Evaluation of T-stress for An Interface Crack between Dissimilar Anisotropic Materials Using the Boundary Element Method

P.D. Shah[1], C.L. Tan[1,2], X. Wang[1]

Abstract: In this paper, the path independent mutual or M-integral for the computation of the T-stress for interface cracks between dissimilar anisotropic, linear elastic solids, is developed. The required auxiliary field solution is derived from the solution of the problem of an anisotropic composite wedge subjected to a point force at its apex. The Boundary Element Method (BEM) is employed for the numerical stress analysis in which special crack-tip elements with the proper oscillatory traction singularity are used. The successful implementation of the procedure for evaluating the T-stress in a bi-material interface crack and its application are demonstrated by numerical examples.

keyword: T-stress, Mutual M-integral, Interface crack, Anisotropic elasticity, Boundary element method (BEM).

1 Introduction

The study of cracks along the interface between dissimilar materials has become increasingly important due to structural integrity and reliability issues in modern engineered materials such as composites, thin film coatings, bi-crystals and other multiphase material systems. At the crack-tip, the stress singularity is oscillatory in nature and it has been a subject of extensive study and discussion over the years [e.g. Williams (1959), Erdogan, (1963), Comninou (1977, 1990), Rice (1988), Ting (1990), Suo (1990)]. The stress intensity factors that characterize this near-tip stress field, K_I and K_{II}, are coupled and they are often written in complex form as $\mathbf{K} = K_I + \boldsymbol{i}K_{II}$, where $\boldsymbol{i} = \sqrt{(-1)}$; schemes for the numerical evaluation of these parameters have also been developed over the years [see, e.g., Wang and Yuan (1983), Matos *et al.* (1989), Yuuki and Cho (1989) , Tan, Gao and Afagh (1992), Mizayaki, *et al*. (1993), Chow and Atluri (1995), Chow *et al.* (1995), Beom and Atluri (1996), Banks-Sills (1997), Sladek *et al.* (2004)] .

In fracture mechanics analysis of cracked bodies, the so-called T-stress is also increasingly being recognized as an important second parameter to characterize the near crack-tip field. It corresponds to the leading non-singular term of the Williams' (1957) eigenfunction series expansion for the elastic stress distribution in the vicinity of a crack-tip and represents the stress acting parallel to the plane of the crack. Numerous studies over the years have found its use in, for example, offering better prediction of the shape and extent of plastic zones, crack path stability as well as fracture toughness of elastic solids under different levels stress constraint around the crack-tip [see, e.g. Larsson and Carlsson (1973), Rice (1974), Cotterell and Rice (1980), Betegon and Hancock (1991), Tvergaard and Hutchinson (1994), Smith, Ayatollahi and Pavier (2001)]. Its consideration as an additional parameter to the traditional parameter, namely, the stress intensity factor or the J-integral, in fracture assessment procedures in engineering design codes (see, e.g., Ainsworth *et al*., 2000) has further reinforced its practical significance.

The means to obtain the T-stress for homogenous, isotropic materials has been quite extensively investigated over the years and several numerical approaches for its evaluation exist in the literature. Works on the evaluation of the T-stress in orthotropic, functionally-graded and anisotropic homogenous materials are, on the other hand, still relatively scarce and more recent. They include those of Ma, *et al*. (1997), Yang and Yuan (2000), Kim and Paulino (2004), Song (2004), and, Shah, Tan and Wang (2006). Among the various schemes considered is the path independent mutual- or M-integral method which was first devised by Kfouri (1986) based on Eshelby's (1975) work, to obtain the T-stress in homogenous isotropic cracked bodies. This approach has been found to be perhaps the most reliable. Unlike many of the other schemes which are based directly on the asymptotic eigenfunction series expansion of the field

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solutions, this method involves field values along the path of a contour sufficiently remote from the crack tip, thereby, averting the effects of the stress singularity near the crack tip.

In comparison with homogenous cracked bodies, studies dealing with the determination of T-stress for a bimaterial interface crack are very scarce indeed. For the two-dimensional (2D) interface crack between isotropic materials, Sladek and Sladek (1997) extended their earlier work for cracks in homogeneous materials [Sladek, Sladek and Fedelinski (1997)] in which they derived the M-integral method using Betti-Rayleigh's reciprocal work theorem and adopted an auxiliary field which has an order of singularity higher than that used by Kfouri (1986). Their development is more suitable for implementation with the Boundary Element Method (BEM). Moon and Earmme (1998) have also investigated, in an analytical approach, the T-stress under in-plane and anti-plane loading for an interface crack between dissimilar isotropic, infinite, bi-materials strips using the M-integral method. Recently, Fett and Rizzi (2004) have obtained the T-stress for isotropic bimaterial interface cracks using the weight function method in conjunction with Finite Element Method (FEM).

Kim, Moon and Earmme (2001) extended the analytical work of Moon and Earmme (1998) mentioned above, to an interface crack between anisotropic bodies of infinite and semi-infinite extents using the M-integral method; no numerical solutions were reported, however. In a recent contribution, Song (2005) has presented a relatively new, semi-analytical boundary element method based on finite elements, called the scaled-boundary finite element method, to obtain T-stress values for cracks at isotropic and non-isotropic bimaterial interfaces. Again, no numerical solution for a generally anisotropic bi-material interface crack problem was presented in the work, even though it can be obtained by the technique. There is indeed paucity of numerical T-stress solutions for the interface crack problem in generally anisotropic bi-materials, and to the authors' knowledge, the use of the BEM to this end has hitherto also not been reported in the open literature.

In the following sections, the M-integral approach implemented by Sladek and Sladek (1997) for obtaining the T-stress for isotropic bimaterial interface cracks in conjunction with BEM is extended to the generally anisotropic case. A relatively simpler approach to obtain the terms required to evaluate the M-integral and thence, the T-stress, will be discussed. The veracity and application of this approach will be illustrated by numerical examples.

2 Bimaterial Interface Crack in 2D Anisotropy

For two-dimensional anisotropic elasticity, the constitutive equation is given by Eq. (1) below:

$$\left\{\begin{array}{c}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{array}\right\}=\left[\begin{array}{ccc}a_{11}&a_{12}&a_{16}\\ a_{12}&a_{22}&a_{26}\\ a_{16}&a_{26}&a_{66}\end{array}\right]\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{array}\right\} \tag{1}$$

where, ε_{ij}, σ_{ij} are the strains and stresses, respectively, and a_{ij} are the compliances. The above relation holds true for plane stress problems. For plane strain conditions, a_{ij} is replaced with b_{ij} as follows,

$$b_{ij}=a_{ij}-a_{i3}a_{j3}/a_{33} \tag{2}$$

Under plane stress or plane strain condition, the elastic field in an anisotropic material can be expressed in terms of complex functions $f_1(z_1)$ and $f_2(z_2)$ where z_i is a generalized complex variable in terms of a complex parameter μ_i, as follows: $z_i = x_1+\mu_i x_2$ (i = 1,2). Here, μ_i are the roots for the characteristic equation, Eq. (3), for the plane anisotropic body in stable equilibrium [Lekhnitskii (1968)].

$$a_{11}\mu^4-2a_{16}\mu^3+(2a_{12}+a_{66})\mu^2-2a_{26}\mu+a_{22}=0 \tag{3}$$

The stresses and displacements can then be expressed in terms of the complex functions and material properties as follows,

$$\sigma_{1i}=-2Re\left[\sum_{j=1}^{2}B_{ij}\mu_j f'_j(z_j)\right] \tag{4a}$$

$$\sigma_{2i}=2Re\left[\sum_{j=1}^{2}B_{ij}f'_j(z_j)\right] \tag{4b}$$

$$u_i=2Re\left[\sum_{j=1}^{2}\overline{A}_{ij}f_j(z_j)\right] \quad (j=1,2) \tag{5}$$

where **Re**[..] denotes the real part of the complex quantities within the parentheses, **B** and $\overline{A}$ are complex matrices the elements of which are dependent upon material properties, given by

$$\mathbf{B} = \begin{bmatrix} -\mu_1 & -\mu_2 \\ 1 & 1 \end{bmatrix} \tag{6}$$

$$\overline{A}_{1j} = a_{11}\mu_j^2 + a_{12} - a_{16}\mu_j \tag{7a}$$

$$\overline{A}_{2j} = a_{21}\mu_j + a_{22}/\mu_j - a_{26} \tag{7b}$$

The stress field near a crack tip can be generally written as

$$\sigma_{ij}^{(m)} = f(\mathbf{K}, r, \gamma, \theta) + C^{(m)}\delta_{i1j1}T + O(r^{\alpha}) \tag{8}$$

where r, θ are polar coordinates with origins at crack tip, γ is the bimaterial constant, m is 1 for material 1 or 2 for material 2 and αg 0. The focus here is on the second term of Eq. (8) which contains the T-stress. The material specific coefficient $C^{(m)}$ may be obtained by the steps as follows.

Along the bonded interface between the materials, displacement $u_i^{(m)}$ being continuous yields,

$$u_{i,1}^{(1)} = u_{i,1}^{(2)} \tag{9}$$

which implies,

$$u_{1,1}^{(1)} = u_{1,1}^{(2)} \tag{10}$$

Along the interface,

$$u_{1,1}^{(m)} \Rightarrow \varepsilon_{11}^{(m)} = a_{11}^{'(m)}\sigma_{11}^{(m)} \tag{11}$$

where $a_{11}^{'(m)}$ is $a_{11}^{(m)}$ transformed in the direction parallel to the crack plane.

From Eqs. (10) and (11)

$$a_{11}^{'(2)}\sigma_{11}^{(2)} = a_{11}^{'(1)}\sigma_{11}^{(1)} \tag{12}$$

$$\sigma_{11}^{(1)} = \frac{a_{11}^{'(2)}}{a_{11}^{'(1)}}\sigma_{11}^{(2)} \tag{13}$$

Thus,

$$C^{(2)} = 1 \tag{14}$$

$$C^{(1)} = \frac{a_{11}^{'(2)}}{a_{11}^{'(1)}} \tag{15}$$

It should be noted that, in the case of a bimaterial interface crack, there are two in-plane T-stresses, depending on which material region it is defined for; the two values are, however, related, as is explained here. In the above development, the T-stress is defined with respect to material 2 (Figure 1). In order to obtain the T-stress with respect to material 1, $C^{(1)}$ should be set to unity and the value of $C^{(2)}$ follows. The T-stress values in material 1 and 2 can be easily shown to be related according to

$$T^{(1)}a_{11}^{'(1)} = T^{(2)}a_{11}^{'(2)} \tag{16}$$

In what follows, the T-stress is defined along the interface in material 2 and will simply be denoted as Tinstead of $T^{(2)}$.

3 *M*-integral for *T*-stress Evaluation

The work of Sladek and Sladek (1997) for isotropic bimaterials is extended to general anisotropy in the present study. The path independent J-integral along contour Γ_0, shown in Figure 1 is given as

$$J = \int_{\Gamma_0} (Wn_1 - t_i u_{i,1})\, d\Gamma \tag{17}$$

where W is the strain energy density as

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{18}$$

Considering two independent equilibrium states, A and aux, the mutual integral, also commonly referred to as the M-integral, is expressed in terms of $J-$integral as follows,

$$M = J^{(A+aux)} - J^{(A)} - J^{(aux)} \tag{19}$$

where

$$J^{(A)} = \int_{\Gamma_0} \left[\frac{1}{2}\left(\sigma_{ij}^{A}\varepsilon_{ij}^{A}n_1\right) - \sigma_{ij}^{A}n_j u_{i,1}^{A}\right] d\Gamma \tag{20a}$$

$$J^{(aux)} = \int_{\Gamma_0} \left[\frac{1}{2}\left(\sigma_{ij}^{aux}\varepsilon_{ij}^{aux}n_1\right) - \sigma_{ij}^{aux}n_j u_{i,1}^{aux}\right] d\Gamma \tag{20b}$$

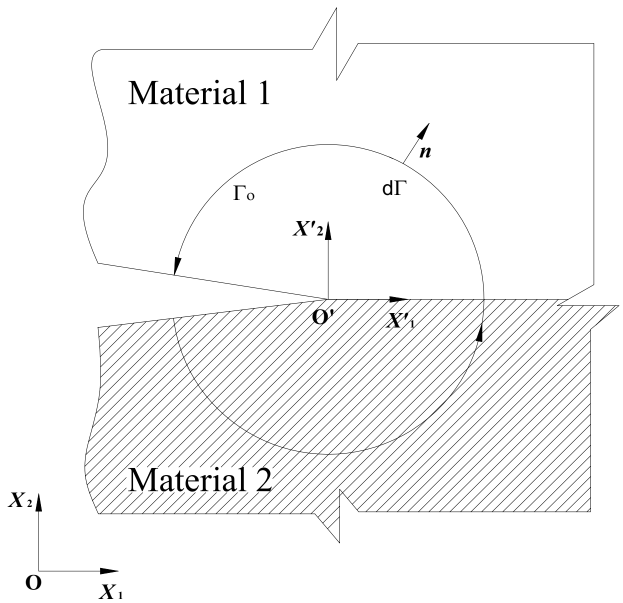


Figure 1 : Contour Γ_0 around the crack tip and material designation of bimaterial interface crack.

Thus,

$$M = \int_{\Gamma_0} \left(\sigma^A_{ij}\varepsilon^{aux}_{ij} n_1 - \sigma^A_{ij} n_j u^{aux}_{i,1} - \sigma^{aux}_{ij} n_j u^A_{i,1} \right) d\Gamma \tag{21}$$

The first state A corresponds to the boundary value problem being analysed. As is taken to be the case for cracked homogeneous body, the second state *aux*, also called the auxiliary field, is chosen to correspond to the solution of a semi-infinite crack loaded by a point (line) force f applied at the crack tip in the direction parallel to the crack plane as given in Figure 8 in the Appendix. This auxiliary field solution, in the case of interface crack between dissimilar anisotropic materials, can be derived from the solution of the problem of an anisotropic composite wedge subjected to a point force at its apex given by Chung and Ting (1995). The derivation is given in the Appendix.

Since the J-integral is path independent, so will the M-integral. It can thus be expressed in terms of an arbitrary circular contour with radius ε shrunk to zero, thus the M-integral takes the form

$$M = \lim_{\varepsilon \to 0} \int_{\Gamma_e} \left(\sigma^A_{ij}\varepsilon^{aux}_{ij} n_1 - \sigma^A_{ij} n_j u^{aux}_{i,1} - \sigma^{aux}_{ij} n_j u^A_{i,1} \right) d\Gamma \tag{22}$$

The boundedness of J-integral and thus M-integral implies that there is no contribution from the singular terms in the stress and displacement fields around the crack tip. The asymptotic stresses and displacements of state A can be split into the singular and non-singular components as follows:

$$\sigma^A_{ij} = \sigma^s_{ij} + \sigma^T_{ij} \tag{23a}$$

$$u^A_{ij} = u^s_{ij} + u^T_{ij} \tag{23b}$$

In Eq. (23), the terms with the superscript s denote those components which are singular in nature and containing the stress intensity factors; those terms with superscript T are the leading non-singular terms of the asymptotic expansion proportional to the T-stress.

The circular contour integral from θ = -π to +π of the angular functions of the singular terms of the auxiliary field in Eq (22) cancel out; this leaves only the non-vanishing contribution from the T-stress. Thus, the M-integral in Eq. (22) reduces to

$$M = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \left(\sigma^T_{ij}\varepsilon^{aux}_{ij}\delta_{1j} - \sigma^T_{ij} u^{aux}_{i,1} - \sigma^{aux}_{ij} u^T_{i,1} \right) n_j d\Gamma \tag{24}$$

where

$$\sigma^T_{ij} = \frac{a'^{(2)}_{11}}{a'^{(1)}_{11}} T\delta_{i1}\delta_{j1}, \quad \text{for } 0 \le \theta \le \pi \tag{25a}$$

$$\sigma^T_{ij} = T\delta_{i1}\delta_{j1}, \quad \text{for } -\pi \le \theta \le 0 \tag{25b}$$

In Eq. (24), the first and the second terms collectively vanish as shown below,

$$\begin{aligned}
&\sigma^T_{ij}\left(\varepsilon^{aux}_{ij}\delta_{1j} - u^{aux}_{i,1}\right) n_j \\
&\quad = \left(\sigma^T_{ij}\varepsilon^{aux}_{i1} - \sigma^T_{ij} u^{aux}_{i,1}\right) n_j \\
&\quad = \left(C^{(a)} T\delta_{i1}\delta_{j1}\varepsilon^{aux}_{i1} - C^{(a)} T\delta_{i1}\delta_{j1} u^{aux}_{i,1}\right) n_j \\
&\quad = C^{(a)}\left(T\delta_{j1}\varepsilon^{aux}_{11} - T\delta_{j1} u^{aux}_{1,1}\right) n_j \\
&\quad = C^{(a)}\left(T\delta_{j1} u^{aux}_{1,1} - T\delta_{j1} u^{aux}_{1,1}\right) n_j = 0
\end{aligned} \tag{26}$$

Equation (24) then becomes

$$\begin{aligned}
M &= -\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \left[\sigma_{ij}^{aux} u_{i,1}^{T}\right] n_j d\Gamma \\
&= -\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \left[\sigma_{ij}^{aux} n_j \varepsilon_{11}^{T}\delta_{i1}\right] d\Gamma \\
&= -\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \left[\sigma_{1j}^{aux} n_j \varepsilon_{11}^{T}\right] d\Gamma \\
&= -\lim_{\varepsilon\to 0}\int_{0}^{\pi} \left[\sigma_{1j}^{aux} n_j a_{11}^{'(1)}\sigma_{11}^{T}\right] \varepsilon d\theta \\
&\quad -\lim_{\varepsilon\to 0}\int_{-\pi}^{0} \left[\sigma_{1j}^{aux} n_j a_{11}^{'(2)}\sigma_{11}^{T}\right] \varepsilon d\theta \\
&= -\lim_{\varepsilon\to 0}\int_{0}^{\pi} \left[\sigma_{1j}^{aux} n_j a_{11}^{'(1)} \frac{a_{11}^{'(2)}}{a_{11}^{'(1)}} T\right] \varepsilon d\theta \\
&\quad -\lim_{\varepsilon\to 0}\int_{-\pi}^{0} \left[\sigma_{1j}^{aux} n_j a_{11}^{'(2)} T\right] \varepsilon d\theta \\
&= T\left\{ -a_{11}^{'(2)} \lim_{\varepsilon\to 0}\int_{0}^{\pi} \left[\sigma_{1j}^{aux} n_j\right] \varepsilon d\theta \right. \\
&\quad \left. -a_{11}^{'(2)} \lim_{\varepsilon\to 0}\int_{-\pi}^{0} \left[\sigma_{1j}^{aux} n_j\right] \varepsilon d\theta \right\} \\
&= -Ta_{11}^{'(2)} \left\{ \lim_{\varepsilon\to 0}\int_{0}^{\pi} \left[\sigma_{1j}^{aux} n_j\right] \varepsilon d\theta \right. \\
&\quad \left. +\lim_{\varepsilon\to 0}\int_{-\pi}^{0} \left[\sigma_{1j}^{aux} n_j\right] \varepsilon d\theta \right\} \\
&= -Ta_{11}^{'(2)} \left\{ \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \left[\sigma_{1j}^{aux} n_j\right] d\Gamma \right\} \\
&= -Ta_{11}^{'(2)} \{-f\} \\
&= Ta_{11}^{'(2)} f
\end{aligned} \tag{27}$$

Thus, the T-stress can be expressed in terms of M-integral as follows,

$$T = \frac{M}{a_{11}^{'(2)} f} \tag{28}$$

In order to evaluate the M-integral in the above relation, Eq. (21) is used. To evaluate T-stress values in plane strain condition, a_{11} should be replaced by b_{11} as in Eq. (2). Note that the terms in Eqs. (21) to (28) are in the local coordinates system X'_i as shown in Figure 1. In the present implementation, the M-integral is first obtained in global coordinates and later transformed into the local coordinates. The coordinate transformation of the M-integral follows similarly that of the J-integral [Kishimoto, Aoki and Sakata (1980)] as follows,

$$M_{(Local)} = M_{1(Global)}\cos\omega + M_{2(Global)}\sin\omega \tag{29}$$

where

$$M_{k(Global)} = \int_{\Gamma_0} \left(\sigma_{ij}^{A}\varepsilon_{ij}^{aux} n_k - \sigma_{ij}^{A} n_j u_{i,k}^{aux} - \sigma_{ij}^{aux} n_j u_{i,k}^{A}\right) d\Gamma \tag{30}$$

and ω is the angle of inclination of the crack with global coordinates X_i.

4 Results and discussion

The M-integral formulations to obtain the T-stress for a crack at an interface between dissimilar anisotropic materials as presented in the sections above have been implemented into the BEM code, based on the quadratic isoparametric element formulation, as used in Tan and Gao (1992) and Shiah and Tan (1999). For the fracture mechanics analysis, special crack-tip elements which represent the proper oscillatory stress singularity for the bi-material interface crack are employed [Tan and Gao (1992)]. The self-regularized Somigliana's identities for interior point solutions which are required for the M-integral contour integrations have also been implemented, as described in Shah, Tan and Wang (2006). In this study, at least two contours with different radii were considered to verify the path independence of the calculated M-integral to obtain T-stress. The contour radii were varied between 0.4 to 0.6 times the modeled crack length and the discrepancies between the T-stress values obtained for the different contour radii were all less than 2% for the cases treated.

Three examples are presented here to demonstrate the veracity and capability of the developed formulations. The first problem considered is a rectangular plate with a centre, interface crack between two dissimilar isotropic materials; numerical solutions of the $T-$stress for this problem are available for comparison with the present approach using the algorithm for anisotropic elasticity. The second example has the same geometry and loading condition as the first problem, except that the two materials are treated as orthotropic. For the third example, a bi-crystal disc with a crack at the interface is analyzed.

4.1 Example 1

Due to paucity of T-stress solutions in the literature for the bi-material interface crack problem in the generally anisotropic case, the first example investigated was a bi-material interface crack problem in isotropy. The example serves as a check of the current algorithm as being also applicable to bi-material isotropy. It was a rectangular plate comprising of dissimilar isotropic materials 1 and 2 with engineering material constants (E_1, ν_1) and (E_2, ν_2), respectively, containing a central crack at the interface, as shown in Figure 2. The centre cracked plate (CCP) was subjected to uniform tensile stress σ_o at the ends and plane strain conditions were assumed. With reference to this figure, the geometric cases considered were a/W = 0.15 and 0.5 and H/W = 2; the material properties analyzed were E_1/E_2 = 1, 2, 5 and 10, with ν_1=ν_2 = 0.3. These cases have been treated by Sladek and Sladek (1997) and Song (2005). Figure 3 shows the typical BEM mesh used in the present study. The normalized results of T/σ_o obtained in the present work are listed in Table 1 together with those obtained by the above-mentioned authors. It should be noted, however, that the numerical values attributed to Sladek and Sladek (1997) here are digitized quantities reported by Song (2005) of the graphical result presented in the formers' paper. From Table 1, it can be seen that the present BEM results are in excellent agreement with those obtained by Song (2005), albeit slightly less so, with those by Sladek and Sladek (1997). It is also worth noting that the T-stress at the crack tips have negative values but they decrease quite rapidly in magnitude with increasing values of the E_1/E_2 ratio, signifying an enhancement of stress triaxiality with this ratio for a given crack size.

4.2 Example 2

The T-stress for an interface crack between dissimilar orthotropic materials was obtained for the same CCP specimen under remote tension, σ_o, as in the previous example, but for the following relative crack lengths a/W = 0.1, 0.2, 0.3, 0.4 and 0.5. The material properties chosen in the analysis are given below.

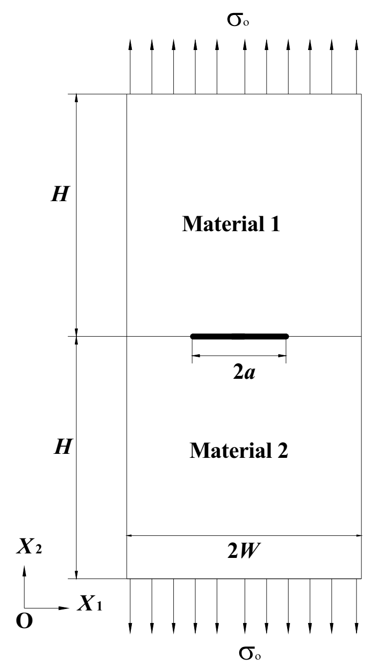


Figure 2 : A centre cracked plate (CCP) under remote load σ_o.

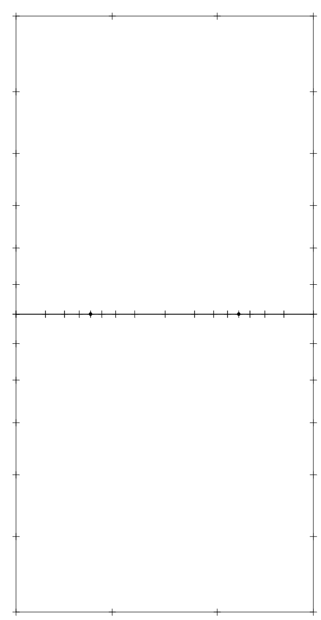


Figure 3 : BEM mesh for Examples 1 and 2: a/W = 0.5

For material 1: E_{11}= 1000; E_{22}= 500, G_{12}=100.1, ν_{12}= 0.3.

For material 2: E_{11}= 200, E_{22}= 60, G_{12}= 15.7, ν_{12}= 0.3.

Plane stress conditions were assumed in this example.

Table 1 : Normalised T-stress, T/σ_o, for a dissimilar isotropic bi-material rectangular CCP specimen with H/W=2.

a/W	$\frac{E_1}{E_2}$	***T/σ$_o$***				
		Present	**Song (2005)**	**% Diff.**	**Sladek & Sladek (1997)**	**% Diff.**
0.15	1	-1.016	-1.015	-0.1	-1.043	2.6
	2	-0.681	-0.681	0	-0.706	3.5
	5	-0.348	-0.347	-0.3	-0.366	4.9
	10	-0.193	-0.192	-0.5	-0.204	5.4
0.5	1	-1.257	-1.260	0.2	-1.272	1.2
	2	-0.846	-0.847	0.1	-0.861	1.7
	5	-0.436	-0.437	0.2	-0.450	3.1
	10	-0.244	-0.244	0	-0.260	6.2

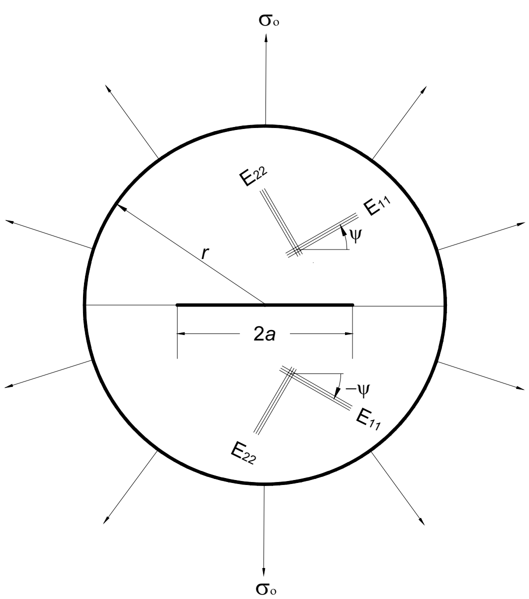


Figure 4 : Silicon [110] disc with dissimilar halves containing a central crack along the interface, under uniform radial tension σ_o.

The numerical results of the normalized t-stress, T/σ_o, are shown in Table 2 for the range of crack lengths considered. As before, the numerical values of the T-stress are negative, their magnitudes increasing gradually with the size of the crack, signifying decreasing stress constraint at the crack-tip. Also shown for comparison in Table 2 are the corresponding results when materials 1 and 2 are isotropic such that the ratio of their Young's moduli, E_1/E_2 =500/60, and their Poisson's ratios being equal in value of 0.3. It is evident that orthotropy of the materials has quite a significant influence on the value of the T-stress.

Table 2 : The normalized T-stress, T/σ_o, for different relative crack lengths, a/W, at the interface crack in an orthotropic bi-material rectangular CCP specimen, H/W=2; corresponding isotropic bi-material results shown for comparison.

a/W	***T/σ$_o$***	
	Orthotropic	**Isotropic**
0.1	-0.522	-0.222
0.2	-0.534	-0.227
0.3	-0.556	-0.238
0.4	-0.591	-0.255
0.5	-0.651	-0.283

4.3 Example 3

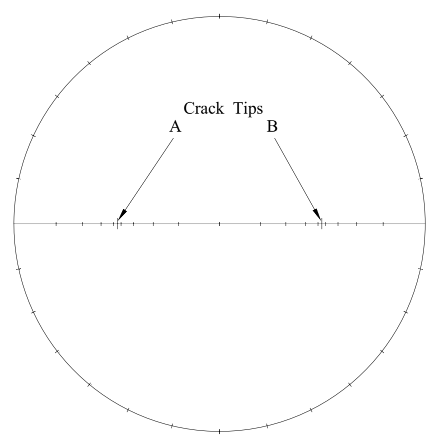


Figure 5 : Boundary element mesh: Example 3, a/r= 0.5

In order to demonstrate the application of the current work in dissimilar generally anisotropic bi-material in-

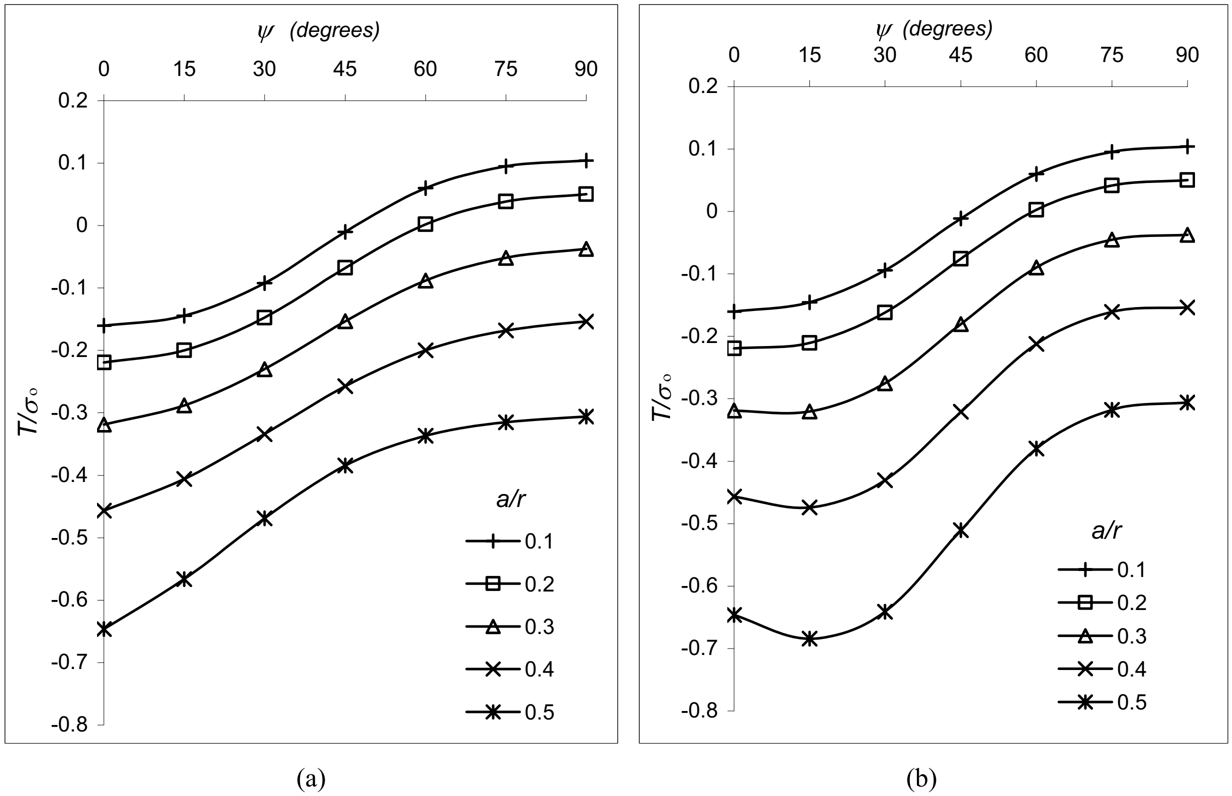


Figure 6 : a) Variation of the normalised T-stress, T/σ_o, at crack tip A with angle ψ in centrally cracked interface silicon [110] disc; b) Variation of the normalised T-stress, T/σ_o, in crack tip B with angle ψ in centrally cracked interface silicon [110] disc.

terface crack problems, a cracked anisotropic bi-material disc subjected to uniform radial tension σ_o, as shown in Figure 4 was investigated. The material chosen was single crystal silicon in the [110] plane. Single crystal silicon is, for example, used widely in electronic devices. The engineering constants in this plane [110] are given below; they were calculated from the elastic compliance constants of the material available in Simmons and Wang (1971).

E_{11} = 169.1 GPa

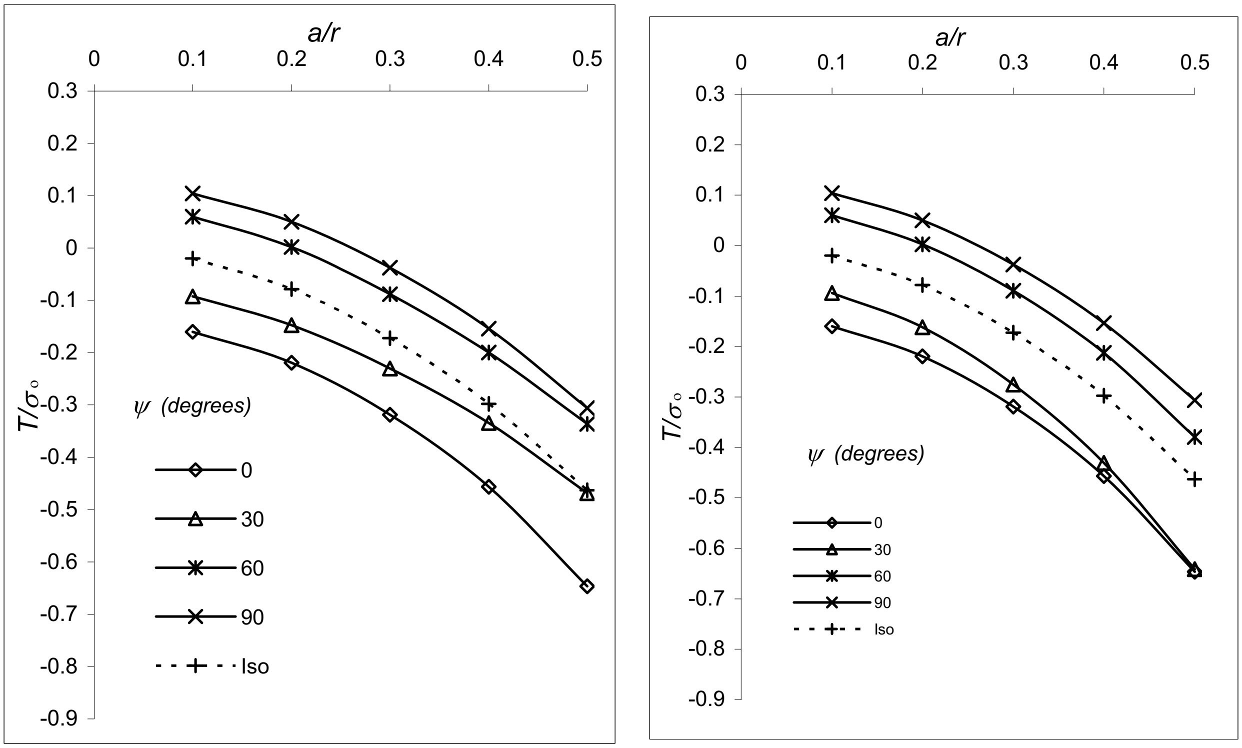
E_{22} = 130.1 GPa

G_{12} = 79.6 GPa

ν_{12} = 0.362

In order to simulate dissimilar anisotropic interface materials, the angle of orientation of the material principal axes with the global Cartesian axes, ψ, was varied from 0^o to 90^o, in 15^o increments, for material 1; for material 2 the material principal axes was rotated in the reverse sense, that is, $-\psi$, in each case by the same magnitude simultaneously. The relative crack lengths analyzed were a/r = 0.1, 0.2, 0.3, 0.4 and 0.5. Plane stress conditions were assumed for this study. For the purpose of comparison, the analysis was also carried out for an isotropic elastic disc with the same geometric parameters and loading condition. A typical BEM mesh used in the analysis is shown in Figure 5.

The effect of ψ on the normalized T-stress values, T/σ_o, for both crack tips are presented in Figures 6a and 6b. Figures 7a and 7b show the corresponding variations of normalized T-stress with a/r for the different angles of ψ analyzed. Also shown in these figures in dotted lines are the corresponding results for an isotropic disc. Although the trends of the variations are similar between the anisotropic and isotropic cases, it is shown here again that the degree of anisotropy clearly has a significant effect on the value of the T-stress.



(a) (b)

Figure 7 : a) Variation of the normalised T-stress,T/σ_o, in crack tip 1 with relative crack size a/r in centrally cracked interface silicon [110] disc; b) Variation of the normalised T-stress, T/σ_o, in crack tip B with relative crack size a/r in centrally cracked interface silicon [110] disc.

5 Conclusions

In this study, an approach to obtain the T-stress for an interface crack between dissimilar anisotropic elastic materials by the M-integral in conjunction with BEM stress analysis has been presented. This approach is general and simpler to implement than what has been developed in the literature thus far for evaluating the $T-$stress in anisotropic bi-material interface crack problems. Numerical examples have been presented to demonstrate the successful implementation and applicability of the formulation. One of these examples involved isotropic bi-materials where numerical solutions are available, and agreement between the present results and those in the literature is excellent. The numerical results have also demonstrated the significant influence material anisotropy has on the magnitudes of the T-stress for a given cracked geometry.

Acknowledgement: The authors would like to express their sincere thanks to Dr. C. Song, University of New South Wales, for verifying some of the new numerical results here using an alternative numerical method, namely, the scaled boundary finite element method.

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Appendix A:

The derivation of auxiliary fields in the M-integral, Eq. (21) is presented here. For the sake of brevity, the superscript ^{aux} is dropped in what follows. The auxiliary fields are obtained with respect to the global coordinates X_i as shown in Figure 8.

With reference to the Stroh's formalism for 2D anisotropic elasticity [Ting (1996)], the displacement u and stress function ϕ for wth wedge of a composite wedge comprising of two anisotropic materials 1 and 2 and subjected to a concentrated force $\mathbf{f}$ at the wedge apex are [Chung and Ting (1995)]

$$u^{(w)} = -\frac{\ln r}{\pi}\mathbf{h}^{(w)} - \mathbf{S}^{(w)}(\theta)\mathbf{h}^{(w)} + \mathbf{H}^{(w)}(\theta)\mathbf{g}^{(w)} + \mathbf{u}_0^{(w)} \quad (31)$$

$$\phi^{(w)} = \frac{\ln r}{\pi}\mathbf{g}^{(w)} + \mathbf{S}^{T(w)}(\theta)\mathbf{g}^{(w)} + \mathbf{L}^{(w)}(\theta)\mathbf{h}^{(w)} \quad (32)$$

where

$\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$ and $\mathbf{L}(\theta)$ are Barnett-Lothe tensors [Ting

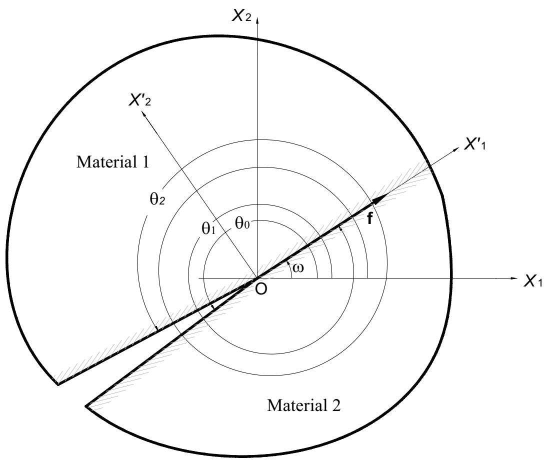

Figure 8 : A composite wedge comprising of two anisotropic materials and point force **f** acting on the apex.

(1996)] given as

$$\mathbf{S}(\theta) = \frac{1}{\pi}\int_0^{\theta} \mathbf{N}_1(\omega)d\omega \tag{33a}$$

$$\mathbf{H}(\theta) = \frac{1}{\pi}\int_0^{\theta} \mathbf{N}_2(\omega)d\omega \tag{33b}$$

$$\mathbf{L}(\theta) = -\frac{1}{\pi}\int_0^{\theta} \mathbf{N}_3(\omega)d\omega \tag{33c}$$

This definition of Barnett-Lothe tensors holds true for degenerate materials. The N_is, elements of the fundamental elasticity matrix **N** [Ingebrigtsen and Tonning (1969)] are given as

$$\mathbf{N}_1(\omega) = -\mathbf{T}^{-1}(\omega)\mathbf{R}^T(\omega), \tag{33d}$$

$$\mathbf{N}_2(\omega) = \mathbf{T}^{-1}(\omega), \tag{33e}$$

$$\mathbf{N}_3(\omega) = \mathbf{R}(\omega)\mathbf{T}^{-1}(\omega)\mathbf{R}^T(\omega) - \mathbf{Q}(\omega) \tag{33f}$$

The matrices **Q**, **R** and **T** are given in terms of the stiffness constants and are 2x2 matrices for 2D plane problem as follows,

$$\mathbf{Q}_{ik} = C_{i1k1},$$

$$\mathbf{Q} = \begin{bmatrix} C_{1111} & C_{1121} \\ C_{1121} & C_{2121} \end{bmatrix} = \begin{bmatrix} \mathrm{C}_{11} & \mathrm{C}_{16} \\ \mathrm{C}_{16} & \mathrm{C}_{66} \end{bmatrix} \tag{34a}$$

$$\mathbf{R}_{ik} = C_{i1k2},$$

$$\mathbf{R} = \begin{bmatrix} C_{1112} & C_{1122} \\ C_{2112} & C_{2122} \end{bmatrix} = \begin{bmatrix} \mathrm{C}_{16} & \mathrm{C}_{12} \\ \mathrm{C}_{66} & \mathrm{C}_{26} \end{bmatrix} \tag{34b}$$

$$\mathbf{T}_{ik} = C_{i2k2},$$

$$\mathbf{T} = \begin{bmatrix} C_{1212} & C_{1222} \\ C_{1222} & C_{2222} \end{bmatrix} = \begin{bmatrix} \mathrm{C}_{66} & \mathrm{C}_{26} \\ \mathrm{C}_{26} & \mathrm{C}_{22} \end{bmatrix} \tag{34c}$$

C_{ijkl} represents the reduced stiffnesses in the constitutive equation as,

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{35}$$

The elastic matrices in rotated coordinate system are given as,

$$\mathbf{Q}(\omega) = \mathbf{Q}cos^2\theta + (\mathbf{R} + \mathbf{R}^T)sin\theta cos\theta + \mathbf{T}sin^2\theta \tag{36a}$$

$$\mathbf{R}(\omega) = \mathbf{R}cos^2\theta + (\mathbf{T} - \mathbf{Q})sin\theta cos\theta - \mathbf{R}^T sin^2\theta \tag{36b}$$

$$\mathbf{T}(\omega) = \mathbf{T}cos^2\theta - (\mathbf{R} + \mathbf{R}^T)sin\theta cos\theta + \mathbf{Q}sin^2\theta \tag{36c}$$

For the wedge, the tractions are null along $\theta = \theta_n$ and $\theta = \theta_0$, thus in Eq. (32)

$$\mathbf{g}^{(w)} = 0. \tag{37}$$

The matrix $\mathbf{h}^{(w)}$ is given as

$$\mathbf{h}^{(w)} = \underline{\mathbf{L}}^{-1}\,\mathbf{f} \tag{38}$$

where

$$\underline{\mathbf{L}} = \frac{1}{\pi}\int_{\theta_0}^{\theta_n} -\mathbf{N}_3(\omega)d\omega \tag{39}$$

and **f** is the point force vector applied at the wedge apex, that is,

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{40}$$

Due to the displacement continuity along the interface, **h** is invariant for all wedges in the composite wedge. Thus, in following discussion, $\mathbf{h}^{(w)}$ will be replaced by **h**.

Equations (31) and (32) then reduce to

$$u^{(w)} = -\frac{\ln r}{\pi}\mathbf{h} - \mathbf{S}^{(w)}(\theta)\mathbf{h} + \mathbf{u}_0^{(w)} \tag{41}$$

$$\phi^{(w)} = \mathbf{L}^{(w)}(\theta)\mathbf{h} \tag{42}$$

The displacement derivatives with respect to coordinates X_i' are obtained as follows,

$$\begin{aligned} u_{i,j}^{(w)} &= \frac{\partial u_i^{(w)}}{\partial r}\frac{\partial r}{\partial X_j} + \frac{\partial u_i^{(w)}}{\partial \theta}\frac{\partial \theta}{\partial X_j} \\ &= \frac{\partial u_i^{(w)}}{\partial r} n_j(\theta) + \frac{\partial u_i^{(w)}}{\partial \theta}\frac{1}{r} m_j(\theta), \\ &= \frac{1}{\pi r}\left\{ -h_i n_j(\theta) - (\mathbf{N}_1(\theta)\mathbf{h})_i m_j(\theta) \right\} \end{aligned} \tag{43}$$

where

$$n_1(\theta) = cos\theta \tag{44a}$$

$$n_2(\theta) = sin\theta \tag{44b}$$

$$m_1(\theta) = -sin\theta \tag{44c}$$

$$m_2(\theta) = cos\theta \tag{44d}$$

The stresses are obtained as follows,

$$\begin{aligned} \sigma_{1j}^{(w)} &= -\phi_{j,2}^{(w)} \\ &= -\frac{\partial \phi_j^{(w)}}{\partial X_2} = -\left[\frac{\partial \phi_j^{(w)}}{\partial r}\frac{\partial r}{\partial X_2} + \frac{\partial \phi_j^{(w)}}{\partial \theta}\frac{\partial \theta}{\partial X_2}\right] \\ &= -\frac{\phi_j^{(w)}}{\partial \theta}\frac{1}{r} m_2(\theta) = \frac{1}{\pi r}[\mathbf{N}_3(\theta)\mathbf{h}]_j m_2(\theta) \end{aligned} \tag{45a}$$

and,

$$\sigma_{2j}^{(w)} = \phi_{j,1}^{(w)} = -\frac{1}{\pi r}[\mathbf{N}_3(\theta)\mathbf{h}]_j m_1(\theta) \tag{45b}$$

In order to utilize the above expressions to obtain the auxiliary field values applicable to the M-integral, Eq. (21), for an interface crack between dissimilar anisotropic materials, the integration in Eq. (39) can be carried out with limits θ_0 to $\theta_1 = \pi + \theta_0$ and θ_1 to $\theta_2 = 2\pi + \theta_0$ (Figure 8). The force applied at the crack tip is taken as unity for simplicity.