Copyright © 2006 Tech Science Press CMES, vol.12, no.3, pp.197-211, 2006

A Group Preserving Scheme for Burgers Equation with Very Large Reynolds Number

Chein-Shan Liu[1]

Abstract: In this paper we numerically solve the Burgers equation by semi-discretizing it at the n interior spatial grid points into a set of ordinary differential equations: $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u},t)$, $\mathbf{u} \in \mathbb{R}^n$. Then, we take the dissipative behavior of Burgers equation into account by considering the magnitude $\|\mathbf{u}\|$ as another component; hence, an augmented quasilinear differential equations system $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ with $\mathbf{X} := (\mathbf{u}^{\mathrm{T}}, \|\mathbf{u}\|)^{\mathrm{T}} \in \mathbb{M}^{n+1}$ is derived. According to a Lie algebra property of $\mathbf{A} \in so(n,1)$ we thus develop a new numerical scheme with the transformation matrix $\mathbf{G} \in SO_o(n,1)$ being an element of the proper orthochronous Lorentz group. The numerical results were in good agreement with exact solutions, and it can be seen that the group preserving scheme is better than other numerical methods. Even for very large Reynolds number the group preserving scheme supplemented with a spatial rescaling technique also provides a reliable result without inducing numerical instability.

keyword: Burgers equation, Lie algebra, Lorentz Group, Group preserving scheme, Spatial rescale.

1 Introduction

In this paper we are concerned with numerical solutions of Burgers equation:

$$u_t + uu_x = \frac{1}{R}u_{xx}, \;\; a < x < b, \;\; 0 < t < T, \tag{1}$$

$$u(a,t) = u_a(t), \;\; u(b,t) = u_b(t), \;\; 0 \le t \le T, \tag{2}$$

$$u(x,0) = f(x), \;\; a \le x \le b, \tag{3}$$

where R is the Reynolds number characterizing the viscosity of fluid. Burgers' equation has been of considerable physical interest because it is an appropriate simplification of the Navier-Stokes equations, and is also the governing equation for a number of one-dimensional flow systems including the convection and diffusion of heat, weak shock propagation, compressible turbulence, and continuum traffic simulation.

The Burgers equation was named after Burgers (1948, 1974), the behavior of which exhibits a delicate balance between advection and diffusion. Moreover, it is one of the few nonlinear partial differential equations that exact solutions are known in terms of the initial values [Cole (1951), Hopf (1950)]. However, when R is large over 100, the computation by means of exact solution is not practical due to the slow convergence of the Fourier series. Sometimes, the numerical methods for $u_t + uu_x = 0$ and their generalization often employ an artifical viscosity or dissipation mechanism to control instability with u_{xx}/R playing a suitable regularization role for large R. In this sense, a numerical method that can treat the computations of Burgers equation with large R becomes significant.

In the past a few decades there were much studies on the numerical solutions of Burgers' equation, for example, Fletcher (1983), Basdevant, Deville and Haldenwang (1986), Arina and Canuto (1993), Özis and Özdes (1996), Hon and Mao (1998), Kutluay, Bahadir and Özdes (1999), Lin and Zhou (2001), Wei and Gu (2002), Özis, Aksan and Özdes (2003), Young (2005) and Young, Hu, Fan and Chen (2006).

As mentioned by Ames (1992), the numerical solution for Burgers equation using explicit method may induce oscillations and ripples when R is very large as to $R =$ 10000, and under more large R, the computer time limitations imposed by stability requirements prevented further use of the explicit method.

In this paper we will propose a new numerical scheme of explicit type for calculating the Burgers' equation under very large Reynolds number even up to $R = 20000$ and only using 100 grid points and with a reasonable time stepsize 10^{-4} sec. In order to get a stable solution of the Burgers' equation with $R = 10000$ by using the predictor-

[1] Department of Mechanical and Mechatronic Engineering, Taiwan Ocean University, Keelung, Taiwan. E-mail: csliu@mail.ntou.edu.tw

corrector method, it requires 200 grid points and with a stepsize 4×10^{-4} sec as reported by Ames (1992).

The proposed scheme is based on the numerical method of line, which is a well-developed numerical method that transforms partial differential equations (PDEs) into a system of ordinary differential equations (ODEs), together with the group preserving scheme (GPS) developed previously by Liu (2001) for ODEs. The GPS method is very effective to treat ODEs with special structures as shown by Liu (2005, 2006a) for stiff equations and ODEs with constraints. Furthermore, Liu (2006b) has applied the backward group preserving scheme (BGPS) on the calculations of backward in time Burgers equation. However, this methodology is not yet applied to the forward Burgers equation in the open literature. On the other hand, the Burgers equation is a useful test medium for investigating various numerical methods on PDEs. It thus deserves our attention to develop an effective numerical method for this specific well-known PDE and to investigate the numerical behavior of this new method based on group properties.

The major contributions of this paper would be employing the group preserving property of the resultant system in the development of numerical scheme and giving a conviction that the proposed scheme is superior to the Euler scheme and the fourth-order Runge-Kutta method (RK4). Specifically, the proposed scheme is easy to implement and time saving than other methods, for example, the multiquadric spatial approximation [Hon and Mao (1998)], which at each time step requires to solve the coefficients iteratively even its grid points can be largely reduced. Usually, when one applies the line method to PDEs, the resultant differential equations system is highly-dimensional for an accuracy consideration, and thus it is desired to use an easily-implemented program with a minimal step and a minimal stage in the numerical method. Of course, for ODEs the forward Euler method is the simplest one to be implemented; however, it would be seen that the Euler scheme can not be applied to the Burgers equation under a reasonable grid spacing length and time stepsize. Through this study, we might have an easy-implementation and explicit-single step group preserving scheme used in the calculations of Burgers' equation, the accuracy of which is much better than the Euler scheme and other methods, and also over the RK4.

2 Numerical method of line

The line method for a given system of PDEs discretizes all but one of the independent variables; see, *e.g.*, Schiesser (1991) and Ames (1992). The semi-discrete procedure yields a coupled ODEs system which is then numerically integrated. For the Burgers equation we discretize the derivatives on spatial coordinate x by the central differences:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=a+i\Delta x} = \frac{u_{i+1}(t)-u_{i-1}(t)}{2\Delta x},$$

$$\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=a+i\Delta x} = \frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{(\Delta x)^2},$$

where $\Delta x=(b-a)/(n+1)$ is a uniform grid spacing length, and $u_i(t):=u(a+i\Delta x,t)$ for simplicity. There are totally n variables $u_i(t)$ at the n interior grid points. Then, from Eq. (1) we obtain a system of ODEs:

$$\dot{u}_i(t) = \frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{R(\Delta x)^2} - u_i(t)\frac{u_{i+1}(t)-u_{i-1}(t)}{2\Delta x}, \tag{4}$$

where the index i runs from 1 to n, and a superimposed dot stands for the differential with respect to t.

If we replace the second u_i on the right-hand side of Eq. (4) by the average $\overline{u}_i := (u_{i+1}+u_i+u_{i-1})/3$, we also have

$$\dot{u}_i(t) = \frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{R(\Delta x)^2} - \overline{u}_i(t)\frac{u_{i+1}(t)-u_{i-1}(t)}{2\Delta x}. \tag{5}$$

Eq. (5) has totally n coupled differential equations for the n variables $u_i(t), i=1,2,\ldots,n$, which are subjected to the initial conditions:

$$u_i(0) = f(a+i\Delta x). \tag{6}$$

The two boundary conditions in Eq. (2) led to $u_0(t)=u_a(t)$ and $u_{n+1}(t)=u_b(t)$.

Aref and Daripa (1984) have compared the two different discretizations (4) and (5), and indicated that the discretization (4) may induce spurious solution and the discretization (5) provides a dissipative mechanism to rule out the spurious solution.

To see this, let us consider homogeneous boundary conditions. From Eq. (1) it can be shown that the kinetic energy of the field u:

$$E_{kin} = \frac{1}{2}\int_a^b u^2 dx \tag{7}$$

satisfies

$$\frac{dE_{kin}}{dt} = -\frac{1}{R}\int_a^b \left(\frac{\partial u}{\partial x}\right)^2 dx < 0. \tag{8}$$

A discrete version is thus

$$\sum_{k=1}^n u_k \frac{du_k}{dt} < 0, \tag{9}$$

which is an one-side constraint that requires the numerical scheme to be a contraction mapping.

Let $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{T}}$ and then Eq. (5) can be expressed as

$$\dot{\mathbf{u}} = \frac{1}{R(\Delta x)^2}\mathbf{C}\mathbf{u} - \frac{1}{6\Delta x}\mathbf{W}\mathbf{u}, \tag{10}$$

where

$$\mathbf{W} := \begin{bmatrix} 0 & u_1+u_2 & 0 & 0 & \cdots & 0 & 0 \\ -(u_1+u_2) & 0 & u_2+u_3 & 0 & \cdots & 0 & 0 \\ 0 & -(u_2+u_3) & 0 & u_3+u_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(u_{n-1}+u_n) & 0 \end{bmatrix} \tag{11}$$

is an $n \times n$ skew-symmetric matrix, and

$$\mathbf{C} := \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & 1 & -2 \end{bmatrix} \tag{12}$$

is an $n \times n$ symmetric center difference matrix.

In above we have imposed the homogeneous boundary conditions $u_0 = u_{n+1} = 0$. The eigenvalues of $\mathbf{C}$ are found to be [Liu (2004)]

$$-4\sin^2\frac{m\pi}{2(n+1)}, \quad m = 1, 2, \ldots, n, \tag{13}$$

which together with the symmetry indicate that $\mathbf{C}$ is negative definite. Thus, from Eq. (10) it follows that $\mathbf{u}\cdot\dot{\mathbf{u}} < 0$, where the dot between two vectors denotes their inner product. That is, the discretization (5) satisfies the requirement specified in Eq. (9). For this reason we will use Eq. (5) as a platform of our numerical integrations of the Burgers equation.

3 The group preserving scheme

Let $\mathbf{f} = (f_1, \ldots, f_n)^{\mathrm{T}}$ denote the corresponding vector fields in Eq. (5), which can be viewed as a system of n ordinary differential equations:

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, t), \tag{14}$$

where $\mathbf{u}$ is an n-dimensional vector, and $\mathbf{f}$ is a vector-valued function of $\mathbf{u}$ and t with components

$$f_i = \frac{1}{R(\Delta x)^2}[u_{i+1} - 2u_i + u_{i-1}] - \frac{1}{2\Delta x}\overline{u}_i[u_{i+1} - u_{i-1}], \quad i = 1, \ldots, n. \tag{15}$$

From Eqs. (10)-(12) it can be seen that $\mathbf{u}$ is dominated by two forces: the first term on the right-hand side of Eq. (10) tends to decrease the magnitude $\|\mathbf{u}\| := \sqrt{u_1^2 + \ldots + u_n^2}$ of $\mathbf{u}$, while the second term is strongly to keeping the magnitude invariant. Especially, when R is very large, the dominion is the second term. Therfore, we prefer a new numerical scheme that can take the variation of $\|\mathbf{u}\|$ into account and keeps the magnitude very slowly decreasing when R is very large.

On the other hand, Eq. (9) shows that the magnitude is an important factor to select a suitable numerical scheme with $d\|\mathbf{u}\|/dt < 0$. It is noteworthy to take the evolution of $\|\mathbf{u}\|$ into account, which by Eq. (14) leads to

$$\frac{d}{dt}\|\mathbf{u}\| = \frac{\mathbf{f}\cdot\mathbf{u}}{\|\mathbf{u}\|}. \tag{16}$$

Combined Eqs. (14) and (16) together, Liu (2001) has derived a single differential equations system:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \tag{17}$$

where

$$\mathbf{X} := \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} \tag{18}$$

is an augmented vector, and

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{u},t)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},t)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \tag{19}$$

is an augmented state matrix.

It is obvious that the first differential equation in Eq. (17) is the same as Eq. (14), but the introduction of the second differential equation led to a Minkowskian structure for the augmented nonlinear system with the augmented variable $\mathbf{X}$ satisfying

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{u}\cdot\mathbf{u} - \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{u}\|^2 = 0, \tag{20}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{21}$$

is a Minkowski metric, and $\mathbf{I}_n$ is the identity matrix of order n. Moreover, $\mathbf{A}$ satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0} \tag{22}$$

is a Lie algebra of the Lorentz group $SO_o(n,1)$.

Therefore, the n-dimensional dynamical system (14) in $\mathbb{R}^n$ can be embedded into an augmented $n+1$-dimensional dynamical system (17) in $\mathbb{M}^{n+1}$, and the cone condition

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0 \tag{23}$$

is a constraint. Even the dimension of the new system is raising one more, the new system with its Lie algebra property (22) allows us to develop a group preserving numerical scheme [Liu (2001, 2004, 2005, 2006a)]:

$$\mathbf{u}^{j+1} = \mathbf{u}^j + \eta_j \Delta t \mathbf{f}^j \tag{24}$$

with the adaptive factor

$$\eta_j := \frac{4\|\mathbf{u}^j\|^2 + 2\Delta t \mathbf{f}^j\cdot\mathbf{u}^j}{4\|\mathbf{u}^j\|^2 - (\Delta t)^2\|\mathbf{f}^j\|^2} \tag{25}$$

varying step-by-step on time. In above $\mathbf{u}^j$ denotes the numerical value of $\mathbf{u}$ at a discrete time t_j, $\Delta t = t_{j+1} - t_j$, and $\mathbf{f}^j := \mathbf{f}(\mathbf{u}^j, t_j)$ for simplicity.

Employing the numerical scheme (24) on Eq. (5) we obtain a new numerical scheme for the Burgers equation. For the heat conduction problems, Liu (2004) has adopted the group preserving scheme for the numerical solutions of both forward and backward problems, and Chang, Liu and Chang (2005) have applied the group preserving scheme for the numerical solutions of the sideways heat equation, finding that the group preserving scheme was able to calculate the numerical solutions within a certain accuracy. In the next section we will apply it to some numerical examples.

Now, we prove that the group preserving scheme provides a contraction mapping under the following conditions about the vector field and time stepsize:

$$\mathbf{f}\cdot\mathbf{u} < 0, \tag{26}$$

$$\Delta t < \frac{-2\mathbf{f}\cdot\mathbf{u}}{\|\mathbf{f}\|^2}. \tag{27}$$

From Eq. (24) we have

$$\|\mathbf{u}^{j+1}\|^2 = \|\mathbf{u}^j\|^2 + 2\Delta t \eta_j \mathbf{u}^j\cdot\mathbf{f}^j + (\Delta t)^2\eta_j^2\|\mathbf{f}^j\|^2. \tag{28}$$

Inserting Eq. (25) for η_j into the above equation and through some manipulations we get

$$\|\mathbf{u}^{j+1}\|^2 = \frac{\|\mathbf{u}^j\|^2[4\|\mathbf{u}^j\|^2 + (\Delta t)^2\|\mathbf{f}^j\|^2 + 4\Delta t \mathbf{f}^j\cdot\mathbf{u}^j]^2}{[4\|\mathbf{u}^j\|^2 - (\Delta t)^2\|\mathbf{f}^j\|^2]^2}. \tag{29}$$

Under condition (27) the time stepsize is smaller than $2\|\mathbf{u}^j\|/\|\mathbf{f}^j\|$, hence $4\|\mathbf{u}^j\|^2 - (\Delta t)^2\|\mathbf{f}^j\|^2 > 0$, and then taking the square roots of both sides of the above equation we obtain

$$\|\mathbf{u}^{j+1}\| = \frac{4\|\mathbf{u}^j\|^2 + (\Delta t)^2\|\mathbf{f}^j\|^2 + 4\Delta t \mathbf{f}^j\cdot\mathbf{u}^j}{4\|\mathbf{u}^j\|^2 - (\Delta t)^2\|\mathbf{f}^j\|^2}\|\mathbf{u}^j\|. \tag{30}$$

The factor before $\|\mathbf{u}^j\|$ satisfies

$$\frac{4\|\mathbf{u}^j\|^2 + (\Delta t)^2\|\mathbf{f}^j\|^2 + 4\Delta t \mathbf{f}^j\cdot\mathbf{u}^j}{4\|\mathbf{u}^j\|^2 - (\Delta t)^2\|\mathbf{f}^j\|^2} < 1, \tag{31}$$

if conditions (26) and (27) hold. Therefore, the scheme (24) is a contraction mapping. However, in the limiting case with $R = \infty$, the new method preserves the magnitude $\|\mathbf{u}\|$ invariant.

It deserves to note that Eqs. (24) and (30) can be expressed neatly in terms of the augmented variable $\mathbf{X}$ by

$$\mathbf{X}^{j+1} = \mathbf{G}\mathbf{X}^j, \tag{32}$$

Table 1 : Comparing the errors of GPS and Euler scheme for R =100 and 200 of Problem 1.

Δx	Δt	Error of GPS	Error of Euler	$R=100$
0.025	1/60	0.03024	0.11199	
	1/70	0.01429	0.10291	
	1/80	0.00331	0.09635	
	1/90	0.00470	0.09141	
	1/100	0.01081	0.08754	
0.02	1/60	0.05579	0.08715	
	1/70	0.03849	0.07907	
	1/80	0.02661	0.07323	
	1/90	0.01796	0.06880	
	1/100	0.01138	0.06533	
				$R=200$
0.025	1/70	0.03805	0.24369	
	1/80	0.00644	0.23810	
	1/90	0.03251	0.23044	
	1/100	0.05054	0.22352	
0.02	1/70	0.08321	0.23086	
	1/80	0.03955	0.21418	
	1/90	0.01331	0.20227	
	1/100	0.00524	0.19304	

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{2(\Delta t)^2 \mathbf{f}^j (\mathbf{f}^j)^{\mathrm{T}}}{4\|\mathbf{u}^j\|^2 - (\Delta t)^2 \|\mathbf{f}^j\|^2} & \frac{4\Delta t \|\mathbf{u}^j\| \mathbf{f}^j}{4\|\mathbf{u}^j\|^2 - (\Delta t)^2 \|\mathbf{f}^j\|^2} \\ \frac{4\Delta t \|\mathbf{u}^j\| (\mathbf{f}^j)^{\mathrm{T}}}{4\|\mathbf{u}^j\|^2 - (\Delta t)^2 \|\mathbf{f}^j\|^2} & \frac{4\|\mathbf{u}^j\|^2 + (\Delta t)^2 \|\mathbf{f}^j\|^2}{4\|\mathbf{u}^j\|^2 - (\Delta t)^2 \|\mathbf{f}^j\|^2} \end{bmatrix} \quad (33)$$

under condition (27) satisfying the following properties:

$$\mathbf{G}^{\mathrm{T}} \mathbf{g} \mathbf{G} = \mathbf{g}, \quad (34)$$

$$\det \mathbf{G} = 1, \quad (35)$$

$$G_0^0 \geq 1. \quad (36)$$

The det is the shorthand of determinant, and G_0^0 is the 00th component of $\mathbf{G}$, which is indeed an element of $SO_o(n,1)$, and the numerical scheme basing on it is called a group preserving scheme (GPS).

In passing we note that the scheme (24) becomes an Euler scheme when $\eta_j = 1$:

$$\mathbf{u}^{j+1} = \mathbf{u}^j + \Delta t \mathbf{f}^j. \quad (37)$$

Under conditions (26) and (27) the above scheme is also of the contraction type. However, we will give numerical examples below to show that GPS is much accurate than the Euler scheme. The latter cannot be applied to the Burgers equation when the Reynolds number is moderately large. Both schemes are explicit and single-step.

4 The problems of Burgers equation

4.1 Problem 1

Let us first consider the Burgers equation (1) with the following boundary conditions and initial condition:

$$u(0,t) = \frac{1}{1+\exp[-Rt/4]},$$

$$u(1,t) = \frac{1}{1+\exp[R/2 - Rt/4]},$$

$$u(x,0) = \frac{1}{1+\exp[Rx/2]}, \quad 0 \leq x \leq 1.$$

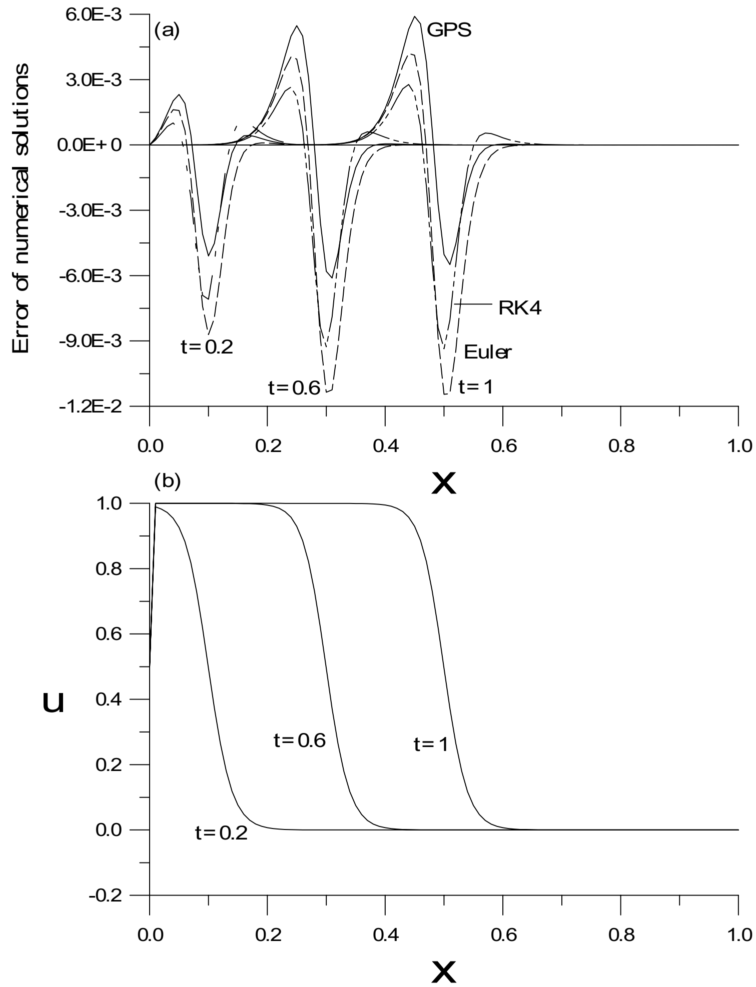

Figure 1 : Comparison of numerical solutions with exact solutions for Problem 1: (a) numerical errors at three different times and (b) the wave fronts at three different times.

The exact solution [Byrne and Hindmarsh (1987)] is given by

$$u(x,t) = \frac{1}{1+\exp[Rx/2 - Rt/4]}. \tag{38}$$

We apply the computational schemes (24) and (37) as well as the RK4 to this problem by letting $n = 99$, $\Delta t = 10^{-3}$ sec, $R = 100$, and $T = 1$ sec. The numerical errors being the differences of numerical solutions and exact solutions were plotted in Fig. 1(a) for GPS, Euler and RK4 at three different times of $t = 0.2, 0.6, 1$ sec, while the wave fronts propagation obtained by GPS were plotted in Fig. 1(b). The exact solutions obtained from Eq. (38) were also drawn on the same figure, but the graphs can not be distinguished due to the closeness of the numerical solutions to the exact ones. The peaks of numerical errors appeared in Fig. 1(a) are caused by the propagation of wave fronts, during which the field variable u undergoes a steep variation from 1 to 0 within a thin spatial region. At these places it can be seen that the numerical errors are still in the order of $O(\Delta t)$ for GPS but the errors of the Euler scheme are large up to the order of 10^{-2}. The errors of GPS are smaller than that of RK4.

In addition, we make a comparison of the numerical errors of GPS and Euler scheme in Table 1 for two values of R =100 and 200 and for different grid spacing lengths and time stepsizes. The errors were defined by the absolute values of the differences of numerical solutions to the exact solution at $x = 0.5$ and $t = 1$ sec, which is $u = 0.5$ for all R by Eq. (38). From Table 1 it can be seen that the errors of GPS are far less than those of the Euler scheme for all cases. From the same table we can observe

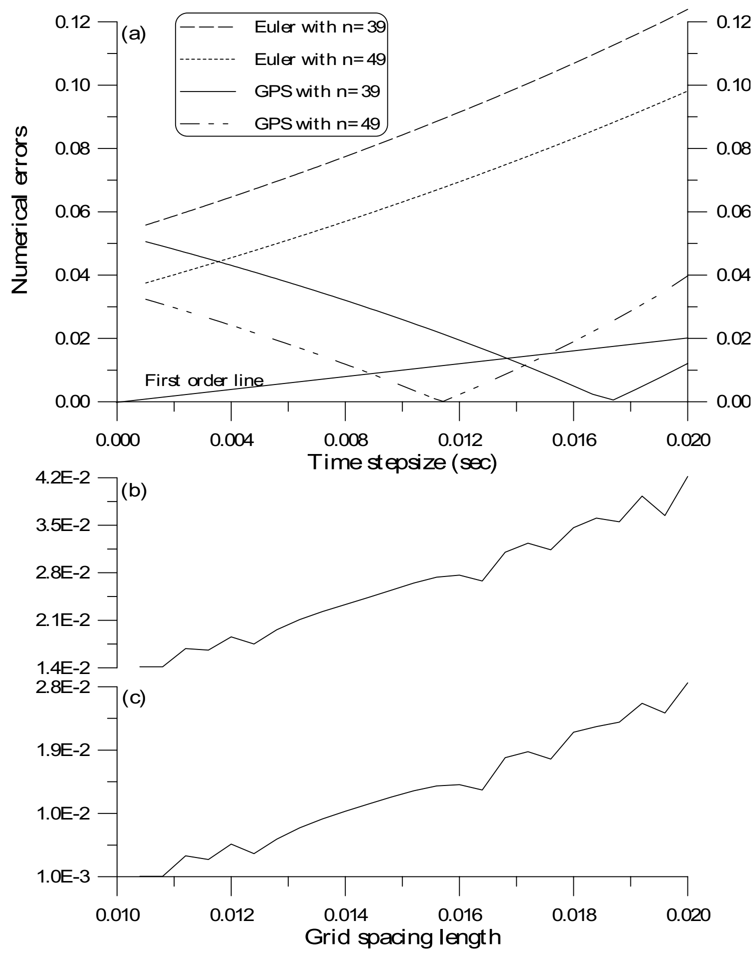

Figure 2 : For Problem 1 the numerical errors of (a) GPS and Euler scheme vs. time stepsize, (b) Euler scheme vs. grid spacing length, and (c) GPS vs. grid spacing length.

that the errors of the Euler scheme are reduced when the time stepsizes decrease or when the grid spacing lengths decrease. But they are not true for GPS. For example, under the same $\Delta x = 0.025$, the error with $\Delta t = 1/80$ is smaller than those with Δt =1/90 and 1/100; and under the same $\Delta t = 1/80$ the error with $\Delta x = 0.025$ is far less than that with $\Delta x = 0.02$. We also observe that the Euler scheme is almost failed applying to the case $R = 200$. But for GPS the errors for these two cases of R =100 and 200 are both in the range of $10^{-3} - 10^{-2}$.

Next, we choose to compare the numerical solutions with the exact solution at $x = 0.2$ and $t = 0.4$ sec, of which $u = 0.5$ for all R by Eq. (38). This is used to explain that the above positive results about GPS are not dependent on x and t. Fixing $R = 100$ and varying the time stepsizes in the range of 0.001-0.02 sec, we plot the numerical errors in Fig. 2(a) with the solid line for GPS and the dashed line for the Euler scheme with both $n = 39$ interior grid points ($\Delta x = 0.025$), and the dashed-dotted line for GPS and the dotted line for the Euler scheme with both $n = 49$ interior grid points ($\Delta x = 0.02$). In all cases, GPS is much better than the Euler scheme. It is only in the limiting case with an almost zero time stepsize, where $\eta = 1$ by Eq. (25), that the Euler scheme is comparable with GPS. The Euler scheme is a *linear scheme*, since its numerical errors are nearly linear functions of time stepsizes. Conversely, the GPS is a *nonlinear scheme*, the numerical errors of which do not depend on the time stepsizes linearly. For both cases n =39 and 49 there exists a time stepsize under which the numerical error of GPS is minimal; it is not of the usual situation that a smaller time stepsize implies a smaller numerical error. For example, the GPS with $n = 49$ can suppress the error to 1.47893×10^{-4} with a time stepsize of $\Delta t = 1.14286 \times 10^{-2}$ sec; but with the time

Table 2 : Comparing the numerical solutions of RK4, GPS and exact solutions for $R = 10$ of Problem 1.

x	t	RK4	GPS	Exact
0.25	0.4	0.43772	0.43780	0.43782
	0.6	0.56201	0.56210	0.56218
	0.8	0.67901	0.67912	0.67918
	1.0	0.77717	0.77730	0.77730
	3.0	0.99808	0.99809	0.99807
0.5	0.4	0.18245	0.18245	0.18243
	0.6	0.26890	0.26892	0.26894
	0.8	0.37738	0.37745	0.37754
	1.0	0.49972	0.49985	0.50000
	3.0	0.99335	0.99337	0.99331
0.75	0.4	0.06013	0.06012	0.06009
	0.6	0.09540	0.09538	0.09535
	0.8	0.14808	0.14806	0.14805
	1.0	0.22264	0.22265	0.22270
	3.0	0.97712	0.97718	0.97702

stepsize of $\Delta t = 4 \times 10^{-4}$ the error increases to 3.3965×10^{-2}.

Then, we fixed the time stepsize to $\Delta t = 0.002$ sec and varied the grid spacing lengths from $\Delta x = 0.01$ to 0.02, and the numerical errors were obtained by comparing the numerical solutions to the closed form solution at an end time $T = 1$ sec and at the midle point $x = 0.5$. The numerical errors of the Euler scheme are plotted in Fig. 2(b), which locate in the range of $(1.4 \times 10^{-2}, 4.2 \times 10^{-2})$. The numerical errors of GPS are plotted in Fig. 2(c), which locate in the range of $(1 \times 10^{-3}, 2.8 \times 10^{-2})$. The accuracy of GPS can be appreciated. It is known that the accuracy of Euler scheme is of the first order. However, the situation is slightly complicated for the Burgers equation with a semi-discretization, since the accuracy is also dependent on the grid spacing length as shown in Fig. 2(b). Under the above grid spacing lengths of $\Delta x = 0.025$ and 0.02 the accuracy of the Euler scheme is always worse than the first order as shown in Fig. 2(a) for its numerical error lines are over the first order line. For GPS the numerical error curves may be under the first order line in a certain range of the time stepsizes as shown in Fig. 2(a), which indicates that GPS has better accuracy than the first order and when using some stepsizes the accuracy may approach to the second order.

Due to the wave front propagation of Burgers' equation, the numerical error heavily depends on the grid point position and the time elapsed, and the comparison needs to consider different positions and times. For this purpose we first compare the numerical solutions of RK4, GPS and exact solutions in Table 2 with $n = 39$, $\Delta t = 0.001$ sec and $R = 10$ at three different grid points of $x = 0.25, 0.5, 0.75$ and at five different times of $t = 0.4, 0.6, 0.8, 1, 3$ sec. It can be seen that for most of the data, GPS is accurate than RK4.

Table 3 : Comparing the global errors of GPS, Euler and RK4 for R =100 of Problem 1.

Δt	Error of GPS	Error of Euler	Error of RK4
1/60	0.76422	1.26882	0.26293
1/70	0.63620	1.21381	0.30629
1/80	0.55849	1.18613	0.34966
1/90	0.51268	1.17578	0.39302
1/100	0.48702	1.17711	0.43639
1/110	0.47556	1.18668	0.47975
1/120	0.47435	1.20228	0.52312

Then, let us consider the following global error:

$$\text{Error} := \sum_{i=1}^{n} \sum_{j=1}^{N} [u_i^j - u(i\Delta x, j\Delta t)]^2, \tag{39}$$

of which u_i^j is a numerical solution at the i-th grid point and at the j-th time step; $u(i\Delta x, j\Delta t)$ is a corresponding exact solution. $N = T/\Delta t$ is the total number of time steps. In Table 3 we compare the global errors of GPS, Euler and RK4 with $R = 100$, $n = 39$ and $T = 1$ sec. No matter which time stepsize is used the error by the Euler scheme is over 1 and much larger than the other two schemes. When the time stepsizes decrease from $1/60$ sec to $1/120$ sec, the errors of GPS stably reduce; conversely, the errors of RK4 increase. The errors of RK4 are smaller than that of GPS at the first five time stepsizes, and then over GPS at the last two time stepsizes.

4.2 Problem 2

For the Burgers equation (1) with the following boundary conditions and initial condition:

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = \sin \pi x,$$

Table 4 : Comparison of exact solutions with the numerical results calculated by GPS and the Galerkin finite element method for $R = 1$ of Problem 2.

x	t	Galerkin	GPS	Exact
0.25	0.10	0.25469	0.25376	0.25364
	0.15	0.15672	0.15672	0.15660
	0.20	0.09619	0.09654	0.09644
	0.25	0.05924	0.05929	0.05921
0.5	0.10	0.37134	0.37177	0.37158
	0.15	0.22674	0.22700	0.22682
	0.20	0.13829	0.13862	0.13847
	0.25	0.08457	0.08464	0.08453
0.75	0.10	0.27102	0.27273	0.27258
	0.15	0.16411	0.16450	0.16437
	0.20	0.09929	0.09954	0.09943
	0.25	0.06036	0.06042	0.06034

the exact solution is obtained by transforming them through the Hopf-Cole transformation [Cole (1951), Hopf (1950)]:

$$u = \frac{-2\phi_x}{R\phi}, \tag{40}$$

into the following heat diffusion equation, boundary conditions and initial condition:

$$\phi_t = \frac{1}{R}\phi_{xx}, \;\; 0 < x < 1, \;\; 0 < t < T,$$

$$\phi_x(0,t) = \phi_x(1,t) = 0,$$

$$\phi(x,0) = \exp\left[\int_0^x \frac{-R}{2}\sin\pi\xi d\xi\right] = \exp\left[\frac{R}{2\pi}(\cos\pi x - 1)\right].$$

Then, applying the method of separation of variables and the Fourier series method to the above linear equation we obtain

$$\phi(x,t) = a_0 + \sum_{k=1}^{\infty} a_k \exp\left[\frac{-(k\pi)^2 t}{R}\right]\cos(k\pi x), \tag{41}$$

where

$$a_0 = \exp\left[\frac{-R}{2\pi}\right]\int_0^1 \exp\left[\frac{R\cos\pi x}{2\pi}\right]dx, \tag{42}$$

$$a_k = 2\exp\left[\frac{-R}{2\pi}\right]\int_0^1 \exp\left[\frac{R\cos\pi x}{2\pi}\right]\cos(k\pi x)dx. \tag{43}$$

Table 5 : Comparison of exact solutions with the numerical results calculated by GPS and the Galerkin finite element method for $R = 10$ of Problem 2.

x	t	Galerkin	GPS	Exact
0.25	0.4	0.31429	0.30889	0.30889
	0.6	0.24373	0.24077	0.24074
	0.8	0.19758	0.19573	0.19568
	1.0	0.16391	0.16264	0.16256
	3.0	0.02743	0.02725	0.02720
0.5	0.4	0.57636	0.56988	0.56963
	0.6	0.45169	0.44745	0.44721
	0.8	0.36245	0.35948	0.35924
	1.0	0.29437	0.29215	0.29192
	3.0	0.04057	0.04028	0.04021
0.75	0.4	0.62952	0.62605	0.62544
	0.6	0.49034	0.48778	0.48721
	0.8	0.37713	0.37438	0.36392
	1.0	0.29016	0.28784	0.28747
	3.0	0.01334	0.02983	0.02977

Substituting Eq. (41) for ϕ into Eq. (40) we can obtain the solution for u:

$$u(x,t) = \frac{2\pi\sum_{k=1}^{\infty} k a_k \exp[-(k\pi)^2 t/R]\sin(k\pi x)}{Ra_0 + R\sum_{k=1}^{\infty} a_k \exp[-(k\pi)^2 t/R]\cos(k\pi x)}. \tag{44}$$

Özis, Aksan and Özdes (2003) have employed the Galerkin finite element method to calculate this problem with $R = 1$, $\Delta x = 0.0125$ and $\Delta t = 10^{-5}$ sec. We first calculate this problem by using $\Delta x = 0.025$ and $\Delta t = 10^{-4}$ sec. The numerical results are then compared with those of Özis, Aksan and Özdes (2003) and with the exact solutions calculated from Eq. (44) at three different grid points of $x = 0.25, 0.5, 0.75$ and at four different times of $t = 0.1, 0.15, 0.2, 0.25$ sec in Table 4. It can be seen that our method is more accurate than that of the Galerkin finite element method, even our time stepsize is larger and the number of grid points is much fewer than that used by Özis, Aksan and Özdes (2003).

Next we consider the case $R = 10$. Comparisons made in Table 5 are the exact solutions, the numerical solutions by Özis, Aksan and Özdes (2003) with $\Delta x = 0.0125$ and $\Delta t = 10^{-4}$ sec, and GPS solutions with $\Delta x = 0.025$ and $\Delta t = 10^{-3}$ sec. The solutions are compared at three different grid points of $x = 0.25, 0.5, 0.75$ and at five dif-

Table 6 : Comparison of exact solutions with the numerical results calculated by GPS, RK4 and the exact-explicit finite difference method for $R = 100$ of Problem 2.

x	t	Exact-explicit	RK4	GPS	Exact
0.25	0.4	0.34164	0.34197	0.34193	0.34191
	0.6	0.26890	0.26900	0.26897	0.26896
	0.8	0.22150	0.22151	0.22149	0.22148
	1.0	0.18825	0.18821	0.18820	0.18819
	3.0	0.07515	0.07512	0.07511	0.07511
0.5	0.4	0.65606	0.66083	0.66079	0.66071
	0.6	0.52658	0.52950	0.52946	0.52942
	0.8	0.43743	0.43919	0.43916	0.43914
	1.0	0.37336	0.37446	0.37443	0.37442
	3.0	0.15015	0.15019	0.15018	0.15018
0.75	0.4	0.90111	0.91053	0.91058	0.91026
	0.6	0.75862	0.76741	0.76739	0.76724
	0.8	0.64129	0.64750	0.64747	0.64740
	1.0	0.55187	0.55620	0.55609	0.55605
	3.0	0.22454	0.22484	0.22483	0.22481

Table 7 : Comparison of exact solutions with the numerical results calculated by GPS, RK4 and the exact-explicit finite difference method for $R = 100$ of Problem 3.

x	t	Exact-explicit	RK4	GPS	Exact
0.25	0.4	0.36185	0.36260	0.36224	0.36226
	0.6	0.28193	0.28229	0.28201	0.28204
	0.8	0.23046	0.23064	0.23042	0.23045
	1.0	0.19474	0.19483	0.19466	0.19469
	3.0	0.07617	0.07616	0.07612	0.07613
0.5	0.4	0.67851	0.68412	0.68385	0.68368
	0.6	0.54508	0.54867	0.54831	0.54832
	0.8	0.45176	0.45399	0.45366	0.45371
	1.0	0.38446	0.38590	0.38561	0.38568
	3.0	0.15215	0.15223	0.15215	0.15218
0.75	0.4	0.91169	0.92128	0.92179	0.92050
	0.6	0.77402	0.78355	0.78349	0.78299
	0.8	0.65617	0.66311	0.66286	0.66272
	1.0	0.56478	0.56961	0.56932	0.56932
	3.0	0.22746	0.22789	0.22778	0.22774

ferent times of $t = 0.4, 0.6, 0.8, 1, 3$ sec. It can be seen that our results are more accurate than that calculated by the Galerkin finite element method, even our time stepsize and grid spacing length are larger than that used by Özis, Aksan and Özdes (2003). Kutluay, Bahadir and Özdes (1999) have calculated this problem by considering the explicit and exact-explicit finite difference methods for the transformed heat diffusion equation from the Burgers equation, and then converted the finite difference solutions of heat diffusion equation to the numerical solutions of the Burgers equation by the Hopf-Cole transformation. The accuracy of our numerical solutions are compatible with the numerical solutions by Kutluay, Bahadir and Özdes (1999). However, their numerical solutions require a lot of computations of integrals and the sum of infinite series.

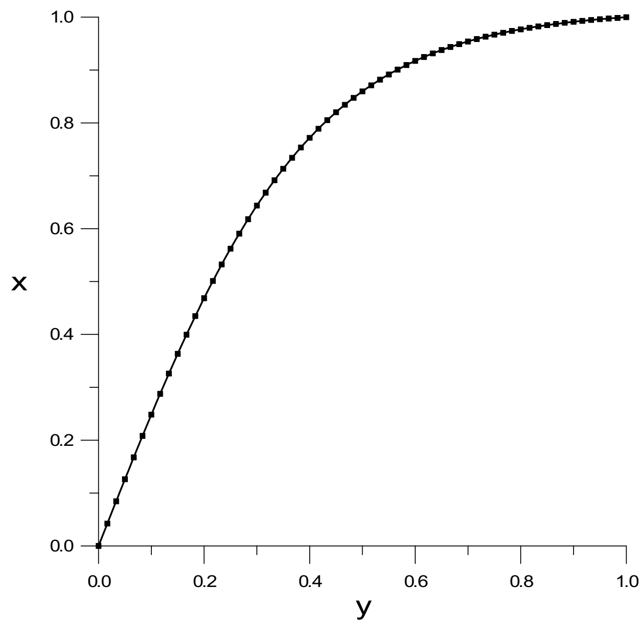

Figure 3 : The relation between x and y with $A = 2.5$.

Then we consider the third case $R = 100$. Comparisons made in Table 6 are the exact solutions, the numerical solutions by Kutluay, Bahadir and Özdes (1999) with exact-explicit finite-difference scheme with $\Delta x = 0.0125$ and $\Delta t = 10^{-4}$ sec, and GPS and RK4 solutions with the same Δx and Δt. The solutions are compared at three different grid points of $x = 0.25, 0.5, 0.75$ and at five different times of $t = 0.4, 0.6, 0.8, 1, 3$ sec. The accuracy of our numerical solutions is better than that calculated by RK4 and is much better than that calculated by Kutluay, Bahadir and Özdes (1999). For this case, their numerical solutions converge slowly.

Through the above comparisons of GPS solutions with the exact solutions for three cases of $R = 1, 10, 100$, it can be seen that GPS is effective for moderately high Reynolds number, and the computational accuracy increases when R increases.

When R is more large, we prefer to consider a scalar transformation of x-coordinate:

$$x = \frac{\tanh(Ay)}{\tanh A}, \tag{45}$$

or its inverse:

$$y = \frac{1}{2A} \ln \frac{1 + x \tanh A}{1 - x \tanh A}. \tag{46}$$

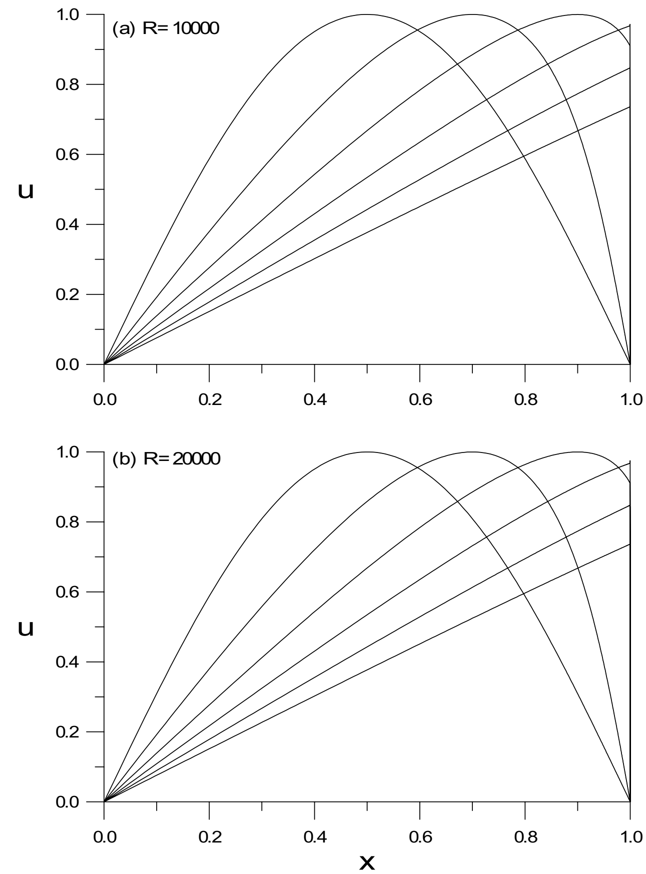

Figure 4 : The GPS solutions of Burgers' equation for (a) $R = 10000$; (b) $R = 20000$.

It can accumulate much grid points in the region where the solution appears large variation, and place a small number of grid points in the region where the solution does not change rapidly. To demonstrate this effect, we have plotted the relation of x and y in Fig. 3 for a given $A = 2.5$. It can be seen that when the grid points are uniformly distributed in the y-coordinate, there appear much grid points near the end $x = 1$ in the x-coordinate, wherein the solution of the Burgers equation with high Reynolds number exhibits large variation.

From Eqs. (1) and (46) it follows that

$$\begin{aligned} u_t &+ \frac{\tanh A}{A[1-\tanh^2(Ay)]} u u_y \\ &= \frac{1}{R}\left[\frac{\tanh^2 A}{A^2[1-\tanh^2(Ay)]^2} u_{yy}\right] \\ &+ \frac{1}{R}\left[\frac{2\tanh^2 A \tanh(Ay)}{A[1-\tanh^2(Ay)]^2} u_y\right]. \end{aligned} \tag{47}$$

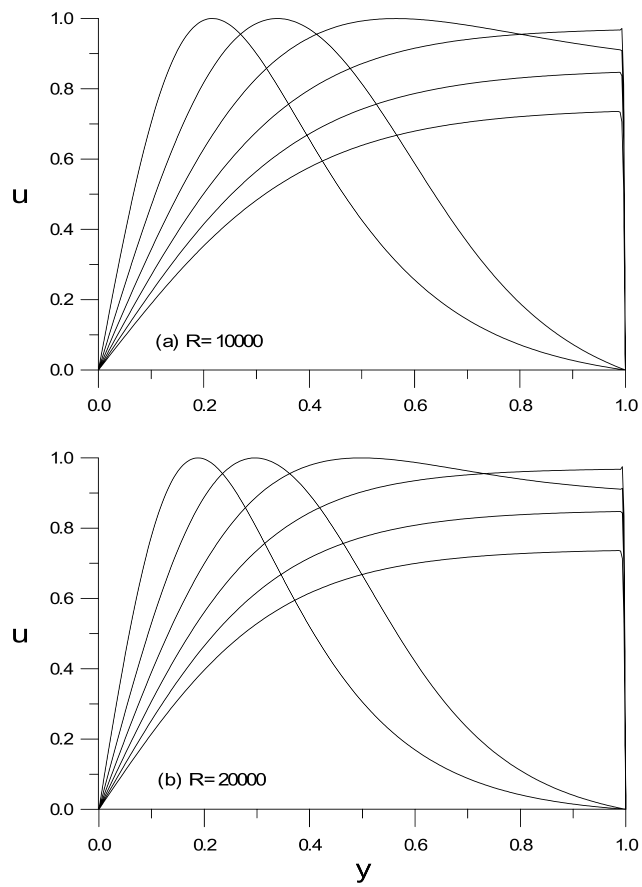

Figure 5 : The GPS solutions of Burgers' equation for (a) $R = 10000$; (b) $R = 20000$ in the y-domain.

Applying the GPS on the above equation by a uniform grid spacing length of $\Delta y = 1/(n+1)$, we can integrate it for large R. Figure 4 displays the numerical results for two cases of $R = 10000, 20000$. The parameters used in these calculations are $n = 99$, $\Delta t = 0.0001$ sec and $A = 3, 3.5$. The six curves represent the solutions at six different times at $t = 0, 0.2, 0.4, 0.6, 0.8, 1$ sec. For the comparison purpose we also plotted these curves in the y-domain as shown in Fig. 5. The very sharp variations as seen in the x-domain are now released in y-domain.

As reported by Ames (1992), in order to get a stable solution of the Burgers' equation with $R = 10000$ by using the predictor-corrector method, it requires 200 grid points and with a stepsize 4×10^{-4} sec. This method is however more complicated than our method.

4.3 *Problem 3*

Let us consider the Burgers equation (1) with the following boundary conditions and initial condition:

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = 4x(1-x).$$

The exact solution can be obtained by a similar way in the previous problem but with the following Fourier coefficients:

$$a_0 = \int_0^1 \exp\left[\frac{-Rx^2}{3}(3-2x)\right] dx, \tag{48}$$

$$a_k = 2\int_0^1 \exp\left[\frac{-Rx^2}{3}(3-2x)\right] \cos(k\pi x) dx, \quad k = 1, 2, 3, \ldots. \tag{49}$$

Comparisons made in Table 7 are exact solutions, the numerical solutions by Kutluay, Bahadir and Özdes (1999) with $\Delta x = 0.0125$ and $\Delta t = 10^{-3}$ sec, and GPS and RK4 solutions with $\Delta x = 0.025$ and the same Δt. The solutions are compared at three different grid points of $x = 0.25, 0.5, 0.75$ and at five different times of $t = 0.4, 0.6, 0.8, 1, 3$ sec. The accuracy of our numerical solutions by GPS is much better than that calculated by Kutluay, Bahadir and Özdes (1999), and is also better than that calculated by RK4.

4.4 *Problem 4*

In this section we extend Problem 1 to a two-dimensional case with the following boundary conditions and initial condition:

$$u(0,y,t) = \frac{1}{1+\exp[Ry/2 - Rt/2]}, \quad 0 \le y \le 1,$$

$$u(1,y,t) = \frac{1}{1+\exp[R/2 + Ry/2 - Rt/2]}, \quad 0 \le y \le 1,$$

$$u(x,0,t) = \frac{1}{1+\exp[Rx/2 - Rt/2]}, \quad 0 \le x \le 1,$$

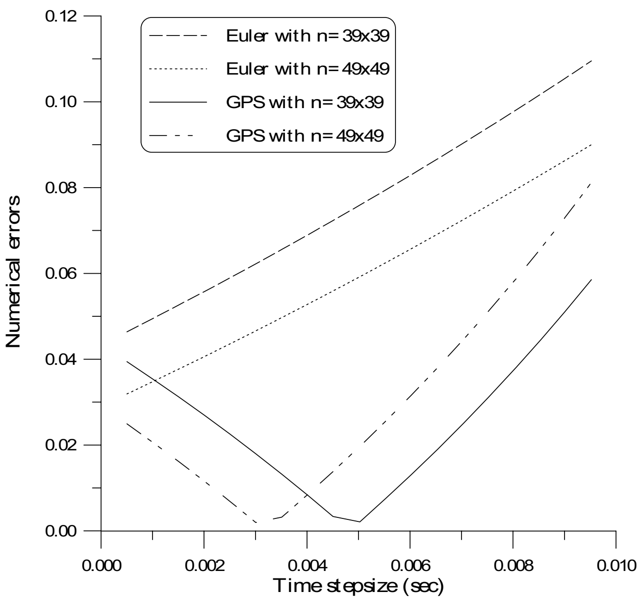

Figure 6 : The numerical errors of GPS and Euler scheme with respect to time stepsize for Problem 4.

$$u(x,1,t)=\frac{1}{1+\exp[R/2+Rx/2-Rt/2]},\ \ 0\le x\le 1,$$

$$u(x,y,0)=\frac{1}{1+\exp[Rx/2+Ry/2]},\ \ 0\le x\le 1,\ \ 0\le y\le 1.$$

The exact solution [Schiesser (1991)] is given by

$$u(x,y,t)=\frac{1}{1+\exp[Rx/2+Ry/2-Rt/2]}. \tag{50}$$

We can apply the GPS on the following discretization:

$$\begin{aligned}\dot{u}_{i,j}(t)=&\frac{u_{i+1,j}(t)-2u_{i,j}(t)+u_{i-1,j}(t)}{R(\Delta x)^2}\\&+\frac{u_{i,j+1}(t)-2u_{i,j}(t)+u_{i,j-1}(t)}{R(\Delta y)^2}\\&-\overline{u}_{i,j}(t)\left[\frac{u_{i+1,j}(t)-u_{i-1,j}(t)}{2\Delta x}+\frac{u_{i,j+1}(t)-u_{i,j-1}(t)}{2\Delta y}\right]\end{aligned} \tag{51}$$

by considering the average $\overline{u}_{i,j}:=(u_{i+1,j}+u_{i,j+1}+u_{i,j}+u_{i-1,j}+u_{i,j-1})/5$, where $\Delta x=\Delta y=1/(N+1)$ is a uniform grid spacing length, and $u_{i,j}(t):=u(i\Delta x, j\Delta y, t)$ for simplicity. There are totally $n=N\times N$ variables $u_{i,j}(t)$ at the totally $N\times N$ interior grid points.

We have compared the numerical solutions with the exact solution at $x=0.5$, $y=0.5$ and $t=1$ sec, of which $u=0.5$ for all R by Eq. (50). Fixing $R=100$ and varying the time stepsizes in the range of 0.001-0.01 sec, we plot the numerical errors in Fig. 6 with the solid line for GPS and the dashed line for the Euler scheme with both 39×39 interior grid points ($\Delta x=\Delta y=0.025$), and the dashed-dotted line for GPS and the dotted line for the Euler scheme with both 49×49 interior grid points ($\Delta x=\Delta y=0.02$). In all cases GPS is much better than the Euler scheme. It is only in the limiting case with an almost zero time stepsize that the Euler scheme is comparable with GPS.

By letting $n=39\times 39$, $\Delta t=2\times 10^{-3}$ sec, $R=150$, and $T=1$ sec, the numerical errors being the differences of numerical solutions and exact solutions were plotted in Fig. 7(a) with the solid lines for GPS and the dashed lines for the Euler scheme at three different times of $t=0.6,0.8,1$ sec and at point $y=0.5$ along the x-axis, while the differences were plotted in Fig. 7(b) with the solid lines for GPS and the dashed lines for the Euler scheme at three different times of $t=0.6,0.8,1$ sec and at point $x=0.25$ along the y-axis. It can be seen that GPS is much better than the Euler scheme.

5 Conclusions

The Burgers equation was calculated by a semi-discretization of the spatial coordinates in conjuction with the group preserving numerical integration scheme. We have taken the dissipative behavior of Burgers equation into account by considering the magnitude $\|\mathbf{u}\|$ as another component, and according to the Lie algebra property we have developed a novel numerical scheme with the transformation matrix $\mathbf{G}$ being a proper orthochronous Lorentz group. Under certain condition on the time stepsize we have proved that the resulting numerical scheme is a contraction mapping, which is congruent with the dissipation behavior of Burgers equation. The computational accuracy of GPS was confirmed by comparing its numerical results with those of other numerical methods and closed-form solutions for several numerical examples. Even for very high Reynolds number our scheme was applicable under a reasonable grid spacing length and time stepsize. The numerical results indicate that the group preserving scheme is efficient to numerically integrating the Burgers

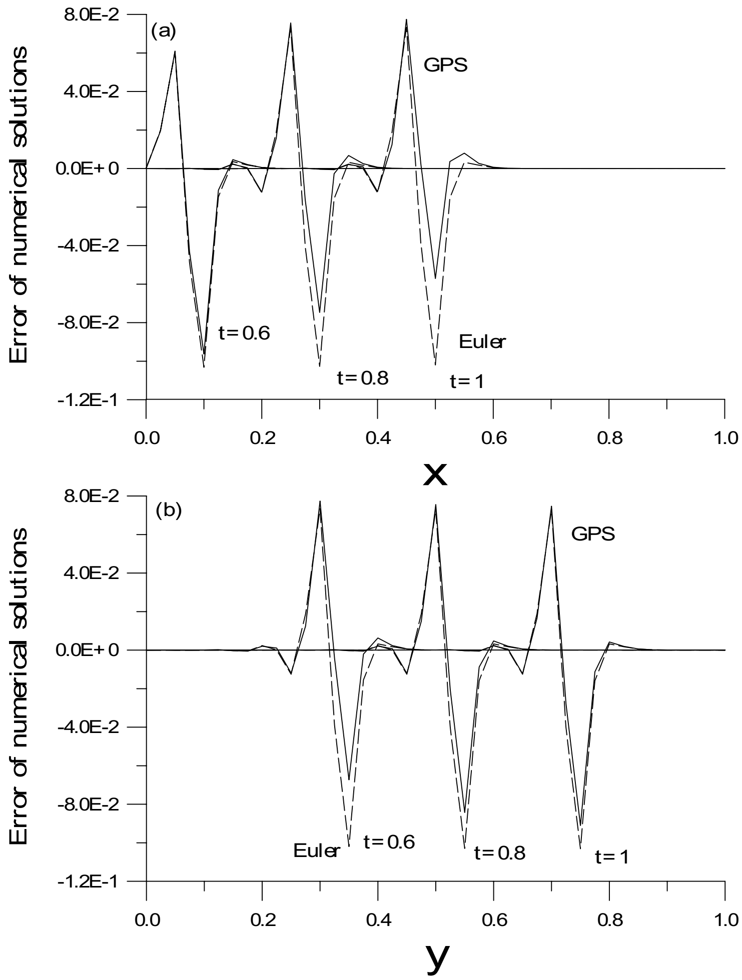

Figure 7 : Comparison of numerical solutions with exact solutions for Problem 4: (a) numerical errors at three different times along x-direction, and (b) numerical errors at three different times along y-direction.

equation. Numerical tests indicated that GPS is also better than RK4. In the plane of numerical error vs. time stepsize an L-curve appeared for GPS, and at the tip point a best time stepsize would make the numerical error minimal. We have verified this point by 1D and 2D Burgers' equations. The particular behavior of GPS is very different from the conventional numerical methods, *e.g.*, the Euler scheme and RK4, of which the numerical errors are reduced when the time stepsizes are decreased. Because under a reasonable grid spacing length and time stepsize, the GPS produced almost the exact values as shown in Tables 2, 5-7, and its implementation is very easy, it is highly recommended to be used in the numerical computations of the Burgers' equation.

References

Ames, W. F. (1992): Numerical Methods for Partial Differential Equations. 3rd ed., Academic Press, New York.

Aref, H.; Daripa, P. K. (1984): Note on finite difference approximations to Burgers' equation. *SIAM J. Sci. Stat. Comp.*, vol. 5, pp. 856-864.

Arina, R.; Canuto, C. (1993): A self-adaptive domain decomposition for the viscous/inviscid coupling. I. Burgers' equation. *J. Comp. Phys.*, vol. 105, pp. 290-300.

Basdevant, C.; Deville, M.; Haldenwang, P. (1986): Spectral and finite difference solutions of the Burgers equation. *Comp. Fluids*, vol. 14, pp. 23-41.

Burgers, J. M. (1948): A mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.*, vol. 1, pp. 171-199.

Burgers, J M. (1974): The Nonlinear Diffusion Equation. Reidel, Boston.

Byrne, G. D.; Hindmarsh, A. C. (1987): Stiff ODE solvers: a review of current and coming attractions. *J. Comp. Phys.*, vol. 70, pp. 1-62.

Chang, C.-W.; Liu, C.-S.; Chang, J.-R. (2005): A group preserving scheme for inverse heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 10, pp. 13-38.

Cole, J. D. (1951): On a quasi linear parabolic equation occurring in aerodynamics. *Quart. Appl. Math.*, vol. 9, pp. 225-236.

Fletcher, C. A. J. (1983): A comparison of finite element and finite difference solutions of the one- and two-dimensional Burgers' equations. *J. Comp. Phys.*, vol. 51, pp. 159-188.

Hon, Y. C.; Mao, X. Z. (1998): An efficient numerical scheme for Burgers' equation. *Appl. Math. Comp.*, vol. 95, 37-50.

Hopf, E. (1950): The partial differential equation $u_t + uu_x = u_{xx}$. *Comm. Pure. Appl. Math.*, vol. 3, pp. 201-230.

Kutluay, S.; Bahadir, A. R.; Özdes, A. (1999): Numerical solution of one-dimensional Burgers equation: explicit and exact-explicit finite difference methods. *J. Comp. Appl. Math.*, vol. 103, pp. 251-261.

Lin, E. B.; Zhou, X. (2001): Connection coefficients on an interval and wavelet solutions of Burgers equation. *J. Comp. Appl. Math.*, vol. 135, pp. 63-78.

Liu, C.-S. (2001): Cone of non-linear dynamical system and group preserving schemes. *Int. J. Non-Linear Mech.*, vol. 36, pp. 1047-1068.

Liu, C.-S. (2004): Group preserving scheme for backward heat conduction problems. *Int. J. Heat Mass Transf.*, vol. 47, pp. 2567-2576.

Liu, C.-S. (2005): Nonstandard group-preserving schemes for very stiff ordinary differential equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 9, pp. 255-272.

Liu, C.-S. (2006a): Preserving constraints of differential equations by numerical methods based on integrating factors. *CMES: Computer Modeling in Engineering & Sciences*, in press.

Liu, C.-S. (2006b): An efficient backward group preserving scheme for the backward in time Burgers equation. *CMES: Computer Modeling in Engineering & Sciences*, in press.

Özis, T.; Özdes, A. (1996): A direct variational methods applied to the Burgers' equation. *J. Comp. Appl. Math.*, vol. 71, pp. 163-175.

Özis, T.; Aksan, E. N.; Özdes, A. (2003): A finite element approach for solution of Burgers' equation. *Appl. Math. Comp.*, vol. 139, pp. 417-428.

Schiesser, W. E. (1991): The Numerical Method of Lines: Integration of Partial Differential Equations. Academic Press, San Diego.

Wei, G. W.; Gu, Y. (2002): Conjugate filter approach for solving Burgers' equation. *J. Comp. Appl. Math.*, vol. 149, pp. 439-456.

Young, D. L. (2005): An Eulerian-Lagrangian method of fundamental solutions for Burgers' equation, in Advances in the Meshless Method, eds. Sladek, J., Sladek, V. and Atluri, S. N., Tech Science Press, USA.

Young, D. L.; Hu, S. P.; Fan, C. M.; Chen, C. W. (2006): Methods of fundamental solutions and Cole-Hopf transformation for Burgers' equations. *Comp. Fluid Dynam. J.*, vol. 14, pp. 454-467.