

Plate Bending Analysis by using a Modified Plate Theory

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Abstract: Since Reissner and Mindlin proposed their classical thick plate theories, many authors have presented refined theories including transverse shear deformation. Most of those plate theories have tended to use higher order power series for displacements and stresses along the thickness in order to achieve the higher accuracy. However, they have not carefully noticed lateral load effect. In this paper, we pay attention to constitution of the lateral loads: a body force and upper and lower surface tractions. Especially we formulate a modified theory for plate bending, in which the effect of a body force is distinguished from that of surface tractions. The present plate theory includes not only transverse shear deformation but also transverse normal stress effect. In this paper, our attention is focused on bending moment behavior of plates.

keyword: Thick plate theory, Transverse shear deformation, Load effect, Transverse normal stress, Body force.

1 Introduction

Since Reissner (1945) and Mindlin (1951) proposed their classical thick plate theories, many authors have presented refined theories including transverse shear deformation. As well-known the Reissner's theory (1945) is an assumed-stress theory and the Mindlin's theory (1951) an assumed-displacement one.

In the Reissner's theory, a parabolic distribution of transverse shear stresses is assumed and we can satisfy the shear-free condition on the upper and lower surfaces of plates. In addition a transverse normal stress is also incorporated in the theory. A cubic distribution of the transverse normal stress is assumed, in which only an upper surface traction is considered. Discussion of strains of plates is supplemented in Reissner (1947).

In the Mindlin's theory, linear distributions of in-plane displacements and a constant deflection are assumed and,

therefore, the transverse shear stresses distribute constantly along the thickness of plates. This apparently contradicts the shear-free condition on the plate surfaces. In order to compensate the contradiction, Mindlin introduced a correction parameter for the transverse shear stresses. In this theory, the transverse normal stress is neglected and difference between the upper and lower surface tractions is adopted as a lateral load.

It is well-known that the above classical theories coincide when the transverse normal stress is neglected and the shear correction parameter is $5/6$. Rational determination of the correction parameter, however, is not presented. In some cases we cannot obtain accurate solutions when the parameter is $5/6$. The solutions depend on not only the shear correction parameter but also constitution of the lateral loads.

Many high order theories have been presented in order to obtain more accurate solutions of the plate bending. Levinson (1980) and Reddy (1984) assume a cubic distribution of in-plane displacements for thick plates. Reissner (1975) and Rehfield (1982 and 1984) consider deflection change along the thickness of plates. Lo, Christensen, and Wu (1977a, 1977b, and 1978) formulate a high-order theory of plates which includes both in-plane and out-of-plane modes of deformation introducing 11 unknown parameters.

The above plate theories are assumed-displacement ones. Alternative theories, which are assumed-stress theories, are also important. Ambartsumyan (1975) presents an assumed-stress high-order plate theory, in which both the transverse shear stresses and the transverse normal stress is incorporated. Voyiadjis and Baluch (1981) consider the transverse normal strain in addition to the transverse stresses. Reissner (1983) also formulate an assumed-stress high-order plate theory.

Hirashima and Muramatsu (1980), and Hirashima and Negishi (1983 1st and 2nd) establish a generalized high-order plate theory that includes the above theories as particular ones. Hirashima and Negishi (1983 1st) also discuss the accuracy of the plate theories in detail. Krenk

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(1981) employs Legendre polynomials for representing the distribution of displacements and stresses along the thickness. Green and Naghdi (1972) also present a multi-director approach for plates and shells, which surpasses the classical theories. Recently, many sophisticated numerical approaches are applied to the analysis of the plate bending, for example, Long and Atluri (2002), and Qian, Batra, and Chen (2003).

In general, the higher the order of the theories is, the higher the accuracy of those is. However, the conciseness of the theories has been lost. On the other hand, the importance of the lateral load effect is not noticed in the above high-order plate theories. The lateral load of plates consists of the upper and lower surface tractions and the body force. In the above theories the body force is not considered and its effect on plate bending is not separated from that of the surface tractions.

In this paper, we pay attention to the constitution of the lateral loads as Suetake and Tomoda (2004) and Suetake (2005), in which a modified bending theory of thick plates is formulated. In the modified theory employed here, the body force is separated from the surface tractions. The present plate theory includes not only transverse shear deformation but also transverse normal stress effect. The present modified theory can be a simple means of comparison for numerical analyses. In this paper, our attention is focused on bending moment behavior of plates and we make sure that the present modified theory gives us excellent results, even though it is as simple as the classical theory.

2 Modified Theory

In this paper a modified bending theory of plates is presented by using the Levinson-Reddy type displacement field [Levinson (1980) and Reddy (1984)]. We pay attention to the constitution of the lateral loads through consideration of the transverse normal stress. Consequently we can treat with the surface tractions and the body force separately.

2.1 Displacement-strain relation

Levinson (1980) and Reddy (1984) assume the 3rd-order displacement field in order to represent distortion of a normal to the mid-surface. The displacement field as-

sumed here is given by

$$\left. \begin{aligned} U &= -\Psi_x z - \frac{4}{3t^2} \Phi_x z^3 ; \Phi_x \equiv \frac{\partial w}{\partial x} - \Psi_x \\ V &= -\Psi_y z - \frac{4}{3t^2} \Phi_y z^3 ; \Phi_y \equiv \frac{\partial w}{\partial y} - \Psi_y \\ W &= w \end{aligned} \right\}, \quad (1)$$

where x and y are in-plane coordinates, z is a coordinate normal to the mid-surface of plates, t is a thickness of plates, Ψ_x and Ψ_y are deflection angles, and w is a deflection of plates. In Eq. (1), if we neglect the 3rd-order terms, we have the Mindlin type displacement field [Mindlin (1951)]. From the displacement field (1), we obtain the following strain distribution:

i) *in-plane strains*

$$\left. \begin{aligned} \varepsilon_x &= -\frac{\partial \Psi_x}{\partial x} z - \frac{4}{3t^2} \frac{\partial \Phi_x}{\partial x} z^3 \\ \varepsilon_y &= -\frac{\partial \Psi_y}{\partial y} z - \frac{4}{3t^2} \frac{\partial \Phi_y}{\partial y} z^3 \\ \gamma_{xy} &= -\left(\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\right) z - \frac{4}{3t^2} \left(\frac{\partial \Phi_x}{\partial y} + \frac{\partial \Phi_y}{\partial x}\right) z^3 \end{aligned} \right\}, \quad (2)$$

ii) *transverse shear strains*

$$\gamma_{xz} = \Phi_x \left(1 - \frac{4}{t^2} z^2\right), \quad \gamma_{yz} = \Phi_y \left(1 - \frac{4}{t^2} z^2\right). \quad (3)$$

Note that Eq.(3) satisfies the shear-free condition on the upper and lower surfaces of plates.

2.2 Constitutive equation

Since we treat with isotropic elastic plates here, we employ the Hooke's law as a constitutive equation. Eliminating the transverse normal strain ε_z from the 3-D Hooke's law, we obtain the following relation:

$$\left. \begin{aligned} \sigma_x &= \frac{E}{1-\nu^2} (\varepsilon_x + \nu \varepsilon_y) + \frac{\nu}{1-\nu} \sigma_z \\ \sigma_y &= \frac{E}{1-\nu^2} (\nu \varepsilon_x + \varepsilon_y) + \frac{\nu}{1-\nu} \sigma_z \\ \tau_{xy} &= G \gamma_{xy}; \quad G \equiv \frac{E}{2(1+\nu)} \end{aligned} \right\}, \quad (4)$$

and

$$\tau_{xz} = G \gamma_{xz}, \quad \tau_{yz} = G \gamma_{yz}, \quad (5)$$

where E is Young's modulus and ν is Poisson's ratio.

Substitution of Eqs. (2) and (3) into Eqs. (4) and (5) gives us

$$\left. \begin{aligned} \sigma_x &= -\frac{E}{1-\nu^2} \left\{ \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right) z \right. \\ &\quad \left. + \frac{4}{3t^2} \left(\frac{\partial \phi_x}{\partial x} + \nu \frac{\partial \phi_y}{\partial y} \right) z^3 \right\} + \frac{\nu}{1-\nu} \sigma_z \\ \sigma_y &= -\frac{E}{1-\nu^2} \left\{ \left(\nu \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) z \right. \\ &\quad \left. + \frac{4}{3t^2} \left(\nu \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right) z^3 \right\} + \frac{\nu}{1-\nu} \sigma_z \\ \tau_{xy} &= -G \left\{ \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) z + \frac{4}{3t^2} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) z^3 \right\} \end{aligned} \right\}, \quad (6)$$

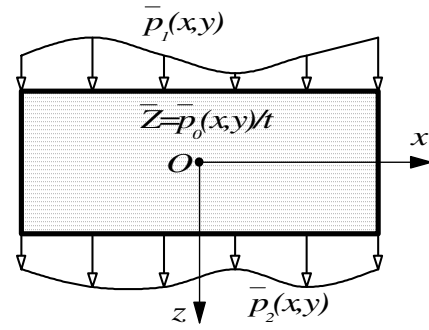


Figure 1 : Lateral Load Constitution of Plate

and

$$\tau_{xz} = G\phi_x \left(1 - \frac{4}{t^2} z^2\right), \quad \tau_{yz} = G\phi_y \left(1 - \frac{4}{t^2} z^2\right). \quad (7)$$

Equation (7) is the same as the shear distribution in the Reissner's theory [Reissner (1945)]. In the Mindlin's theory [Mindlin (1951)], the 2nd- and 3rd-order terms are neglected and the effect of the transverse normal stress is not incorporated.

2.3 Equilibrium condition

Linear equilibrium conditions for 3-D bodies are given by

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \bar{X} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \bar{Y} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \bar{Z} &= 0 \end{aligned} \right\}, \quad (8)$$

where \bar{X} , \bar{Y} , and \bar{Z} are body forces and, in this paper, we consider only \bar{Z} , that is, we set $\bar{X} = \bar{Y} = 0$.

By using the 3rd equilibrium condition in Eqs. (8), we can determine the distribution of σ_z . Before doing that, we should pay attention to the constitution of the lateral loads. The lateral load of a plate consists of the body force $\bar{Z} = \bar{p}_0(x,y)/t$ and the upper and lower surface tractions, $\bar{p}_1(x,y)$ and $\bar{p}_2(x,y)$, as shown in Fig.1. Therefore we have the following traction boundary conditions:

$$\sigma_z(x,y, -\frac{t}{2}) = -\bar{p}_1, \quad \sigma_z(x,y, \frac{t}{2}) = \bar{p}_2. \quad (9)$$

Substituting Eq. (7) into the 3rd of Eq. (8) and integrat-

ing it with respect to z , in view of Eq. (10), we have

$$\begin{aligned} \sigma_z &= \frac{Gt}{3} \left(\nabla^2 w - \frac{\partial \psi_x}{\partial x} - \frac{\partial \psi_y}{\partial y} \right) \left(1 - \frac{3}{t} z + \frac{4}{t^3} z^3 \right) \\ &\quad + \frac{\bar{p}_0}{2} \left(1 - \frac{2}{t} z \right) + \bar{p}_2. \end{aligned} \quad (10)$$

When we apply the traction boundary condition (9) to Eq.(10), we obtain

$$\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} = \nabla^2 w + \frac{3}{2Gt} \bar{p} \quad ; \quad \bar{p} \equiv \bar{p}_0 + \bar{p}_1 + \bar{p}_2. \quad (11)$$

Consequently we obtain the distribution of σ_z as

$$\sigma_z = -\frac{\bar{p}}{2} \left(1 - \frac{3}{t} z + \frac{4}{t^3} z^3 \right) + \frac{\bar{p}_0}{2} \left(1 - \frac{2}{t} z \right) + \bar{p}_2. \quad (12)$$

Integrating the 1st and 2nd of Eq. (8) with respect to z , we obtain the ordinary moment equilibrium equations.

2.4 Governing equation

Integration of Eqs.(6) and (7) with respect to z , in view of Eq. (12), gives us the following moment and shear force expressions:

$$\begin{aligned} M_x &= -D \left\{ \frac{4}{5} \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right) + \frac{1}{5} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right\} \\ &\quad + \frac{\nu t^2}{60(1-\nu)} \{ \bar{p}_0 + 6(\bar{p}_1 + \bar{p}_2) \}, \end{aligned} \quad (13)$$

$$\begin{aligned} M_y &= -D \left\{ \frac{4}{5} \left(\nu \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) + \frac{1}{5} \left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \right\} \\ &\quad + \frac{\nu t^2}{60(1-\nu)} \{ \bar{p}_0 + 6(\bar{p}_1 + \bar{p}_2) \}, \end{aligned} \quad (14)$$

$$M_{xy} = -\frac{D}{2}(1-\nu)\left\{\frac{4}{5}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right) + \frac{2}{5}\frac{\partial^2 w}{\partial x\partial y}\right\}, \quad (15) \quad \nabla^4\psi_y = \frac{1}{D}\frac{\partial}{\partial y}[\bar{p}_0 + \bar{p}_1 + \bar{p}_2$$

$$+ \frac{t^2}{60(1-\nu)}\nabla^2\{(3+\nu)\bar{p}_0 + 3(1+2\nu)(\bar{p}_1 + \bar{p}_2)\}]. \quad (22)$$

and

$$Q_x = \frac{2}{3}Gt\left(\frac{\partial w}{\partial x} - \psi_x\right), \quad Q_y = \frac{2}{3}Gt\left(\frac{\partial w}{\partial y} - \psi_y\right), \quad (16)$$

where D is the bending rigidity of plates: $D \equiv Et^3/12(1-\nu^2)$. Substituting Eqs. (13) to (16) into the ordinary moment equilibrium equations, we have

$$D\left\{\frac{4}{5}\left(\frac{\partial^2\psi_x}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\right) + \frac{1}{5}\frac{\partial}{\partial x}(\nabla^2 w)\right\}$$

$$+ \frac{2}{3}Gt\left(\frac{\partial w}{\partial x} - \psi_x\right) - \frac{\nu t^2}{60(1-\nu)}\left\{\frac{\partial\bar{p}_0}{\partial x} + 6\left(\frac{\partial\bar{p}_1}{\partial x} + \frac{\partial\bar{p}_2}{\partial x}\right)\right\}$$

$$= 0, \quad (17)$$

and

$$D\left\{\frac{4}{5}\left(\frac{\partial^2\psi_y}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}\right) + \frac{1}{5}\frac{\partial}{\partial y}(\nabla^2 w)\right\}$$

$$+ \frac{2}{3}Gt\left(\frac{\partial w}{\partial y} - \psi_y\right) - \frac{\nu t^2}{60(1-\nu)}\left\{\frac{\partial\bar{p}_0}{\partial y} + 6\left(\frac{\partial\bar{p}_1}{\partial y} + \frac{\partial\bar{p}_2}{\partial y}\right)\right\}$$

$$= 0. \quad (18)$$

Equations (11), (17), and (18) are the present governing equations for the plate bending. We can rewrite Eqs. (17) and (18), in view of Eq.(11), as

$$\left(\nabla^2 - \frac{10}{t^2}\right)\left(\frac{\partial\psi_x}{\partial y} - \frac{\partial\psi_y}{\partial x}\right) = 0, \quad (19)$$

$$\nabla^4 w = \frac{1}{D}[\bar{p}_0 + \bar{p}_1 + \bar{p}_2$$

$$- \frac{t^2}{6(1-\nu)}\nabla^2\left\{\frac{12-\nu}{10}\bar{p}_0 + \frac{3}{5}(2-\nu)(\bar{p}_1 + \bar{p}_2)\right\}] \quad (20)$$

If we assume that $\partial\psi_x/\partial y = \partial\psi_y/\partial x$, Eq. (19) can be satisfied a priori. In that case, we can derive the governing equations for ψ_x and ψ_y from Eqs. (11) and (17) to (19):

$$\nabla^4\psi_x = \frac{1}{D}\frac{\partial}{\partial x}[\bar{p}_0 + \bar{p}_1 + \bar{p}_2$$

$$+ \frac{t^2}{60(1-\nu)}\nabla^2\{(3+\nu)\bar{p}_0 + 3(1+2\nu)(\bar{p}_1 + \bar{p}_2)\}], \quad (21)$$

3 Fourier Analysis

For numerical calculations we employ here the Fourier series analysis. In this section we explain the Fourier analyses of plates and 3-D bodies. Two plate analyses are presented here; one is based on the present theory and the other on the classical one. We can also employ alternative approaches for the analyses.

3.1 Plate analysis based on the present theory

Elastic plates adopted here as numerical examples are simply supported rectangular plates subjected to lateral loads. The coordinates of the plate model is shown in Fig.2.

The deflection and the deflection angles are expressed by the following trigonometric double series here:

$$\left. \begin{aligned} w &= \sum_n \sum_m W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \psi_x &= \sum_n \sum_m \Phi_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \psi_y &= \sum_n \sum_m \Psi_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \end{aligned} \right\}. \quad (23)$$

Note that Eqs. (23) satisfy the boundary condition of simply supported plates. In addition, we expand the load functions into the following Fourier double series:

$$\bar{p}_i = \sum_n \sum_m \bar{P}_{mn}^{(i)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b};$$

$$\bar{P}_{mn}^{(i)} = \frac{4}{ab} \int_0^b \int_0^a \bar{p}_i(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy. \quad (24)$$

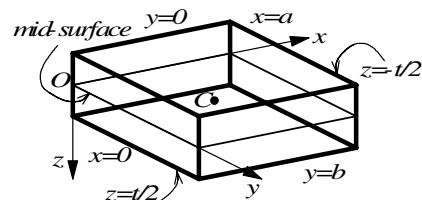


Figure 2 : Rectangular Plate and Coordinates

Substituting Eqs. (23) into Eqs. (20) to (22), we can easily determine the coefficients W_{mn} , Φ_{mn} , and Ψ_{mn} as

$$W_{mn} = \frac{1}{\lambda_{mn}^4 D} \left[\bar{P}_{mn}^{(0)} + \bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)} + \frac{\lambda_{mn}^2 t^2}{6(1-\nu)} \left\{ \frac{12-\nu}{10} \bar{P}_{mn}^{(0)} + \frac{3}{5}(2-\nu)(\bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)}) \right\} \right], \quad (25)$$

$$\Phi_{mn} = \frac{1}{\lambda_{mn}^4 D} \cdot \frac{m\pi}{a} \left[\bar{P}_{mn}^{(0)} + \bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)} - \frac{\lambda_{mn}^2 t^2}{60(1-\nu)} \left\{ (3+\nu)\bar{P}_{mn}^{(0)} + 3(1+2\nu)(\bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)}) \right\} \right], \quad (26)$$

$$\Psi_{mn} = \frac{1}{\lambda_{mn}^4 D} \cdot \frac{n\pi}{b} \left[\bar{P}_{mn}^{(0)} + \bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)} - \frac{\lambda_{mn}^2 t^2}{60(1-\nu)} \left\{ (3+\nu)\bar{P}_{mn}^{(0)} + 3(1+2\nu)(\bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)}) \right\} \right], \quad (27)$$

where $\lambda_{mn}^2 \equiv (m\pi/a)^2 + (n\pi/b)^2$.

3.2 Plate analysis based on the classical theory

At this stage it is significant to review the classical plate theories. The governing equations of the static Mindlin's theory [Mindlin (1951)] are as follows:

$$\nabla^4 w = \frac{1}{D} \left\{ 1 - \frac{t^2}{6(1-\nu)\kappa} \nabla^2 \right\} \bar{p}, \quad (28)$$

$$\nabla^4 \psi_x = \frac{1}{D} \frac{\partial \bar{p}}{\partial x}, \quad \nabla^4 \psi_y = \frac{1}{D} \frac{\partial \bar{p}}{\partial y}, \quad (29)$$

where κ is the shear correction parameter. In Eq. (28), if we

set $\kappa = 5/3(2-\nu)$, the static Mindlin's theory coincides with the Reissner's one.

If we use the trigonometric series (23) again in the classical bending analysis of plates, the coefficients W_{mn} , Φ_{mn} , and Ψ_{mn} can be determined as

$$W_{mn} = \frac{\bar{P}_{mn}^{(0)} + \bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)}}{\lambda_{mn}^4 D} \left\{ 1 + \frac{\lambda_{mn}^2 t^2}{6(1-\nu)\kappa} \right\}, \quad (30)$$

$$\Phi_{mn} = \frac{1}{\lambda_{mn}^4 D} \cdot \frac{m\pi}{a} (\bar{P}_{mn}^{(0)} + \bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)}), \quad (31)$$

$$\Psi_{mn} = \frac{1}{\lambda_{mn}^4 D} \cdot \frac{n\pi}{b} (\bar{P}_{mn}^{(0)} + \bar{P}_{mn}^{(1)} + \bar{P}_{mn}^{(2)}). \quad (32)$$

As well-known, we can calculate the bending and twisting moments of plates by using the following expressions instead of Eqs. (13) to (15):

$$M_x = -D \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right), \quad (33)$$

$$M_y = -D \left(\nu \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right), \quad (34)$$

$$M_{xy} = -\frac{D}{2} (1-\nu) \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right). \quad (35)$$

3.3 3-D analysis

In order to evaluate the accuracy of the present plate theory, we perform a 3-D elastic analysis of the plate model. We employ the Fourier series again for the 3-D analysis. We explain the approach briefly in this subsection. Geometrical boundary conditions of the model as a 3-D body is given by

$$\left. \begin{aligned} V(0, y, z) = V(a, y, z) = W(0, y, z) = W(a, y, z) = 0 \\ U(x, 0, z) = U(x, b, z) = W(x, 0, z) = W(x, b, z) = 0 \end{aligned} \right\}, \quad (36)$$

which corresponds to the conditions for simply supported plates. To satisfy these conditions, we employ the following trigonometric series

$$\left. \begin{aligned} U &= \sum_n \sum_m u_{mn}(z) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ V &= \sum_n \sum_m v_{mn}(z) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ W &= \sum_n \sum_m w_{mn}(z) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \right\}. \quad (37)$$

Traction boundary conditions employed here are

$$\left. \begin{aligned} \sigma_z(x, y, -\frac{t}{2}) &= -\bar{p}_1(x, y) \\ \sigma_z(x, y, \frac{t}{2}) &= \bar{p}_2(x, y), \\ \tau_{xz}(x, y, \pm \frac{t}{2}) &= \tau_{yz}(x, y, \pm \frac{t}{2}) = 0. \end{aligned} \right\}. \quad (38)$$

In addition, the body forces of the model are represented by

$$\bar{X} = \bar{Y} = 0, \quad \bar{Z} = \frac{1}{t} \bar{p}_0(x, y). \quad (39)$$

Substitution of Eq. (37) into the Navier’s equation, which is a governing equation for 3-D elastic problems, yields an ordinary differential equation system with respect to the unknown functions $u_{mn}(z)$, $v_{mn}(z)$, and $w_{mn}(z)$. When we solve the differential equation system under Eqs. (38) and (39), we can determine those three unknown functions.

4 Numerical Models

As numerical examples we adopt simple plate bending problems of a square plate. The plate model is simply supported along the all edges. The width-thickness ratio $\mu \equiv t/a$ is changed within the range of $0.001 \leq \mu \leq 0.5$. Poisson’s ratio is $\nu = 0.3$. In the Fourier analysis, we take $200 \times 200 = 40000$ terms in the double series.

A constitution of lateral loads adopted here is shown in Fig.3. The plate model is subjected to a constant body force and a partially distributed constant load on the upper surface.

The Fourier coefficients of the load functions are given by

$$\bar{P}_{mn}^{(0)} = \begin{cases} \frac{16\bar{p}_0}{\pi^2(2j-1)(2k-1)} & \text{(m and n: odd)} \\ 0 & \text{(m or n: even)} \end{cases}, \quad (40)$$

$$\bar{P}_{mn}^{(1)} = \frac{4\bar{p}_1}{\pi^2 mn} \left\{ \cos \frac{m\pi x_0}{a} - \cos \frac{m\pi(x_0 + a^*)}{a} \right\} \times \left\{ \cos \frac{n\pi y_0}{b} - \cos \frac{n\pi(y_0 + b^*)}{b} \right\}. \quad (41)$$

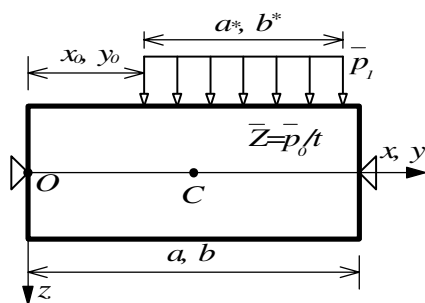


Figure 3 : Figure 3: Lateral Load Constitution

Two different constitutions of lateral loads are adopted in the present numerical calculations. One is a symmetric distribution on the upper surface; the other a

non-symmetric distribution. Numerical properties of the loads are as follows:

i) *symmetric distribution*

$$\hat{\chi}_0 = 0.8, \hat{\chi}_1 = 1.25, x_0/a = y_0/b = 0.3, a^*/a = b^*/b = 0.4, \quad (42)$$

ii) *non-symmetric distribution*

$$\hat{\chi}_0 = 0.6, \hat{\chi}_1 = 40, x_0/a = y_0/b = 0.2, a^*/a = b^*/b = 0.1, \quad (43)$$

where the non-dimensional load parameter $\hat{\chi}_i$ is defined by

$$\hat{\chi}_i = \frac{\hat{p}_i a^3}{\mu D} = \frac{12(1-\nu^2)}{E\mu^4} \hat{p}_i; \quad \mu = \frac{t}{a}. \quad (44)$$

5 Numerical Results

In this paper, our attention is focused on bending moment behavior of plates. In the numerical calculation, we estimate the error of the present plate analysis. Results of the 3-D elastic analysis are employed as the standard values. Classical analysis based on the Mindlin’s theory is also performed in order to confirm the accuracy of the present analysis.

Since we adopt the Levinson-Reddy type displacement field, Eq.(1), it is not so easy to predict local behaviors for the large thickness of plates. This is a further issue to be improved.

5.1 Symmetric surface traction

In this subsection, we discuss the case of the symmetric surface traction. As mentioned before, the plate model is subjected to not only a constant body force but also a partially distributed constant load on the upper surface. In this case, the surface traction is symmetric.

Numerical results are shown in Fig.4, in which the errors of the present and the classical plate analyses, ϵ , are plotted against the width-thickness ratio of the model, μ . In Fig.4, closed circles indicate the results of the present analysis and closed triangles that of the classical analysis, respectively.

It follows from Fig.4 that the present modified plate theory approximates the 3-D analysis with high accuracy. Especially, we should note that the error of the present

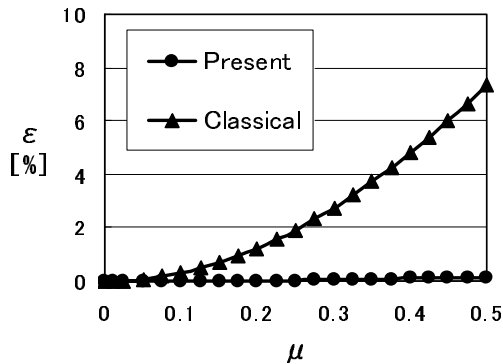


Figure 4 : Error of Moment at Center Point (Symmetric Model)

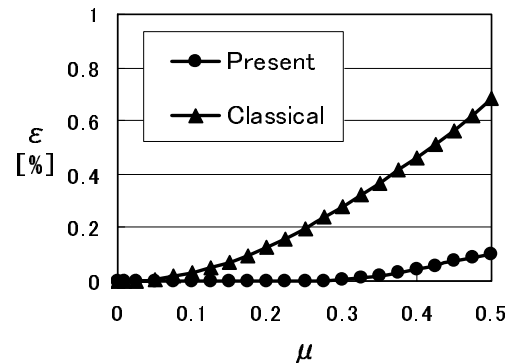


Figure 5 : Error of Moment at Center Point (Non-symmetric Model)

analysis remains quite small even in the thick plate region near $\mu = 0.5$. On the other hand, the error of the classical analysis increases rapidly with the increase of the width-thickness ratio.

The excellent approximation of the deflection behavior has already been confirmed [Suetake 2005]. The present investigation shows the efficiency of the modified plate theory also in the moment analysis.

5.2 Non-symmetric surface traction

The results of the non-symmetric surface traction model are presented in this subsection. The load adopted here consists of a constant body force and a non-symmetrically distributed constant load on the upper surface.

Numerical results of the moment error at the center point of the model are shown in Fig.5, which is depicted in the same manner as Fig.4. It can be seen also from Fig.5 that the present modified plate theory gives us excellent results. In this case, however, the classical theory also maintains practically sufficient accuracy.

6 Concluding Remarks

The following conclusions may be drawn from the present investigation:

- 1) A modified plate bending theory is proposed, in which the effect of lateral loads is carefully considered.
- 2) The new theory gives us excellent approximations for moments even in thick plate region, while the classical one maintains practically sufficient accuracy within

a moderately thick plate region.

- 3) The constitution of the lateral loads plays a key role in the plate bending analyses of thick plates. In particular, when the constitution of loads is not simple, we should use the modified theory instead of the classical one.

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