stresses can be written as

$$\sigma_{ij}^{-1}=D_{ijkl}^{\varepsilon}\frac{\partial u_k^0}{\partial y_l} \tag{19a}$$

$$\sigma_{ij}^{0}=D_{ijkl}^{\varepsilon}\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right) \tag{19b}$$

$$\sigma_{ij}^{1}=D_{ijkl}^{\varepsilon}\left(\frac{\partial u_k^1}{\partial x_l}+\frac{\partial u_k^2}{\partial y_l}\right) \tag{19c}$$

The elastic problem with a periodic microstructure is described by

$$\sigma_{ij,j}^{\varepsilon}+f_i=0 \qquad \text{in } \Omega \tag{20a}$$

$$\sigma_{ij}^{\varepsilon}n_j=T_i \qquad \text{on } \Gamma_t \tag{20b}$$

$$u_i^{\varepsilon}=\tilde{u}_i \qquad \text{on } \Gamma_u \tag{20c}$$

where f_i, T_i and $\tilde{u}_i$ are specific body force, tractions and boundary displacements. Substituting Eq.(17) into Eq.(20), and equaling the powers of ε, the equilibrium equations can be rewritten as

$$\frac{\partial\sigma_{ij}^{-1}}{\partial y_j}=0 \tag{21a}$$

$$\frac{\partial\sigma_{ij}^{-1}}{\partial x_j}+\frac{\partial\sigma_{ij}^{0}}{\partial y_j}=0 \tag{21b}$$

$$\frac{\partial\sigma_{ij}^{0}}{\partial x_j}+\frac{\partial\sigma_{ij}^{1}}{\partial y_j}+f_i=0 \tag{21c}$$

For solving the system of Eq.(21), an important result is introduced here. For a Y periodic function

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial\Phi}{\partial y_j}\right)=F \tag{22}$$

has a unique solution if the mean value of F, defined by

$$\overline{F}=\frac{1}{|Y|}\int_Y F\,\mathrm{d}Y \tag{23}$$

equals zero, where $|Y|$ is the periodicity or the volume of the RVE. Application of this condition to Eq.(21a) leads to

$$\sigma_{ij}^{-1}=0 \tag{24}$$

This implies that the stresses are independent of $1/\varepsilon$. Then from Eq.(18) and (15a), one has

$$u_i^0(x,y)=u_i^0(x) \tag{25}$$

This shows that u_i^0 is a function of the global coordinate x only. Therefore, the expansion of displacement field can be rewritten as

$$u_i=u_i^{\varepsilon}(x)=u_i^0(x)+\varepsilon u_i^1(x,y)+\varepsilon^2u_i^2(x,y)+\cdots \tag{26}$$

We can regard u_i^0 as the macroscopic displacement, while $u_i^1,u_i^2,\cdots$ are the microscopic displacements. The physical interpretation of Eq.(26) is that the real displacement u_i is rapidly oscillating around the mean displacement u_i^0 due to the inhomogeneity from the microscopic point of view. $u_i^1,u_i^2,\cdots$ are the perturbing displacements according to the microstructure.

The final microscopic equilibrium equations are reduced as, through substituting (24) into Eqs.(21a) and (21b),

$$\frac{\partial\sigma_{ij}^{0}}{\partial y_j}=0 \quad \text{in } \Omega \tag{27}$$

Taking the mean of Eq. (21c) on Ω and applying the condition (23) in the second term, $\frac{\partial\sigma_{ij}^{1}}{\partial y_j}$, leads to the macroscopic equilibrium equations

$$\frac{\partial\overline{\sigma}_{ij}^{0}}{\partial x_j}+f_i=0 \quad \text{in } \Omega \tag{28}$$

where $\overline{\sigma}^0_{ij}$ are macroscopic stresses.

For evaluation of the stresses σ^0_{ij}, we must know the displacements u^0_i and u^1_i, as shown in Eq.(19b). For this purpose, we assume that the displacement fields u^0_i and u^1_i are related by

$$u^1_i = -\psi^{kl}_i(x,y)\frac{\partial u^0_k}{\partial x_l} \tag{29}$$

Then, substituting Eq.(29) into (19b) yields the stresses

$$\sigma^0_{ij} = \left(D_{ijkl} - D_{ijmn}\frac{\partial \psi^{kl}_m}{\partial y_n}\right)\frac{\partial u^0_k}{\partial x_l} \tag{30}$$

Then integrating over the domain Ω leads to the effective stress-strain relations for an elastic medium

$$\overline{\sigma}^0_{ij} = \overline{D}_{ijkl}\frac{\partial u^0_k}{\partial x_l} \tag{31}$$

where the macroscopic stresses, , and the effective stiffness coefficients, $\overline{D}_{ijkl}$, are expressed by

$$\overline{\sigma}^0_{ij} = \frac{1}{V}\int_\Omega \sigma^0_{ij}(x,y)\mathrm{d}\Omega \tag{32}$$

$$\overline{D}_{ijkl} = \frac{1}{V}\int_\Omega \left[D_{ijkl} - D_{ijmn}\frac{\partial \psi^{kl}_m}{\partial y_n}\right]\mathrm{d}\Omega \tag{33}$$

It can been seen from Eq.(33) that the function $\psi(x,y)$ must be known before the determination of the homogenized stiffness coefficients. Generally, evaluation of $\psi(x,y)$ can be achieved by a finite element method.

Remark 2: *With Eq.(30), we can see that the micro-stresses consist of two parts: first part is uniform stresses caused by the uniform micro-strains and second part is the perturbing stresses caused by the perturbing strains. From the homogenization process appearing in Eq.(31) to (33), it is clear that the principle of the two-scale expansion method is completely the same as the direct average method of the stresses and strains. The definition of the effective stiffness of the composites is also entirely identical between two methods.*

For application of the two-scale expansion method in conjunction with the finite element method, a variational form of the microscopic equilibrium equations should be established. Here, another important result is introduced. For a Y-periodic function $\phi(y)$, we define a mean operator as follows

$$\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon}\phi\left(\frac{x}{\varepsilon}\right)\mathrm{d}\Omega = \frac{1}{|Y|}\int_\Omega\int_Y \phi(y)\mathrm{d}Y\mathrm{d}\Omega \tag{34}$$

The variational form for Eq.(21a) is

$$\int_{\Omega^\varepsilon}\frac{\partial \sigma^{-1}_{ij}}{\partial y_j}\delta u^0_i \mathrm{d}\Omega = \int_{\Omega^\varepsilon}\left(D^\varepsilon_{ijkl}\frac{\partial u^0_k}{\partial y_l}\right)_{,j}\delta u^0_i\mathrm{d}\Omega = 0 \tag{35}$$

where δu^0_i are arbitrary virtual displacements. Since the homogenization method consists of finding the limit of the solutions to Eqs.(21a)-(21c) as ε tends to zero, so we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon}\left(D^\varepsilon_{ijkl}\frac{\partial u^0_k}{\partial y_l}\right)_{,j}\delta u^0_i\mathrm{d}\Omega\\ &= \frac{1}{|Y|}\int_\Omega\int_Y\left(D_{ijkl}\frac{\partial u^0_k}{\partial y_l}\right)_{,j}\delta u^0_i\mathrm{d}Y\mathrm{d}\Omega = 0\end{aligned} \tag{36}$$

Using the divergence theorem on Eq.(36) yields:

$$\begin{aligned}&\frac{1}{|Y|}\int_\Omega\int_Y\left(D_{ijkl}\frac{\partial u^0_k}{\partial y_l}\right)_{,j}\delta u^0_i\mathrm{d}Y\mathrm{d}\Omega\\ &= \frac{1}{|Y|}\int_\Omega\oint_s D_{ijkl}\frac{\partial u^0_k}{\partial y_l}n_j\delta u^0_i\mathrm{d}s\mathrm{d}\Omega = 0\end{aligned} \tag{37}$$

where S is the boundary of the RVE. Thus we have

$$\frac{\partial u^0_k}{\partial y_j} = 0 \tag{38}$$

It is shown again that u^0_i is only a function of x.

Substituting Eq. (19b) into the variational form, with arbitrary virtual displacements δu^1_i, of Eq. (21b) yields:

$$\int_{\Omega^\varepsilon}\frac{\partial \sigma^0_{ij}}{\partial y_j}\delta u^1_i\mathrm{d}\Omega = \int_{\Omega^\varepsilon}D^\varepsilon_{ijkl}\left(\frac{\partial u^0_k}{\partial x_l}+\frac{\partial u^1_k}{\partial y_l}\right)_{,j}\delta u^1_i\mathrm{d}\Omega = 0 \tag{39}$$

Then using Eq.(34), Eq.(39) becomes

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon}D^\varepsilon_{ijkl}\left(\frac{\partial u^0_k}{\partial x_l}+\frac{\partial u^1_k}{\partial y_l}\right)_{,j}\delta u^1_i\mathrm{d}\Omega\\ &= \frac{1}{|Y|}\int_\Omega\int_Y D_{ijkl}\left(\frac{\partial u^0_k}{\partial x_l}+\frac{\partial u^1_k}{\partial y_l}\right)_{,j}\delta u^1_i\mathrm{d}Y\mathrm{d}\Omega = 0\end{aligned} \tag{40}$$

Integrating by parts, and noting that virtual displacements $\delta u_i^1 = 0$ at the boundary of RVE, and u_i^0 is a function of x only, we have the relations between the displacements u_i^0 and u_i^1 in integration form

$$\int_\Omega \frac{\partial u_k^0}{\partial x_l}\left(\int_Y D_{ijkl}\frac{\partial \delta u_i^1}{\partial y_j}\mathrm{d}Y\right)\mathrm{d}\Omega + \int_\Omega \int_Y D_{ijkl}\frac{\partial u_k^1}{\partial y_l}\frac{\partial \delta u_i^1}{\partial y_j}\mathrm{d}Y\mathrm{d}\Omega = 0 \tag{41}$$

Substantially, Eq.(41) is equivalent to Eq.(29) which connects the displacements u_i^0 and u_i^1. In fact, we introduce the function $\psi(x,y)$, which satisfies

$$\int_Y D_{ijpq}\frac{\partial \psi_p^{kl}}{\partial y_q}\frac{\partial \delta u_i^1}{\partial y_j}\mathrm{d}Y = \int_Y D_{ijkl}\frac{\partial \delta u_i^1}{\partial y_j}\mathrm{d}Y \tag{42}$$

Substituting Eq.(42) into Eq.(41) yields

$$\int_\Omega \frac{\partial u_k^0}{\partial x_l}\int_Y D_{ijpq}\frac{\partial \psi_p^{kl}}{\partial y_q}\frac{\partial \delta u_i^1}{\partial y_j}\mathrm{d}Y\mathrm{d}\Omega + \int_\Omega \int_Y D_{ijkl}\frac{\partial u_k^1}{\partial y_l}\frac{\partial \delta u_i^1}{\partial y_j}\mathrm{d}Y\mathrm{d}\Omega = 0 \tag{43}$$

Applying the divergence theorem in Eq.(43) leads to

$$\int_\Omega \oint_S D_{ijpq}\psi_p^{kl} n_q \frac{\partial u_k^0}{\partial x_l}\frac{\partial \delta u_i^1}{\partial y_j}\mathrm{d}s\mathrm{d}\Omega + \int_\Omega \oint_S D_{ijpq} u_p^1 n_q \frac{\partial \delta u_i^1}{\partial y_j}\mathrm{d}s\mathrm{d}\Omega = 0 \tag{44}$$

It is clear that this equation is the integration form of Eq.(29).

Actually, Eq.(42) provides a basic equation for solving the function $\psi(x,y)$ by the finite element method. The interpolation of the finite element form for the function $\psi(x,y)$ can be expressed by

$$\psi_i^{kl} = (N_\alpha \psi_\alpha)_i^{kl} = (\mathbf{N}\psi)_i^{kl} \quad ,\alpha = 1,\cdots,M \tag{45}$$

where $\mathbf{N}$ is a shape function and ψ stands for the nodal generalized coordinates, M is the total number of freedom degrees of the system. Then the derivative of the function $\psi(x,y)$ can be expressed by

$$\frac{\partial \psi_p^{kl}}{\partial y_q} = (\mathbf{B}_q\psi)_p^{kl} \tag{46}$$

and the derivative of the displacements is

$$\frac{\partial \delta u_i^1}{\partial y_j} = (\mathbf{B}_j\psi)_i^{kl}\frac{\partial u_k^0}{\partial x_l} \tag{47}$$

where $\mathbf{B}_i$ corresponds to the derivatives of the shape function $\mathbf{N}$ with respect to y_i. It is noted that the function u_i^0 is independent of y.

Then we can rewrite Eq.(42) in the standard form of a finite element

$$\left(\int_Y \mathbf{B}^T\mathbf{D}\mathbf{B}\mathrm{d}Y\right)\psi^{kl} = \int_Y \mathbf{B}^T\mathbf{D}^{kl}\mathrm{d}Y \tag{48}$$

where $\mathbf{D}$ is the stress-strain matrix of the components of the composite, $\mathbf{B}$ is the discrete displacement-strain matrix depending on the element shape functions. $\mathbf{D}^{kl}$ is a vector of column kl (kl=11,22,33,23,31,12) of the stress-strain matrix $\mathbf{D}$. ψ^{kl} is the characteristic displacement vector associated with the kl mode. Six equations should be solved for different strain states. A conventional finite element can be used to calculate Eq.(48).

Therefore, the homogenized elastic constants defined by Eq.(43) can be expressed by

$$\overline{\mathbf{D}} = \frac{1}{|Y|}\int_Y \mathbf{D}(\mathbf{I} - \mathbf{B}\psi)\mathrm{d}Y \tag{49}$$

where

$$\psi = (\psi^{11},\psi^{22},\psi^{33},\psi^{23},\psi^{31},\psi^{12}) \tag{50}$$

In summary, the two-scale expansion method provides a procedure to calculate the effective properties of the composite with a periodic microstructure. ψ^{kl} in Eq.(48) is solved by a finite element method and then the effective properties can be calculated from Eq.(49).

4 Approximate estimation of effective properties

In this section, let us estimate approximately the effective properties based on the two-scale expansion method. Two specific cases, equal strain model and equal stress model, are considered here.

By analyzing the basic assumption made in the two-scale expansion method, Eq.(29), and the effective stiffness, Eq. (33), we can see that the first term in Eq. (33) is the well-known rule of mixture, while the second term is a correction with the perturbing strain due to the heterogeneity of the microstructure.

In the equal strain model, it is assumed that the strains undergone in each phase are of the same values. Thus the perturbing parts of the displacement do not exist. As a result, Eq.(29) becomes

$$u_i^1 = -\psi_i^{kl}(x,y)\frac{\partial u_k^0}{\partial x_l} = 0 \tag{51}$$

Then the effective stiffness coefficients according to Eq.(33) can be reduced as

$$\begin{aligned}\overline{D}_{ijkl} &= \frac{1}{V}\int_\Omega \left[D_{ijkl} - D_{ijmn}\frac{\partial \psi_m^{kl}}{\partial y_n}\right]\mathrm{d}\Omega \\ &= \frac{1}{V}\int_\Omega D_{ijkl}\mathrm{d}\Omega \end{aligned}\tag{52}$$

This is the known rule of mixture. The simple expression under uniaxial state is

$$\overline{E}_{11} = \frac{1}{V}\int_\Omega E_{11}\mathrm{d}\Omega = v_1 E_{11}^{(1)} + v_2 E_{11}^{(2)} + \cdots \tag{53}$$

where E_{11} is the Young's modulus, v_i and $E_{11}^{(i)}$ are the volume fraction and Young's modulus of phase i, respectively. Eq.(53) is referred to as the Voigt's approximation (Voigt, 1889) and usually is used to predict the effective axial modulus of the unidirectional fiber composite material. It is noted that Eq.(53) gives the upper bound of the elastic modulus.

The equal stress state means that the stresses in each phase are uniform and equal everywhere. In this case, the perturbing part of the stress does not exist and the macro-stress is equal to the applied stress. Then from Eq.(19), one has

$$\sigma_{ij}^{-1} = 0, \quad \sigma_{ij}^1 = 0 \tag{54}$$

and

$$\sigma_{ij}^0 = \overline{\sigma}_{ij} \tag{55}$$

According to the local elastic stress-strain relations, the strains are

$$\varepsilon_{ij}^{-1} = 0, \quad \varepsilon_{ij}^1 = 0 \tag{56}$$

and

$$\varepsilon_{ij}^0 = C_{ijkl}^\varepsilon \sigma_{kl}^0 = C_{ijkl}^\varepsilon \overline{\sigma}_{kl} \tag{57}$$

Taking the integration over the domain Ω, one can obtain

$$\int_\Omega \varepsilon_{ij}^0 \mathrm{d}\Omega = \int_\Omega C_{ijkl}^\varepsilon \overline{\sigma}_{kl}\mathrm{d}\Omega \tag{58}$$

Then, dividing both sides in Eq.(58) by volume of the domain yields homogenized stress-strain relations

$$\overline{\varepsilon}_{ij} = \overline{C}_{ijkl}\overline{\sigma}_{kl} \tag{59}$$

with the homogenized compliance coefficients

$$\overline{C}_{ijkl} = \frac{1}{V}\int_\Omega C_{ijkl}\mathrm{d}\Omega \tag{60}$$

The equations (60) can be interpreted as the rule of mixture for the compliance coefficients. For the uniaxial state, the Young's modulus can be expressed by

$$\frac{1}{\overline{E}_{22}} = \frac{1}{V}\int_\Omega \frac{1}{E_{22}}\mathrm{d}\Omega = \frac{v_1}{E_{22}^{(1)}} + \frac{v_2}{E_{22}^{(2)}} + \cdots \tag{61}$$

This equation is referred to as the Reuss's approximation (Reuss, 1929) and usually is used to predict the transverse modulus of the unidirectional fiber composite materials. It is verified that this equation gives a simple lower bound of the effective elastic modulus of a composite.

It should be noted that the Voigt's and Reuss's approximations provide generous upper and lower bounds. They are the most simply cases of the Hashin and Strikman's variational solutions (Hashin and Shtrikman, 1962).

Remark 3: *The rule of mixture for the effective properties described by Eq.(52),(53) or Eq. (60),(61), can be easily derived with the direct average method if the equal stress or equal strain model is assumed, respectively. This shows that the direct average method and two-scale expansion method give the same upper and lower bound values.*

5 Periodic boundary conditions

The periodicity is one of the main characteristics of the composites. The microscopic displacement and stress fields are the Y periodic solutions and a RVE is a Y periodic cell. In the homogenized properties calculation, the periodic boundary conditions must be imposed on the RVE to reflect the repeatability of the microstructure.

Without lose of generality, the periodic conditions of the displacement and stress fields can be expressed by

$$u_i(\mathbf{y}) = u_i(\mathbf{y}+\mathbf{Y}) \quad \forall \mathbf{y} \in \Omega \tag{62}$$

$$\sigma_{ij}(\mathbf{y}) = \sigma_{ij}(\mathbf{y}+\mathbf{Y}) \quad \forall \mathbf{y} \in \Omega \tag{63}$$

where $\mathbf{Y} = (Y_1, Y_2, Y_3)$ is the periodicity. Here domain Ω represents a RVE and Γ is its boundary. For arbitrary $\mathbf{y}^0 \in \Gamma$, the periodic displacement boundary conditions of the RVE can be expressed by [Havner 1971]

$$u_i(\mathbf{y}^0) = u_i(\mathbf{y}^0+\mathbf{Y}), \quad \forall \mathbf{y}^0 \in \Gamma \tag{64}$$

The stress periodicity requires anti-periodic traction boundary conditions

$$T_i(\mathbf{y}^0) = -T_i(\mathbf{y}^0+\mathbf{Y}), \quad \forall \mathbf{y}^0 \in \Gamma \tag{65}$$

where $\mathbf{y}^0+\mathbf{Y}$ is the boundary of the periodic RVE.

For a 2D square or rectangular RVE, as shown in Fig.4, the periodic displacement boundary conditions can be expressed by

$$u_1(y_1^0, y_2) = u_1(y_1^0+Y_1, y_2) \tag{66a}$$

$$u_2(y_1^0, y_2) = u_2(y_1^0+Y_1, y_2) \tag{66b}$$

on the left and right opposite sides and

$$u_1(y_1, y_2^0) = u_1(y_1, y_2^0+Y_2) \tag{67a}$$

$$u_2(y_1, y_2^0) = u_2(y_1, y_2^0+Y_2) \tag{67b}$$

on the upper and lower opposite sides. The anti-periodicity of the traction boundary conditions leads to

$$\sigma_{11}(y_1^0, y_2) = -\sigma_{11}(y_1^0+Y_1, y_2) \tag{68a}$$

$$\sigma_{12}(y_1^0, y_2) = -\sigma_{12}(y_1^0+Y_1, y_2) \tag{68b}$$

on the left and right sides and

$$\sigma_{22}(y_1, y_2^0) = -\sigma_{22}(y_1, y_2^0+Y_2) \tag{69a}$$

$$\sigma_{21}(y_1, y_2^0) = -\sigma_{21}(y_1, y_2^0+Y_2) \tag{69b}$$

on the upper and lower sides.

In the present study, only the symmetric rectangular RVE is considered. This case can reflect many model composites that the inclusion may have, in a 2D state, a shape of circle, ellipse, rectangle and so on. The work for the general case of the periodic boundary conditions will be reported elsewhere.

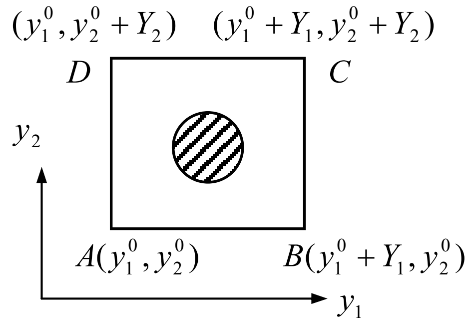


Figure 3 : Periodic and symmetric RVE

Firstly, consider the normal (extension and compression) deformation modes of the RVE. The periodicity and symmetry of the RVE (see Fig.3) yields

$$\begin{aligned} u_1(y_1^0, y_2) &= u_1(y_1^0+Y_1, y_2) \\ &= -u_1(y_1^0+Y_1, y_2) \\ &= 0 \end{aligned} \tag{70}$$

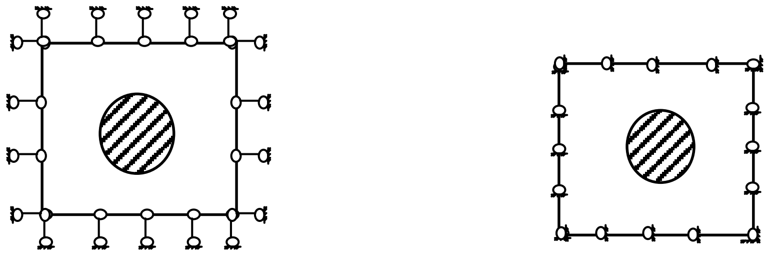
on the left and right opposite sides and

$$\begin{aligned} u_2(y_1, y_2^0) &= u_2(y_1, y_2^0+Y_2) \\ &= -u_2(y_1, y_2^0+Y_2) \\ &= 0 \end{aligned} \tag{71}$$

on the upper and lower opposite sides. Eqs.(70) and (71) imply that the normal displacements for the all external edges of the RVE are fixed, as shown in Fig.4a. Clearly, these constraints can satisfy the anti-periodic and symmetric requirements of the traction boundary conditions.

Secondly, let us consider the pure shear deformation of the RVE. An anti-symmetric deformation mode happens in this case. Then we can obtain

$$\begin{aligned} u_2(y_1^0, y_2) &= u_2(y_1^0+Y_1, y_2) \\ &= -u_2(y_1^0+Y_1, y_2) \\ &= 0 \end{aligned} \tag{72}$$



(a) (b)

Figure 4 : The constraints on the periodic and symmetric RVE for normal deformation (a) and pure shear deformation (b)

on the left and right opposite sides and

$$u_1(y_1,y_2^0) = u_1(y_1,y_2^0+Y_2) = -u_1(y_1,y_2^0+Y_2) = 0 \quad (73)$$

on the upper and lower opposite sides. Eq.(72) and Eq.(73) mean that the tangent displacements on the boundary of the RVE are fixed, as shown in Fig.4b.

The periodic and symmetric boundary conditions can be used in the two-scale expansion method where the initial strains are loaded (Hassani and Hinton, 1998). For the symmetric RVE, only one quarter of the RVE is analyzed.

6 Numerical results

The above mentioned analysis has shown that the direct methods and two-scale expansion method are the same in principle. Some numerical examples are given in this section in order to compare these methods numerically.

The effective properties are calculated for three kinds of the composite materials, e.g. transversely isotropic fiber composite, void solid and rigid inclusion medium. The numerical results are compared for three homogeneous methods: the direct average method of stress and strain, the direct average method of strain energy density and the two-scale expansion method.

A plane strain model is considered here and the in-plane or transverse properties of the composites are calculated. For the direct average methods of stress, strain and strain energy density, there are two classes of the boundary conditions of the RVE, the uniform traction- and uniform displacement-loading boundary conditions that result in different deformations and effective stiffness coefficients (Miehe and Koch, 2002). Here the boundary conditions with specific displacements are imposed and then the finite element method is applied in calculation of the average stresses, strains and strain energy density on a RVE with a uniaxial strain state. The resulting effective stiffness coefficients of the plane strain problem are used to calculated the engineering constants by

$$\mu = \frac{D_{11}}{D_{11}+D_{12}} \quad (74a)$$

$$E = \frac{D_{11}(1+\mu)(1-2\mu)}{1-\mu} \quad (74b)$$

$$G = D_{66} \quad (74c)$$

where the subscripts take the compact form, that is $D_{11} = D_{1111}$, $D_{12} = D_{1122}$, $D_{66} = D_{1212}$.

6.1 FE mesh dependence

Three kinds of mesh are investigated in order to examine the mesh-dependence. The first case, Mesh 1 has 39 elements and 140 nodes. The Mesh 2 has 64 elements, 221 nodes and the Mesh 3 has 85 elements and 288 nodes. The meshes are illustrated in Fig. 5. Here the 8-node quadratic isoparameter elements are used in the finite element analysis.

The numerical results for different FE meshes are listed in Table 1. Here three methods are used. ASS denotes the direct average of stress and strain, ASE the direct average of strain energy density and TEM the two-scale expansion method. It is shown that the effective stiffness coefficients are not sensitive to the FE meshes. Only the shear stiffness coefficients slightly vary with the FE

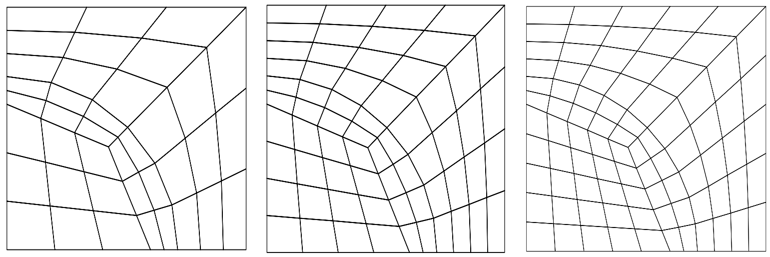


Figure 5 : Three kinds of mesh for test the mesh-dependence

Table 1 : Stiffness coefficients for different FE meshes

	D_{11} (GPa)			D_{12} (GPa)		D_{33} (GPa)		
	ASS	ASE	TEM	ASS	TEM	ASS	ASE	TEM
Mesh 1	10.4874	10.4874	10.4873	4.2879	4.2879	2.2315	2.2315	2.2315
Mesh 2	10.4874	10.4874	10.4875	4.2879	4.2880	2.2314	2.2314	2.2314
Mesh 3	10.4873	10.4873	10.4874	4.2879	4.2879	2.2313	2.2313	2.2314

Table 2 : Transverse stiffness coefficients for fiber composite

	D_{11} (GPa)			D_{12} (GPa)		D_{66} (GPa)		
	ASS	ASE	TEM	ASS	TEM	ASS	ASE	TEM
0.1	6.3147	6.3147	6.3148	3.2920	3.2920	1.4726	1.4726	1.4727
0.2	7.3218	7.3218	7.3218	3.6171	3.6171	1.6824	1.6824	1.6824
0.3	8.6606	8.6606	8.6606	3.9511	3.9511	1.9255	1.9255	1.9255
0.4	10.4873	10.4873	10.4874	4.2879	4.2879	2.2313	2.2313	2.2314
0.5	13.0754	13.0755	13.0758	4.6347	4.6346	2.6549	2.6550	2.6551
0.6	17.0605	17.0606	17.0608	5.0817	5.0817	3.3356	3.3356	3.3357

meshes. The richest mesh, Mesh 3, is used in the following calculations though the results are not sensitive to the FE meshes.

6.2 Transversely isotropic composite

A composite reinforced by the unidirectional continuous fibers demonstrates the transversely isotropic properties. For this case, the following material data is used:

E-glass fiber: the Young's modulus is 73.1 GPa, the Poisson's ratio is 0.22.

Epoxy Matrix: the Young's modulus is 3.45 GPa, the Poisson's ratio is 0.35.

The effective transverse stiffness coefficients of the transversely isotropic composite are listed in Table 2. It is shown that the three methods give completely identical stiffness coefficients. This is not surprising because of the same homogenization principle used in all of the three methods. The engineering constants can be found by Eq.(74) for the comparison with the approximate bounds and experimental data. Fig.6 illustrates the transverse Young's modulus E_{22} as a function of the fiber volume fraction. The lower bound was calculated by

Eq.(61). It is shown that ASS, ASE and TEM provide good agreeable results with the experimental data (Tsai, 1964).

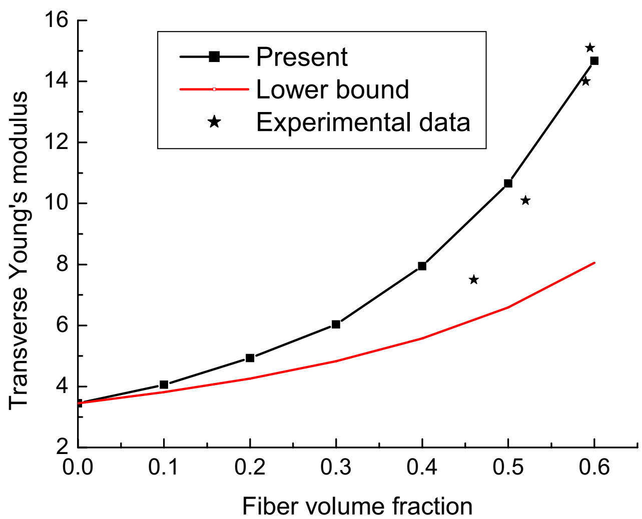


Figure 6 : Transverse Young's modulus

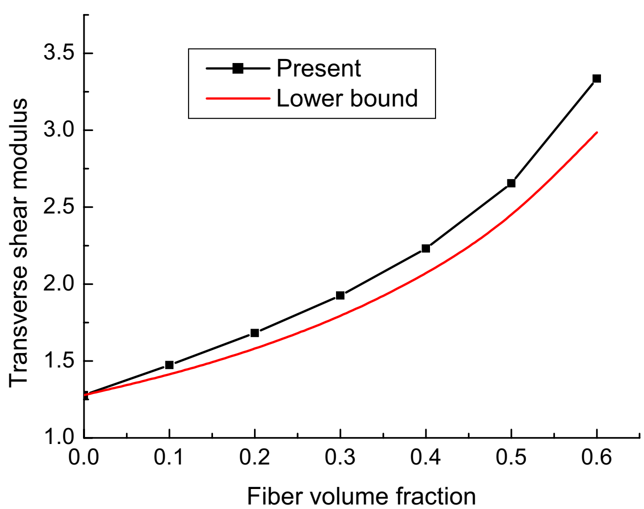


Figure 7 : Transverse shear modulus

Fig.7 shows the transverse shear modulus G of the composite with different fiber volume fraction. No experimental data for the transverse shear modulus is available for comparison. An approximate estimation for the transverse shear modulus by

$$\frac{1}{G} = \frac{v_m}{G_m} + \frac{v_f}{G_f} \tag{75}$$

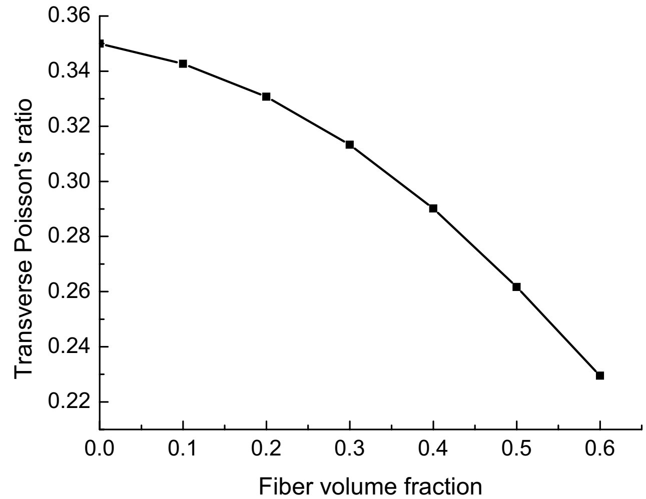


Figure 8 : Transverse Poisson's ratio

is plotted in Fig.7. It is easy to prove that Eq.(75) provides a lower bound for the shear modulus.

The transverse Poisson's ratio is shown in Fig.8. The nonlinear relation between effective transverse Poisson's ratio and the fiber volume fraction is demonstrated. No appropriate bounds and experimental data are available for comparison.

6.3 Rigid inclusion medium

An inclusion with very large elastic modulus is used to model a rigid inclusion. The elastic modulus of the inclusion is 10^4 times that of the modulus of the matrix. The Poisson's ratio of the matrix is 0.35. The present results have been normalized by the modulus of the matrix. The effective stiffness coefficients of rigid inclusion medium are listed in Table 3 for a detailed precision comparison. It is shown that the three methods give mostly identical results except for the slight differences in a few cases.

6.4 Void solid

An inclusion with very small elastic modulus is used to model a void in an isotropic solid. The elastic modulus of the void inclusion is 10^{-6} times that of the modulus of the matrix. The Poisson's ratio of the matrix is 0.35. Table 4 lists normalized effective stiffness coefficients of this medium. Same, the identical results are obtained by the three methods.

Remark 4: *The example provides numerical illustration*

Table 3 : Stiffness coefficients for rigid inclusion medium

v_1	D_{11}/E_0			D_{12}/E_0		D_{66}/E_0		
	ASS	ASE	TEM	ASS	TEM	ASS	ASE	TEM
0.1	1.8560	1.8560	1.8673	0.9678	0.9678	0.4324	0.4311	0.4325
0.2	2.1900	2.1900	2.1967	1.0791	1.0784	0.4952	0.4968	0.5007
0.3	2.6504	2.6504	2.6577	1.1957	1.1986	0.5721	0.5738	0.5821
0.4	3.3123	3.3123	3.3464	1.3128	1.3150	0.6638	0.6724	0.6797
0.5	4.3297	4.3297	4.4308	1.4206	1.4195	0.8096	0.8140	0.8273
0.6	6.1427	6.1427	6.2106	1.5056	1.5047	1.0548	1.0548	1.0766

that the direct average method of stresses, strains and strain energy density is the same as two-scale expansion method. The effective stiffnesses calculated by the three methods are not only completely identical but also in good agreements with the experimental data.

7 Conclusions

The paper gives the detailed comparison for three homogenization concepts, e.g. direct average method of stress and strain, direct average method of strain energy density and two-scale expansion method. It is shown in principle and numerically that direct average methods are the same as the two-scale expansion method. Although they have different expressions in mathematics, the identical calculation results are obtained for the composites with a large range of elastic mismatch of the components. The effective properties obtained from the three methods are in agreements with the experimental data. Future work will include the prediction and comparison of composites with non-anisotropic inclusion and non-symmetric RVE.

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Table 4 : Stiffness coefficients for void solid

V_1	D_{11}/E_0			D_{12}/E_0		D_{66}/E_0		
	ASS	ASE	TEM	ASS	TEM	ASS	ASE	TEM
0.1	1.1314	1.1314	1.1314	0.5381	0.5381	0.2763	0.2763	0.2763
0.2	0.8405	0.8405	0.8405	0.3459	0.3459	0.1919	0.1919	0.1919
0.3	0.6388	0.6388	0.6388	0.2221	0.2221	0.1235	0.1235	0.1235
0.4	0.4863	0.4863	0.4863	0.1385	0.1385	0.0731	0.0731	0.0731
0.5	0.3625	0.3625	0.3625	0.0811	0.0811	0.0389	0.0389	0.0389
0.6	0.2537	0.2537	0.2537	0.0413	0.0413	0.0171	0.0171	0.0171

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