

A Virtual Crack Closure-Integral Method (VCCM) for Three-Dimensional Crack Problems Using Linear Tetrahedral Finite Elements

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Abstract: In this paper, a three-dimensional VCCM (Virtual Crack Closure-Integral Method) for evaluating the energy release rate and the stress intensity factor is presented. Many engineers and researchers believe that hexahedral finite elements should be used to perform three-dimensional fracture analyses. Previous VCCM formulations assume the use of hexahedral finite elements. In present study, the authors have been developing a VCCM that works with tetrahedral finite elements. In the field of large-scale computation, the use of tetrahedral finite elements has becoming very popular as high performance mesh generation programs became available. Therefore, building a large and complex analysis model with hexahedral finite elements is a much more difficult task than with tetrahedral elements. The outcomes of present research would make three-dimensional fracture mechanics analysis on complex shaped solid/structure a tractable task to do. In this paper, some preliminary results are presented.

keyword: Fracture Mechanics, Stress Intensity Factor, Energy Release Rate, VCCM (Virtual Crack Closure-Integral Method), FEM (Finite Element Method)

1 Introduction

Structural integrity analyses for infrastructures, power plant structures/components, pressure vessels and pipings, etc. are very important. In many cases, they are carried out based on fracture mechanics analyses with the stress intensity factors that are evaluated from the results of finite element (FE) or of boundary element (BE) analyses. Among them, the finite element method (FEM) is especially popular as there are many commercial program packages available. It is noted here that though we assume to employ a conventional finite element method to the fracture mechanics problems, researchers are/have been spending efforts to develop new approaches such as MLPG (Meshless Local Petrov-

Galerkin Method), SGBEM (Symmetric Galerkin BEM), Finite Element Alternating Method, etc. that can completely/partly eliminates the needs for spatial discretization (i.e., finite element meshes). Also, researches to simplify model generation processes by using “element overlay technique” was presented. The readers are referred to Han and Atluri (2004, 2003, 2002) and Okada, Endoh and Kikuchi (2005).

Presently, there are several popular ways to evaluate the stress intensity factors from the results of finite element analysis. They are displacement and stress methods, stiffness derivative method and other energetic approaches. In displacement and stress method, those that are evaluated by the finite element calculation are used to compute the stress intensity factors by assuming their asymptotic distributions at the vicinity of the crack tip/front. These methodologies have successfully been applied to crack problems [see, for example, Chan, Tuba and Wilson (1970) and Broek (1986)]. In energetic methods, the energy release rate is first evaluated and then the stress intensity factor is computed. There are three popular ways to evaluate the energy release rate. They are: J-integral method [Rice (1968)], stiffness derivative method [virtual crack extension (VCE); Parks (1974), Hellen (1975)] and virtual crack closure-integral method (VCCM) [Rybicki and Kanninen (1977) and Shivakumar, Tan and Newman, Jr. (1988)]. VCE technique that computes the energy difference when crack extends for a small amount had evolved to be equivalent domain integral (EDI) method [deLorenzi (1982), deLorenzi (1985), Li, Shih and Needleman (1985), and Nikishkov and Atluri (1987)]. EDI has been adopted in many commercial software. VCCM was proposed by Rybicki and Kanninen (1977) for two-dimensional crack problem and was later extended to three-dimensional cases by Shivakumar, Tan and Newman, Jr. (1988). VCCM is very simple and is able to split the energy release rate into its Mode I, II and III contributions (for two-dimensional problem, Mode I and II contributions) without any further compli-

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cations. Though it is assumed in this paper that material of our interest be isotropic and homogeneous, stress intensity factor/energy release rate computations for interface crack/dissimilar materials have been reported [see, for example, So, Lau and Ng. (2004), Xie, Waas, Shahwan, Schroeder and Boeman (2004)].

When the extension of VCCM from two- to three-dimensional formulation was carried out in Shivakumar, Tan and Newman, Jr. (1988), a thickness was added to the two-dimensional plane, making a three-dimensional solid. This extension process implicitly assumed that hexahedral finite elements be used to model a cracked body. Also, rigorously speaking, the faces of finite elements across the crack front must be the same sized rectangles, as shown in Figs. 1 and 2. However, when the crack front has some curvature, such a requirement can not be satisfied. Fawas (1998) demonstrated that the energy release rate could be computed accurately if the area of element faces across the crack front were the same even though they are not rectangles. de Roeck, and Abdel Wahab (1995) proposed a way to account the change of element face areas across the crack front for linear hexahedral element. Okada, Higashi, Kikuchi, Fukui and Kumazawa (2005) proposed a general framework that can allow the elements be skewed at and their faces are not symmetrically arranged across the crack front. What was proposed by Fawas (1998) is included in the framework of Okada, Higashi, Kikuchi, Fukui and Kumazawa (2005). This methodology can be applied to any types of hexahedral finite elements (i.e., eight node, 20 node serendipity, 27 node Lagrange, etc.) and was demonstrated for 20 node serendipity element.

VCCMs so far assume the use of hexahedral finite element. Presently popular finite element mesh generation software is not able to automatically create a complex shaped model with hexahedral finite elements. This means that fracture analysis using VCCM would be almost prohibitive for complex shaped three-dimensional structures and structural/mechanical components, since the model generation processes need to be carried out manually for some extent.

In this paper, we propose a VCCM for tetrahedral finite elements, as an extension of Okada, Higashi, Kikuchi, Fukui and Kumazawa (2005). A conventional VCCM is first presented briefly and then a newly proposed one for tetrahedral finite elements is presented. Hence, proposed methodology is demonstrated in some simple example

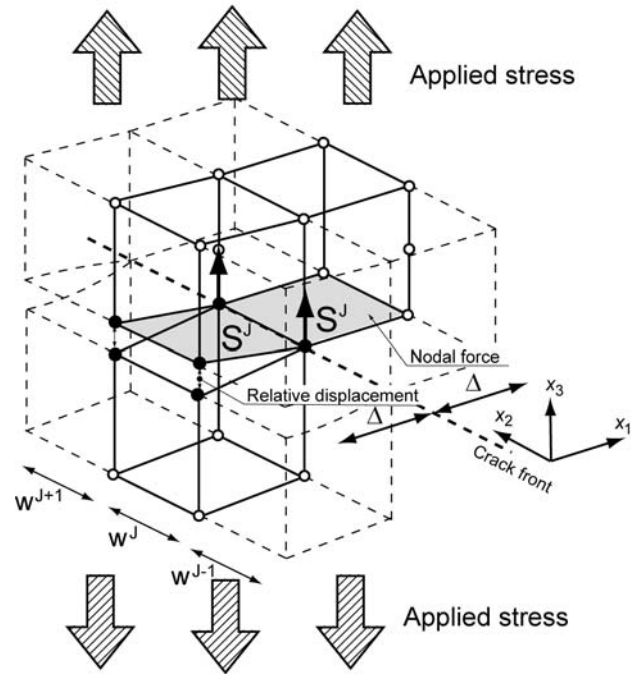


Figure 1 : Virtual crack closure-integral method for linear hexahedral finite element.

problems, i.e. through crack in a flat panel and semi-elliptical/circular embedded cracks. Though the formulations are developed for linear tetrahedral finite element, they can be extended to the case of quadratic element.

2 Brief review on VCCM for three-dimensional crack problems

In this section, a brief review of three-dimensional VCCM [Shivakumar, Tan and Newman, Jr. (1988)] is presented. It is, in general, assumed that the finite element mesh in the crack plane is arranged in an orthogonal manner, as shown in Fig. 1. The sizes of mesh across the crack front are the same. Energy release rate can be calculated by following the crack closure-integral method of Irwin (1958).

$$G_{Total} = \frac{1}{2\Delta w^J} \int_{S^J} \sigma_{3i}(x_1) v_i (\Delta - x_1) dS^J \quad (1)$$

where Δ and w^J are the length and the width of element perpendicular to and parallel to the crack front. J designates the segment number along the crack front. σ_{3i} are the cohesive stresses in the plane of crack ahead the crack front. v_i are the crack opening displacements, which are the functions of distance from the crack front. S^J is the

area of face of a finite element at the crack front segment J . It is noted that x_1 and x_2 coordinate directions are perpendicular and parallel to the crack front, and x_3 axis is normal to the plane of crack. The subscript “Total” refers the total energy release rate.

When 8 node linear element (8 node hexahedral element) is employed, the calculation of VCCM can be carried out by using nodal forces P_i^I and the differences v_i^I of displacements between upper and lower crack faces at nodes.

$$G_{Total} = \frac{1}{2\Delta w^J} \sum_{I=1}^2 C^I v_i^I P_i^I \quad (2)$$

where superscript I ($=1,2$) refers the nodal points on the plane of crack, as depicted in Fig. 2. Nodal forces P_i^I are calculated as consistent nodal force vector (see, for example, Bathe (1996), for the calculation of consistent nodal force vector). Constants C^I are given by:

$$C^1 = \frac{w^J}{w^J + w^{J-1}}, C^2 = \frac{w^J}{w^{J+1} + w^J} \quad (3)$$

For mixed mode problems, the total energy release rate G_{Total} can be split into its components G_I , G_{II} and G_{III} due to mode I, II and III crack loadings, as:

$$\begin{aligned} G_I &= \frac{1}{2\Delta w^J} \sum_{I=1}^2 C^I v_3^I P_3^I, \\ G_{II} &= \frac{1}{2\Delta w^J} \sum_{I=1}^2 C^I v_1^I P_1^I, \\ G_{III} &= \frac{1}{2\Delta w^J} \sum_{I=1}^2 C^I v_2^I P_2^I \end{aligned} \quad (4)$$

Thus, their corresponding stress intensity factors are evaluated, by:

$$K_I = \sqrt{E'G_I}, \quad K_{II} = \sqrt{E'G_{II}}, \quad K_{III} = \sqrt{2\mu G_{III}} \quad (5)$$

Here, $E' = E$ when a plane stress condition is assumed and $E' = E/(1 - \nu^2)$ for the case of plane strain condition. E and ν are the Young’s modulus and Poisson’s ratio of the material. μ is the shear modulus.

Discussions given above imply that the finite elements at the crack front must be hexahedral or their degenerated wedge-type elements. The faces of the elements across the crack front must be symmetrically arranged rectangles, as depicted in Fig. 2. The use of tetrahedral finite elements whose faces are triangles is not considered.

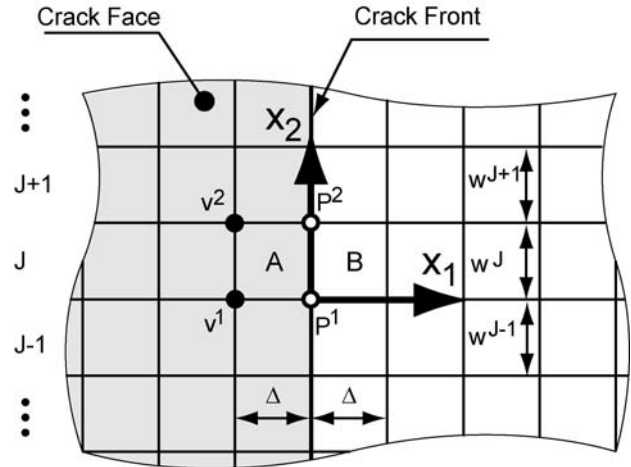


Figure 2 : The rectangular arrangement of element faces for energy release rate computation by VCCM with hexahedral finite elements.

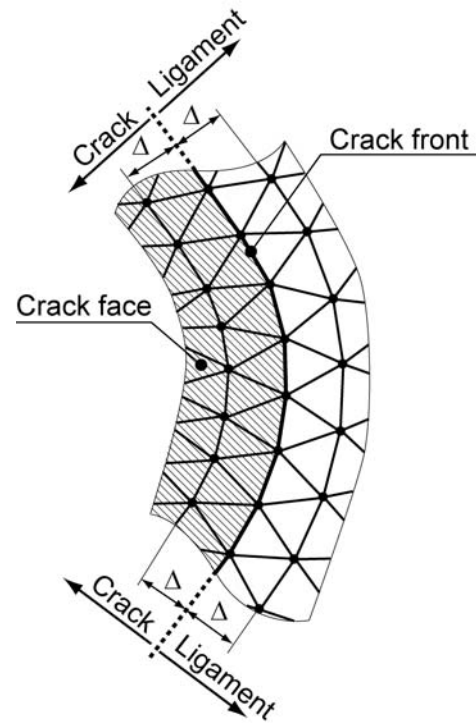


Figure 3 : Arrangement of element faces of tetrahedral finite elements for the computation of energy release rate using proposed VCCM with tetrahedral finite elements.

3 VCCM for three-dimensional crack problems using tetrahedral finite elements

3.1 Crack closure-integral and energy release rate

In this section, we consider the use of tetrahedral finite elements to three-dimensional crack problems with ar-

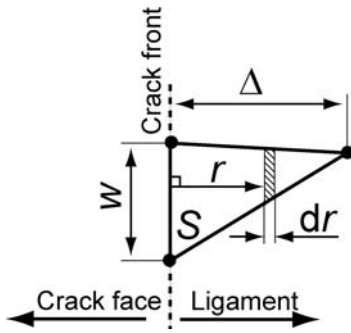


Figure 4 : A face of tetrahedral finite element for VCCM computation.

bitrary curved crack front and the evaluation of energy release rates by using the virtual crack closure-integral method (VCCM). We pose several assumptions on finite element discretization at the crack front. They are (i) the faces of tetrahedral elements exist on the extended plane of the crack and (ii) the widths (Δ) of element faces across the crack front are the same, as shown in Fig. 3. The energy release rates are then computed from energy that is required when a triangle shaped area is opened. In the followings, discussions on the mode I energy release rate are given. The same can be carried out for mode II and III and total energy release rate computations.

First, we compute the energy spent to open a face of a tetrahedral finite element, as shown in Fig. 4, by assuming the asymptotic solutions of the crack opening displacement v_3 and the normal stress σ_{33} in x_3 direction. They are written to be:

$$v_3(r) = \frac{4K_I}{E'} \sqrt{\frac{2r}{\pi}}, \quad \sigma_{33}(r) = \frac{K_I}{\sqrt{2\pi r}} \quad (6)$$

where K_I is the mode I stress intensity factor and r is the distance measured from the crack front on the crack face and its extended plane. Amount of energy δW that is required to open the area S , can be defined by extending the concept of crack closure-integral approach of Irvin (1958).

$$\delta W = \int_S \frac{1}{2} v_3(\Delta - r) \sigma_{33}(r) dS \quad (7)$$

Integral of Eq. 7 can be written by using the distance r , as:

$$\delta W = \int_0^\Delta \frac{1}{2} v_3(\Delta - r) \sigma_{33}(r) \left(1 - \frac{r}{\Delta}\right) w dr \quad (8)$$

where Δ and w are the sizes of the element face perpendicular to and parallel to the crack front as shown in Fig. 4. Eq. 8 can be rewritten by using the asymptotic expressions [Eq. 6] for the crack opening displacement and the stress.

$$\delta W = \int_0^\Delta \frac{2K_I}{E'} \sqrt{\frac{2(\Delta - r)}{\pi}} \frac{K_I}{\sqrt{2\pi r}} \left(1 - \frac{r}{\Delta}\right) w dr \quad (9)$$

We then compute the integral.

$$\begin{aligned} \delta W &= \int_0^\Delta \frac{2K_I}{E'} \sqrt{\frac{2(\Delta - r)}{\pi}} \frac{K_I}{\sqrt{2\pi r}} \left(1 - \frac{r}{\Delta}\right) w dr \\ &= \frac{2wK_I^2}{\pi E'} \int_0^\Delta \left\{ \sqrt{\frac{\Delta - r}{r}} - \frac{1}{\Delta} \sqrt{r(\Delta - r)} \right\} dr \\ &= \frac{3}{4} \frac{\Delta w K_I^2}{E'} \end{aligned} \quad (10)$$

Here, the area S of the triangular region is:

$$S = \frac{\Delta w}{2} \quad (11)$$

Therefore, we have:

$$\delta W = \frac{3}{2} \frac{SK_I^2}{E'} \quad (12)$$

Since the relationship between the energy release rate G_I and the stress intensity factor K_I is given by $G_I = K_I^2/E'$, we can express the energy release rate, as:

$$G_I = \frac{K_I^2}{E'} = \frac{2\delta W}{3S} \quad (13)$$

3.2 Computation for crack closure integral using the results of finite element analysis

In the finite element computation, energy δW that is released when the crack is opened for the area S can be expressed in terms of nodal forces \bar{P}_3^I , as:

$$\delta W = \frac{1}{2} \sum_{I=1}^2 v_3^I \bar{P}_3^I \quad (14)$$

where \bar{P}_3^I are the nodal forces arising from the cohesive stress on the area S and v_3^I are the respective nodal displacements, as shown in Fig. 5.

When a finite element program computes the nodal reaction forces P_3^I , they are computed based on element

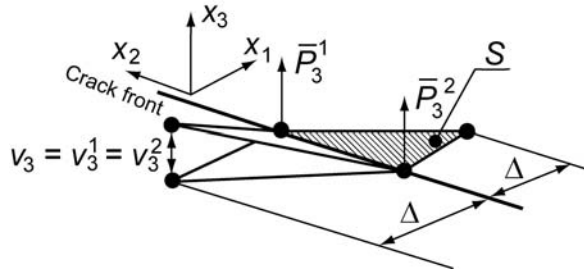


Figure 5 : The reaction forces \bar{P}_3^I due to the cohesive stress on area S and their respective nodal opening displacements v_3^I .

stresses. Therefore, they are not the same as \bar{P}_3^I that are used in Eq. 14. Nodal reaction forces P_3^I that are computed by using ordinary finite element procedures contain the contributions of the cohesive forces on the neighboring element faces. We thus partition P_3^I appropriately and \bar{P}_3^I are written to be:

$$\bar{P}_3^I = \bar{C}^I P_3^I \quad (15)$$

We call \bar{C}^I to be partitioning constants in this paper. For example, as shown in Fig. 6, node A at the crack front belongs to three different element faces. To appropriately partition the nodal reaction forces, we define the representative weights $W^{S^{I-1}}$, $W^{S^{I-1}-1}$ and $W^{S^{I-2}-2}$ of finite element faces S^{I-2} , S^{I-1} and S^I . Here, the superscript such as $S^I - 1$ indicates the weight associated with the 1st node of the I -th element face that is shown in Fig. 6. The weights are computed with assuming the asymptotic distribution ($\propto 1/\sqrt{x_1}$) of cohesive stress at the vicinity of the crack front. Thus, we compute the weights by the following equations.

$$W^{S^I-1} = \int_{S^I} \frac{N_1}{\sqrt{x_1}} dS^I = \frac{16S^I}{15\sqrt{x_1^I}} \quad (16)$$

$$W^{S^{I-1}-1} = \int_{S^{I-1}} \frac{N_1}{\sqrt{x_1}} dS^{I-1} = \frac{16S^{I-1}}{15\left(\sqrt{x_1^{I-1}} + \sqrt{x_1^I}\right)} \quad (17)$$

$$W^{S^{I-2}-1} = \int_{S^{I-2}} \frac{N_2}{\sqrt{x_1}} dS^{I-2} = \frac{16S^{I-2}}{15\sqrt{x_1^{I-1}}} \quad (18)$$

where shape functions associated with the respective element faces associated with the node A are shown by symbols such as N_1 . x_1^{I-1} and x_1^I are the coordinate values of the nodes, as depicted in Fig. 6.

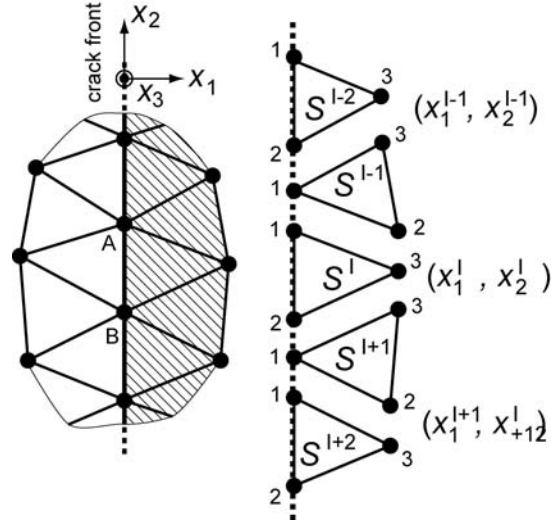


Figure 6 : Computation of partitioning constants for the VCCM computation using tetrahedral finite elements.

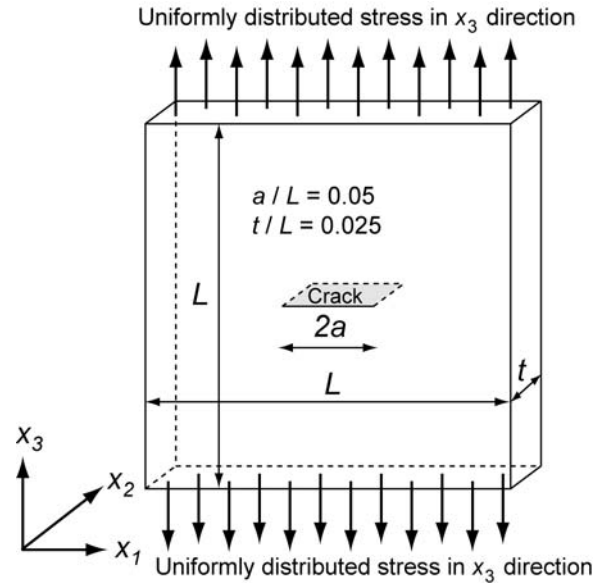
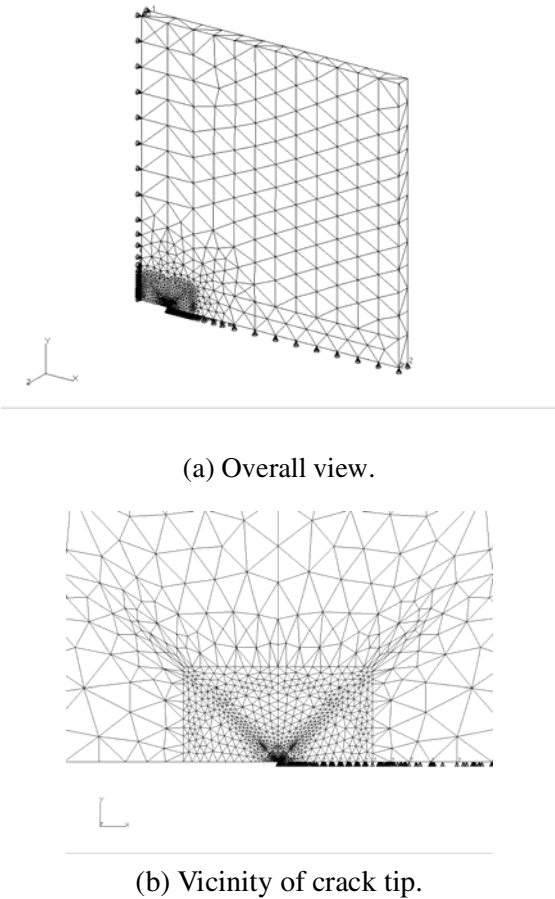


Figure 7 : A through crack in a tension panel.

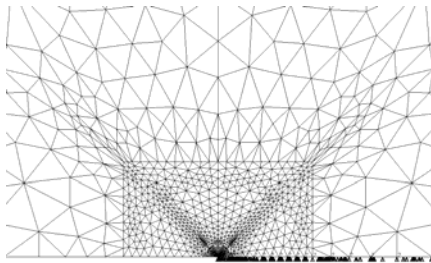
Constants \bar{C}^I are defined, as:

$$\begin{aligned} \bar{C}^1 &= \frac{W^{S^I-1}}{W^{S^I-1} + W^{S^{I-1}-1} + W^{S^{I-2}-2}}, \\ \bar{C}^2 &= \frac{W^{S^I-2}}{W^{S^I-2} + W^{S^{I+1}-1} + W^{S^{I+2}-1}} \end{aligned} \quad (19)$$

When the node A belongs to only one element face, respective partitioning constant \bar{C}^I is one.



(a) Overall view.



(b) Vicinity of crack tip.

Figure 8 : Finite element discretization for the tension panel of Fig. 7.

Thus, the energy release rate can be computed, by:

$$G_I = \frac{1}{3S} \sum_{l=1}^2 \bar{C}^l P_3^l v_3^l \quad (20)$$

Mode II and III energy release rates can be evaluated in the same manner. Hence, the stress intensity factors can be computed by using Eq. 5.

4 Numerical demonstrations

4.1 Through crack in a flat panel

First, the problem of a through crack in a flat panel is presented. The configurations of the panel, crack and boundary conditions are depicted in Fig. 7. Finite element model is shown in Fig. 8. There are a total of 580879 elements and 115219 nodes. The size of crack

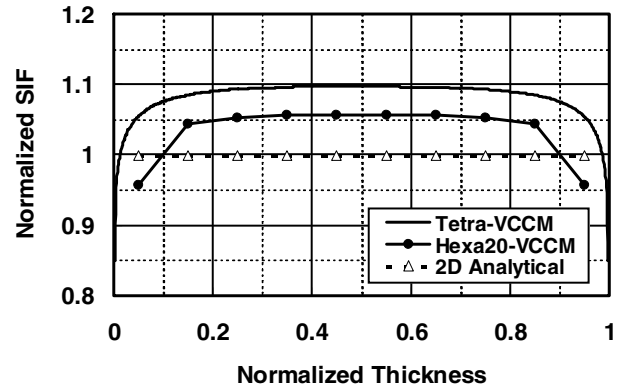


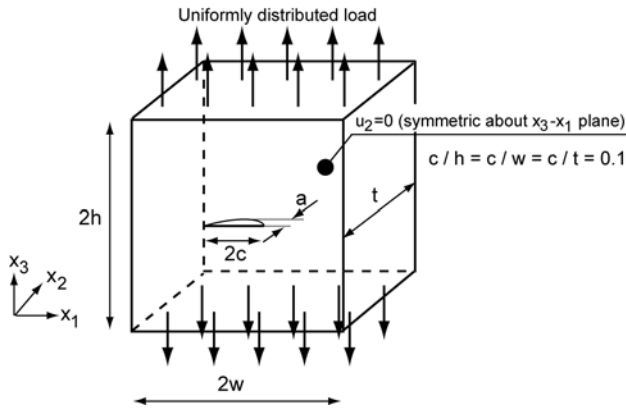
Figure 9 : Normalized stress intensity factors that were evaluated by proposed VCCM for tetrahedral finite elements and by VCCM with 20 node hexahedral elements.

front element is 1/1000 of the total length of the crack. The Young's modulus and Poisson's ratio are set to be 210 GPa and 0.3.

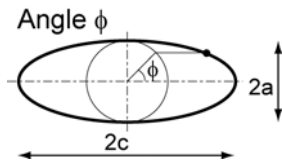
In Fig. 9, comparisons between the stress intensity factors of two-dimensional analytical solution and those computed by the three-dimensional VCCMs using hexahedral and tetrahedral finite elements, are presented. All the data in Fig. 9 is normalized by two-dimensional analytical solution [see, for example, Murakami et al. (1987)]. That is why two-dimensional analytical solution is unity in Fig. 9. The solution of hexahedral finite elements is taken from the author's previous work [Okada, Higashi, Kikuchi, Fukui and Kumazawa (2005)] and is used as a reference solution. In Okada, Higashi, Kikuchi, Fukui and Kumazawa (2005), 20 node hexahedral element is employed and the size of element at the crack front is 8/1000 of total crack length. In the computation of stress intensity factor, the plane strain condition is assumed except for at the surface of the panel. It is, indeed, the common practice as done in Fawaz (1998) also. Both the results based on tetrahedral and hexahedral finite elements have maximum values at the mid-surface of the panel. The trends are very similar to each other. However, the one that is computed based on tetrahedral finite elements have a larger value than that based on hexahedral finite elements. The difference is about 4%.

4.2 Embedded elliptical/ circular crack problem

The problems of embedded elliptical/circular cracks are solved and the results are compared with the analytical



(a) Overall view of the block, crack configuration and BCs.



(b) Definition of angle ϕ .

Figure 10 : The problem of embedded elliptical/circular crack.

solutions of those in an infinite elastic body. In Fig. 10, the problem configuration is depicted. In Fig. 11, a typical finite element model is shown. There are a total of 413112 elements and 72091 nodes. Young's modulus and Poisson's ratio are set to be 210 GPa and 0.3, respectively. The plane strain condition is assumed when the stress intensity factor is computed from the energy release rate.

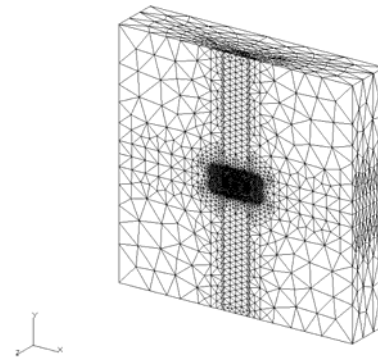
In Fig. 12, the results of present analyses are compared with the analytical solutions. In the figures, the solutions are normalized by $Q(a/c) \sqrt{\pi a}$ where $Q(a/c)$ is the shape factor [see, for example, Murakami et al. (1987)]. $Q(a/c)$ is given by:

$$Q\left(\frac{a}{c}\right) = \sqrt{\int_0^{\pi/2} [1 - k^2 \sin^2 \theta]^{1/2} d\theta} \approx 1 + 1.464 \left(\frac{a}{c}\right)^{1.65}$$

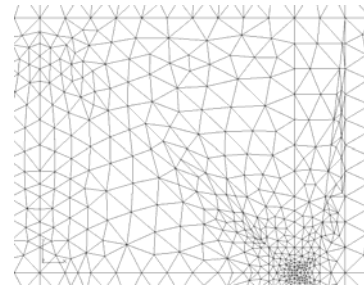
$$k = \left(1 - \frac{a^2}{c^2}\right)^{\frac{1}{2}} \quad (c \geq a),$$

$$k = \left(1 - \frac{c^2}{a^2}\right)^{\frac{1}{2}} \quad (c < a) \quad (21)$$

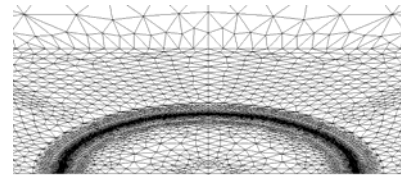
For all the cases that are presented in Fig. 12, the stress



(a) Overall view.



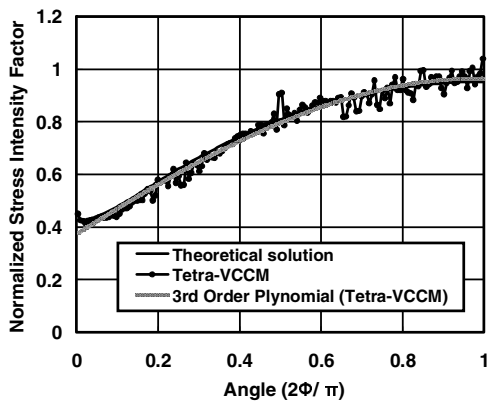
(b) A close-up of crack front.



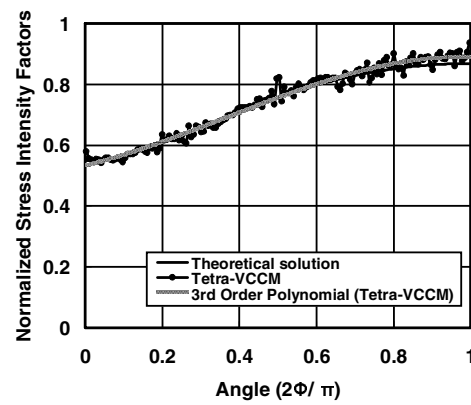
(c) A view of plane of crack.

Figure 11 : A typical finite element mesh discretization for the problem of embedded elliptical/circular crack (aspect ratio $a/c = 0.4$).

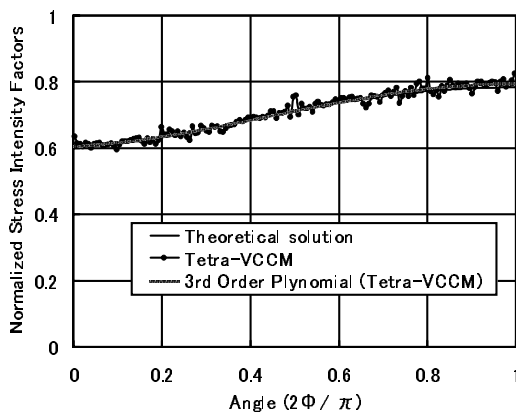
intensity factor that was computed by the present approach has oscillatory behavior for all the aspect ratios. We then approximated the variations of the stress intensity factor by the third-order polynomial. Thus, as seen in the figures, the polynomial approximations and the analytical solutions are very close. The differences are within 2%.



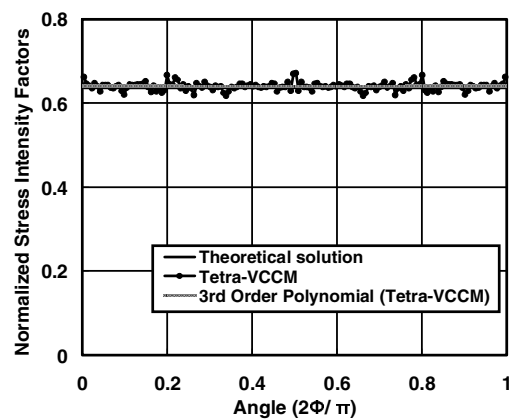
(a) Aspect ratio $a/c = 0.2$.



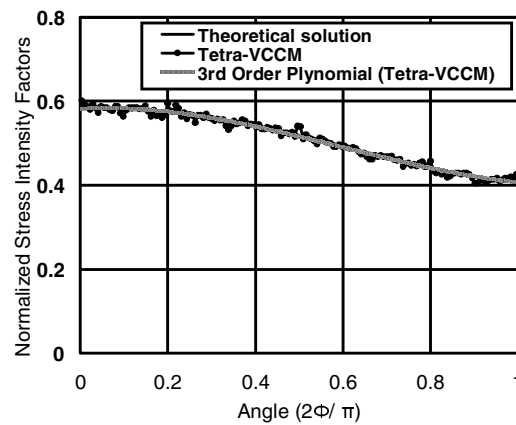
(b) Aspect ratio $a/c = 0.4$



(c) Aspect ratio $a/c = 0.6$.



(d) Aspect ratio $a/c = 1.0$.



(e) Aspect ratio $a/c = 2.0$.

Figure 12 : Variations of normalized stress intensity factor for the problems of embedded elliptical/circular cracks computed by proposed VCCM for tetrahedral finite elements.

5 Concluding remarks

In this paper, some preliminary results of stress intensity factor/energy release rate calculations by using VCCM with linear tetrahedral finite element are presented. The results indicate that the accuracy of present approach is not satisfactory compared with those computed by VCCM with twenty-node hexahedral finite elements [see, for example, Okada, Higashi, Kikuchi, Fukui and Kumazawa (2005)]. This means that the method is still premature and that it requires many improvements. In the engineering practices, the use of quadratic tetrahedron is a very popular way to compromise the accuracy and efficiency of analysis. Therefore, the most important improvement that the authors should try is to extend present formulation to the case of quadratic tetrahedral element.

Though we still need to make many improvements, the outcomes of present course of study so far indicate a possibility that tetrahedral finite elements could replace hexahedral elements in fracture mechanics analysis.

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