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Linear Buckling Analysis of Shear Deformable Shallow Shells by the Boundary Domain Element Method

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Abstract: In this paper the linear buckling problem of elastic shallow shells by a shear deformable shell theory is presented. The boundary domain integral equations are obtained by coupling two dimensional plane stress elasticity with boundary element formulation of Reissner plate bending. The buckling problem is formulated as a standard eigenvalue problem, in order to obtain directly critical loads and buckling modes as part of the solution. The boundary is discretised into quadratic isoparametric elements while in the domain quadratic quadrilateral cells are used. Several examples of cylindrical shallow shells (curved plates) with different dimensions and boundary conditions are analysed. The results are compared with finite element solutions, and very good agreement is obtained.

keyword: Shallow Shell, Buckling, Shear Deformable Theory, Boundary Element Method.

1 Introduction

The behavior of curved plates under compression loads is of major concern in areas such as aerospace, in which the design requirements of weight critical applications usually leads to thin panels with stability problems.

The first study of stability of shells can be traced back in 1911 by Lorenz, in which he presented solutions for cylinders subjected to axial compression. More complete descriptions of shell stability have been presented by Timoshenko and Gere (1961) for several classical problems, and by Brush and Almorth (1975) for nonlinear theories. Other works dealing with shell buckling can be found in Gerard and Becker (1957), giving an overall view of shell buckling problems; Nash's review [Nash (1966)] with several hundred papers on shell buckling; Singer (1982) who reports on experimental investigations and Bushnell (1985) who concentrates on numerical procedures for the modeling and solution of complex nonlinear problems. Other more recent and useful bibliographies and review papers can be found in works by Noor (1990), Teng (1996) and Knight and Starnes (1997).

The applications of the Boundary Element Method (BEM) to stability problems for plate and shallow shell structures have been investigated since the 80's. Manolis, Beskos, and Pineros (1986) developed a direct boundary element formulation dealing with linear elastic stability analysis of Kirchhoff plates. More recently, Syngellakis and Elzein (1994) presented an extended boundary element formulation to incorporate any combination of loading and support conditions; Nerantzaki and Katsikadelis (1996) presented a boundary element formulation for buckling of plates with variable thickness; and Lin, Duffield, and Shih (1999) described a general boundary element formulation for different boundary conditions and arbitrary planar shapes.

In the case of post-buckling formulations for thin elastic plates O'Donoghue and Atluri (1987) introduced the first boundary element approach to nonlinear plate analysis; while for thin shallow shells, Zhang and Atluri (1988) presented a boundary element formulation applied to the analysis of snap-through phenomena. In all these cases, plate and shallow shell BEM formulations have used the Classical or Kirchhoff-Love theory.

Although for most practical applications the classical theory is sufficient; it has been shown by Reissner (1947) that Kirchhoff theory of thin plates is not in agreement with the experimental results for problems with stress concentrations (stresses at an edge of a hole when the hole diameter became so small as to be of the order of magnitude of the plate thickness); or also in the case of composite shells, where the ratio of Young's modulus to shear modulus can be very large (low transverse shear modulus compared to isotropic materials). Therefore, shell theories accounting transverse shear deformation overcome problems associated with the application of the classical theory, and additionally can be used for the analysis of thin and thick shells.

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Recent developments with shear deformable plate theory by BEM include de works of Purbolaksono and Aliabadi (2005b) for large deformation; Wen, Aliabadi, and Young (2006) for post-buckling; and Supriyono and Aliabadi (2006) for combined large deflection and plasticity.

Application of shear deformable theory to the linear elastic buckling problem of plates using the boundary element method has been presented recently by Purbolaksono and Aliabadi (2005a); and to the best knowledge of the authors, no study on the linear elastic buckling analysis of shallow shell structures by BEM has been reported, for classical (Kirchhoff-Love) or shear deformable (Reissner or Mindlin) theories. Other works on plates and shells by BEM can be found in Beskos (1991), and recent advances in BEM and their solid mechanics applications are in Aliabadi (2002).

The present paper reports on the investigation of a new boundary domain element formulation for the buckling analysis of shear deformable shallow shells. Initially, basic concepts of shear deformable shallow shells, and boundary integral equations are described. The buckling problem is formulated as a standard eigenvalue problem, to provide direct evaluation of critical load factors and buckling modes. Numerical procedure to solve the formulation is presented, first the in plane stresses at domain points are calculated and subsequently the boundary integral equations for the buckling problem are solved. Several examples in which results from the proposed BEM formulation are compared with FEM results and good agreements are obtained.

2 Boundary domain integral equations for shear deformable shallow shells

Consider a shallow shell of an isotropic linear elastic material, with uniform thickness h, Young's modulus E, Poisson ratio v and shear modulus $G=E/2(1+v)$, with a quadratic middle surface defined by R_1 and R_2, which are principal curvatures of the shell in the x_1- and x_2- directions, respectively. The indicial notation used throughout this paper is as follows: the Greek indices (α,β,γ) will vary from 1 to 2 and Roman indices (i,j,k) from 1 to 3.

Equilibrium equations for shear deformable plate bending (Reissner-Mindlin) and 2D elasticity for shallow shells can be written in indicial notation as follows Aliabadi (2002):

$$M_{\alpha\beta,\beta}-Q_\alpha=0; \tag{1}$$

$$Q_{\alpha,\alpha}-k_{\alpha\beta}N_{\alpha\beta}+q_3=0; \tag{2}$$

$$N_{\alpha\beta,\beta}+q_\alpha=0 \tag{3}$$

where $k_{11}=1/R_1$, $k_{22}=1/R_2$ and $k_{12}=k_{21}=0$; $(\ )_{,\beta}=\partial(\ )/\partial x_\beta$. $N_{\alpha\beta}$ denote membrane stresses, $M_{\alpha\beta}$ represent bending moments, Q_α is the shear forces for plate bending and q_i are the body forces.

Generalized displacements are represented as w_i and u_α, where w_α denotes rotations of the middle surface (w_1 and w_2), w_3 denotes the out-of-plane displacement, and u_α denotes in-plane displacements (u_1 and u_2). The generalized tractions are denoted as p_i and t_α, where p_α denotes tractions due to the stress couples (p_1 and p_2), p_3 denotes the traction due to shear stress resultant (p_3) and t_α denotes tractions due to membrane stress resultants (t_1 and t_2) as shown in figure 1.

The constitutive equations based on Reissner's variational theorem of elasticity Reissner (1950) can be written as follows:

$$M_{\alpha\beta}=D\frac{1-\nu}{2}\left(w_{\alpha,\beta}+w_{\beta,\alpha}+\frac{2\nu}{1-\nu}w_{\gamma,\gamma}\delta_{\alpha\beta}\right) \tag{4}$$

$$Q_\alpha=C(w_\alpha+w_{3,\alpha}) \tag{5}$$

$$N_{\alpha\beta}=B\frac{1-\nu}{2}\left(u_{\alpha,\beta}+u_{\beta,\alpha}+\frac{2\nu}{1-\nu}u_{\gamma,\gamma}\delta_{\alpha\beta}\right)+B\left[(1-\nu)k_{\alpha\beta}+\nu\delta_{\alpha\beta}k_{\phi\phi}\right]w_3 \tag{6}$$

where $B(=\ Eh/\left(1-\nu^2\right))$ is the tension stiffness; $D(=\ Eh^3/\left[12\left(1-\nu^2\right)\right])$ is the bending stiffness; $C(=\left[D\left(1-\nu\right)\lambda^2\right]/2)$ is the shear stiffness; $\lambda=\sqrt{10}/h$ is called the shear factor; and $\delta_{\alpha\beta}$ is the Kronecker delta function. The term $N_{\alpha\beta}$ is separated into $N_{\alpha\beta}^{(i)}$, which are due to in-plane displacements, and $N_{\alpha\beta}^{(ii)}$, which are due to curvature and out-of-plane displacements.

The integral equations for shear deformable shallow shell problems are derived by considering the integral representations of the governing equations (1)-(3) from the following integral identities:

$$\int_\Omega\left[\left(M_{\alpha\beta,\beta}-Q_\alpha\right)W_\alpha^*+\left(Q_{\alpha,\alpha}-k_{\alpha\beta}N_{\alpha\beta}+q_3\right)W_3^*\right]d\Omega=0 \tag{7}$$

and

$$\int_\Omega\left(N_{\alpha\beta,\beta}+q_\alpha\right)U_\alpha^*d\Omega=0 \tag{8}$$

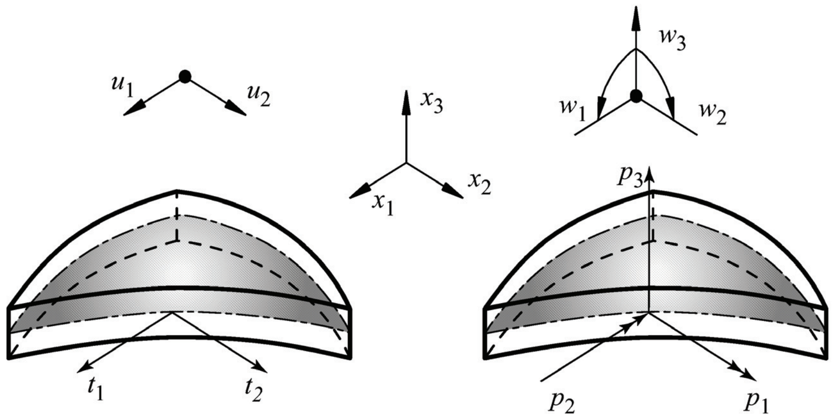

Figure 1 : Sign convection for generalised displacement and tractions.

where U^*_α and $W^*_i(i=\alpha,3)$ are weighting functions and Ω is the projected domain of a shell on x_1-x_2 plane, bounded by boundary Γ. Equation (7) is an integral representation related to the governing equations for bending and transverse shear stress resultants, while equation (8) is an integral representation related to the governing equations for membrane stress resultants.

2.1 Rotations and out of plane integral equations

The boundary domain integral representation related to the governing equations for bending and transverse shear stress resultants of a boundary source point $\mathbf{x}'$ are derived by using the weighted residual method as shown in Dirgantara and Aliabadi (1999) and Dirgantara (2002). After taking into account all the limits and the jump terms:

$$
\begin{aligned}
&c_{ij}(\mathbf{x}')w_j(\mathbf{x}') + ⨍_\Gamma P^*_{ij}(\mathbf{x}',\mathbf{x})w_j(\mathbf{x})\,d\Gamma(\mathbf{x}) \\
&= \int_\Gamma W^*_{ij}(\mathbf{x}',\mathbf{x})p_j(\mathbf{x})\,d\Gamma(\mathbf{x}) \\
&- \int_\Gamma W^*_{i3}(\mathbf{x}',\mathbf{x})\,k_{\alpha\beta}B\frac{1-\nu}{2}\times \\
&\left[u_\alpha(\mathbf{x})n_\beta + u_\beta(\mathbf{x})n_\alpha + \frac{2\nu}{1-\nu}u_\gamma(\mathbf{x})n_\gamma\delta_{\alpha\beta}\right]d\Gamma(\mathbf{x}) \\
&+ \int_\Omega k_{\alpha\beta}B\frac{1-\nu}{2}\left[u_\alpha(\mathbf{X})\,W^*_{i3,\beta}(\mathbf{x}',\mathbf{X})\right. \\
&+ u_\beta(\mathbf{X})\,W^*_{i3,\alpha}(\mathbf{x}',\mathbf{X}) \\
&\left.+ \frac{2\nu}{1-\nu}u_\gamma(\mathbf{X})\,W^*_{i3,\gamma}(\mathbf{x}',\mathbf{X})\,\delta_{\alpha\beta}\right]d\Omega(\mathbf{X}) \\
&- \int_\Omega W^*_{i3}(\mathbf{x}',\mathbf{X})\,k_{\alpha\beta}B\times \\
&\left[(1-\nu)k_{\alpha\beta}+\nu\delta_{\alpha\beta}k_{\gamma\gamma}\right]w_3(\mathbf{X})\,d\Omega(\mathbf{X}) \\
&+ \int_\Omega W^*_{i3}(\mathbf{x}',\mathbf{X})q_3(\mathbf{X})d\Omega(\mathbf{X})
\end{aligned}
\tag{9}
$$

where $⨍$ denotes a Cauchy principal value integral, $\mathbf{x}',\mathbf{x}\in\Gamma$, $\mathbf{X}\in\Omega$ are source and field points respectively, $c_{ij}(\mathbf{x}')$ are the jump terms, n_β are the components of the outward normal vector to the shell boundary. The value of $c_{ij}(\mathbf{x}')$ is equal to $\frac{1}{2}\delta_{ij}$ when $\mathbf{x}'$ is located on a smooth boundary and equal to δ_{ij} when collocation is at domain points $\mathbf{X}$.

W^*_{ij} and P^*_{ij} are the displacement and traction fundamental solutions respectively, derived by Vander Weeen (1982) and $W^*_{i3,\beta}$ is the derivative of the displacement fundamental solutions with respect to the field point $\mathbf{X}$. All these kernels are given in appendix A.

2.2 In plane displacement integral equations

In the same way, the boundary domain integral equation related to the governing equations for membrane stress resultants of a boundary source point $\mathbf{x}'$ can be written as Dirgantara and Aliabadi (1999), Dirgantara (2002):

$$
\begin{aligned}
&c_{\theta\alpha}(\mathbf{x}')\,u_\alpha(\mathbf{x}') + ⨍_\Gamma T^{*(i)}_{\theta\alpha}(\mathbf{x}',\mathbf{x})u_\alpha(\mathbf{x})d\Gamma(\mathbf{x}) \\
&+ \int_\Omega U^*_{\theta\alpha,\beta}(\mathbf{x}',\mathbf{X})B\left[k_{\alpha\beta}(1-\nu)+\nu\delta_{\alpha\beta}k_{\gamma\gamma}\right]w_3(\mathbf{X})d\Omega(\mathbf{X}) \\
&= \int_\Gamma U^*_{\theta\alpha}(\mathbf{x}',\mathbf{x})t_\alpha(\mathbf{x})d\Gamma(\mathbf{x}) + \int_\Omega U^*_{\theta\alpha}(\mathbf{x}',\mathbf{X})q_\alpha(\mathbf{X})d\Omega(\mathbf{X})
\end{aligned}
\tag{10}
$$

where $U^*_{\theta\alpha}$ and $T^{(i)*}_{\theta\alpha}$ are the well known fundamental solutions for in-plane displacements and membrane tractions respectively, while $U^*_{\theta\alpha,\beta}$ is the derivative of the displacement fundamental solution with respect to the field point $\mathbf{X}$. These kernels are also given in appendix A. The upper index (i) on $T^*_{\theta\alpha}$ refers to the in plane displacement, as it was explained with $N^{(i)}_{\alpha\beta}$.

Equations (9) and (10) represent the five boundary-domain integral equations for shear deformable shallow shell theory, the first two are in (9) $(i=1,2)$ and are for rotations (w_1 and w_2), the third $(i=3)$ also in (9) is for the out-of-plane displacement (w_3). The last two are in (10) $(\theta=1,2)$ and are for in-plane displacements (u_1 and u_2).

It is important to mention that due to the curvature terms (containing $k_{\alpha\beta}$), equations (9) and (10) have to be solved simultaneously and not only for collocation on boundary points $\mathbf{x}$ but also on domain points $\mathbf{X}$.

3 Governing integral equations for linear buckling of shear deformable shallow shells

In this section the buckling phenomenon of shallow shells is studied, initially the membrane stress resultants in the domain are considered to be unknown due to external loads on the boundary; therefore, determination of membrane stress resultants in the domain is the first step solution for the analysis. Next, the shell buckling equations are obtained by introducing multiplication factors of body forces or transverse loads (λ) in the governing integral equations.

3.1 Integral equations for in plane stresses

The membrane stress resultants at domain points $\mathbf{X}'$ can be evaluated from the derivative of equation (10) and by using the relationship in equation (6), resulting in the following boundary-domain integral equation:

$$
\begin{aligned}
N_{\alpha\beta}\left(\mathbf{X}'\right) &= \int_\Gamma U^*_{\alpha\beta\gamma}(\mathbf{X}',\mathbf{x})t_\gamma(\mathbf{x})d\Gamma(\mathbf{x}) \\
&-\int_\Gamma T^*_{\alpha\beta\gamma}(\mathbf{X}',\mathbf{x})u_\gamma(\mathbf{x})d\Gamma(\mathbf{x}) \\
&-\int_\Omega U^*_{\alpha\beta\gamma,\theta}(\mathbf{X}',\mathbf{X})B\left[k_{\gamma\theta}(1-\nu)+\nu\delta_{\gamma\theta}k_{\phi\phi}\right]\times \\
&w_3(\mathbf{X})d\Omega(\mathbf{X})+f_{\alpha\beta}(w_3\left(\mathbf{X}'\right)) \\
&+\int_\Omega U^*_{\alpha\beta\gamma}(\mathbf{X}',\mathbf{X})q_\gamma(\mathbf{X})d\Omega(\mathbf{X}) \\
&+B\left[(1-\nu)k_{\alpha\beta}+\nu\delta_{\alpha\beta}k_{\phi\phi}\right]w_3\left(\mathbf{X}'\right)
\end{aligned}
\tag{11}
$$

The kernels $U^*_{\alpha\beta\gamma}$ and $T^*_{\alpha\beta\gamma}$ in equation (11) are linear combination of the first derivatives of $U^*_{\alpha\beta}$ and $T^*_{\alpha\beta}$ with respect to the source point $\mathbf{X}'$ and can be found in Dirgantara and Aliabadi (1999), Dirgantara (2002) and are also listed in appendix A. $U^*_{\alpha\beta\gamma,\theta}$ is the derivative of $U^*_{\alpha\beta\gamma}$ with respect to the field point $\mathbf{X}$ and is given in appendix A.

The term $f_{\alpha\beta}$ in equation (11) arises from the integration of the curvature term over the surface Γ' centered at the load point $\mathbf{X}'$. Details of the procedure to obtain this term are given in appendix B.

Another approach that could be used for the evaluation of stresses at internal points consist of a numerical differentiation of displacements by means of the shape functions of the domain cells, after internal displacement have been found, Zhang and Atluri (1988). The boundary-domain integral equation (11) although computational more time consuming and mathematically more cumbersome, gives more accurate results and therefore it was adopted in this work.

3.2 Integral formulation for the linear buckling problem

Appropriate forms of the linearized buckling problem can be derived by transforming the shell integral equation (9) into an equivalent shell buckling formulation and introducing a critical load factor λ, resulting in a group of equation in terms of the prebuckling membrane stresses and the buckled shell displacements, as follows:

$$
\begin{aligned}
&c_{ij}(\mathbf{x}')w_j(\mathbf{x}')+\fint_\Gamma P^*_{ij}(\mathbf{x}',\mathbf{x})w_j(\mathbf{x})\,d\Gamma(\mathbf{x}) \\
&=\int_\Gamma W^*_{ij}(\mathbf{x}',\mathbf{x})p_j(\mathbf{x})\,d\Gamma(\mathbf{x}) \\
&-\int_\Gamma W^*_{i3}\left(\mathbf{x}',\mathbf{x}\right)k_{\alpha\beta}B\frac{1-\nu}{2}\times \\
&\left[u_\alpha(\mathbf{x})\,n_\beta+u_\beta(\mathbf{x})\,n_\alpha+\frac{2\nu}{1-\nu}u_\gamma(\mathbf{x})\,n_\gamma\delta_{\alpha\beta}\right]d\Gamma(\mathbf{x}) \\
&+\int_\Omega k_{\alpha\beta}B\frac{1-\nu}{2}\Big[u_\alpha(\mathbf{X})\,W^*_{i3,\beta}\left(\mathbf{x}',\mathbf{X}\right) \\
&+u_\beta(\mathbf{X})\,W^*_{i3,\alpha}\left(\mathbf{x}',\mathbf{X}\right) \\
&+\frac{2\nu}{1-\nu}u_\gamma(\mathbf{X})\,W^*_{i3,\gamma}\left(\mathbf{x}',\mathbf{X}\right)\delta_{\alpha\beta}\Big]\,d\Omega(\mathbf{X}) \\
&-\int_\Omega W^*_{i3}\left(\mathbf{x}',\mathbf{X}\right)k_{\alpha\beta}B\left((1-\nu)k_{\alpha\beta}+\nu\delta_{\alpha\beta}k_{\phi\phi}\right)\times \\
&w_3(\mathbf{X})\,d\Omega(\mathbf{X})
\end{aligned}
$$

$$+\lambda\int_{\Omega}W_{i3}^{*}(\mathbf{x}',\mathbf{X})q_3(\mathbf{X})d\Omega(\mathbf{X})$$
$$+\lambda\int_{\Omega}W_{i3}^{*}(\mathbf{x}',\mathbf{X})(N_{\alpha\beta}w_{3,\beta})_{,\alpha}(\mathbf{X})d\Omega(\mathbf{X}) \quad (12)$$

where $(N_{\alpha\beta}w_{3,\beta})_{,\alpha}$ is a body term due to the large deflection of $w_3(\mathbf{X})$. This term is the common extra term that appears in the nonlinear equilibrium equations of plates and shells (e.g. see the nonlinear equilibrium equation 6.10 in Brush and Almorth (1975)).

It is important to notice again that because of the presence of the curvature terms in (12), this equation has to be solve simultaneously with equation (10).

The deflection equation w_3 at the domain points $\mathbf{X}'$ is required as the additional equation to arrange an eigenvalue equation, as follows:

$$w_3(\mathbf{X}') = \int_{\Gamma}W_{3j}^{*}(\mathbf{X}',\mathbf{x})p_j(\mathbf{x})\,d\Gamma(\mathbf{x})$$
$$-\int_{\Gamma}P_{3j}^{*}(\mathbf{X}',\mathbf{x})w_j(\mathbf{x})\,d\Gamma(\mathbf{x})$$
$$-\int_{\Gamma}W_{33}^{*}(\mathbf{X}',\mathbf{x})\,k_{\alpha\beta}B\frac{1-\nu}{2}\times$$
$$\left[u_\alpha(\mathbf{x})n_\beta+u_\beta(\mathbf{x})n_\alpha+\frac{2\nu}{1-\nu}u_\gamma(\mathbf{x})n_\gamma\delta_{\alpha\beta}\right]d\Gamma(\mathbf{x})$$
$$+\int_{\Omega}k_{\alpha\beta}B\frac{1-\nu}{2}\left[u_\alpha(\mathbf{X})W_{33,\beta}^{*}(\mathbf{X}',\mathbf{X})\right.$$
$$+u_\beta(\mathbf{X})W_{33,\alpha}^{*}(\mathbf{X}',\mathbf{X})$$
$$\left.+\frac{2\nu}{1-\nu}u_\gamma(\mathbf{X})W_{33,\gamma}^{*}(\mathbf{X}',\mathbf{X})\,\delta_{\alpha\beta}\right]d\Omega(\mathbf{X})$$
$$-\int_{\Omega}W_{33}^{*}(\mathbf{X}',\mathbf{X})\,k_{\alpha\beta}B\left((1-\nu)k_{\alpha\beta}+\nu\delta_{\alpha\beta}k_{\phi\phi}\right)\times$$
$$w_3(\mathbf{X})\,d\Omega(\mathbf{X})$$
$$+\lambda\int_{\Omega}W_{33}^{*}(\mathbf{X}',\mathbf{X})q_3(\mathbf{X})d\Omega(\mathbf{X})$$
$$+\lambda\int_{\Omega}W_{33}^{*}(\mathbf{X}',\mathbf{X})(N_{\alpha\beta}w_{3,\beta})_{,\alpha}(\mathbf{X})d\Omega(\mathbf{X}) \quad (13)$$

To arrange an eigenvalue equation, the derivatives $w_{3,\beta}(\mathbf{X})$ and $w_{3,\alpha\beta}(\mathbf{X})$ have to be expressed in terms of $w_3(\mathbf{X})$, see section 4.3. Therefore the last integrals in equations (12) and (13), can be expressed as follows:

$$c_{ij}(\mathbf{x}')w_j(\mathbf{x}')+\fint_{\Gamma}P_{ij}^{*}(\mathbf{x}',\mathbf{x})w_j(\mathbf{x})\,d\Gamma(\mathbf{x})$$
$$=\int_{\Gamma}W_{ij}^{*}(\mathbf{x}',\mathbf{x})p_j(\mathbf{x})\,d\Gamma(\mathbf{x})$$
$$-\int_{\Gamma}W_{i3}^{*}(\mathbf{x}',\mathbf{x})\,k_{\alpha\beta}B\frac{1-\nu}{2}\times$$
$$\left[u_\alpha(\mathbf{x})n_\beta+u_\beta(\mathbf{x})n_\alpha+\frac{2\nu}{1-\nu}u_\gamma(\mathbf{x})n_\gamma\delta_{\alpha\beta}\right]d\Gamma(\mathbf{x})$$
$$+\int_{\Omega}k_{\alpha\beta}B\frac{1-\nu}{2}\left[u_\alpha(\mathbf{X})W_{i3,\beta}^{*}(\mathbf{x}',\mathbf{X})\right.$$
$$+u_\beta(\mathbf{X})W_{i3,\alpha}^{*}(\mathbf{x}',\mathbf{X})$$
$$\left.+\frac{2\nu}{1-\nu}u_\gamma(\mathbf{X})W_{i3,\gamma}^{*}(\mathbf{x}',\mathbf{X})\,\delta_{\alpha\beta}\right]d\Omega(\mathbf{X})$$
$$-\int_{\Omega}W_{i3}^{*}(\mathbf{x}',\mathbf{X})\,k_{\alpha\beta}B\left((1-\nu)k_{\alpha\beta}+\nu\delta_{\alpha\beta}k_{\phi\phi}\right)\times$$
$$w_3(\mathbf{X})\,d\Omega(\mathbf{X})$$
$$+\lambda\int_{\Omega}W_{i3}^{*}(\mathbf{x}',\mathbf{X})f_b(\mathbf{X})d\Omega(\mathbf{X}) \quad (14)$$

and,

$$w_3(\mathbf{X}') = \int_{\Gamma}W_{3j}^{*}(\mathbf{X}',\mathbf{x})p_j(\mathbf{x})\,d\Gamma(\mathbf{x})$$
$$-\int_{\Gamma}P_{3j}^{*}(\mathbf{X}',\mathbf{x})w_j(\mathbf{x})\,d\Gamma(\mathbf{x})$$
$$-\int_{\Gamma}W_{33}^{*}(\mathbf{X}',\mathbf{x})\,k_{\alpha\beta}B\frac{1-\nu}{2}\times$$
$$\left[u_\alpha(\mathbf{x})n_\beta+u_\beta(\mathbf{x})n_\alpha+\frac{2\nu}{1-\nu}u_\gamma(\mathbf{x})n_\gamma\delta_{\alpha\beta}\right]d\Gamma(\mathbf{x})$$
$$+\int_{\Omega}k_{\alpha\beta}B\frac{1-\nu}{2}\left[u_\alpha(\mathbf{X})W_{33,\beta}^{*}(\mathbf{X}',\mathbf{X})\right.$$
$$+u_\beta(\mathbf{X})W_{33,\alpha}^{*}(\mathbf{X}',\mathbf{X})$$
$$\left.+\frac{2\nu}{1-\nu}u_\gamma(\mathbf{X})W_{33,\gamma}^{*}(\mathbf{X}',\mathbf{X})\,\delta_{\alpha\beta}\right]d\Omega(\mathbf{X})$$
$$-\int_{\Omega}W_{33}^{*}(\mathbf{X}',\mathbf{X})\,k_{\alpha\beta}B\left((1-\nu)k_{\alpha\beta}+\nu\delta_{\alpha\beta}k_{\phi\phi}\right)\times$$
$$w_3(\mathbf{X})\,d\Omega(\mathbf{X})$$
$$+\lambda\int_{\Omega}W_{33}^{*}(\mathbf{X}',\mathbf{X})f_b(\mathbf{X})d\Omega(\mathbf{X}) \quad (15)$$

where:

$$f_b = q_3+N_{\alpha\beta,\alpha}\mathbf{f}(r)_{,\beta}\mathbf{F}^{-1}w_3+N_{\alpha\beta}\mathbf{f}(r)_{,\alpha}\mathbf{f}(r)_{,\beta}\mathbf{F}^{-1}w_3 \quad (16)$$

4 Numerical implementation

In order to solve the integral equations presented in the previous section, the boundary Γ and the domain Ω must be discretized. Generally, BEM formulations rely on continuous boundary elements and cells. However, because of the presence of singular integrals in the domain during the evaluation of the stresses $N_{\alpha\beta}$, it was decided to implement continuous and discontinuous (partially or totally) quadrilateral internal cells; while on the boundary semi-discontinuous elements are used for corners to avoid difficulties with discontinuity of the tractions at corners.

From the implementation point of view, the use of discontinuous or semi discontinuous elements requires the consideration of two different meshes: the geometric mesh, defined by the geometric nodes which always lie on the boundary of the element; and the functional mesh, defined by the functional nodes which can exist anywhere within the element boundaries. In the domain this distinction allows the use of a whole range of quadrilateral cell elements by just defining an appropriate set of parameters, as it will be explained later.

4.1 Discretization

In the present study, quadratic isoparametric boundary elements are used to describe the geometry and the function along Γ. In the same way for the domain, quadratic quadrilateral isoparametric elements are used to describe the geometry and the function over Ω.

Equation (9) can be rewritten in a discretized form as:

$$
\begin{aligned}
&c_{ij}(\mathbf{x}')w_j(\mathbf{x}') + \sum_{n=1}^{N_e}\sum_{m=1}^{3} w_j^{nm}\int_{\xi=-1}^{\xi=+1} P_{ij}^{*}(\mathbf{x}',\mathbf{x})\Phi^m(\xi)J_n(\xi)d\xi \\
&= \sum_{n=1}^{N_e}\sum_{m=1}^{3} p_j^{nm}\int_{\xi=-1}^{\xi=+1} W_{ij}^{*}(\mathbf{x}',\mathbf{x})\Phi^m(\xi)J_n(\xi)d\xi \\
&- \sum_{n=1}^{N_e}\sum_{m=1}^{3} k_{\alpha\beta}B\frac{1-\nu}{2}\Big(u_\alpha^{nm}n_\beta^{nm} + u_\beta^{nm}n_\alpha^{nm} \\
&+ \frac{2\nu}{1-\nu}u_\gamma^{nm}n_\gamma^{nm}\delta_{\alpha\beta}\Big) \times \int_{\xi=-1}^{\xi=+1} W_{i3}^{*}\left(\mathbf{x}',\mathbf{x}\right)\Phi^m(\xi)J_n(\xi)d\xi \\
&+ \sum_{k=1}^{N_c}\sum_{l=1}^{9} k_{\alpha\beta}B\frac{1-\nu}{2}u_\alpha^{kl}\int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3,\beta}^{*}\left(\mathbf{x}',\mathbf{X}\right) \times \\
&\Psi^l(\xi,\eta)J_k(\xi,\eta)d\xi d\eta \\
&+ \sum_{k=1}^{N_c}\sum_{l=1}^{9} k_{\alpha\beta}B\frac{1-\nu}{2}u_\beta^{kl}\int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3,\alpha}^{*}\left(\mathbf{x}',\mathbf{X}\right) \times \\
&\Psi^l(\xi,\eta)J_k(\xi,\eta)d\xi d\eta \\
&+ \sum_{k=1}^{N_c}\sum_{l=1}^{9} k_{\alpha\alpha}B\nu u_\gamma^{kl}\int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3,\gamma}^{*}\left(\mathbf{x}',\mathbf{X}\right) \times \\
&\Psi^l(\xi,\eta)J_k(\xi,\eta)d\xi d\eta \\
&- \sum_{k=1}^{N_c}\sum_{l=1}^{9} k_{\alpha\beta}B\left[(1-\nu)k_{\alpha\beta} + \nu\delta_{\alpha\beta}k_{\gamma\gamma}\right]w_3^{kl} \times \\
&\int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3}^{*}\left(\mathbf{x}',\mathbf{X}\right)\Psi^l(\xi,\eta)J_k(\xi,\eta)d\xi d\eta \\
&+ \sum_{k=1}^{N_c}\sum_{l=1}^{9} q_3^{kl}\int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3}^{*}\left(\mathbf{x}',\mathbf{X}\right) \times \\
&\Psi^l(\xi,\eta)J_k(\xi,\eta)d\xi d\eta
\end{aligned}
\qquad (17)
$$

where N_e and N_c are number of boundary elements and internal cells respectively. Φ^m are the boundary shape functions. Ψ^l are the domain shape functions. ξ and η are local coordinates. J_n and J_k are the Jacobian of transformation for boundary elements and internal cells respectively. A complete description of the boundary elements and domain cells used in this work is given in Appendix C.

Equations (10,14 and 15) have also to be expressed in the same way as equation (17), but for the sake of space the complete expression will not be shown here.

In the case of the domain integral that contains f_b in equations (14 and 15), when $q_3 = 0$, the integral can be discretized as follows:

$$
\begin{aligned}
&\int_\Omega W_{i3}^{*}(\mathbf{x}',\mathbf{X})f_b(\mathbf{X})d\Omega(\mathbf{X}) \\
&= \sum_{k=1}^{N_c}\sum_{l=1}^{9} w_3^{kl}\cdot f_{bw}^{kl}\int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3}^{*}\left(\mathbf{x}',\mathbf{X}\right) \times \\
&\Psi^l(\xi,\eta)J_k(\xi,\eta)d\xi d\eta
\end{aligned}
\qquad (18)
$$

where $f_{bw}^{kl} = N_{\alpha\beta,\alpha}^{kl}\mathbf{f}(r)_{,\beta}\mathbf{F}^{-1} + N_{\alpha\beta}^{kl}\mathbf{f}(r)_{,\alpha}\mathbf{f}(r)_{,\beta}\mathbf{F}^{-1}$.

4.2 Treatment of the integrals

Generally speaking, two different kinds of integrals can be defined for both the boundary and domain. Depending on the integrands, integrals can be classified as: Regular, in which case they can be evaluated using the standard gauss quadrature rule or; Singular, when the collocation point belongs to the element over which the integration is performed, in this case special techniques must be used.

All the singular integrals appearing in the displacement and internal stress integral equations are dealt with by using well established techniques and are treated separately based on their order of singularity.

On the boundary, near singular integrals (when the collocation node is close to the integration element) are treated with the element subdivision technique Aliabadi (2002). Weakly singular integrals $O(\ln r)$ are treated using a nonlinear coordinate transformation as reported by Telles Telles (1987). Strong singular integrals $O(1/r)$ are computed indirectly by considering the generalized rigid body motion, as explained in Dirgantara (2002).

The domain singular integrals can also be separated in weakly $O(1/r)$ and strong $O(1/r^2)$. Weak singular integrals are treated by a simple technique such as polar coordinate transformation, followed by a regular procedure Aliabadi (2002). Strong singular integrals require special techniques such as the ones described by Leitao Leitao (1994).

4.3 Evaluation of the derivative terms

Several derivative terms in the domain have to be obtained, and although this procedure could be performed with the polynomial interpolation of the domain cells, it was considered more convenient the use of radial basis functions $f(r)$ in order to establish the same frame of comparison with an only boundary formulation that will be developed in a future publication.

The derivative terms $w_{3,\beta}(\mathbf{X})$ and $w_{3,\alpha\beta}(\mathbf{X})$ are approximated as follows:

$$w_3(x_1,x_2) = \sum_{m=1}^{L} f(r)^m \Psi^m \tag{19}$$

where the radial basis function is chosen as $f(r) = \sqrt{c^2+r^2}$ and c^2 =2. L is the total number of selected points in the domain, which are the same domain points used in the domain integration process. As it can be seen from the integral equations (14) and (15), some of the integrals are on the domain Ω, for which quadratic isoparametric quadrilateral domain cells of 9 nodes are used.

The distance r in equation (19) is given by the following expression:

$$r = \sqrt{(x_1 - x_1^m)^2 + (x_2 - x_2^m)^2} \tag{20}$$

The Ψ^m are coefficients determined by values at the L domain points:

$$\boldsymbol{\Psi} = \mathbf{F}^{-1}\mathbf{w}_3 \tag{21}$$

Consequently, the first derivative of deflection $w_{3,\beta}$ can be expressed as the product of the first derivative of the radial basis function and the coefficients Ψ^m, as follows:

$$w_{3,\beta}(x_1,x_2) = \mathbf{f}(r)_{,\beta}\mathbf{F}^{-1}\mathbf{w}_3 \tag{22}$$

In the same way, the second derivative of deflection $w_{3,\alpha\beta}$ can be written as:

$$w_{3,\alpha\beta}(x_1,x_2) = \mathbf{f}(r)_{,\beta}\mathbf{f}(r)_{,\alpha}\mathbf{F}^{-1}\mathbf{w}_3 \tag{23}$$

Similar to the above expressions, the derivative of in-plane stress resultants $N_{\alpha\beta,\alpha}$ can be expressed as:

$$N_{\alpha\beta,\alpha}(x_1,x_2) = \mathbf{f}(r)_{,\alpha}\mathbf{F}^{-1}\mathbf{N}_{\alpha\beta} \tag{24}$$

5 Numerical procedure

In this section a numerical procedure developed to solve the equations shown in the previous sections is explained. The solution steps towards the linear buckling solution are given as follows:

- Initially, solution of the linear shallow shell boundary integral equations (9) and (10) is obtained.
- After boundary and domain displacements as well as boundary tractions are known, the membrane stresses at domain nodes $N_{\alpha\beta}(\mathbf{X}')$ are obtained from equation (11).
- Approximated derivatives of membrane stresses and out of plane displacement are obtained, as explained in the subsection 4.3.
- Finally, the boundary integral equations of the buckling problem (10), (14) and (15) are assembled and solved; obtaining buckling modes and buckling load factors.

This procedure is explained with more detail in the following subsections.

5.1 Shell in plane stresses

After discretized equations (9) and (10) as explained in section 4.1, and point collocation the following linear system of equations is obtained for every node:

$$\begin{bmatrix} \cdots & c+\mathbf{H}^p & \mathbf{H}^p & \mathbf{H}^p_w & \mathbf{H}^u & \mathbf{H}^u & \cdots \\ \cdots & \mathbf{H}^p & c+\mathbf{H}^p & \mathbf{H}^p_w & \mathbf{H}^u & \mathbf{H}^u & \cdots \\ \cdots & \mathbf{H}^p & \mathbf{H}^p & c+\mathbf{H}^p_w & \mathbf{H}^u & \mathbf{H}^u & \cdots \\ \cdots & 0 & 0 & \mathbf{H}^w & c+\mathbf{H}^s & \mathbf{H}^s & \cdots \\ \cdots & 0 & 0 & \mathbf{H}^w & \mathbf{H}^s & c+\mathbf{H}^s & \cdots \end{bmatrix}_{5\times5(N+L)} \begin{Bmatrix} \vdots \\ w_1 \\ w_2 \\ w_3 \\ u_1 \\ u_2 \\ \vdots \end{Bmatrix}_{5(N+L)}$$

$$= \begin{bmatrix} \cdots & \mathbf{G}^p & \mathbf{G}^p & \mathbf{G}^p & 0 & 0 & \cdots \\ \cdots & \mathbf{G}^p & \mathbf{G}^p & \mathbf{G}^p & 0 & 0 & \cdots \\ \cdots & \mathbf{G}^p & \mathbf{G}^p & \mathbf{G}^p & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \mathbf{G}^s & \mathbf{G}^s & \cdots \\ \cdots & 0 & 0 & 0 & \mathbf{G}^s & \mathbf{G}^s & \cdots \end{bmatrix}_{5\times 15NE} \times \begin{Bmatrix} \vdots \\ p_1 \\ p_2 \\ p_3 \\ t_1 \\ t_2 \\ \vdots \end{Bmatrix}_{15NE} + \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ g_1 \\ g_2 \end{Bmatrix} \tag{25}$$

where b_i are the product of the bending displacement fundamental solutions with the domain load q_3, which in this study is set to zero ($q_3 = 0 \rightarrow b_i = 0$). Similarly g_α are the product of the membrane displacement fundamental solutions with the in plane domain loads q_α, which are also set to zero in this analysis ($q_\alpha = 0$). **H** and **G** are boundary element matrices for tractions and displacements fundamental solutions, respectively. The indexes p and s on **H** and **G** refer to plate bending and plane stress formulations respectively; while the indexes u and w are coupled terms between plate bending and plane stress formulations. N, L and NE are number of boundary nodes, domain points and boundary elements, respectively.

After performing all the collocation process, the known and unknown quantities in equation (25) can be arranged as a set of linear algebraic equation:

$$[\mathbf{A}]_{5(N+L)\times 5(N+L)}\{\mathbf{X}\}_{5(N+L)} = \{\mathbf{F}\}_{5(N+L)} \tag{26}$$

where $[\mathbf{A}]$ is the system matrix, $\{\mathbf{X}\}$ contains the unknowns displacements and tractions on the boundary N, as well as all the displacement in the domain L. The vector $\{\mathbf{F}\}$ is obtained by multiplying the related matrices of **H** or **G** by the known values of w_i, u_α or p_i, t_α (because the body forces where set to zero, $q_i = 0$).

Once equation (26) has been solved, in-plane stresses N_{11}, N_{12}, and N_{22} in the domain are calculated from equation (11). They are required to solve the shell buckling problem.

5.2 Shell buckling problem

After discretization of the equations (10) and (14) as explained in section 4.1, the linear system of algebraic equations for every collocation node on the boundary can be written similar to the linear shell solution:

$$\begin{bmatrix} \cdots & c+\mathbf{H}^p & \mathbf{H}^p & \mathbf{H}^p_w & \mathbf{H}^u & \mathbf{H}^u & \cdots \\ \cdots & \mathbf{H}^p & c+\mathbf{H}^p & \mathbf{H}^p_w & \mathbf{H}^u & \mathbf{H}^u & \cdots \\ \cdots & \mathbf{H}^p & \mathbf{H}^p & c+\mathbf{H}^p_w & \mathbf{H}^u & \mathbf{H}^u & \cdots \\ \cdots & 0 & 0 & \mathbf{H}^w & c+\mathbf{H}^s & \mathbf{H}^s & \cdots \\ \cdots & 0 & 0 & \mathbf{H}^w & \mathbf{H}^s & c+\mathbf{H}^s & \cdots \end{bmatrix}_{5\times(5N+3L)} \begin{Bmatrix} \vdots \\ w_1 \\ w_2 \\ w_3 \\ u_1 \\ u_2 \\ \vdots \end{Bmatrix}_{5N+3L}$$

$$= \begin{bmatrix} \cdots & \mathbf{G}^p & \mathbf{G}^p & \mathbf{G}^p & 0 & 0 & \cdots \\ \cdots & \mathbf{G}^p & \mathbf{G}^p & \mathbf{G}^p & 0 & 0 & \cdots \\ \cdots & \mathbf{G}^p & \mathbf{G}^p & \mathbf{G}^p & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \mathbf{G}^s & \mathbf{G}^s & \cdots \\ \cdots & 0 & 0 & 0 & \mathbf{G}^s & \mathbf{G}^s & \cdots \end{bmatrix}_{5\times 15NE} \begin{Bmatrix} \vdots \\ p_1 \\ p_2 \\ p_3 \\ t_1 \\ t_2 \\ \vdots \end{Bmatrix}_{15NE} + \lambda \begin{bmatrix} \cdots & Q_1 & \cdots \\ \cdots & Q_2 & \cdots \\ \cdots & Q_3 & \cdots \\ \cdots & 0 & \cdots \\ \cdots & 0 & \cdots \end{bmatrix}_{5\times L} \begin{Bmatrix} \vdots \\ w_3 \\ \vdots \end{Bmatrix}_L \tag{27}$$

and for every collocation node on the domain, the linear system of algebraic equations will be given for only the last tree equations in (27); which correspond to w_3, u_1 and u_2 displacements. Body force terms in equation (27) were considered zero ($q_i = 0$), and are not shown.

In equation (27), **Q** is a node influence formed by the following domain integral:

$$Q_i = \int_\Omega W^*_{i3}(\mathbf{x}', \mathbf{X}) f_{bw}(\mathbf{X}) d\Omega(\mathbf{X}) \tag{28}$$

Equation (27) can be arranged in a similar manner as equation (26), and give:

$$[\mathbf{B}]_{(5N+3L)\times(5N+3L)}\{\mathbf{Y}\}_{5N+3L} = \lambda[\mathbf{K}]_{(5N+3L)\times L}\{\mathbf{w}_3\}_L \tag{29}$$

In order to arrange an eigenvalue formulation, equation

(15) can also be written in matrix form, similar to equation (29):

$$[\mathbf{I}]\{\mathbf{w}_3\}_L = [\mathbf{BB}]_{L\times(5N+3L)}\{\mathbf{Y}\}_{5N+3L} + \lambda[\mathbf{KK}]_{L\times L}\{\mathbf{w}_3\}_L \quad (30)$$

where the matrices $[\mathbf{B}]$ and $[\mathbf{BB}]$ contain coefficient matrices related to the fundamental solutions. Matrix $[\mathbf{I}]$ is the identity matrix. Vector $\{\mathbf{Y}\}$ represents the unknown boundary conditions ($w_i(\mathbf{x}), u_\alpha(\mathbf{x})$ or $p_i(\mathbf{x}), t_\alpha(\mathbf{x})$) and the unknown domain displacements ($u_\alpha(\mathbf{X}), w_3(\mathbf{X})$). Vector $\{\mathbf{w}_3\}$ contains the unknown out of plane displacement $w_3(\mathbf{X})$. Matrices $[\mathbf{K}]$ and $[\mathbf{KK}]$ are obtained by multiplication of the fundamental solutions with the prebuckling in plane stresses $N_{\alpha\beta}(\mathbf{X})$ and approximation functions $f(r)$.

As it can be seen in equation (29) the only load considered in this transformed linearized buckling equation is the transverse body load ($(N_{\alpha\beta}w_{3,\beta})_{,\alpha}$) multiplied by the critical load factor λ, implying that all the known values of w_i, u_α or p_i, t_α (boundary conditions), are set to zero. For this reason, there is no vector such as $\{\mathbf{F}\}$ (from equation (26)) in equations (29) and (30).

Equation (29) can be rearranged in term of the unknown vector $\{\mathbf{Y}\}_{5N+3L}$,

$$\{\mathbf{Y}\}_{5N+3L} = \lambda[\mathbf{B}]^{-1}_{(5N+3L)\times(5N+3L)}[\mathbf{K}]_{(5N+3L)\times L}\{\mathbf{w}_3\}_L \quad (31)$$

where matrix $[\mathbf{B}]^{-1}$ is the inverse of matrix $[\mathbf{B}]$.

The substitution of equation (31) into equation (30) yields:

$$[\mathbf{I}]_{L\times L}\{\mathbf{w}_3\}_L = \lambda[\mathbf{BB}]_{L\times(5N+3L)}[\mathbf{B}]^{-1}_{(5N+3L)\times(5N+3L)}\times [\mathbf{K}]_{(5N+3L)\times L}\{\mathbf{w}_3\}_L + \lambda[\mathbf{KK}]_{L\times L}\{\mathbf{w}_3\}_L \quad (32)$$

Equation (32) can be written as a standard eigenvalue problem equation as follows:

$$([\psi] - \frac{1}{\lambda}[\mathbf{I}])\{\mathbf{w}_3\}_L = 0 \quad (33)$$

Buckling analysis of shear deformable shallow shell has been presented as a standard eigenvalue problem; buckling modes $\{\mathbf{w}_3\}$ and buckling load factors λ can be obtained by solving equation (33). This standard eigenvalue problem was solved with LAPACK Anderson, Bai, Bischof, Blackford, Demmel, Dongarra, Du Croz, Greenbaum, Hammarling, McKenney, and Sorensen (1999) which is freely available on the internet.

6 Numerical examples

The proposed technique is applied to several benchmark problems to assess its accuracy and efficiency. Rectangular cylindrical shallow shells with different curvature parameters, aspect ratios (a/b) and boundary conditions will be presented. Analytical solutions for buckling of cylindrical shells under the action of uniform axial compression have been given by Timoshenko and Gere (1961) or Flugge (1964) which are base on a set of three equilibrium equations or Donnell (1933) who gives a single eighth order partial differential equation in the radial displacement.

For axial compression of curved sheet panels (see figure 2), the same method as in the case of a circular cylindrical tube axially compressed has been used for calculating the critical stresses. The analytical solutions for any of the buckling equations given so far in the literature, are based on the substitution of trigonometric functions which satisfy some specific boundary conditions and assumed buckling mode. On the other hand, they are based on classical shell theory, where shear deformable effects are not considered. Therefore, it seems more convenient to compare our results with finite element solutions.

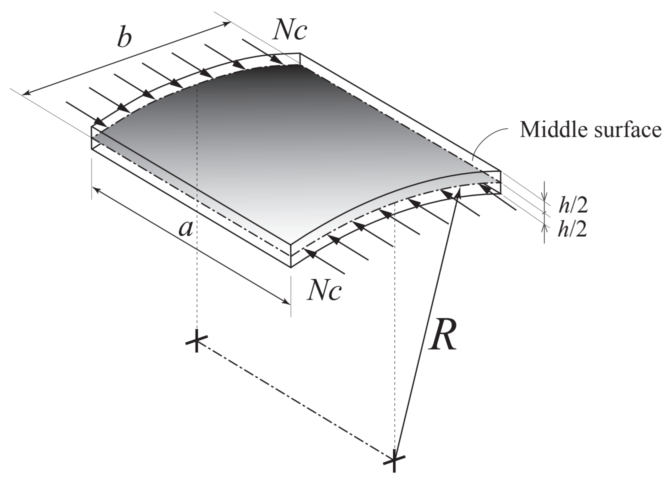

Figure 2 : Cylindrical shallow shell subjected to uniform axial compression.

The buckling coefficients (K) and the curvature parameters (Z) for curved plate structures are defined as follows:

$$K = \frac{N_{cr}\cdot 12(1-\nu^2)\cdot b^2}{\pi^2\cdot E\cdot h^2} \quad (34)$$

and,

$$Z = \frac{b^2 \cdot \sqrt{1-\nu^2}}{R \cdot h} \tag{35}$$

where N_{cr} represent the critical in plane stress, obtained from the multiplication of the buckling factors λ with the actual applied stress, and b denotes the length of the curved side of the shell.

In the present work, two different sets of boundary conditions will be considered. The simply supported condition refers to zero out of plane displacement ($w_3 = 0$) while the clamped condition is based on zero out of plane displacement and zero rotations ($w_i = 0$). The in plane displacements in both cases were set free ($u_\alpha \neq 0$). These boundary conditions are the same through all the boundary.

6.1 Convergency study

First of all, a convergency study was performed for the simple supported case of a shallow shell subjected to uniform axial compression, as shown in figure 2. The shell considered has an aspect ratio $a/b = 2$ and width $b = 2in$, thickness $h = 0.03in$, Young's modulus $E = 1.05 \cdot 10^7 psi$ and Poisson ratio $\nu = 0.3$. The radius of the shell is $R = 6.5in$ that correspond for a curvature parameter of $Z = 19.568$.

Table 1 : Buckling coefficients for different meshes.

No.	Boundary Mesh	Domain Mesh	K	K/Ko
1	12 elements	2X4 cells	7.212	0.942
2	18 elements	3X6 cells	7.614	0.995
3	24 elements	4X8 cells	7.640	0.998
4	30 elements	5X10 cells	7.653	1.000

As is shown in table I, four different meshes were considered. Ko is given by the buckling coefficient of the most refined mesh (5x10 cells). In figure 3, the normalized buckling coefficients for the four meshes are shown, and it is clear that good convergency is achieved with a mesh of just 18 boundary elements and 18 domain cells, less than 1% difference compared with the most refined mesh.

The boundary domain meshes used, are shown in figure 4. In this figure can also easily be seen the partially discontinuous domain cells and the semi discontinuous boundary elements. As it was explained in section 4.2, due to the order of the singularities in the domain $O(1/r^2)$, the functional domain mesh is slightly moved inwards, away from the boundary Γ; in this way singularity problems can be avoided during the domain and boundary integration. Semi discontinuous boundary elements are used next to the corners, in order to avoid difficulties with discontinuity of the tractions.

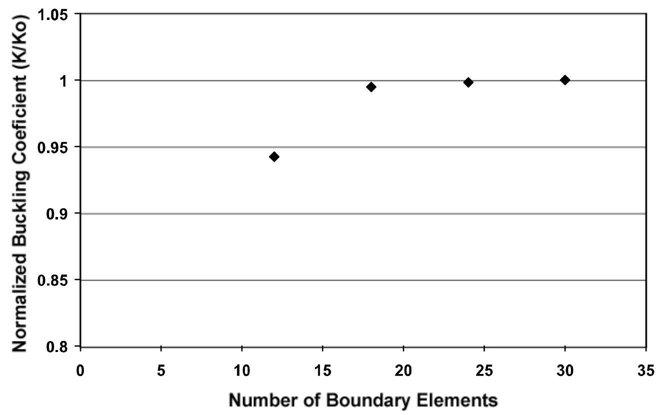

Figure 3 : Normalized buckling coeficients for different meshes.

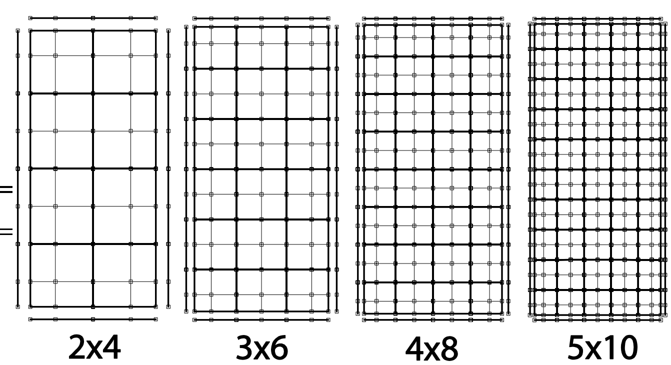

Figure 4 : Boundary domain meshes used for the convergency study.

6.2 Cylindrical shallow shells with different curvature

The BEM results were obtained with a model of 18 boundary elements and 18 domain cells (91 internal nodes and 36 boundary nodes) while for the FEM mesh 72 quadratic elements were used (253 nodes). In figure 5 the buckling coefficients for different radius of a cylindrical shallow shells under compressive loads are given. The aspect ratio is $a/b = 2$, width $b = 2in$, thickness $h = 0.03in$, Young's modulus $E = 1.05 \cdot 10^7 psi$ and Poisson ratio $\nu = 0.3$. As expected, an increase in the curvature also increase the buckling coefficients, while the decrease of the curvature converge to the flat plate solutions.

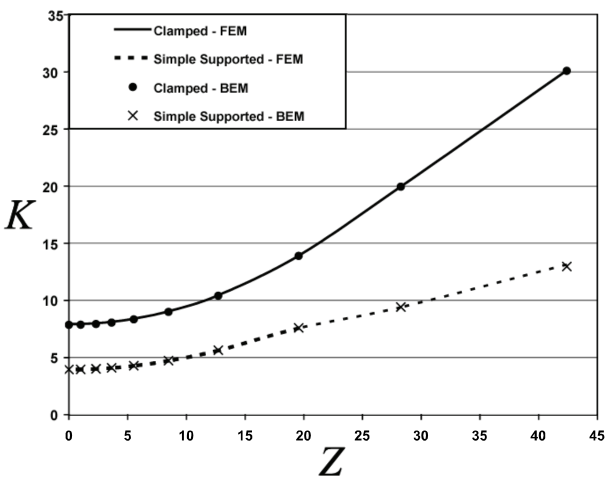

Figure 5 : Buckling coefficients for simple supported and clamped cylindrical shallow shell.

Results shown in figure 5, are also given in table II. It can be seen that the differences between boundary element results and finite element results are very small, less than 1%, except for the smallest radius of the simple supported case, where the difference reaches 1.315%.

Buckling modes for a radius of $R = 6.5in$ are shown in figure 6. The 2 and 3 half waves in the simple supported and clamped buckling modes agree with the expected number of half waves that are found on the buckling of a rectangular curved plate with aspect ratio $a/b = 2$.

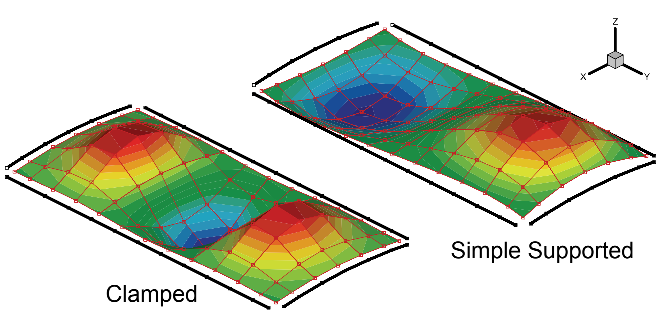

Figure 6 : Buckling modes for simple supported and clamped shallow cylindrical shells.

6.3 *Cylindrical shallow shells with different aspect ratios (a/b)*

In this last example, different aspects ratios a/b are considered. The material properties (E, ν), width b and thickness h used in the previous examples are also used here. A constant curvature parameter of $Z = 12.719$ $(R = 10in)$ is considered. As it can be seen from figure 7, boundary and finite element solutions agree very well, with differences less than 2% for the clamped boundary condition and less than 1% for the simply supported case.

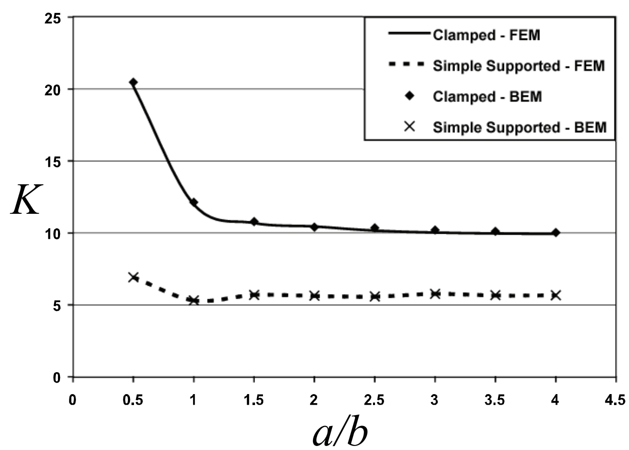

Figure 7 : Buckling coeficients for differents aspects ratios and boundary conditions for cylindrical shallow shells under axial uniform compression.

7 Conclusions

In this work, a boundary domain element formulation for the solution of the linear elastic buckling problem of shear deformable shallow cylindrical shells under uniform compressive load was developed. Buckling equations were obtained by introducing multiplication factors of transverse loads (due to the large deflection of the out of plane displacement) into the governing linear integral equations. Membrane stresses $(N_{\alpha\beta})$ are obtained from the prebuckling state, resulting in a set of linear buckling equations in terms of the buckling deflection $(w_3 = 0)$ and the buckling factors (λ). Buckling equations were presented as a standard eigenvalue problem, making possible to obtaining critical load factors and buckling modes in a single procedure.

From the examples it is clear that the present boundary

Table 2 : Buckling coefficients for different curvatures and boundary conditions.

		Simple Supported			Clamped		
Radius (*in*)	Z	BEM	FEM	%Diff.	BEM	FEM	%Diff.
3.0	42.397	12.968	13.141	1.315	30.082	30.129	0.157
4.5	28.265	9.420	9.405	-0.161	19.946	19.966	0.104
6.5	19.568	7.614	7.591	-0.302	13.887	13.920	0.234
10.0	12.719	5.642	5.621	-0.372	10.402	10.453	0.494
15.0	8.479	4.734	4.716	-0.399	8.985	9.048	0.694
23.0	5.530	4.291	4.274	-0.398	8.334	8.403	0.817
35.0	3.634	4.100	4.084	-0.394	8.060	8.131	0.876
55.0	2.313	4.012	3.997	-0.391	7.936	8.009	0.904
126.0	1.009	3.964	3.948	-0.390	7.868	7.943	0.942
Plate	0.000	3.952	3.937	-0.388	7.852	7.925	0.924

domain integral equations requires fewer number of degrees of freedom compare with FEM solutions in order to achieved a good level of accuracy. This observation agrees well with the pioneer results presented by Zhang and Atluri (1988). Finally, although it is also well known that fully populated and non symmetric matrices are obtained with the present formulation, from the authors experience during the durations of this work, computer times are slightly shorter for the present method.

Base on the results, the presented boundary domain method can be used as an effective tool to solve buckling problems of cylindrical shallow shells with different geometries and boundary conditions under axial compressive loads.

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Appendix A: Appendix

The expressions for the kernels W^*_{ij} and P^*_{ij} are given by Vander Weeen (1982) as follows:

$$
\begin{aligned}
W^*_{\alpha\beta} &= \frac{1}{8\pi D(1-\nu)}\{[8B(z)-(1-\nu)(2\ln z-1)]\delta_{\alpha\beta} \\
&\quad -[8A(z)+2(1-\nu)]r_{,\alpha}r_{,\beta}\} \\
W^*_{\alpha 3} &= -W^*_{3\alpha} = \frac{1}{8\pi D}(2\ln z-1)rr_{,\alpha} \\
W^*_{33} &= \frac{1}{8\pi D(1-\nu)\lambda^2}[(1-\nu)z^2(\ln z-1)-8\ln z] \quad (36)
\end{aligned}
$$

and

$$P^*_{\gamma\alpha} = \frac{-1}{4\pi r}[(4A(z)+2zK_1(z)+1-\nu)(\delta_{\alpha\gamma}r_{,n}+r_{,\alpha}n_\gamma) + (4A(z)+1+\nu)r_{,\gamma}n_\alpha - 2(8A(z)+2zK_1(z)+1-\nu)r_{,\alpha}r_{,\gamma}r_{,n}]$$

$$P^*_{\gamma 3} = \frac{\lambda^2}{2\pi}[B(z)n_\gamma - A(z)r_{,\gamma}r_{,n}]$$

$$P^*_{3\alpha} = \frac{-(1-\nu)}{8\pi}\left[\left(2\frac{(1+\nu)}{(1-\nu)}\ln z - 1\right)n_\alpha + 2r_{,\alpha}r_{,n}\right]$$

$$P^*_{33} = \frac{-1}{2\pi r}r_{,n} \tag{37}$$

where

$$A(z) = K_0(z) + \frac{2}{z}\left[K_1(z) - \frac{1}{z}\right]$$

$$B(z) = K_0(z) + \frac{1}{z}\left[K_1(z) - \frac{1}{z}\right] \tag{38}$$

in which $K_0(z)$ and $K_1(z)$ are modified Bessel functions of the second kind, $z=\lambda r$, r is the absolute distance between the source and the field points, $r_{,\alpha} = r_\alpha/r$, where $r_\alpha = x_\alpha(\mathbf{x}) - x_\alpha(\mathbf{x}')$ and $r_{,n} = r_{,\alpha}n_\alpha$.As it can be seen, $A(z)$ is a smooth function, whereas, $B(z)$ is a weakly singular $O(\ln r)$. Therefore W^*_{ij} is weakly singular and P^*_{ij} has a strong (Cauchy principal value) singularity $O(1/r)$.

The expressions for the kernels $U^*_{\theta\alpha}$ and $T^*_{\theta\alpha}$ are the well known (Kelvin solution) for two-dimensional plane stress problems, and are given as Dirgantara and Aliabadi (1999) :

$$U^*_{\theta\alpha} = \frac{1}{4\pi B(1-\nu)}\left[(3-\nu)\ln\left(\frac{1}{r}\right)\delta_{\theta\alpha} + (1+\nu)r_{,\theta}r_{,\alpha}\right] \tag{39}$$

$$T^{(i)*}_{\theta\alpha} = -\frac{1}{4\pi r}\{r_{,n}[(1-\nu)\delta_{\theta\alpha} + 2(1+\nu)r_{,\theta}r_{,\alpha}] + (1-\nu)[n_\theta r_{,\alpha} - n_\alpha r_{,\theta}]\} \tag{40}$$

where $U^*_{\theta\alpha}$ are weakly singular kernels of order $O\left(\ln\frac{1}{r}\right)$ and $T^*_{\theta\alpha}$ are strongly singular of order $O(1/r)$.

Derivatives of the displacement fundamental solutions with respect to the field point ($\mathbf{X}$) are given as follows:

$$W^*_{\gamma 3,\alpha} = \frac{1}{8\pi D}[(2\ln z - 1)\delta_{\alpha\gamma} + 2r_{,\gamma}r_{,\alpha}] \tag{41}$$

$$W^*_{33,\alpha} = \frac{r_{,\alpha}}{8\pi D(1-\nu)\lambda}[z(1-\nu)(2\ln z - 1) - \frac{8}{z}] \tag{42}$$

$$U^*_{\alpha\beta,\gamma} = \frac{1+\nu}{4\pi B(1-\nu)r} \times \left[-\frac{(3-\nu)}{(1+\nu)}r_{,\gamma}\delta_{\alpha\beta} + \delta_{\alpha\gamma}r_{,\beta} + \delta_{\beta\gamma}r_{,\alpha} - 2r_{,\beta}r_{,\gamma}r_{,\alpha}\right] \tag{43}$$

The kernel $W^*_{\gamma 3,\alpha}$ is regular, while $W^*_{33,\alpha}$ and $U^*_{\alpha\beta,\gamma}$ are weakly singular in the domain, singularity $O(1/r)$.

The expressions for the kernels $U^*_{\alpha\beta\gamma}$ and $T^*_{\alpha\beta\gamma}$ are Dirgantara (2002):

$$U^*_{\alpha\beta\gamma} = \frac{1}{4\pi r}\left[(1-\nu)\left(\delta_{\gamma\alpha}r_{,\beta} + \delta_{\gamma\beta}r_{,\alpha} - \delta_{\alpha\beta}r_{,\gamma}\right) + 2(1+\nu)r_{,\alpha}r_{,\beta}r_{,\gamma}\right] \tag{44}$$

$$T^{(i)*}_{\alpha\beta\gamma} = \frac{B(1-\nu)}{4\pi r^2}\{2r_{,n}[(1-\nu)\delta_{\alpha\beta}r_{,\gamma} + \nu(\delta_{\gamma\alpha}r_{,\beta} + \delta_{\gamma\beta}r_{,\alpha}) - 4(1+\nu)r_{,\alpha}r_{,\beta}r_{,\gamma}] + 2\nu(n_\alpha r_{,\beta}r_{,\gamma} + n_\beta r_{,\alpha}r_{,\gamma}) + (1-\nu)(2n_\gamma r_{,\alpha}r_{,\beta} + n_\beta\delta_{\alpha\gamma} + n_\alpha\delta_{\beta\gamma}) - (1-3\nu)n_\gamma\delta_{\alpha\beta}\} \tag{45}$$

Finally, the expression for $U^*_{\alpha\beta\gamma,\theta}$ is given by:

$$U^*_{\alpha\beta\gamma,\theta} = \frac{1}{4\pi r^2}\{2(1+\nu)[\delta_{\theta\alpha}r_{,\beta}r_{,\gamma} + \delta_{\beta\theta}r_{,\alpha}r_{,\gamma} + \delta_{\theta\gamma}r_{,\beta}r_{,\alpha} - 4r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,\theta}] - 2r_{,\theta}(1-\nu)[\delta_{\gamma\alpha}r_{,\beta} + \delta_{\gamma\beta}r_{,\alpha} - \delta_{\alpha\beta}r_{,\gamma}] + (1-\nu)[\delta_{\gamma\alpha}\delta_{\beta\theta} + \delta_{\gamma\beta}\delta_{\alpha\theta} - \delta_{\alpha\beta}\delta_{\gamma\theta}]\} \tag{46}$$

Appendix B: Appendix

For the non curvature terms in equation (10), the differentiation can be applied directly to the tensors related to the fundamental solutions, whereas in the case of the curvature integral, which already have been differentiated once ($U^*_{\theta\alpha,\beta}$), special considerations are necessary.

Let's represent the curvature integral in equation (10) on a more formal manner:

$$V_\theta = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} U^*_{\theta\alpha,\beta}(\mathbf{X}',\mathbf{X})B\left[k_{\alpha\beta}(1-\nu) + \nu\delta_{\alpha\beta}k_{\phi\phi}\right] \times w_3(\mathbf{X})d\Omega(\mathbf{X}) \tag{47}$$

where Ω_ε is the domain that remains after removed a circle of radius ε centred at the point ($\mathbf{X}'$) from the domain Ω.

The derivative of V_θ with respect to the coordinate x_γ of point $(\mathbf{X}')$ can be written as:

$$\frac{\partial V_\theta}{\partial x_\gamma} = \lim_{\varepsilon\to 0}\left\{\frac{\partial}{\partial x_\gamma}\int_{\Omega_\varepsilon} U^*_{\theta\alpha,\beta}(\mathbf{X}',\mathbf{X})B \right. \\ \left. \left[k_{\alpha\beta}(1-\nu)+\nu\delta_{\alpha\beta}k_{\phi\phi}\right] w_3(\mathbf{X})d\Omega(\mathbf{X})\right\} \quad (48)$$

The derivative of this domain integral, must be carried out by using the Leibnitz formula, that is given by the following expression:

$$\frac{d}{d\alpha}\int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} F(x,\alpha)dx = \int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} \frac{F(x,\alpha)}{d\alpha}dx \\ -F(\varphi(\alpha),\alpha)\frac{d\varphi_1(\alpha)}{d\alpha}+F(\varphi(\alpha),\alpha)\frac{d\varphi_2(\alpha)}{d\alpha} \quad (49)$$

After the Leibnitz formula have been applied to equation (48), one concludes that:

$$\frac{\partial V_\theta}{\partial x_\gamma} = \int_\Omega \frac{U^*_{\theta\alpha,\beta}(\mathbf{X}',\mathbf{X})}{\partial x_\gamma}B\left[k_{\alpha\beta}(1-\nu)+\nu\delta_{\alpha\beta}k_{\phi\phi}\right]\times \\ w_3(\mathbf{X})d\Omega(\mathbf{X}) \\ -B\left[k_{\alpha\beta}(1-\nu)+\nu\delta_{\alpha\beta}k_{\phi\phi}\right]w_3(\mathbf{X}')\times \\ \int_{\Gamma'} U^*_{\theta\alpha,\beta}(\mathbf{X}',\mathbf{x})r_{,\gamma}d\Gamma'(\mathbf{x}) \quad (50)$$

where the first integral in the right hand side is in the Cauchy principal value sense and the second one is calculated for a circle of radius $\varepsilon\to 0$ centred at point $(\mathbf{X}')$.

In order to solve the integral on Γ', the following relationships have to be considered:

$$r=\varepsilon;\quad r_{,n}=1;\quad d\Gamma'=\varepsilon d\varphi$$

$$r_{,1}=cos\varphi;\quad r_{,2}=sin\varphi$$

Now, it is possible to obtain the derivative of equation (10):

$$\frac{u_\theta(\mathbf{X}')}{\partial\gamma}+\int_\Gamma \frac{T^{*(i)}_{\theta\alpha}(\mathbf{X}',\mathbf{x})}{\partial\gamma}u_\alpha(\mathbf{x})d\Gamma(\mathbf{x}) \\ +\int_\Omega \frac{U^*_{\theta\alpha,\beta}(\mathbf{X}',\mathbf{X})}{\partial x_\gamma}B\left[k_{\alpha\beta}(1-\nu)+\nu\delta_{\alpha\beta}k_{\phi\phi}\right]\times \\ w_3(\mathbf{X})d\Omega(\mathbf{X}) \\ -\frac{w_3(\mathbf{X}')}{8(1-v)}\left\{\left[k_{\theta\gamma}(1-\nu)+\nu\delta_{\theta\gamma}k_{\phi\phi}\right](3v-5)\right. \\ \left.+\left[k_{\theta\gamma}(v-1)+\delta_{\theta\gamma}k_{\phi\phi}\right](1+v)\right\} \\ =\int_\Gamma \frac{U^*_{\theta\alpha}(\mathbf{X}',\mathbf{x})}{\partial\gamma}t_\alpha(\mathbf{x})d\Gamma(\mathbf{x}) \\ +\int_\Omega \frac{U^*_{\theta\alpha}(\mathbf{X}',\mathbf{X})}{\partial\gamma}q_\alpha(\mathbf{X})d\Omega(\mathbf{X}) \quad (51)$$

Finally, by introducing equation (51) into equation (6), the following expression can be obtained:

$$N_{\alpha\beta}(\mathbf{X}') = \int_\Gamma U^*_{\alpha\beta\gamma}(\mathbf{X}',\mathbf{x})t_\gamma(\mathbf{x})d\Gamma(\mathbf{x}) \\ -\int_\Gamma T^*_{\alpha\beta\gamma}(\mathbf{X}',\mathbf{x})u_\gamma(\mathbf{x})d\Gamma(\mathbf{x}) \\ -\int_\Omega U^*_{\alpha\beta\gamma,\theta}(\mathbf{X}',\mathbf{X})B\left[k_{\gamma\theta}(1-\nu)+\nu\delta_{\gamma\theta}k_{\phi\phi}\right]w_3(\mathbf{X})d\Omega(\mathbf{X}) \\ +\int_\Omega U^*_{\alpha\beta\gamma}(\mathbf{X}',\mathbf{X})q_\gamma(\mathbf{X})d\Omega(\mathbf{X}) \\ +B\left[(1-\nu)k_{\alpha\beta}+\nu\delta_{\alpha\beta}k_{\phi\phi}\right]w_3(\mathbf{X}') \\ +\frac{Bw_3(\mathbf{X}')}{8}\left\{\left[k_{\alpha\beta}(1-\nu)+\nu\delta_{\alpha\beta}k_{\phi\phi}\right](-v-5)\right. \\ \left.+\left[k_{\alpha\beta}(\nu-1)+\delta_{\alpha\beta}k_{\phi\phi}\right](1-3v)\right\} \quad (52)$$

Appendix C: Appendix

The quadratic continuous shape functions for the boundary are defined as:

$$\Phi^1(\xi)=\frac{1}{2}\xi(\xi-1) \\ \Phi^2(\xi)=(1-\xi)(1+\xi) \\ \Phi^3(\xi)=\frac{1}{2}\xi(\xi+1) \quad (53)$$

For the case of semi-discontinuous boundary elements:

$$\Phi^1_{S1}(\xi)=\frac{9}{10}\xi(\xi-1) \\ \Phi^1_{S3}(\xi)=\frac{6}{10}\xi\left(\xi-\frac{2}{3}\right) \\ \Phi^2_{S1}(\xi)=-\frac{3}{2}(\xi-1)\left(\xi+\frac{2}{3}\right) \\ \Phi^2_{S3}(\xi)=-\frac{3}{2}(\xi+1)\left(\xi-\frac{2}{3}\right) \\ \Phi^3_{S1}(\xi)=\frac{6}{10}\xi\left(\xi+\frac{2}{3}\right) \\ \Phi^3_{S3}(\xi)=\frac{9}{10}\xi(\xi+1) \quad (54)$$

where Φ^m_{S1} correspond to nodes placed at $\xi=-\frac{2}{3},0,+1$, while Φ^m_{S3} is for nodes placed at $\xi=-1,0,+\frac{2}{3}$. See figure Appendix B:. The position of the internal node in semi-discontinuous element is chosen arbitrarily at $-\frac{2}{3}$

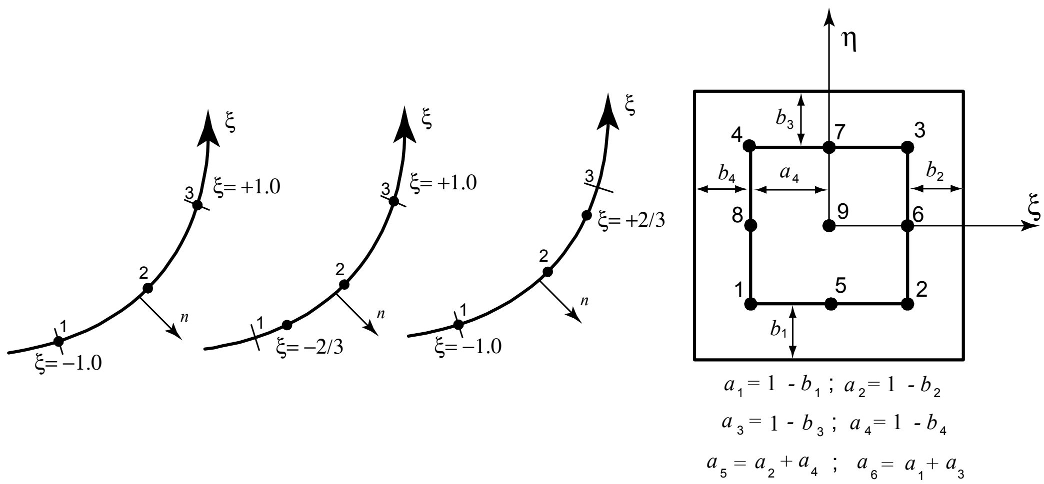

Figure 8 : Types of elements used.

or $+\frac{2}{3}$, not very close to the element end point to avoid near singularity problems.

J_n is the Jacobian of transformation for boundary elements, and is defined as:

$$J_n(\xi) = \sqrt{\frac{\partial x_\theta(\xi)}{\partial \xi}\frac{\partial x_\theta(\xi)}{\partial \xi}} \tag{55}$$

where $\frac{\partial x_\theta(\xi)}{\partial \xi}$ is the derivative of the global coordinates x_θ with respect to the local coordinate ξ.

The quadratic quadrilateral shape functions for the domain cells are given as:

$$\begin{aligned}
\Psi^1(\xi,\eta) &= \xi\eta\frac{(-a_2+\xi)(-a_3+\eta)}{a_4a_5a_1a_6}\\
\Psi^2(\xi,\eta) &= \xi\eta\frac{(a_4+\xi)(-a_3+\eta)}{a_2a_5a_1a_6}\\
\Psi^3(\xi,\eta) &= \xi\eta\frac{(a_4+\xi)(a_1+\eta)}{a_2a_5a_3a_6}\\
\Psi^4(\xi,\eta) &= \xi\eta\frac{(-a_2+\xi)(a_1+\eta)}{a_4a_5a_3a_6}\\
\Psi^5(\xi,\eta) &= \eta\left((a_2-\xi)(a_4+\xi)\right)\frac{(-a_3+\eta)}{a_2a_4a_1a_6}\\
\Psi^6(\xi,\eta) &= \xi\left((a_3-\eta)(a_1+\eta)\right)\frac{(a_4+\xi)}{a_1a_3a_2a_5}\\
\Psi^7(\xi,\eta) &= \eta\left((a_2-\xi)(a_4+\xi)\right)\frac{(a_1+\eta)}{a_2a_4a_3a_6}\\
\Psi^8(\xi,\eta) &= \xi\left((a_3-\eta)(a_1+\eta)\right)\frac{(-a_2+\xi)}{a_1a_3a_4a_5}\\
\Psi^9(\xi,\eta) &= \left((a_2-\xi)(a_4+\xi)\right)\frac{((a_3-\eta)(a_1+\eta))}{a_2a_4a_1a_3}
\end{aligned} \tag{56}$$

The parameters a in equation 56 are defined in figure 8. The continuous case is obtained when, $a_1 = a_2 = a_3 = a_4 = 1.0$. The discontinuous case requires values in the range 0.0 to 1.0, to be chosen for a_1, a_2, a_3, a_4. In this work the following parameter were chosen for the totally discontinuous case: $a_1 = a_2 = a_3 = a_4 = 2/3$. A whole range of transitional cells can be generated by varying one or more of these parameters (semi discontinuous cells).

The Jacobian of transformation for cell elements is defined as:

$$J_k(\xi,\eta) = \sqrt{\left(N_{31}^2 + N_{32}^2 + N_{33}^2\right)} \tag{57}$$

where N_{ij} is a minor of

$$\begin{bmatrix}
\frac{\partial x_1(\xi,\eta)}{\partial \xi} & \frac{\partial x_2(\xi,\eta)}{\partial \xi} & \frac{\partial x_3(\xi,\eta)}{\partial \xi}\\
\frac{\partial x_1(\eta)}{\partial \eta} & \frac{\partial x_2(\xi,\eta)}{\partial \eta} & \frac{\partial x_3(\xi,\eta)}{\partial \eta}\\
1 & 1 & 1
\end{bmatrix} \tag{58}$$