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An Efficient Backward Group Preserving Scheme for the Backward in Time Burgers Equation

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Abstract: In this paper we are concerned with the numerical integration of Burgers equation backward in time. We construct a one-step backward group preserving scheme (BGPS) for the semi-discretization of Burgers equation. The one-step BGPS is very effectively to calculate the solution at an initial time $t = 0$ from a given final data at $t = T$, which with a time stepsize equal to T and with a suitable grid length produces a highly accurate solution never seen before. Under noisy final data the BGPS is also robust to against the disturbance. When the solution appears steep gradient, several steps BGPS can be used to retrieve the desired initial data.

keyword: Past cone dynamics, Backward group preserving scheme, Backward Burgers equation, Ill-posed problem.

1 Introduction

In this paper we are concerned with the numerical integration of backward in time Burgers equation:

$$u_t + uu_x = \frac{1}{R}u_{xx}, \;\; a < x < b, \;\; 0 < t < T,$$

$$u(a,t) = u_a(t), \;\; u(b,t) = u_b(t), \;\; 0 \le t \le T,$$

$$u(x,T) = f(x), \;\; a \le x \le b,$$

where R is the Reynolds number characterizing the viscosity of fluid. Given a velocity function $f(x)$ at the final time $t = T$, the problem is retrieving the past history and initial profile of the fluid velocity.

Burgers' equation has been of considerable physical interest because it is an appropriate simplification of the Navier-Stokes equations, and is also the governing equation for a number of one-dimensional flow systems including the convection and diffusion of heat, weak shock propagation, compressible turbulence, and continuum traffic simulation.

The Burgers equation was first appeared in a paper by Bateman (1915) and was named after Burgers (1948,1974). The behavior of Burgers equation exhibits a delicate balance between advection and diffusion. Moreover, it is one of the few nonlinear partial differential equations that exact and complete solutions are known in terms of the initial values [Cole (1951); Hopf (1950)]. Thus, the Burgers equation is a useful test medium for investigating various numerical methods on partial differential equations. In the past several decades there were much studies on the numerical solutions of Burgers' equation, for example, Fletcher (1983), Basdevant, Deville and Haldenwang (1986), Arina and Canuto (1993), Özis and Özdes (1996), Hon and Mao (1998), Kutluay, Bahadir and Özdes (1999), Lin and Zhou (2001), Wei and Gu (2002), Özis, Aksan and Özdes (2003), and references therein.

Here we propose a new numerical scheme for solving the well-known Burgers equation backward in time. However, while most papers are concerned with the numerical integrations of the forward problems of Burgers' equation, there are only a few papers which are devoted to the inverse problems of Burgers equation, for example, Carasso (1977) and Marbán and Palencia (2002).

We would develop a backward group preserving scheme for the backward problems of Burgers equation. It is an extension of the work of Liu (2001, 2004) by taking the time backward of equations into account. Numerical schemes adopted for backward problems are usually implicit. The explicit schemes that have been applied to solving the backward problems are apparently not very effective up to now. As mentioned by Ames and Epperson (1997), because the backward problems are ill-posed, they are necessary ill conditioned from a numerical point of view, and the problem must be regularized before any approximation can be constructed. Obviously, most people assert that the backward problems are impossible to

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solve using classical numerical methods and require special techniques to be employed.

Our proposed scheme is based on the numerical method of line which is a well-developed numerical method that transforms partial differential equations into a system of ordinary differential equations. The major contributions of this paper are applying the group preserving property of the resultant system in the development of numerical scheme and giving a conviction that the proposed scheme is workable to the backward in time Burgers equation. Specifically, the proposed scheme is easy to implement and time saving. Usually, the resultant differential equations system is highly-dimensional for an accuracy consideration when one applies the line method to partial differential equations, and thus it is desired to use an easily-implemented program with a minimal step and a minimal stage in the numerical method. Of course, for the ordinary differential equations the Runge-Kutta method is the most popular one to implement; however, it would be seen that the Runge-Kutta scheme can not be applied to the backward in time Burgers equation due to its fast divergence of the numerical solutions.

Through this study, we may have an easy-implementation and explicit-one step backward group preserving scheme (BGPS) used in the calculations of backward in time Burgers equation, the accuracy of which is much better than other schemes. The algorithm is also applicable to other semilinear parabolic problems, and it is an important contribution for calculating the backward problems of semilinear parabolic type.

2 Forward problems and GPS

2.1 Dynamics on future cone

Group-preserving scheme (GPS) can preserve the internal symmetry group of the considered system. Although we do not know previously the symmetry group of nonlinear differential equations systems, Liu (2001) has embedded them into the augmented dynamical systems, which concern with not only the evolution of state variables but also the evolution of the magnitude of state variables vector. That is, for an n ordinary differential equations system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \tag{1}$$

we can embed it to the following $n+1$-dimensional augemented dynamical system:

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x} \\ \|\mathbf{x}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{x},t)}{\|\mathbf{x}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{x},t)}{\|\mathbf{x}\|} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \|\mathbf{x}\| \end{bmatrix}. \tag{2}$$

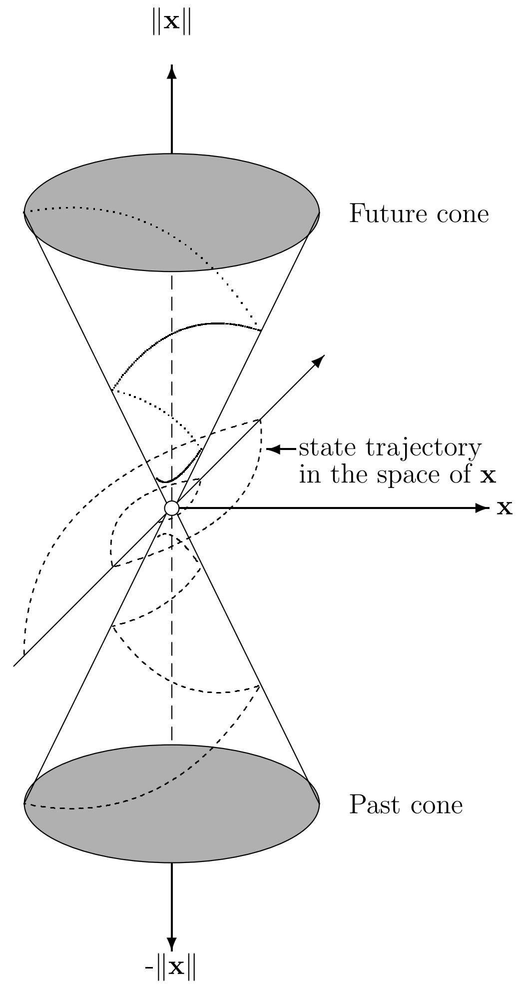

Figure 1 : The construction of deleted cones in the Minkowski space for forward and backward problems signifies a conceptual breakthrough. The trajectory observed in the state space $\mathbf{x}$ is a parallel projection of the trajectory in the null cones along the $\|\mathbf{x}\|$ or $-\|\mathbf{x}\|$-axis.

It is obvious that the first row in Eq. (2) is the same as the original equation (1), but the inclusion of the second row in Eq. (2) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{x}^{\mathrm{T}}, \|\mathbf{x}\|)^{\mathrm{T}}$ satisfying a future cone condition as shown in Fig. 1:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \tag{3}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{4}$$

is a Minkowski metric, $\mathbf{I}_n$ is the identity matrix of order n, and the superscript T stands for the transpose. In terms of $(\mathbf{x}, \|\mathbf{x}\|)$, Eq. (3) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{x}\cdot\mathbf{x} - \|\mathbf{x}\|^2 = \|\mathbf{x}\|^2 - \|\mathbf{x}\|^2 = 0, \tag{5}$$

where the dot between two n-dimensional vectors denotes their Euclidean inner product. The cone condition is thus the most natural constraint that we can impose on the dynamical system (2).

Consequently, we have an $n+1$-dimensional augmented system:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} \tag{6}$$

with a constraint (3), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{x},t)}{\|\mathbf{x}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{x},t)}{\|\mathbf{x}\|} & 0 \end{bmatrix}, \tag{7}$$

satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{8}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$. This fact prompts us to devise the so-called group-preserving scheme, whose discretized mapping $\mathbf{G}$ exactly preserves the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{9}$$

$$\det \mathbf{G} = 1, \tag{10}$$

$$G_0^0 > 0, \tag{11}$$

where G_0^0 is the 00th component of $\mathbf{G}$. Such $\mathbf{G}$ is an element of the proper orthochronous Lorentz group denoted by $SO_o(n,1)$. The term orthochronous used in the special relativity theory is referred to the preservation of time orientation. However, it should be understood here as the preservation of the sign of $\|\mathbf{x}\|$.

Remarkably, the original n-dimensional dynamical system (1) in $\mathbb{E}^n$ can be embedded naturally into an augmented $n+1$-dimensional dynamical system (6) in $\mathbb{M}^{n+1}$. These two systems are mathematically equivalent. Although the dimension of the new system is raising one more, it has been shown that under the Lipschitz condition of

$$\|\mathbf{f}(\mathbf{x},t) - \mathbf{f}(\mathbf{y},t)\| \le \mathcal{L}\|\mathbf{x}-\mathbf{y}\|, \ \ \forall\, (\mathbf{x},t), (\mathbf{y},t) \in \mathbb{D}, \tag{12}$$

where $\mathbb{D}$ is a domain of $\mathbb{R}^n \times \mathbb{R}$, and $\mathcal{L}$ is known as a Lipschitz constant, the new system has the advantage of allowing us to devise group-preserving numerical scheme as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell)\mathbf{X}_\ell, \tag{13}$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at the discrete time t_ℓ, and $\mathbf{G}(\ell) \in SO_o(n,1)$ is the group value of $\mathbf{G}$ at time t_ℓ.

2.2 GPS for forward differential equations system

The Lie group generated from $\mathbf{A} \in so(n,1)$ is known as a proper orthochronous Lorentz group. An exponential mapping of $\mathbf{A}(\ell)$ admits a closed-form representation:

$$\exp[\Delta t\mathbf{A}(\ell)] = \begin{bmatrix} \mathbf{I}_n + \frac{(a_\ell - 1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell\mathbf{f}_\ell^{\mathrm{T}} & \frac{b_\ell \mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell \mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{14}$$

where

$$a_\ell := \cosh\left(\frac{\Delta t\|\mathbf{f}_\ell\|}{\|\mathbf{x}_\ell\|}\right), \ \ b_\ell := \sinh\left(\frac{\Delta t\|\mathbf{f}_\ell\|}{\|\mathbf{x}_\ell\|}\right). \tag{15}$$

Substituting the above $\exp[\Delta t\mathbf{A}(\ell)]$ for $\mathbf{G}(\ell)$ into Eq. (13) and taking its first row, we obtain

$$\begin{aligned} \mathbf{x}_{\ell+1} &= \mathbf{x}_\ell + \eta_\ell \mathbf{f}_\ell \\ &= \mathbf{x}_\ell + \frac{b_\ell\|\mathbf{x}_\ell\|\|\mathbf{f}_\ell\| + (a_\ell - 1)\mathbf{f}_\ell\cdot\mathbf{x}_\ell}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell. \end{aligned} \tag{16}$$

From $\mathbf{f}_\ell\cdot\mathbf{x}_\ell \ge -\|\mathbf{f}_\ell\|\|\mathbf{x}_\ell\|$ we can prove that

$$\eta_\ell \ge \left[1 - \exp\left(-\frac{\Delta t\|\mathbf{f}_\ell\|}{\|\mathbf{x}_\ell\|}\right)\right]\frac{\|\mathbf{x}_\ell\|}{\|\mathbf{f}_\ell\|} > 0, \ \ \forall \Delta t > 0. \tag{17}$$

This scheme is group properties preserved for all $\Delta t > 0$.

3 Backward problems and BGPS

3.1 Dynamics on past cone

Corresponding to the initial value problems (IVPs) governed by Eq. (1) with a specified initial value $\mathbf{x}(0)$ at a zero time, for many systems in the engineering applications, the final value problems (FVPs) may happen due to the fact that one wants to retrieve the past histories of states exhibited in the physical models. These time backward problems can be described as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t), \;\; \mathbf{x} \in \mathbb{R}^n, \;\; t \in \mathbb{R}^-. \tag{18}$$

With a specified final value $\mathbf{x}(0)$ at $t = 0$, we intend to recover the past values of $\mathbf{x}$ in the past time of $t < 0$.

We can embed Eq. (18) into the following $n+1$-dimensional augemented dynamical system:

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x} \\ -\|\mathbf{x}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & -\frac{\mathbf{f}(\mathbf{x},t)}{\|\mathbf{x}\|} \\ -\frac{\mathbf{f}^{\mathrm{T}}(\mathbf{x},t)}{\|\mathbf{x}\|} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ -\|\mathbf{x}\| \end{bmatrix}. \tag{19}$$

It is obvious that the first equation in Eq. (19) is the same as the original equation (18), but the inclusion of the second equation gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{x}^{\mathrm{T}}, -\|\mathbf{x}\|)^{\mathrm{T}}$ satisfying a past cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{x}\cdot\mathbf{x} - \|\mathbf{x}\|^2 = \|\mathbf{x}\|^2 - \|\mathbf{x}\|^2 = 0. \tag{20}$$

Here, we should stress that the cone condition imposed on the dynamical system (2) is a future cone as shown in Fig. 1, and that for the dynamical system (19) the cone condition (20) imposed is a past cone as shown in Fig. 1.

Consequently, we have an $n+1$-dimensional augmented system:

$$\dot{\mathbf{X}} = \mathbf{B}\mathbf{X} \tag{21}$$

with a constraint (20), where

$$\mathbf{B} := \begin{bmatrix} \mathbf{0}_{n\times n} & -\frac{\mathbf{f}(\mathbf{x},t)}{\|\mathbf{x}\|} \\ -\frac{\mathbf{f}^{\mathrm{T}}(\mathbf{x},t)}{\|\mathbf{x}\|} & 0 \end{bmatrix} \tag{22}$$

satisfying

$$\mathbf{B}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{B} = \mathbf{0}, \tag{23}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$. Here, the term orthochronous should be understood as the preservation of the sign of $-\|\mathbf{x}\|$.

According to the above Lie algebra property of $\mathbf{B}$ we can derive a backward group-preserving scheme as Eq. (13) for Eq. (6):

$$\mathbf{X}_{\ell-1} = \mathbf{G}(\ell)\mathbf{X}_\ell. \tag{24}$$

The above is a backward single-step numerical scheme. Below we derive a group-preserving scheme for Eq. (21).

3.2 BGPS for backward differential equations system

An exponential mapping of $\mathbf{B}(\ell)$ admits a closed-form representation:

$$\exp[\Delta t\mathbf{B}(\ell)] = \begin{bmatrix} \mathbf{I}_n + \frac{(a_\ell-1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell\mathbf{f}_\ell^{\mathrm{T}} & -\frac{b_\ell\mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ -\frac{b_\ell\mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{25}$$

where a_ℓ and b_ℓ were defined by Eq. (15).

Substituting the above $\exp[\Delta t\mathbf{B}(\ell)]$ for $\mathbf{G}(\ell)$ into Eq. (24) and taking its first row, we obtain

$$\begin{aligned}\mathbf{x}_{\ell-1} &= \mathbf{x}_\ell + \eta_\ell\mathbf{f}_\ell \\ &= \mathbf{x}_\ell + \frac{-b_\ell\|\mathbf{x}_\ell\|\|\mathbf{f}_\ell\| + (a_\ell-1)\mathbf{f}_\ell\cdot\mathbf{x}_\ell}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell.\end{aligned} \tag{26}$$

From $\mathbf{f}_\ell\cdot\mathbf{x}_\ell \le \|\mathbf{f}_\ell\|\|\mathbf{x}_\ell\|$ it follows that

$$\eta_\ell \le \left[\exp\left(-\frac{\Delta t\|\mathbf{f}_\ell\|}{\|\mathbf{x}_\ell\|}\right) - 1\right]\frac{\|\mathbf{x}_\ell\|}{\|\mathbf{f}_\ell\|} < 0, \;\; \forall \Delta t > 0. \tag{27}$$

This scheme is group properties preserved for all $\Delta t > 0$.

Comparing Eqs. (26) and (16) they have the same form in addition that the sign before $b_\ell\|\mathbf{x}_\ell\|\|\mathbf{f}_\ell\|$ in the numerators. In the later we will call this numerical scheme the backward group preserving scheme (BGPS), which is slightly different from the group preserving scheme (GPS) introduced in Section 2 for the forward differential dynamics.

Previously, Liu, Chang and Chang (2006) have applied the BGPS method to the backward in time heat conduction problems, and found that this method is rather promising. Here we extend this method to the nonlinear Burgers equation.

4 Solving Backward Burgers equation by BGPS

4.1 Semi-Discretization

The numerical method of line is simple in concept that for a given system of partial differential equations discretize all but one of the independent variables. The semi-discrete procedure yields a coupled system of ordinary differential equations which are then numerically integrated.

For the one-dimensional backward in time Burgers equation:

$$u_t + uu_x = \frac{1}{R} u_{xx}, \;\; a < x < b, \;\; 0 < t < T, \tag{28}$$

$$u(a,t) = u_a(t), \;\; u(b,t) = u_b(t), \;\; 0 \le t \le T, \tag{29}$$

$$u(x,T) = f(x), \;\; a \le x \le b, \tag{30}$$

we adopt the numerical method of line to discretize the spatial coordinate x by

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=a+i\Delta x} = \frac{u_{i+1}(t) - u_{i-1}(t)}{2\Delta x}, \tag{31}$$

$$\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=a+i\Delta x} = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{(\Delta x)^2}, \tag{32}$$

where Δx is a uniform discretization spacing length, and $u_i(t) = u(a + i\Delta x, t)$, such that Eq. (28) can be approximated by

$$\begin{aligned} \frac{\partial u_i(t)}{\partial t} &= \frac{1}{R(\Delta x)^2}[u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)] \\ &\quad - u_i(t)\frac{u_{i+1}(t) - u_{i-1}(t)}{2\Delta x}. \end{aligned} \tag{33}$$

The next step is to advance the solution from the final condition to the desired time $t = 0$. Really, in Eq. (33) there are totally n coupled nonlinear differential equations for the n variables $u_i(t), i = 1, 2, \ldots, n$, which can be numerically integrated to obtain the numerical solutions.

4.2 One-step BGPS

Applying scheme (26) to Eq. (33) and using the boundary conditions: $u_0(t) = u_a(t)$ and $u_{n+1}(t) = u_b(t)$, we can compute the solution of the backward in time Burgers equation by BGPS.

Starting from a final condition $\mathbf{X}_K = \mathbf{X}(T)$ we attempt to calculate the value $\mathbf{X}(0)$ at the desired time $t = 0$. Suppose that the total time T is divided by K steps, that is, the time stepsize we use in the BGPS is $\Delta t = T/K$. By Eq. (24) we can obtain

$$\mathbf{X}_0 = \mathbf{G}_1(\Delta t)\cdots\mathbf{G}_K(\Delta t)\mathbf{X}_K, \tag{34}$$

where $\mathbf{X}_0$ approximates the real $\mathbf{X}(0)$ within a certain accuracy depending on Δt. However, let us recall that each $\mathbf{G}_i$, $i = 1, \ldots, K$, is an element of the Lie group, and by the closure and transtive properties of Lie group $\mathbf{G}_1(\Delta t)\cdots\mathbf{G}_K(\Delta t)$ is also an element of the Lie group denoted by $\mathbf{G}$. Hence, we have

$$\mathbf{X}_0 = \mathbf{G}(K\Delta t)\mathbf{X}_K = \mathbf{G}(T)\mathbf{X}_K. \tag{35}$$

This is a one-step transformation from $\mathbf{X}(T)$ to $\mathbf{X}(0)$.

The most simple method to calculate $\mathbf{G}(T)$ is given by

$$\begin{aligned} \mathbf{G}(T) &= \exp[T\mathbf{B}(K)] \\ &= \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\mathbf{f}_K\|^2}\mathbf{f}_K\mathbf{f}_K^{\mathrm{T}} & -\frac{b\mathbf{f}_K}{\|\mathbf{f}_K\|} \\ -\frac{b\mathbf{f}_K^{\mathrm{T}}}{\|\mathbf{f}_K\|} & a \end{bmatrix}, \end{aligned} \tag{36}$$

where

$$a := \cosh\left(\frac{T\|\mathbf{f}_K\|}{\|\mathbf{x}_K\|}\right), \;\; b := \sinh\left(\frac{T\|\mathbf{f}_K\|}{\|\mathbf{x}_K\|}\right). \tag{37}$$

That is, we use the final $\mathbf{x}(T)$ to calculate $\mathbf{G}(T)$. Then from Eq. (26) we obtain a one-step BGPS:

$$\mathbf{x}_0 = \mathbf{x}_K + \frac{-b\|\mathbf{x}_K\|\|\mathbf{f}_K\| + (a-1)\mathbf{f}_K\cdot\mathbf{x}_K}{\|\mathbf{f}_K\|^2}\mathbf{f}_K. \tag{38}$$

The accuracy and efficiency are demonstrated by numerical examples given below.

4.3 Example 1

For the Burgers equation (28) with the following boundary conditions and initial condition:

$$u(0,t) = u(1,t) = 0, \tag{39}$$

$$u(x,0) = \sin \pi x, \tag{40}$$

the exact solution is obtained by transforming them through the Hopf-Cole transformation [Hopf (1950); Cole (1951)]:

$$u = \frac{-2\phi_x}{R\phi}, \tag{41}$$

to the following heat conduction equation, boundary conditions and initial condition:

$$\begin{aligned}
&\phi_t = \frac{1}{R}\phi_{xx}, \quad 0 < x < 1, \quad 0 < t < T,\\
&\phi_x(0,t) = \phi_x(1,t) = 0,\\
&\phi(x,0) = \exp\left[\int_0^x \frac{-R}{2}\sin \pi\xi d\xi\right]\\
&= \exp\left[\frac{R}{2\pi}(\cos \pi x - 1)\right].
\end{aligned} \tag{42}$$

Then, applying the method of separation of variables and the Fourier transform to the above linear equation we obtain

$$\phi(x,t) = a_0 + \sum_{k=1}^{\infty} a_k \exp\left[\frac{-(k\pi)^2 t}{R}\right]\cos(k\pi x), \tag{43}$$

where

$$a_0 = \exp\left[\frac{-R}{2\pi}\right]\int_0^1 \exp\left[\frac{R\cos \pi x}{2\pi}\right]dx, \tag{44}$$

$$a_k = 2\exp\left[\frac{-R}{2\pi}\right]\int_0^1 \exp\left[\frac{R\cos \pi x}{2\pi}\right]\cos(k\pi x)dx. \tag{45}$$

Substituting Eq. (43) for ϕ into Eq. (41) we obtain the solution for u:

$$u(x,t) = \frac{2\pi\sum_{k=1}^{\infty} k a_k \exp[-(k\pi)^2 t/R]\sin(k\pi x)}{Ra_0 + R\sum_{k=1}^{\infty} a_k \exp[-(k\pi)^2 t/R]\cos(k\pi x)}. \tag{46}$$

In order to calculate the backward in time Burgers equation we let $t = T$ in Eq. (46) to be our final data:

$$u(x,T) = \frac{2\pi\sum_{k=1}^{\infty} k a_k \exp[-(k\pi)^2 T/R]\sin(k\pi x)}{Ra_0 + R\sum_{k=1}^{\infty} a_k \exp[-(k\pi)^2 T/R]\cos(k\pi x)}. \tag{47}$$

In practice, we calculate the above series with 100 terms, and each a_k is calculated by the ten-point Gaussian quadrature.

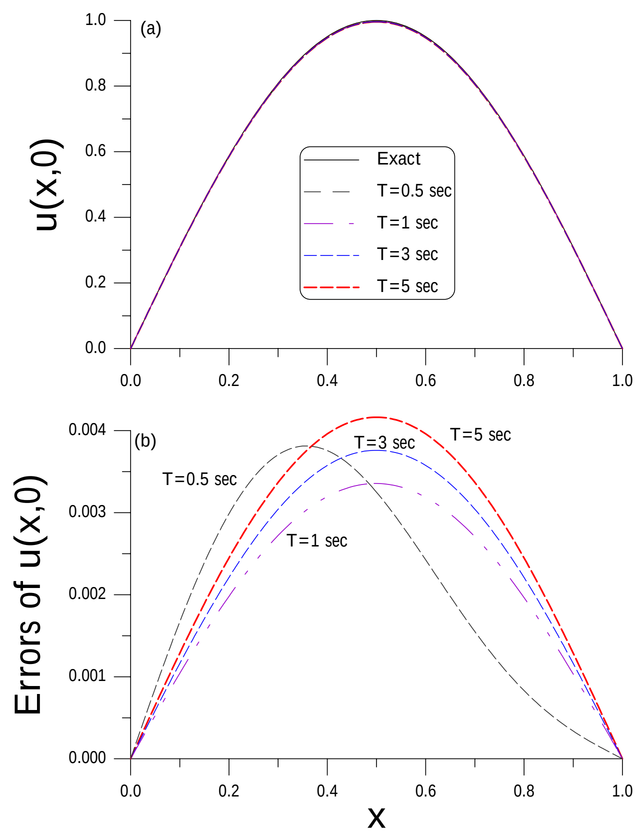

Figure 2 : The comparison of exact solutions and numerical solutions for Example 1 of backward Burgers' equation with different final times: $T = 0.5, 1, 3, 5$ sec.

We first consider a small solution case with $R = 1$. In Fig. 2 we show the numerical results and numerical errors for different final times of $T = 0.5, 1, 3$ sec. The numerical results are calculated by BGPS with one step but keeping $\Delta x = 1/200$. Let us further investigate a very severely ill-posed case of this problem, where $T = 5$ sec was employed, such that when the final data is in the order of $O(10^{-22})$ we want to use BGPS to retrieve the desired initial data $\sin \pi x$, which is in the order of $O(1)$. Even for this severe case up to $T = 5$ sec, our computation is stable, and the maximum error occurring at $x = 0.5$ is about 0.004.

In order to enhance our perception about this difficult problem, we apply the fourth-order Runge-Kutta method (RK4) to this problem with a variable transformation $s = T - t$ as that used by Liu (2004) for the heat conduction equation, such that Eq. (28) becomes

$$u_s = uu_x - u_{xx}. \tag{48}$$

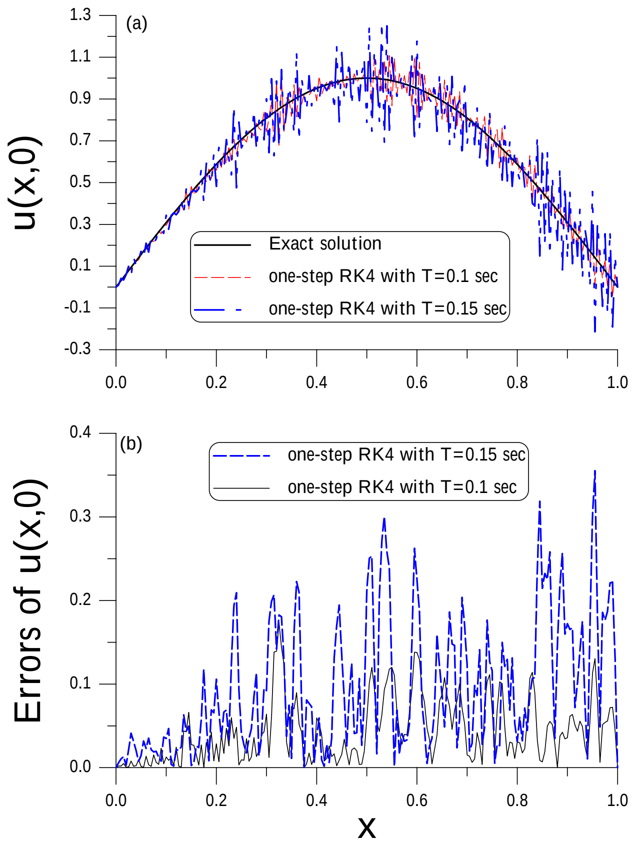

Figure 3 : The comparison of exact solutions and numerical solutions for Example 1 by one-step RK4 with different final times: $T = 0.1, 0.15$ sec.

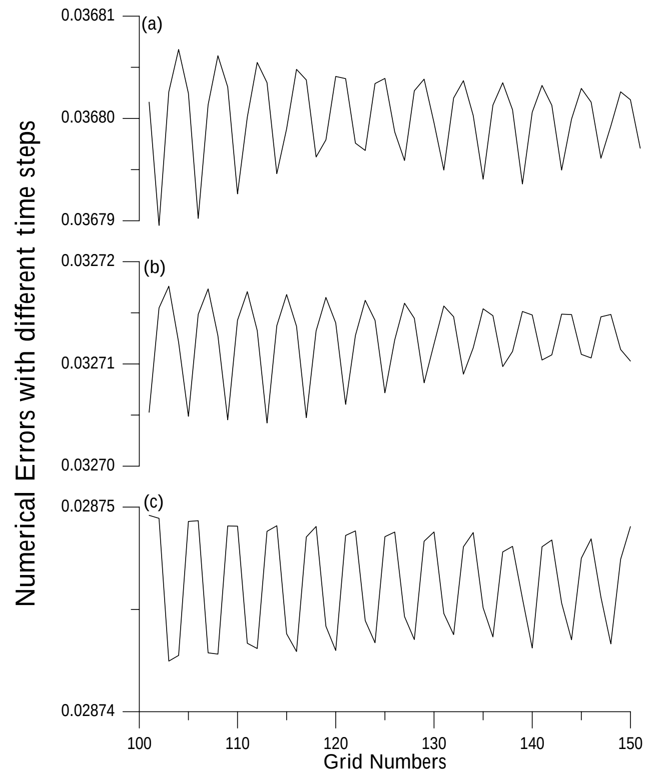

Figure 4 : The influence of time steps on the numerical errors with $T = 0.1$ sec: (a) one-step, (b) two-step, and (c) three-step.

For the prescribed boundary conditions (39) and the final time condition (47) which becomes an initial condition of the above equation with the initial value given at $s = 0$. Spatially discretized the terms u_x and u_{xx} as that given by Eqs. (31) and (32), we obtain a set of ordinary differential equations which can be integrated by the RK4.

We consider two cases of $T = 0.1, 0.15$ sec, and when applying the RK4 in the calculations with a fixed $\Delta x = 1/200$ and two time stepsizes of $\Delta t = 0.1, 0.15$ sec we attempt to recover the data given by Eq. (40) with one-step RK4. Two-step RK4 calculations are already led to the divergence of numerical solutions for both cases. From Fig. 3(a) it can be seen that the one-step RK4 makes oscillatory solutions, and the accuracy is very poor as shown in Fig. 3(b).

Really, in the backward numerical integration of the Burgers equation a simple employment of the finite difference or finite element method with negative time steps is numerically unstable. Even a one-step calculation can alleviate the problem of stability, there also has the problem of accuracy.

With this in mind, we can appreciate the stable behavior of BGPS, which is stable even for several time steps. To demonstrate this fact let us consider the same case $T = 0.1$ sec as that calculated by RK4. In Fig. 4 we plot the maximum numerical errors at the initial time as functions of the number of grid points calculated respectively by one-step, two-step and three-step BGPS. The errors of one-step BGPS are in the range of [0.03679, 0.03681], the errors of two-step BGPS are in the range of [0.03270, 0.03272], and the errors of three-step BGPS are in the range of [0.02874, 0.02875]. It can be seen that the grid number affects the accuracy very little, and the errors are much smaller than that calculated by the RK4. When compare the numerical results for $T = 0.1$ sec with the results for the cases of $T = 0.5, 1, 3, 5$ as shown in Fig. 2(b), it can be found that the one-step BGPS is more effective when T is larger. We can conclude that the BGPS is ac-

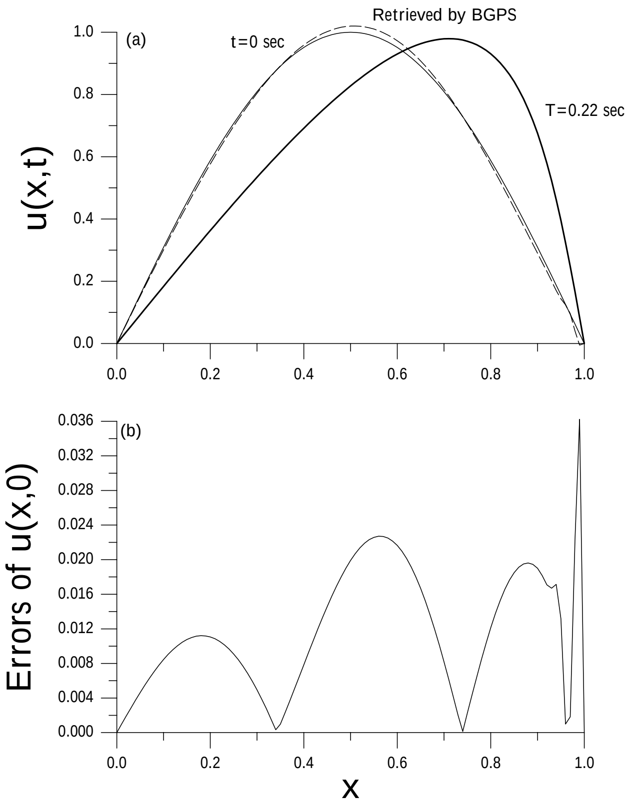

Figure 5 : The comparison of exact solution and numerical solutions for Example 1 with a large $R = 100$.

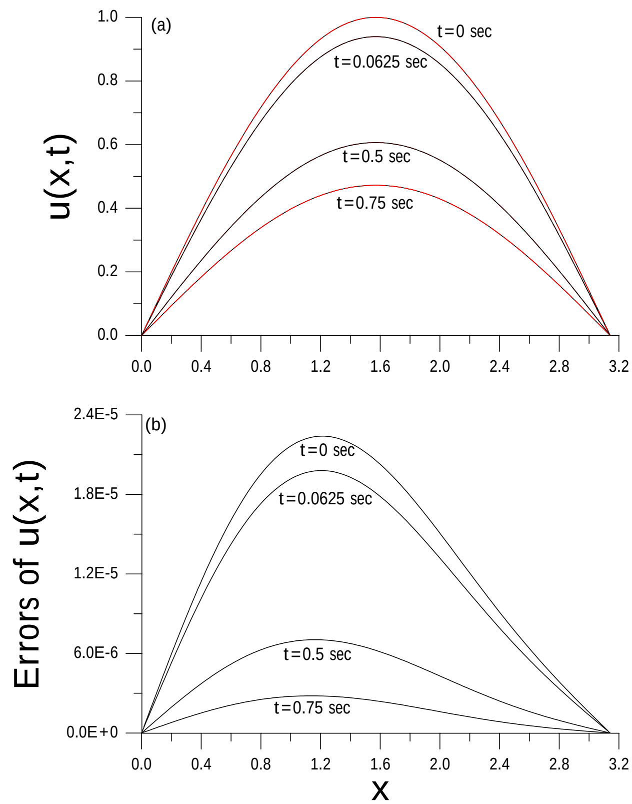

Figure 6 : The comparison of exact solutions and numerical solutions for Example 2 with data at different times been retrieved: $t = 0.75, 0.5, 0.0625, 0$ sec.

curate for the integration of backward in time Burgers equation, which is much accurate than the RK4.

From the above calculations it can be seen that the method of BGPS is useful for small solutions with small $R = 1$. We also observe that in several numerical experiments, the distance backward in time where significant accuracy can be attained is much larger than would be expected on the basis of our calculations; for example, for the case of $T = 5$ sec the final data is very small in the order of $O(10^{-22})$ almost uncomputable by PC, and we can still retrieve the initial data very accurate.

Let us turn to a large $R = 100$. This problem appears a steep gradient of the solution as shown in Fig. 5(a) with a solid thick line when $T = 0.22$ sec. For this case the series solution converges very slowly and instead of we use the GPS developed in Section 2 to calculate the required data at a final time $T = 0.22$ sec. In this calculation the grid length was fixed to be $\Delta x = 1/100$, and the time stepsize used for GPS was $\Delta t = 0.001$ sec. Then we retrieve those final data by BGPS to the data at the initial time $t = 0$ as shown in Fig. 5(a) with a dashed line, from which it can be seen that the result is rather good and close to the exact solution as shown by the solid line in Fig. 5(a). The maximum numerical error as shown in Fig. 5(b) is smaller than 0.036. The time stepsize used in the BGPS was given by $\Delta t = 0.22/12$ sec. Because the final data exhibits a steep variation near to the end point $x = 1$, it is hard to use a one-step BGPS to recover the initial data. However, 12 steps BGPS calculations can approximate the desired initial data.

4.4 Example 2

Next, we consider a Burgers equation with a time-dependent source:

$$u_t + uu_x = u_{xx} + \frac{1}{2}e^{-2t}\sin(2x), \quad 0 < x < \pi, \quad 0 < t < T, \tag{49}$$

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad 0 \le t \le T, \tag{50}$$

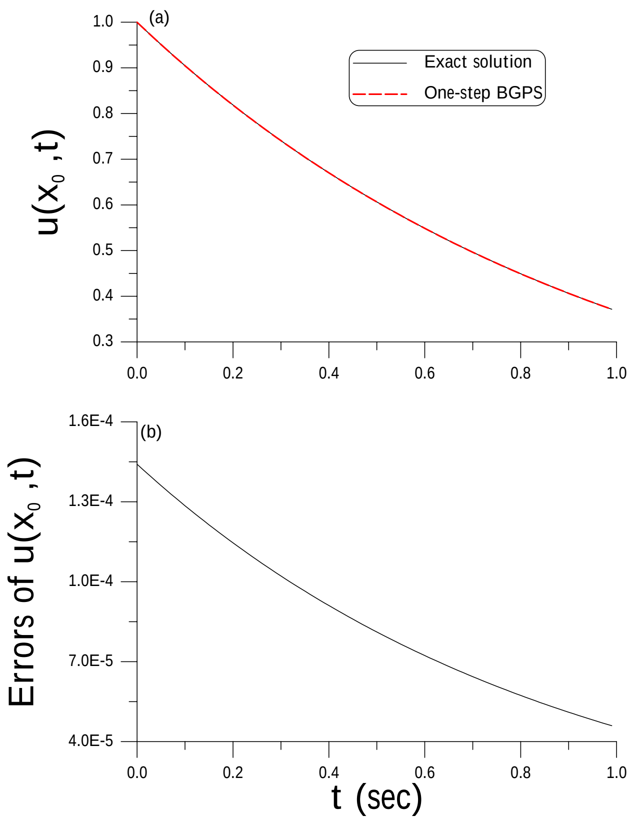

Figure 7 : The comparison of exact solution and numerical solution for Example 2 with data at a fixed point x_0.

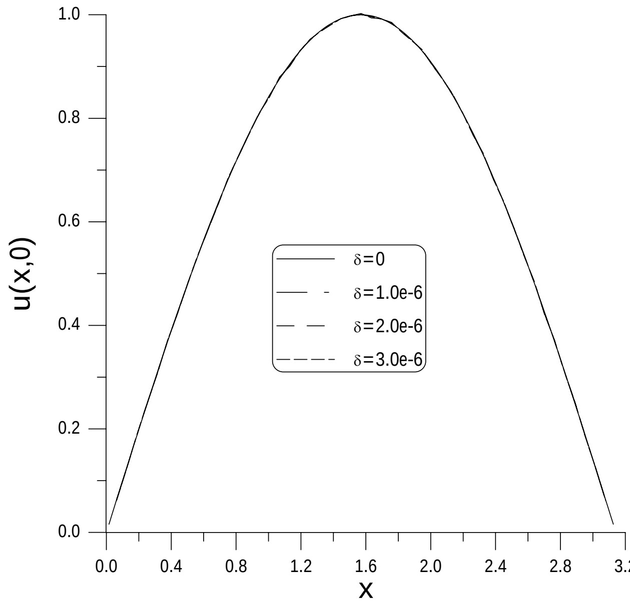

Figure 8 : The comparison of numerical solutions for Example 2 with different levels of noise.

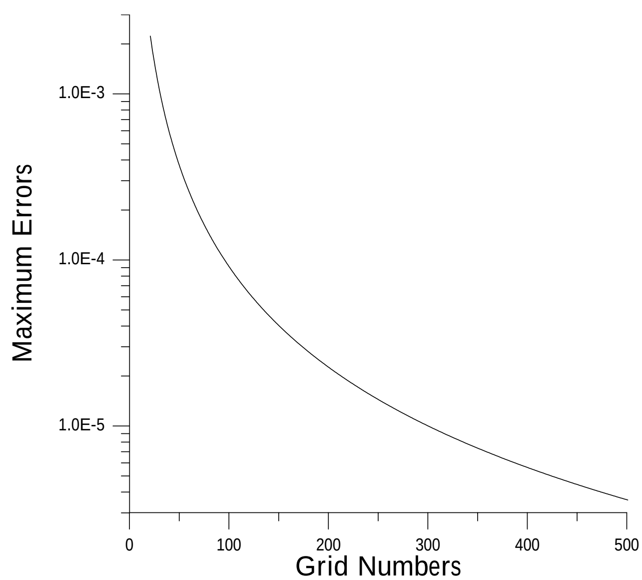

Figure 9 : The maximum error as a function of the number of grid points.

$$u(x,T) = e^{-T}\sin x, \quad 0 \le x \le \pi. \tag{51}$$

The source term is chosen such that $u(x,t) = e^{-t}\sin x$ is a solution of the above equations.

In the calculation we fix $T = 1$ sec and the final data is $u(x,1) = e^{-1}\sin x$. Applying the one-step BGPS for this problem with a fixed $\Delta x = \pi/200$, we plot the exact solutions and numerical solutions in Fig. 6(a) at different times $t = 0.75, 0.5, 0.0625, 0$ sec to be retrieved; however, they are coincident very well. Therefore, we plot the numerical errors in Fig. 6(b). It can be seen that the errors are very small in the order of $O(10^{-5})$. In Table 1 we compare the maximum errors with that obtained by Marbán and Palencia (2002). It can be seen that our results are much better than that obtained by Marbán and Palencia (2002).

In Fig. 7 we use the one-step BGPS technique to recover the past history of u at the point $x_0 = \pi/2$ with $t < T = 1$ sec, wherein the exact solution is e^{-t}. From Fig. 7(a) it is hard to distinct the exact solution and numerical solution, and thus we plot the numerical error in Fig. 7(b). The error is very small in the order of $O(10^{-4})$.

In the case when the input final measured data are contaminated by random noise, we are concerned with the stability of BGPS, which is investigated by adding the different levels of random noise on the final data. We

Table 1 : Comparing the errors of one-step BGPS and that obtained by Marbán and Palencia (2002) of Example 2.

Evaluation past times	Error of Marbán and Palencia	Error of BGPS
$t = T/16$	1.324940E-02	1.979825E-05
$t = T/8$	8.925938E-03	1.742685E-05
$t = T/4$	3.251045E-03	1.329425E-05
$t = T/2$	2.424795E-04	7.038631E-06
$t = 3T/4$	7.889683E-06	2.806019E-06

use the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i)$, where $R(i)$ are random numbers in $[-1,1]$. The numerical results at the initial time $t = 0$ sec were compared with the numerical result without considering random noise in Fig. 8. The noise is obtained by multiplying $R(i)$ by a factor δ. It can be seen that the noise levels with $\delta = 10^{-6}, 2\times10^{-6}, 3\times10^{-6}$ disturb the numerical solutions deviating from the exact solution rather small.

Next we investigate the influence of the number of grid points on the accuracy to recover the initial data with $T = 1$ sec in Fig. 9. It can be seen that more grid points can increase the accuracy up to the order $O(10^{-6})$. However, it was found that even for small numbers of grid point the one-step BGPS can also produce a rather accurate numerical solution with error in the order $O(10^{-3})$.

The above two computational examples supported that we may use a one-step BGPS with a finer grid length to compute the backward in time Burgers problem with small R. On the other hand, there are four reasons for a one-step BGPS: (a) the BGPS is group properties preserved for all $\Delta t > 0$; (b) a one-step computation is much time saving; (c) there has no error propagation in the one-step computation; (d) it can increase the spatial resolution by increasing the number of grid points. Problems where steep gradients occur require considerably more precision in measurement, and the BGPS needs more steps to retrieve the desired intial data.

5 Conclusions

In this paper we were concerned with the numerical integration problem of backward in time Burgers equation. The key point was the construction of a past cone and a backward group preserving scheme. It is a first time that we could construct a geometry (past cone), algebra (Lie algebra) and group (Lie group) description of the backward problems governed by differential equations. By employing the BGPS we can recover all past times data with a high order accuracy. Two numerical examples of the backward in time Burgers equation were work out, which show that our numerical integration methods are applicable to these problems, even for the very strongly ill-posed ones. Under the noised final data the BGPS was also robust enough to retrieve the initial data. In the computations, a one-step BGPS was applicable to recover the initial data, having a higher accuracy with a suitable finer grid length. The efficiency of one-step BGPS was rooted in the closure and transtive properties of the Lie group that we used it to construct the numerical method for backward in time Burgers equation.

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