 CMES, vol.7, no.2, pp.119-132, 2005

The method of fundamental solution for solving multidimensional inverse heat conduction problems

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Abstract: We propose in this paper an effective meshless and integration-free method for the numerical solution of multidimensional inverse heat conduction problems. Due to the use of fundamental solutions as basis functions, the method leads to a global approximation scheme in both the spatial and time domains. To tackle the ill-conditioning problem of the resultant linear system of equations, we apply the Tikhonov regularization method based on the generalized cross-validation criterion for choosing the regularization parameter to obtain a stable approximation to the solution. The effectiveness of the algorithm is illustrated by several numerical two- and three-dimensional examples.

keyword: Inverse heat conduction problem, fundamental solution method, Tikhonov regularization

1 Introduction

A standard inverse heat conduction problem (IHCP) is to compute the unknown temperature and heat flux at an unreachable boundary from scattered temperature measurements at reachable interior or boundary of the domain. In solving direct heat conduction problems, the errors induced from boundary or interior measurements are reduced due to the diffusive characteristic of heat conduction process. These errors, however, are extrapolated and amplified due to the extremely ill-posedness of the inverse heat conduction problems. In other words, a small error in the measurement can induce enormous error in computing the unknown solution at the unreachable boundary.

Several techniques have been proposed for solving a one-dimensional IHCP [Beck, Blackwell, and Clair (1985); Chantasiriwan (1999); Lesnic, Elliott, and Ingham (1996); Lesnic and Elliott (1999); Jonas and Louis (2000); Shen (1999)]. Among the methods proposed for higher dimensional IHCP, boundary element [Chantasiriwan (2001); Kurpisz and Nowak (1992)], finite difference [Guo and Murio (1991); Khalidy (1998)] and finite element [Hsu, Sun, Chen, and Gong (1992); Reinhardt (1991)] have been widely adopted for problems in two-dimension. Besides, the sequential function specification method [Beck, Blackwell, and Clair (1985); Chantasiriwan (2001)] and mollification method [Murio (1993)] have also been used in solving the IHCP. There is, however, still a need on numerical scheme for multidimensional IHCP.

The traditional mesh-dependent finite difference and finite element methods require dense meshes, and hence tedious computational time, to give a reasonable approximation to the solution of IHCP and suffers from numerical instability problem. The use of boundary element method (BEM) reduces the computational time and storage requirement but the problem of numerical stability still persists. Since there is no need on domain discretization in the BEM, the location of interior points, where the temperature data are collected, can be chosen in a quite arbitrary way [Chantasiriwan (2001)].

In this paper a new meshless computational method is proposed to approximate the solution of a multidimensional IHCP under arbitrary geometry. In recent years, meshless methods, in particular the Meshless Local Petrov-Galerkin(MLPG), have attracted great attention in the scientific community [Atluri and Shen (2002); Atluri, Han, and Shen (2003); Atluri, Han, and Rajendran (2004)]. A good reference on MLPG can be found from the book of [Atluri (2004)]. The proposed method uses the fundamental solution of the corresponding heat equation to generate a basis for approximating the solution of the problem. Comparing with the mesh-dependent methods like FEM and BEM, the proposed method does not require any domain or bound-

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ary discretization. This meshless advantage makes the method feasible to solve high dimensional IHCPs under arbitrary geometry. Furthermore, unlike BEM which requires an interpolation of the unknown function on the boundary for complicated boundary integrations, the proposed method gives a global numerical solution on the entire time-spatial domain. The required surface temperature and heat flux on the unreachable boundary can easily be computed without any extra quadratures. To tackle the numerical instability in the resultant ill-conditioned linear system, we use Tikhonov regularization technique and the generalized cross-validation criterion [Hansen (1992, 1994)] to obtain a stable numerical solution for the multi-dimensional IHCP problem.

In the recent rapid development of meshless computational schemes, the radial basis function (RBF) has been successfully developed as an efficient meshless scheme for solving various kinds of partial differential equations (PDEs). Most of the problems are still confined to direct problems [Hon and Mao (1999); Hon, Cheung, Mao, and Kansa (1999); Hon, Lu, Xue, and Zhou (1999); Hon and Chen (2003); Kansa and Hon (2000)]. In [Hon and Wu (2000)], Hon and Wu first applied the meshless RBF to solve a Cauchy problem for Laplace equation, which is a severely ill-posed problem. In [Hon and Wei (2002)], Hon and Wei combined the meshless RBF with the method of fundamental solution (MFS) to successfully solve an one-dimensional inverse heat conduction problem. This approach can be interpreted as an extension of the MFS for treating elliptic problems, for examples, the Laplace equation [Bogomonlny (1985); Mathon and Johnston (1977)], the biharmonic equations [Karageorghis and Fairweather (1987)], elastostatics problems [Poullikkas, Karageorghis, and Georgiou (2002)], and wave scattering problems [Kondapalli, Shippy, and Fairweather, (1992a, 1992b]. The MFS was also applied to solve nonhomogeneous linear and nonlinear Poisson equations [Balakrishnan and Ramachandran (1999, 2000, 2001); Golberg (1995); Partridge and Sensale (2002)]. More details of the MFS can be found in the review papers of [Fairweather and Karageorghis (1998)] and [Golberg and Chen (1998)]. It is noted here that most of these research works focus on well-posed problems in which the Dirichlet or Neumann data are given on the whole boundary. In the IHCP problem, the boundary conditions are usually complicated and incomplete. We expect that the combination of the meshless RBF and the MFS will extend its application to solve higher dimensional inverse problems under irregular geometry.

In [Frankel and Keyhani (1997)], Frankel et al. gave an unified treatment for the IHCP by using the Chebyshev polynomials as basis functions. The use of fundamental solution as the basis functions in this paper provides a global approximation to the solution in both the spatial and time variables. Since IHCP problem is severely ill-posed and its solution is extremely sensitive to any perturbation of given data, enormous error will be accumulated from using any finite difference scheme for discretizing the time variable [Cho, Golberg, Muleshkov, and Li]. The proposed method has a definite advantage over most of the existing numerical methods in solving these kinds of time-dependent inverse problems. The use of Laplace transform for the time variable will reduce this time discretization error but result in solving a Cauchy problem for inhomogeneous modified Helmholtz equation with an extra parameter in the resulting equation [Cho, Golberg, Muleshkov, and Li]. For a slightly larger parameter, the Laplace transform method fails to give a stable and accurate approximation. In fact, it is even more difficult to solve a Cauchy problem for modified Helmholtz than solving the original IHCP. For large scale problems, an efficient numerical method for treating the ill-conditioning discrete problem is still needed for improvement. The recently developed iterative algorithms based on Lanczos bidiagonalization will be a consideration for future work.

2 Methodology

Let Ω be a simply connected domain in R^d, $d = 2, 3$ and Γ_1, Γ_2, Γ_3 be three parts of the boundary $\partial\Omega$. Suppose that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \partial\Omega$, Γ_1 or Γ_2 can be empty set. The IHCP to be investigated in this paper is to determine the temperature and heat flux on boundary Γ_3 from given Dirichlet data on Γ_1, Neumann data on Γ_2 and scattered measurement data at some interior points.

Consider the following heat equation:

$$u_t(x,t) = a^2 \Delta u(x,t), \qquad x \in \Omega, \quad t \in (0, t_{max}), \tag{1}$$

under the initial condition

$$u(x,0) = \varphi(x), \qquad x \in \overline{\Omega}, \tag{2}$$

and the boundary conditions

$$u(x,t) = f(x,t), \qquad x \in \Gamma_1, \quad t \in (0, t_{max}], \tag{3}$$

and

$$\frac{\partial u}{\partial \nu}(x,t) = g(x,t), \qquad x \in \Gamma_2, \quad t \in (0, t_{max}], \tag{4}$$

where ν is the outer unit normal with respect to Γ_2.

Let $\{x_i\}_{i=1}^{M} \subset \overline{\Omega}$ be a set of locations with noisy measured data $\widetilde{h}_i^{(k)}$ of exact temperature $u(x_i, t_i^{(k)}) = h_i^{(k)}$, $i = 1, 2, \cdots, M$, $k = 1, 2, \cdots, I_i$ where $t_i^{(k)} \in (0, t_{max}]$ are discrete times. The absolute error between the noisy measurement and exact data is assumed to be bounded, i.e. $|\widetilde{h}_i^{(k)} - h_i^{(k)}| \leq \delta$ for all measurement points at all measured times. Here, the constant δ is called the noisy level of input data.

The IHCP is now formulated as: Reconstruct $u|_{\Gamma_3}$ and $\frac{\partial u}{\partial \nu}|_{\Gamma_3}$ from (1)–(4) and the scattered noisy measurements $\widetilde{h}_i^{(k)}$, $i = 1, 2, \cdots, M$, $k = 1, 2, \cdots, I_i$.

Remark 2.1 *Most of the existing numerical methods for solving the IHCP problem require a continuous time measurement $u(x_i, t) = h_i(t)$, which is not realistic in real life.*

The fundamental solution of (1) is given by

$$F(x,t) = \frac{1}{(4\pi a^2 t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4a^2 t}} H(t), \tag{5}$$

where $H(t)$ is the Heaviside function. Assuming that $T > t_{max}$ is a constant, the following function

$$\phi(x,t) = F(x, t+T) \tag{6}$$

is a general solution of (1) in the solution domain $\Omega \times [0, t_{max}]$.

We denote the measurement points to be $\{(x_j, t_j)\}_{j=1}^{m}, m = \sum_{i=1}^{M} I_i$ so that a point at the same location but with different time is treated as two distinct points. The corresponding measured noisy data and exact data are denoted by $\widetilde{h}_j$ and h_j. The collocation points are then chosen as $\{(x_j, t_j)\}_{j=m+1}^{m+n}$ on the initial region $\overline{\Omega} \times \{0\}$, $\{(x_j, t_j)\}_{j=m+n+1}^{m+n+p}$ on surface $\Gamma_1 \times (0, t_{max}]$ and $\{(x_j, t_j)\}_{j=m+n+p+1}^{m+n+p+q}$ on surface $\Gamma_2 \times (0, t_{max}]$. Here, n, p, q denote the total number of collocation points for the initial condition (2), Dirichlet boundary condition (3) and Neumann boundary condition (4) respectively. The only requirement on the collocation points are pairwisely distinct in the $(d+1)$-dimensional space (x, t).

Following the idea of the MFS, an approximation $\widetilde{u}$ to the solution of the IHCP under the conditions (2)-(4) with the noisy measurements $\widetilde{h}_j$ can be expressed by the following linear combination:

$$\widetilde{u}(x,t) = \sum_{j=1}^{n+m+p+q} \widetilde{\lambda}_j \phi(x - x_j, t - t_j), \tag{7}$$

where $\phi(x,t)$ is given by (6) and $\widetilde{\lambda}_j$ are unknown coefficients to be determined.

For this choice of basis function ϕ, the approximate solution $\widetilde{u}$ automatically satisfies the original heat equation (1). Using the initial condition (2) and collocating at the boundary conditions (3) and (4), we then obtain the following system of linear equations for the unknown coefficients $\widetilde{\lambda}_j$:

$$A\widetilde{\lambda} = \widetilde{b}, \tag{8}$$

where

$$A = \begin{pmatrix} \phi(x_i - x_j, t_i - t_j) \\ \frac{\partial \phi}{\partial \nu}(x_k - x_j, t_k - t_j) \end{pmatrix} \tag{9}$$

and

$$\widetilde{b} = \begin{pmatrix} \widetilde{h}_i \\ \varphi(x_i, t_i) \\ f(x_i, t_i) \\ g(x_k, t_k) \end{pmatrix} \tag{10}$$

where $i = 1, 2, \cdots, (m+n+p)$, $k = (m+n+p+1), \cdots, (m+n+p+q)$, $j = 1, 2, \cdots, (n+m+p+q)$ respectively.

The solvability of the system (8) depends on the non-singularity of the matrix A, which is still an open research problem. It is not surprise that the resultant matrix A is extremely ill-conditioned due to the ill-posed nature of the IHCP. The following sections shows that the use of regularization technique can produce a stable and accurate solution for the unknown solution $\widetilde{\lambda}$ of the matrix equation (8), and hence the solution for the IHCP is obtained.

3 Regularization Techniques

The ill-conditionedness of the coefficient matrix A indicates that the numerical result is sensitive to the noise of right hand side $\widetilde{b}$ and the number of collocation points. In fact, the condition number of the matrix A increases dramatically with respect to the total number of collocation points. The singular value decomposition (SVD) usually works well for direct problem [Ramachandran (2002)] but still fails to provide a stable and accurate solution to the system (8). A number of regularization methods have been developed for solving this kind of ill-conditioning problem [Hansen (1992)]. In our computation we apply the Tikhonov regularization technique [Tikhonov and Arsenin (1977)] based on SVD with the generalized cross-validation criterion for choosing a good regularization parameter to solve the matrix equation (8).

Denote $N = n+m+p+q$. The SVD of the $N \times N$ matrix A is a decomposition of the form:

$$A = W\Sigma V' = \sum_{i=1}^{N} w_i \sigma_i v_i' \tag{11}$$

where $W = (w_1, w_2, \cdots, w_N)$ and $V = (v_1, v_2, \cdots, v_N)$ satisfying $W'W = V'V = I_N$. Here, the superscript $'$ represents the transpose of a matrix. It is known that $\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_N)$ has non-negative diagonal elements satisfying:

$$\sigma_1 \geq \cdots \sigma_N \geq 0. \tag{12}$$

The values σ_i are called the singular values of A and the vectors w_i and v_i are called the left and right singular vectors of A respectively.

For the matrix A arising from the discretization of the MFS, the singular values decay rapidly to zero and the ratio between the largest and the smallest nonzero singular values is often huge. Thus the linear system (8) is a discrete ill-posed problem in the sense defined in Hansen's paper [Hansen (1992)].

Based on the singular value decomposition, it is easy to know that the solution for the linear equations (8) is given by

$$\widetilde{\lambda} = \sum_{i=1}^{N} \frac{w_i' \widetilde{b}}{\sigma_i} v_i. \tag{13}$$

The corresponding solution with exact data is

$$\lambda = \sum_{i=1}^{N} \frac{w_i' b}{\sigma_i} v_i \tag{14}$$

where b is calculated by (10) from the exact data h_j.

The difference between these two solutions is then given by

$$\widetilde{\lambda} - \lambda = \sum_{i=1}^{N} \frac{w_i' e}{\sigma_i} v_i, \tag{15}$$

where $e = \widetilde{b} - b$, $\|e\| \leq \delta$. The terms in the difference with small values of σ_i will be large since the noisy level is generally greater than the smallest nonzero singular value. This is the reason why the following Tikhonov regularization method is proposed.

The Tikhonov regularized solution $\widetilde{\lambda}_\alpha$ for equation (8) is defined to be the solution to the following least square problem:

$$\min_{\widetilde{\lambda}} \{ \|A\widetilde{\lambda} - \widetilde{b}\|^2 + \alpha^2 \|\widetilde{\lambda}\|^2 \}, \tag{16}$$

where $\|\cdot\|$ denotes the usual Euclidean norm and α is called the regularization parameter. The Tikhonov regularized solution based on SVD can the be expressed as:

$$\widetilde{\lambda}_\alpha = \sum_{i=1}^{N} f_i \frac{w_i' \widetilde{b}}{\sigma_i} v_i, \tag{17}$$

where $f_i = \sigma_i^2 / (\sigma_i^2 + \alpha^2)$ are called the filter factors, $i = 1, 2, \cdots, N$. The difference between $\widetilde{\lambda}_\alpha$ and λ is then given by

$$\widetilde{\lambda}_\alpha - \lambda = \sum_{i=1}^{N} (f_i - 1) \frac{w_i' b}{\sigma_i} v_i + \sum_{i=1}^{N} f_i \frac{w_i' e}{\sigma_i} v_i. \tag{18}$$

The norm of this error vector is

$$\|\widetilde{\lambda}_\alpha - \lambda\| = \left(\sum_{i=1}^{N} \left((f_i - 1) \frac{w_i' b}{\sigma_i} \right)^2 + \left(f_i \frac{w_i' e}{\sigma_i} \right)^2 \right)^{1/2}. \tag{19}$$

Hence, the approximate solution $\widetilde{u}_\alpha$ with noisy measurement for the IHCP is given by

$$\widetilde{u}_\alpha(x,t) = \sum_{j=1}^{N} (\widetilde{\lambda}_\alpha)_j \phi(x - x_j, t - t_j), \tag{20}$$

where $(\widetilde{\lambda}_\alpha)_j$ is the j-th entry of vector $\widetilde{\lambda}_\alpha$.

Let $u_N = \sum_{j=1}^{N} \lambda_j \phi(x - x_j, t - t_j)$ be an approximate solution with exact data given in the IHCP. An error estimation is then given as follow:

$$|\widetilde{u}_\alpha - u_N\|_{L^2(D)} \leq \|\widetilde{\lambda}_\alpha - \lambda\| \sum_{j=1}^{N} \|\phi(x - x_j, t - t_j)\|_{L^2(D)}, \tag{21}$$

where $D = \Omega \times (0, t_{max})$ and the value $\|\widetilde{\lambda}_\alpha - \lambda\|$ is given by formula (22). The determination of a suitable value for the regularization parameter α is crucial and is still under intensive research [Tikhonov and Arsenin (1977)]. In this paper, we use the generalized cross-validation criterion to choose a good regularization parameter.

The Generalized Cross-validation (GCV) [Hansen (1992)] is a strategy that give a good regularization parameter by minimizing the following GCV function

$$G(\alpha) = \frac{\|A\widetilde{\lambda}_\alpha - \widetilde{b}\|^2}{(trace(I_N - AA^I))^2}, \quad \alpha > 0 \tag{22}$$

where A^I is a matrix which produces the regularized solution when multiplied with $\widetilde{b}$, i.e., $\widetilde{\lambda}_\alpha = A^I\widetilde{b}$. In the Tikhonov regularization method, the denominator of (22) is simply $trace(I_N - AA^I) = N - \sum_{i=1}^{N} f_i$.

In our computation, the Matlab code developed by [Hansen (1994)] was used to obtain the optimal choice of regularization parameter α^* for solving the discrete ill-conditioned system (8). The corresponding approximate solution for the problem (1)–(4) with noisy measurement data is then given by

$$\widetilde{u}_{\alpha^*}(x,t) = \sum_{j=1}^{N} (\widetilde{\lambda}_{\alpha^*})_j \phi(x - x_j, t - t_j). \tag{23}$$

The temperature and heat flux at surface Γ_3 can also be calculated respectively.

4 Numerical Examples

For numerical verification, we assume that the heat conduction coefficient $a = 1$ and $t_{max} = 1$ for all the following examples. In the cases when the input data contain noises, we use the function rand given in Matlab to generate the noisy data $\widetilde{h}_i = h_i + \delta\, rand(i)$ where h_i is the exact data and $rand(i)$ is a random number in $[-1, 1]$. The magnitude δ indicates the noisy level of the measurement data.

To test the accuracy of the approximate solution, we compute the Root Mean Square Error (RMSE) by

$$E(u) = \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} ((\widetilde{u}_{\alpha^*})_i - u_i)^2}, \tag{24}$$

at sufficiently total N_t number of testing points in the domain $\Gamma_3 \times [0,1]$ where $(\widetilde{u}_{\alpha^*})_i$ and u_i are respectively the approximate and exact temperature at a test point. The RMSE for the heat flux $E(\frac{\partial u}{\partial \nu})$ is also similarly defined.

Consider the following three examples: Two-dimensional IHCPs:

Example 1: The exact solution of (1) is given by

$$u(x_1, x_2, t) = 2t + \frac{1}{2}(x_1^2 + x_2^2). \tag{25}$$

Example 2: The exact solution of (1) is given by

$$u(x_1, x_2, t) = e^{-4t}(cos(2x_1) + cos(2x_2)). \tag{26}$$

Both examples are computed for the following three configurations:

Case 1: Let
$\Omega = \{\ (x_1, x_2) \mid 0 < x_1 < 1,\ 0 < x_2 < 1\}$,
$\Gamma_1 = \{\ (x_1, x_2) \mid x_1 = 1,\ 0 < x_2 < 1\}$,
$\Gamma_2 = \{\ (x_1, x_2) \mid 0 < x_1 < 1,\ x_2 = 1\}$,
$\Gamma_3 = \partial\Omega \setminus \{\Gamma_1 \cup \Gamma_2\}$.

Case 2: Let
$\Omega = \{\ (x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$,
$\Gamma_1 = \emptyset$,
$\Gamma_2 = \{\ (x_1, x_2) \mid x_1^2 + x_2^2 = 1,\ x_1 > 0\ \}$.
$\Gamma_3 = \partial\Omega \setminus \{\Gamma_1 \cup \Gamma_2\}$.

Case 3: Let Ω be the same as Case 1, $\Gamma_1 = \Gamma_2 = \emptyset$, $\Gamma_3 = \partial\Omega \setminus \{\Gamma_1 \cup \Gamma_2\}$.

Locations of the internal measurements and collocation points over Ω for the three cases are shown in Figure 1.

The boundary data $f(x,t)$, $g(x,t)$ and initial temperature $\varphi(x,t)$ are obtained from the given exact solutions. The values of temperature at measurement points will be used by the exact and noisy data respectively. Numerical results are obtained by taking the constant $T = 1.8$ for Example 1 and $T = 1.6$ for Example 2 in all different

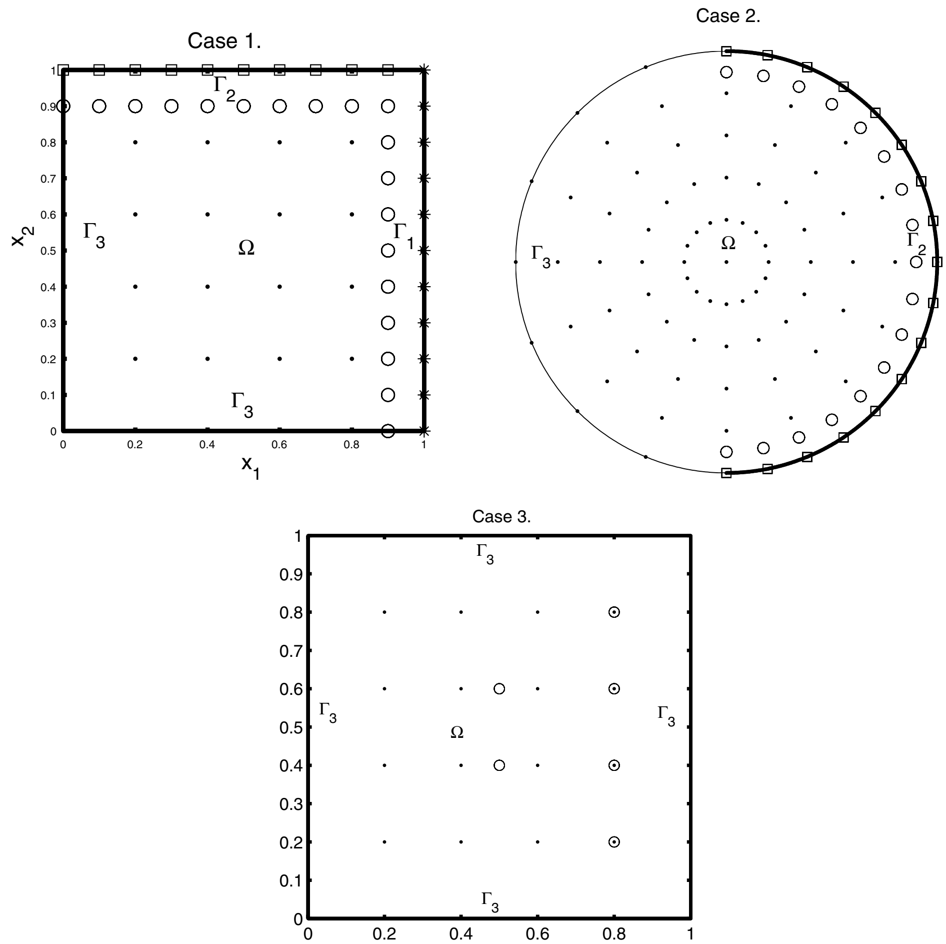


Figure 1 : Distribution of measurement points and collocation points. Here, star represents collocation point matching Dirichlet data, square represents collocation point matching Neumann data, dot represents collocation point matching initial data, and circle represents point with sensor for internal measurement.

cases. The total numbers of various collocation points and measurement points are $n = 36$, $m = 100$, $p = 55$, $q = 50$ in Case 1 for Examples 1-2. In this situation the total number of points and testing points are $N = 241$ and $N_t = 882$. For Case 2, the total numbers are $n = 96$, $m = 85$, $p = 0$, $q = 85$, $N = 266$, $N_t = 693$ and for Case 3, $n = 36$, $m = 30$, $p = 0$, $q = 0$, $N = 66$, $N_t = 1764$ respectively.

Numerical results by using only SVD for the solution (14) and u_N are presented in Table 1. The computed RMSEs from the experiments show that even for exact input data the direct method cannot produce an acceptable solution to this kind of ill-conditioned linear systems. In fact, the condition numbers in Case 1 and Case 2 lie between 10^{33}-10^{35} which are too large to obtain an

Table 1 : RMSEs in domain $\Gamma_3 \times [0,1]$ with exact data. No regularization technique.

	Example 1	Example 2
Case 1	E(u)=5.9122 E($\frac{\partial u}{\partial \nu}$)=2.2155	E(u)= 0.4909 E($\frac{\partial u}{\partial \nu}$)=0.2325
Case 2	E(u)= 2.2742 E($\frac{\partial u}{\partial \nu}$)=0.5040	E(u)=7.9616 E($\frac{\partial u}{\partial \nu}$)= 1.7736
Case 3	E(u)=0.0102 E($\frac{\partial u}{\partial \nu}$)= 0.0543	E(u)= 0.0011 E($\frac{\partial u}{\partial \nu}$)=0.0063

Table 2 : RMSEs in domain $\Gamma_3 \times [0,1]$ with exact data by using Tikhonov regularization technique with GCV parameter α_*.

	Example 1	Example 2
Case 1	α_*= 3.6411e-14 E(u)= 9.2406e-5 E($\frac{\partial u}{\partial \nu}$)=2.5043e-4	α_*=4.2389e-14 E(u)= 1.7062e-5 E($\frac{\partial u}{\partial \nu}$)= 6.1320e-5
Case 2	α_*=3.4730e-14 E(u)=4.2827e-4 E($\frac{\partial u}{\partial \nu}$)= 0.0012	α_*= 3.9698e-14 E(u)= 1.9647e-5 E($\frac{\partial u}{\partial \nu}$)=6.0150e-5
Case 3	α_*= 1.6529e-14 E(u)= 3.5346e-4 E($\frac{\partial u}{\partial \nu}$)=0.0021	α_*= 3.9698e-14 E(u)=0.0021 E($\frac{\partial u}{\partial \nu}$)=0.0124

Table 3 : RMSEs in domain $\Gamma_3\times$ [0,1] with noisy data (δ = 0.01) by using Tikhonov regularization with GCV parameter α_*.

	Example 1	Example 2
Case 1	α_*=5.4088e-8 E(u)= 0.0048 E($\frac{\partial u}{\partial \nu}$)=0.0140	α_*=1.4704e-6 E(u)= 0.0049 E($\frac{\partial u}{\partial \nu}$)=0.0134
Case 2	α_*=6.7604e-9 E(u)=0.0108 E($\frac{\partial u}{\partial \nu}$)= 0.0294	α_*=4.2725e-7 E(u)=0.0043 E($\frac{\partial u}{\partial \nu}$)= 0.0115
Case 3	α_*=4.6697e-8 E(u)= 0.0057 E($\frac{\partial u}{\partial \nu}$)= 0.0229	α_*= 1.6930e-8 E(u)=0.0150 E($\frac{\partial u}{\partial \nu}$)=0.0506

accurate solution without the use of any regularization. For Case 3, the condition number $4.5\cdot 10^{19}$ in Example 1 and $7.9\cdot 10^{18}$ in Example 2 are much smaller and hence the RMSEs in Case 3 looks better. The numerical results given in Table 1 indicate that the IHCP is a severely ill-posed problem. The discretization by using the MFS also leads to a highly ill-conditioned discrete problem. The use of the regularization technique as shown in the following numerical results will give a stable and much more accurate approximation to the solution of the IHCP.

Table 2 gives the RMSEs on the temperature and heat flux in domain $\Gamma_3 \times [0,1]$ with exact data by using Tikhonov regularization method with GCV choice for the regularization parameter α_*. It can be observed from Table 2 that the RMSEs have much been reduced compared to Table 1. It is remarked here that the GCV method works well in searching the crucial regularization parameter.

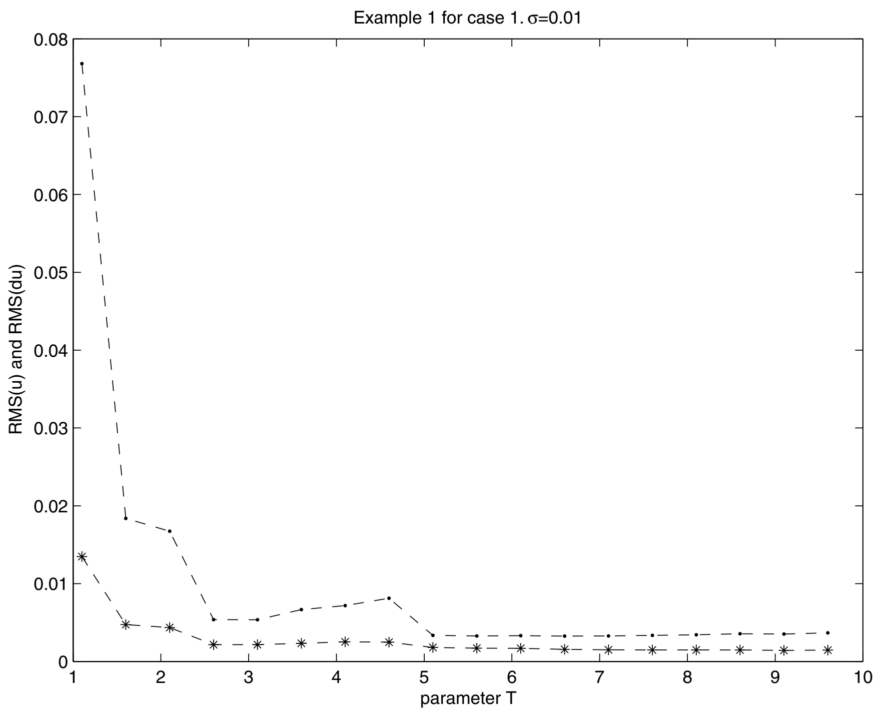
Numerical results for the three cases with noisy data (noisy level $\delta = 0.01$ in all cases) are shown in Table 3. It can be observed that the regularization method with GCV technique provides an acceptable approximation to the solution of the IHCP whilst the direct method completely fails. The problem to choose an optimal regularization parameter is still an open question to researchers.

It is noted here that the setting for Case 3 comes from a real-life problem produced by a steel company. In fact, the solution for the IHCP may not unique but our computational results demonstrate that the proposed method is flexible enough to give a reasonable approximation to the solution of the IHCP under insufficient information.

The relationship between the RMSEs and the value of constant T for Examples 1-2 under Case 1 with noisy data ($\delta = 0.01$) is displayed in Figure 2. Here, the regularization parameter α_* is obtained again by using the GCV method. It is shown that the numerical results are quite stable to the value of the parameter T. It is, however, observed from Figure 2 that the RMSEs decrease with respect to the increasing value of T in Example 1 but increase in Example 2. This interesting behavior is one of the focuses of our future research works.

To further extend the application of the proposed method, we investigate the following sample problem given in [Chantasiriwan (2001)]:

Example 3: Let
Ω be defined as in Case 1.
$\Gamma_1 = \emptyset$,
$\Gamma_2 = \{\ (x_1,x_2) \mid x_1 = 1,\ 0 < x_2 < 1\} \cup \{\ (x_1,x_2) \mid x_2 = 1,\ 0 < x_1 < 1\}$,



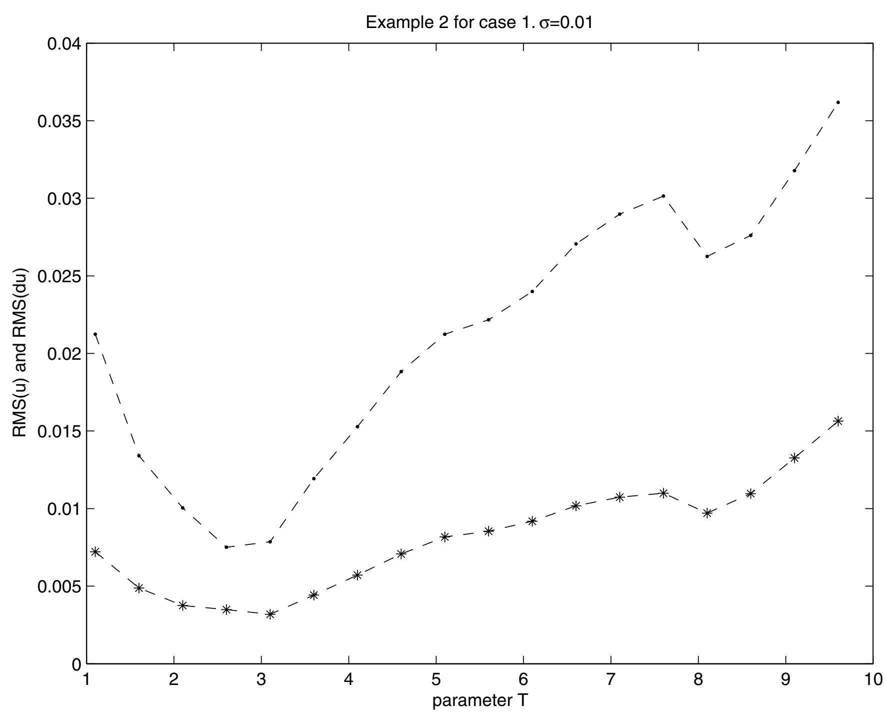


Figure 2 : RMSEs of temperature and heat flux on $\Gamma_3 \times [0,1]$ with respect to parameter T.

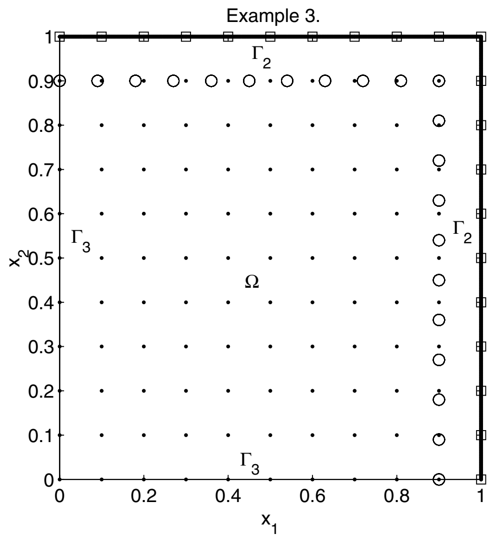


Figure 3 : Distribution of measurement and collocation points. Square represents point with Neumann data, dot represents collocation point for initial temperature, and circle represents point with sensor.

$\Gamma_3 = \partial\Omega \setminus \{\Gamma_1 \cup \Gamma_2\}$.

The locations of measurement points are shown in Figure 3. The temperature distribution is given by

$$u(x_1,x_2,t) \qquad (27)$$
$$= \begin{cases} U(x_1,x_2,t), \\ for \quad t \le 0.5, \\ U(x_1,x_2,t) - 2U(x_1,x_2,t-0.5), \\ for \quad 0.5 < t \le 1.0, \\ U(x_1,x_2,t) - 2U(x_1,x_2,t-0.5) \\ +U(x_1,x_2,t-1.0), \\ for \quad t > 1.0, \end{cases}$$

where

$$U(x_1,x_2,t) =$$
$$1.5t^2 + t(0.5x_1^2 - x_1 + x_2^2 - 2x_2 + 1)$$
$$-4\sum_{j=1}^{\infty} \frac{1}{(j\pi)^4}(0.5cos(j\pi x_1) + cos(j\pi x_2))(1 - e^{-j^2\pi^2 t}).$$

The exact temperature data at the sensor locations are given by $u(x_1,x_2,t)$ and the noisy data are randomly generated as before. In this computation, we take $n = 121, m = 189, p = 0, q = 189, N = 499, N_t = 882$. The plots of temperature and heat flux with respect to time

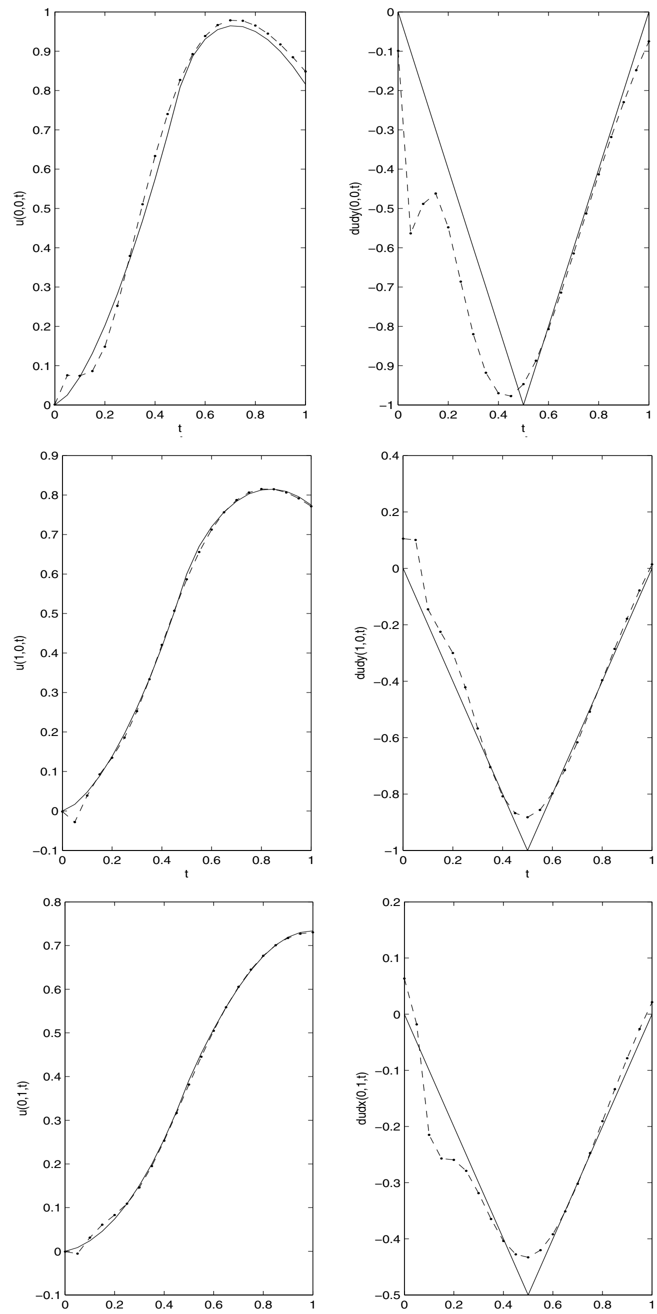


Figure 4 : Plots of temperature and heat flux versus time at points $(0,0),(1,0),(0,1)$. Solid line represents the exact value and dotted-line represents the computed value.

at points $(0,0),(1,0),(0,1)$ are displayed in Figure 4 in which $T = 1.2$ and $\delta = 0.01$, $\alpha_* = 2.0594e-9$. As shown in Figure 4, although the basis functions generated by the MFS are sufficient smooth over the solution domain and the solution to Example 3 does not have a continuous second order time derivative, the computed temperature match the exact data excellently. The computed heat flux looks less accurate but is also comparable to the results given in paper [Chantasiriwan (2001)]. It is also noted that the measurement points in this example are far away from the unspecified boundary whereas these points are close to the boundary in [Chantasiriwan (2001)].

Finally, we consider the following three-dimensional IHCP:

Example 4: Let
$\Omega = \{\ (x_1,x_2,x_3)\mid 0<x_i<1,\ i=1,2,3\}$,
$\Gamma_1 = \{\ (x_1,x_2,x_3)\mid 0<x_1<1,\ 0<x_2<1,\ x_3=1\}$,
$\Gamma_2 = \{\ (x_1,x_2,x_3)\mid 0<x_1<1,\ 0<x_2<1,\ x_3=0\}$,
$\Gamma_3 = \partial\Omega \setminus \{\Gamma_1 \cup \Gamma_2\}$.

The locations of measurement points and collocation points in the domain Ω are shown in Figure 5. In this computation, we take $n=245, m=250, p=180, q=180, N=855, N_t=5324$.

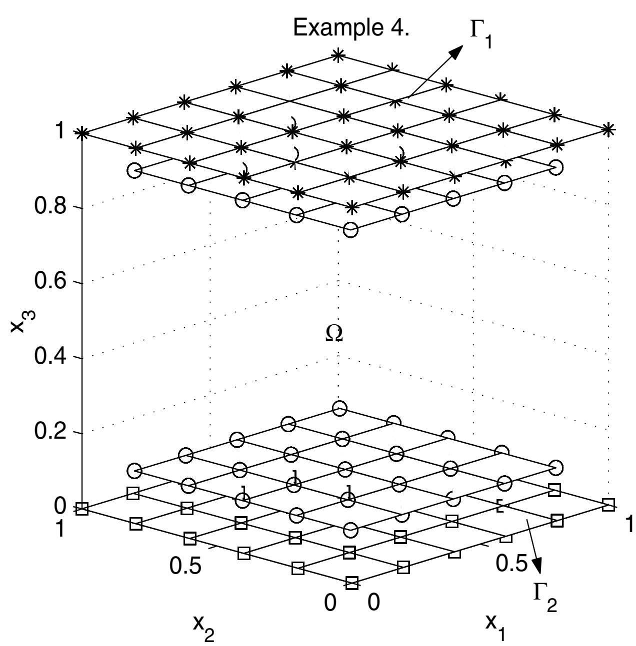


Figure 5 : Locations of measurement and collocation points. Star represents measurement point with Dirichlet data, square represents measurement point with Neumann data, and circle represents point with sensor.

The exact solution for Example 4 is given by

$$u(x_1,x_2,x_3,t) = e^{(-4t)}(cos(2x_1)+cos(2x_2)+cos(2x_3)). \tag{28}$$

In the computation, the value of the parameter T is 2.3 and the noisy level is set to be $\delta = 0.01$. The errors between the exact solution and the approximate solution for the temperature and heat flux on boundary Γ_3 at time $t=1$ are shown in Figure 6 and Figure 7 respectively in which $\alpha_* = 6.1961e-9$. Note that the L^2 norm of u and $\frac{\partial u}{\partial \nu}$ over $\Gamma_3 \times [0,1]$ are about 0.62 and 0.52 respectively. Their relative errors are approximately double the values given in Figures 6 and Figure 7. These small relative errors show that the proposed scheme is effective for solving the three-dimensional inverse heat conduction problem of which very little numerical result has been given so far.

5 Conclusions

The universal approach for solving time-dependent problems involves a time-marching procedure, i.e., advancing each time step and solving the remained problem in the spatial domain. The approach proposed in this paper gives a global approximate solution in both time and spatial domain. Numerical results indicate that the method of fundamental solution the MFS combined with the regularization technique provides an efficient and accurate approximation to this highly ill-posed inverse heat conduction. The use of the generalized cross validation criterion for a suitable regularization parameter stabilizes the resultant ill-conditioned system but still needed to be further investigated for optimal convergence. The proposed approach is potentially valuable to solve the inverse problems in multidimensional space but for large-scale problems, other iterative algorithms based on Lanczos bidiagonalization may be more suitable. This will be our future work.

Acknowledgement: The work described in this paper was partially supported by a grant from CityU (Project No. 7001461). We wish to thank the referees for their constructive comments which helped us much in improving the presentation of this paper.

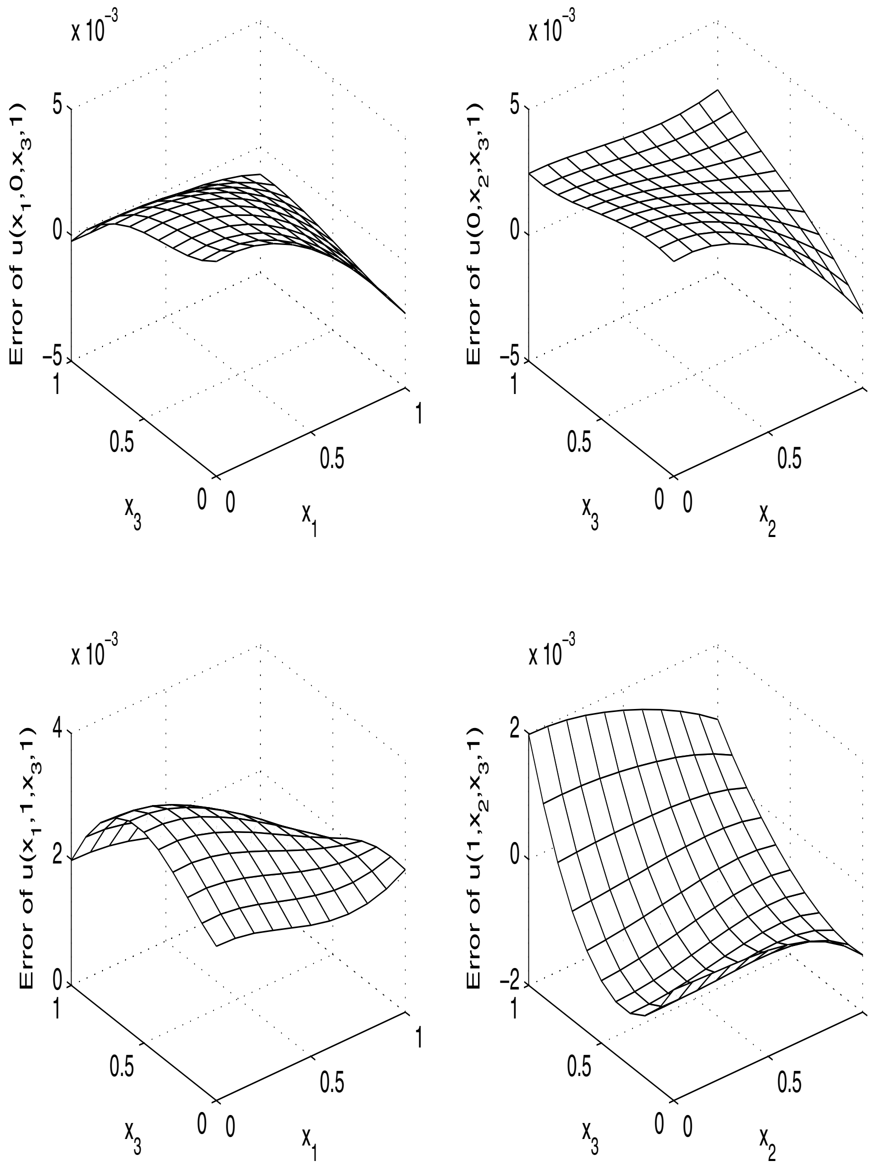


Figure 6 : Surface plots of errors to temperature on boundary Γ_3.

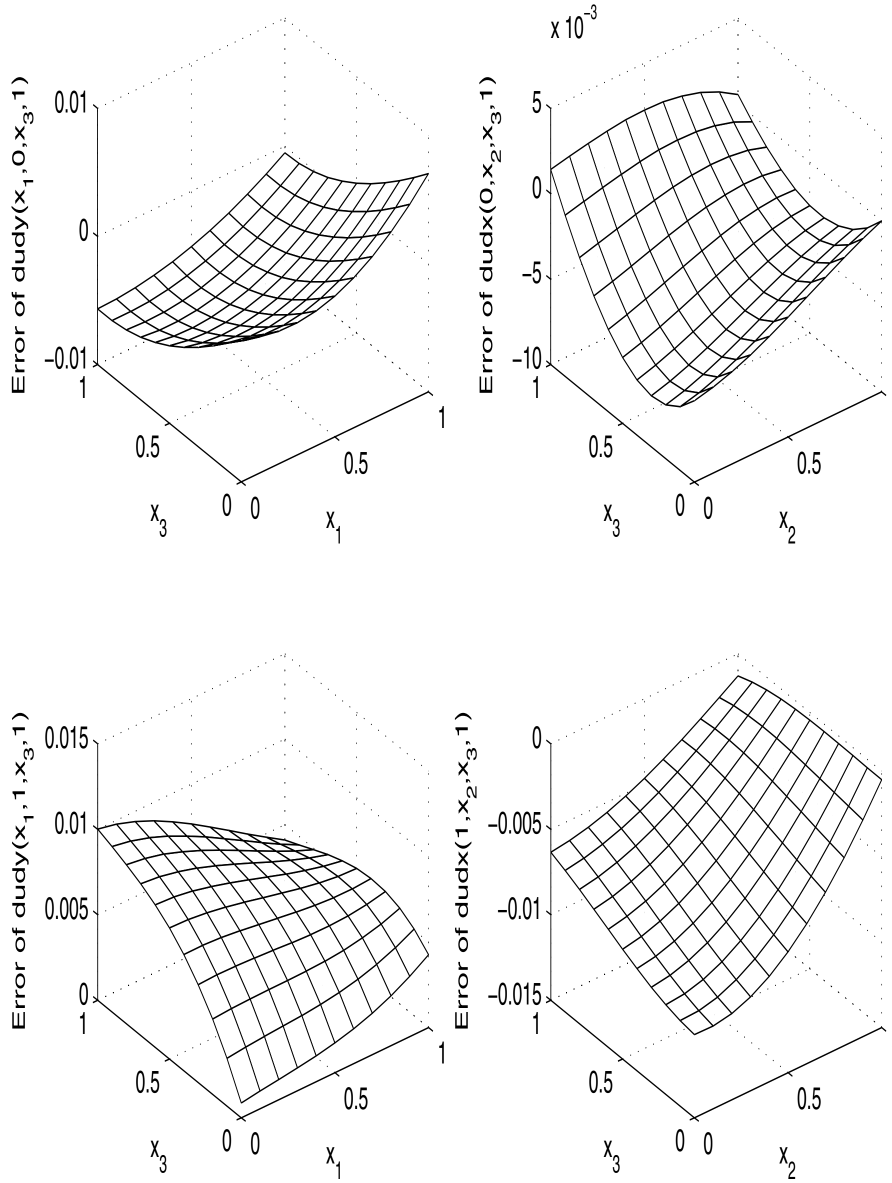


Figure 7 : Surface plots of errors to heat flux on boundary Γ_3.

References

Atluri, S. N. (2004): The meshless(MLPG) for domain & BIE discretizations, Tech Science Press.

Atluri, S. N.; Han, Z. D.; Shen, S. (2003): Meshless Local Petrov-Galerkin (MLPG) Approaches for solving the weakly-singular traction & Displacement Boundary Integral Equations, *CMES: Computer Modeling in Engineering & Sciences*, 4, no. 5, 507-518.

Atluri, S. N.; Han, Z. D.; Rajendran, A. M. (2004): A new implementation of the meshless finite volume method, through the MLPG "mixed" approach, *CMES: Computer Modeling in Engineering & Sciences*, 6, no. 6, 491-514.

Atluri, S. N.; Shen S. (2002): The Meshless Local Petrov-Galerkin (MLPG) Method: A Simple and Less-costly Alternative to the Finite Element and Boundary Element Methods, *CMES: Computer Modeling in Engineering & Sciences*, 3, no. 1, 11-52.

Balakrishnan, K.; Ramachandran, P. A. (1999): A particular solution Trefftz method for non-linear Poisson problems in heat and mass transfer, *Journal of computational physics*, 150, 239–267.

Balakrishnan, K.; Ramachandran, P. A. (2000): The method of fundamental solutions for linear diffusion-reaction equations, *Mathematical and Computer Modelling*, 31, 221–237.

Balakrishnan, K.; Ramachandran, P. A. (2001): Osculatory Interpolation in the Method of Fundamental Solution for Nonliear Possion Problems, *Journal of Computational Physics*, 172, 1-18.

Beck, J. V.; Blackwell, B.; Clair, Ch. R. St. (1985): Inverse heat conduction, Ill-posed Problems, Wiley-Interscience Publication, New York.

Bogomonlny, A. (1985): Fundamental solutions method for elliptic boundary value problems, *SIAM Journal on Numerical Analysis*, 22, 644–669.

Chantasiriwan, S. (1999): Comparison of three sequential function specification algorithms for the inverse heat conduction problem, *International Communications in Heat and Mass Transfer*, 26, no. 1 , 115-124.

Chantasiriwan, S. (2001): An algorithm for solving multidimensional inverse heat conduction problem, *International Journal of Heat and Mass Transfer*, 44, 3823–3832.

Cho, H. A.; Golberg, M. A.; Muleshkov, A. S.; Li, X. : Trefftz methods for time dependent partial differential equations (preprint).

Frankel, J. I.; Keyhani, M. (1997): A global time treatment for inverse heat conduction problems, *Journal of Heat Transfer*, 119, 673–683.

Fairweather, G.; Karageorghis, A. (1998): The method of fundamental solutions for elliptic boundary value problems, Advances in Computational Mathematics, 9, 69–95.

Golberg, M. A. (1995): The method of fundamental solutions for Poisson's equation, *Engineering Analysis with Boundary Elements*, 16, 205–213.

Golberg, M. A.; Chen, C. S. (1998): The method of fundamental solutions for potential, Helmholtz and diffusion problems. In: M. A. Golberg editor. Boundary integral methods-numerical and mathematical aspects, Sounthampton: Computational Mechanic Publications, 103–176.

Guo, L.; Murio, D. A. (1991): A mollified space-marching finite-difference algorithm for the two-dimensional inverse heat conduction problem with slab symmetry, *Inverse problem*, 7, 247–259.

Hansen, P. C. (1992): Analysis of discrete ill-posed problems by means of the *L*-curve, *SIAM Review*, 34, 561–580.

Hansen, P. C. (1994): Regularization tools: a Matlab package for analysis and solution of discrete ill-posed problems, *Numerical Algorithms*, 6, 1–35.

Hon, Y. C.; Chen, W. (2003): Boundary knot method for 2D and 3D Helmholtz and convection-diffusion problems with complicated geometry, *International Journal for Numerical Methods in Engineering*, 56, 1931–1948.

Hon, Y. C.; Cheung, K. F. (1999): Mao, X.Z. and Kansa, E.D., Multiquadric solution for shallow water equations, *ASCE Journal of Hydraulic Engineering*, 125, 524–533.

Hon, Y. C.; Lu, M. W.; Xue, W. M.; Zhou, X. (1999): A new formulation and computation of the triphasic model for mechano-electrochemical mixtures, *Computational Mechanics*, 24, 155–165.

Hon, Y. C.; Mao, X. Z. (1999): A radial basis function method for solving options pricing models, *The Journal of Financial Engineering*, 8, 31–49.

Hon, Y. C.; Wei, T. (2002): A Meshless Computational Method for Solving Inverse Heat Conduction Problem, *International Series on Advances in Boundary Elements*, 13, 135–144.

Hon, Y. C.; Wu, Z. M. (2000): A numerical computation for inverse boundary determination problem, *Engineering Analysis with Boundary Elements*, 24, 599–606.

Hsu, T. R.; Sun, N. S. (1992): Chen, G.G. and Gong, Z.L., Finite element formulation for two-dimensional inverse heat conduction analysis, *Transactions of the ASME, Journal of Heat Transfer*, 114, 553–557.

Jonas, P.; Louis, A. k. (2000): Approximate inverse for a one-dimensional inverse heat conduction problem, *Inverse Problems*, 16, 175–185.

Kansa, E. J.; Hon, Y. C. (2000): Circumventing the ill-conditioning problem with multiquadric radial basis functions: applications to elliptic partial differential equations, *Computers and Mathematics with Applications*, 39, 123–137.

Karageorghis A.; Fairweather G. (1987): The method of fundamental solutions for the numerical solution of the biharmonic equation, *J. Comput. Phys.*, 69, 433–459.

Khalidy, N. A. (1998): A general space marching algorithm for the solution of two-dimensional boundary inverse heat conduction problems, *Numerical Heat Transfer, Part B*, 34, 339–360.

Kondapalli, P. S.; Shippy, D. J.; Fairweather, G. (1992a): Analysis of acoustic scattering in fluids and solids by the method of fundamental solutions, *J. Acoust. Soc. Amer.*, 91, 1844–1854.

Kondapalli, P. S.; Shippy, D. J.; Fairweather, G. (1992b): The method of fundamental solutions for transmission and scattering of elastic waves, *Comput. Methods Appl. Mech. Engrg.*, 96, 255–269.

Kurpisz, K.; Nowak, A. J. (1992): BEM approach to inverse heat conduction problems, *Engineering Analysis with Boundary Elements*, 10, 291–297.

Lesnic, D.; Elliott, L. (1999): The decomposition approach to inverse heat conduction, *Journal of Mathematical Analysis and Application*, 232, 82–98.

Lesnic, D.; Elliott, L.; Ingham, D. B. (1996): Application of the boundary element method to inverse heat conduction problems, *International Communications in Heat and Mass Transfer*, 39, No. 7, 1503–1517.

Mathon, R.; Johnston, R. L. (1977): The approximatical solution of elliptic boundary-value problems by fundamental solutions, *SIAM J. Numer. Anal.*, 14, 638–650.

Murio, D. A. (1993): The mollification method and the numerical solution of ill-posed problems, A Wiley-Interscience Publication.

Partridge, P. W.; Sensale, B. (2002): The method of fundamental solutions with dual reciprocity for diffusion and diffusion-convection using subdomains, *Engineering Analysis with Boundary Elements*, 24, 633–641.

Poullikkas, A.; Karageorghis, A.; Georgiou, G. (2002): The method of fundamental solutions for three-dimensional elastostatics problems, *Computers and Structures*, 80, 365–370.

Ramachandran, P. A. (2002): Method of fundamental solutions: singular value decomposition analysis, *Communications in numerical methods in engineering* 18, 789–801.

Reinhardt, H. J. (1991): A numerical method for the solution of two-dimensional inverse heat conduction problems, *International Journal for Numerical Methods in Engineering*, 32, 363–383.

Shen, S. Y. (1999): A numerical study of inverse heat conduction problems, *Computers and Mathematics with Applications*, 38, 173–188.

Tikhonov, A. N.; Arsenin, V. Y. (1977): On the solution of ill-posed problems, John Wiley and Sons, New York.