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Optimal Tetrahedralization for Small Polyhedron: A New Local Transformation Strategy for 3-D Mesh Generation and Mesh Improvement

Liu Jianfei[1,2], **Sun Shuli**[1,3], **and Wang Dachuan**[1]

Abstract: Local transformation, or topological re-connection, is one of effective procedures of mesh improvement method, especially in three-dimensional situation. The commonly used local transformations for tetrahedral mesh involve changing in mesh topology (i.e. node-element connectivity relationship) within a relatively small region composed of several tetrahedra, such as 2-3 flip, 3-2 flip, 2-2 flip, 4-4 flip, etc. Although these local transformations are easy to implement and effective in removing poorly-shaped tetrahedra, it is still possible to improve the quality of mesh further by expanding the space of transformation region. In this paper, the concept of *optimal tetrahedralization for small polyhedron* and corresponding *small polyhedron re-connection* (or *SPR* for abbreviating) approach are presented. As a new local transformation scheme and a potential substitute for the existing ones, the presented method seeks for the optimal tetrahedralization of a polyhedron with a certain number of vertexes and faces (typically composed of 20 to 40 tetrahedral elements) rather than simply making a selection from several possible configurations within a small region that consists of several tetrahedra, and therefore will give better results than existing ones. Despite of quite high time complexity of the optimal searching algorithm, the presented approach can be significantly speeded up by some deliberate strategies. Experimental investigation and results on tetrahedral finite element mesh show that the SPR approach is quite effective in improvement of mesh quality with acceptable time cost, and more suitable for combining with smoothing approach. Although further researches are required for a more definite conclusion, the presented approach can be utilized as a powerful and effective tool for tetrahedral mesh generation and mesh improvement. We believe that the superior performance of the SPR approach makes it worthy to be further studied.

keyword: Mesh improvement, Tetrahedral mesh, Local transformation, Elementary flips, Optimal tetrahedralization, Small polyhedron re-connection (SPR)

1 Introduction

The quality of mesh has been known to affect both the efficiency and the accuracy of the numerical solution, especially for those engineering problems with complex three-dimensional geometric domain. Although great efforts have been made to ensure a good configuration of nodes and elements in all of mesh generating methods, for instance in some recent researches [Li, Teng and Wan (2001); Liu (2003); Chung, Choi and Kim (2003); Lo and Wang (2005)], it is still possible and necessary to improve the quality of mesh further. Lots of works of mesh improvement have been done for three-dimensional case[Joe (1991a, 1991b, 1995); Dari and Buscaglia (1994); Zavattieri, Dari and Buscaglia (1996); Freitag and Ollivier-Gooch (1997); Lo (1997); Freitag and Plassmann (2000); Sun and Liu (2003)]. Of course, additional computational efforts are needed for mesh optimization or improvement; however, researches have indicated that the cost of mesh improvement is significantly lower than the cost of solving the problem on a poorer quality mesh [Freitag and Ollivier-Gooch (2000)]. An alternative approach to avoid the influence of poor quality meshes may be the meshless method, see for example in [Zhu, Zhang and Atluri (1998); Atluri and Shen (2002)], which eliminates the tedious step of mesh generation.

Basically there are two main categories of mesh improvement procedure. The first is *topological optimization*, also called *local transformation* or *re-connection* [Joe (1991a, 1995); Zavattieri, Dari and Buscaglia (1996); Freitag and Ollivier-Gooch (1997); Lo (1997)]. The second is *geometrical optimization*, also called *node repositioning* or *smoothing* [Zavattieri, Dari and Buscaglia

[1] LTCS, Department of Mechanics and Aerospace Engineering, College of Engineering, Peking University, Beijing 100871, China
[2] National Laboratory on Machine Perception, Peking University, Beijing, 100871, China
[3] Corresponding author, email: SUNSL@mech.pku.edu.cn, phone: 86-10-62757418

(1996); Lo (1997); Freitag and Ollivier-Gooch (1997, 2000); Sun and Liu (2003); Chen, Tristano, and Kwok (2004)]. This paper will focus on the former, *local transformation*.

Local transformation or re-connection changes the topology of a mesh, i.e. node-element connectivity relationship. The most frequently used and most effective operations of re-connection for tetrahedral mesh are so-called basic or elementary flips [George and Borouchaki (2003)], e.g. 2-3 flip, 3-2 flip, 2-2 flip, 4-4 flip. These topological transformations are usually called "local", since only a small number of tetrahedra (typically fewer than 5) are removed or introduced by a single transformation. Such flips are simple, easy to implement, but effective in removing poorly-shaped tetrahedra and improving the quality of mesh [Joe (1991a); Freitag and Ollivier-Gooch (1997); Lo (1997)]. However, since these basic local transformations only simply make a selection from several possible configurations within a relatively small region composed of several tetrahedra, the effect for quality improvement is limited.

In addition to these elementary flips, some more sophisticated local transformations were also discussed in literature [Joe (1995); George and Borouchaki (2003)]. In fact most of these operations are combinations of the elementary flips, and can be implemented as sequences of 2-3 flip, 3-2 flip, 2-2 flip, and 4-4 flip. Although studies show that the combinations of two or more basic local transformations are much more effective [Joe (1995)], the effect for quality improvement is similarly restricted duo to the limitation of small transformation region. It is still possible to improve the quality of mesh further by expanding the scope of transformation region.

In order to break such a limitation and improve the quality of mesh further, this paper presents the strategy of *optimal tetrahedralization for small polyhedron* and corresponding *small polyhedron re-connection* (*SPR*) approach, which seeks for the optimal tetrahedralization of a polyhedron with a certain number of vertexes and faces instead of choosing the best configuration from several possibilities within a small region that consists of a small number of tetrahedra. For a SPR operation, since the concerned local region which is usually composed of 20 to 40 tetrahedral elements is much larger than that in existing local transformations, better result in quality improvement is expected. Up to now, to the best knowledge of the authors, no relevant studies have been reported in the literature before.

The efficiency of optimal searching algorithm is the key to success of the presented method, and it can be significantly enhanced by some deliberate strategies. Experimental investigation and results on tetrahedral finite element mesh show that the SPR approach is quite effective at improvement of mesh quality with acceptable time cost.

With further enhancement in efficiency of optimal searching algorithm, the presented SPR approach can be utilized as a powerful and effective tool for mesh improvement and recovery of geometric boundary.

This paper is organized as follows. Measurement of mesh quality and some commonly used local transformations are briefly reviewed in section 2, respectively. Section 3 introduces the concept of optimal tetrahedralization for small polyhedron and corresponding small polyhedron re-connection (SPR) operation. In section 4, the framework of SPR approach and a recursive enumerative searching algorithm are presented. Some testing results and discussions are given in section 5. The paper ends with conclusion and future work in section 6.

2 Measurement of mesh quality and commonly used local transformations

2.1 Measurement of mesh quality

Local transformations and other techniques are usually applied in local area of a mesh to expect an improvement of quality. In last 20 years, many measures or criteria have been suggested to evaluate the quality of tetrahedron from different points of view, such as minimum solid angle [Joe (1991b); Lo (1997)], radius ratio ρ [Lo (1997)], the γ coefficient [Lo (1991, 1997)], the Q coefficient [Zavattieri , Dari and Buscaglia (1996)], the condition number of the Jacobian matrix [Knupp (2000)], solution-based mesh quality indicator [Berzins (2000)], just to name a few. Many meaningful researches have been done for investigating the behavior of the mesh quality measures [Liu and Joe (1994); Berzins (1999); Shewchuk (2002)]. Some of the quality measures are considered to be equivalent in some weak sense, for example, in work of Liu and Joe (1994). However, recent studies [Sun and Liu (2003); Nie, Liu and Sun (2003)] indicate that using different measures to evaluate change of element shape will probably lead to inconsistent result in some circumstances, i.e. according to one quality measure, the mesh quality is improved, but according to another measure, the quality is deteriorated. Further studies should be carried out on influence of quality measures on optimization process; however, we do not make detailed investigation here and choose the γ coefficient as the quality measure for tetrahedral element in testing hereinafter which is defined as

$$\gamma = \frac{72\sqrt{3}v}{\left(\sum\limits_{1\leq i<j\leq 4} l_{ij}^2\right)^{1.5}} \quad , \tag{1}$$

where v denotes the volume of tetrahedron with vertex P_1,P_2,P_3,P_4, and l_{ij} represents the length of the edge joining P_i and P_j. The γ coefficient takes a maximum value of unity for the equilateral tetrahedron, and approaches zero for degenerated tetrahedra with volume close to zero.

In authors' experiences, the "bad" elements (in sense of quality) produced in mesh generation only accounts for a small part in the whole mesh. This small part of bad elements will often greatly deteriorate the accuracy of solution. Therefore, unlike some studies [Kennon and Dulikravich (1986); Zhang and Trepanier (1994); Lo (1997)], in this paper the quality of a mesh is defined as the quality value of the "worst" element in the mesh and the quality improvement begins from the worst element and its adjacent elements.

2.2 Commonly used local transformations

In two-dimensional situation, the well-known "diagonal swap" is often applied to optimize adjacent triangles. The desired configuration is the one in which the minimum angle of triangles is larger (see Figure 1).

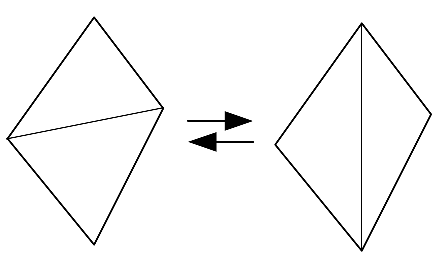

Figure 1 : Diagonal swap in two-dimension

The natural extension of diagonal swap to three-dimension is called 2-2 flip (Figure 2), which can be regarded as a special case of 2-3 flip (Figure 3).

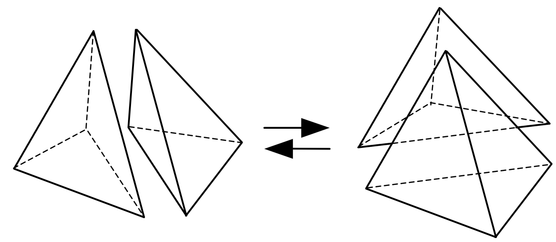

Figure 2 : 2-2 flip

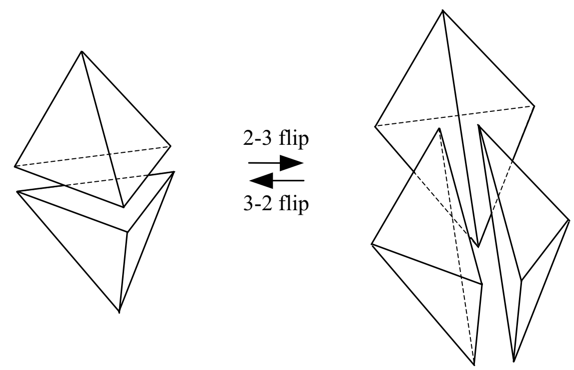

Figure 3 : 2-3 flip and 3-2 flip

As illustrated in Figure 3, 2-3 flip considers 2 tetrahedra sharing a face and replaces these elements by 3 tetrahedra sharing the edge whose endpoints are opposite the common face. 3-2 flip is the inverse of 2-3 flip, and replaces the 3 tetrahedra sharing an edge by means of 2 tetrahedra sharing a face [George and Borouchaki (2003)].

4-4 flip is another basic local transformation that considers a polyhedron in the shape of a rhombus with eight faces. As shown in Figure 4, there are three configurations to divide the polyhedron into four tetrahedra [Lo (1997)].

The above so-called basic or elementary flips are most frequently used and most effective in practice. While applied to mesh improvement, these local transformations will be preformed many times, and each transformation operation only modifies a small local region of a mesh. The procedure stops when none of the elements can be improved in quality by the flip operation. However, by the basic local flips, we can't gain sufficient amounts of improvement in mesh quality, since these basic local transformations only simply make a selection from several possible configurations within a relatively small region. For example, the most frequently used 2-3/3-2 flips

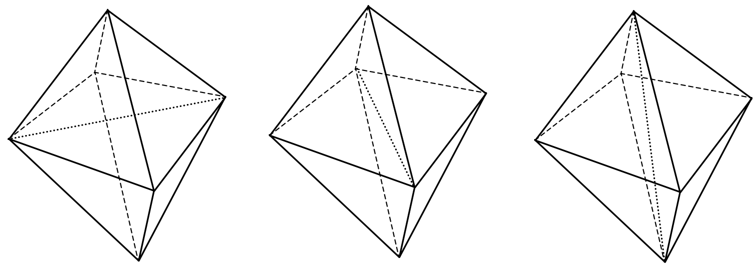

Figure 4 : Three configurations in 4-4 flip [Lo (1997)]

only choose the best triangulation between two configurations in a very small area composed of 2 or 3 elements.

In addition to these elementary flips, some more sophisticated local transformations were also discussed in literature, such as composite transformation operations [Joe (1995)], the general edge flip [George and Borouchaki (2003)]. In fact most of these operations are combinations of the elementary flips, and can be implemented as sequences of 2-3 flip, 3-2 flip, 2-2 flip, and 4-4 flip.

Studies report that the combinations of two or more basic local transformations are much more effective [Joe (1995)], although they still suffer the restriction of small transformation region. This is suggested that further improvement in quality of mesh may be achieved by expanding the scope of transformation region.

3 Optimal tetrahedralization for small polyhedron

3.1 Description of SPR operations

In order to break the limitation of previous local transformations discussed above, a new local re-connection strategy, *optimal tetrahedralization for small polyhedron*, which is illustrated in form of two-dimensional case in Figure 5, is proposed in this section. Rather than simply making a selection from several possible configurations within a small region that consists of a small number of tetrahedra as previous local transformation usually does, the new re-connection strategy seeks for the optimal tetrahedralization of a polyhedron with a certain number of vertexes and faces. There may be a lot of tetrahedralization ways for an enlarged small polyhedron, and the new strategy will find out the best. Since the concerned local region which is usually composed of 20 to 40 tetrahedral elements is much larger than that in previous local transformations, better result of quality improvement is expected. Up to now, to the best knowledge of the authors, no relevant studies have been reported in current literature.

According to the new strategy, two kinds of small polyhedron re-connection (SPR) operations are defined as follows.

SPR 1: For a given polyhedron with a certain number of triangles on boundary, seeks its optimal tetrahedralization without Steiner nodes added.

SPR 2: For a given polyhedron with a certain number of triangles on boundary, seeks its optimal tetrahedralization without Steiner nodes added under some extra geometric restrictions.

Note that number of triangles on boundary, *S*, is taken here to denote the size of the polyhedron instead of number of tetrahedral elements.

The initial triangulation for a polyhedron is not necessary for the SPR operation, however, if the initial triangulation for a polyhedron already exists, the optimal searching process can be greatly speeded up. This is the usual case for mesh improving.

By the way, some more general and sophisticated local transformations, such as composite transformation operations [Joe (1995)], the general edge flip [George and Borouchaki (2003)], may be considered as special cases of the presented SPR operation.

3.2 Consideration for the size of the small polyhedron

Obviously, the larger the size of small polyhedron *S* is, the more profit gained for mesh quality will be. However, the SPR operation involves too much computational efforts in comparison with previous local transformations. Even the problem of deciding whether or not a triangulation exists for such a polyhedron is known to be N-P hard. Generally the SPR operation has to deal with hundreds and thousands of possible triangulations. The time

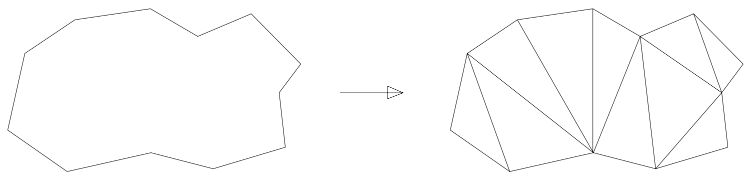

Figure 5 : Two-dimensional illustration for optimal tetrahedralization for small polyhedron

cost may not be afforded if S is too large, say, greater than 100. This may be the reason that, we guess, why no relevant studies have been reported in the literature before.

The efficiency of optimal searching algorithm is closely related to success of the SPR method. Thus determination of a suitable size S of local small polyhedron is a compromise between profit and efficiency. In our test, we set $S \leq 50$ without any strictly theoretical support. Even so, the computational cost still seems too high. Fortunately, most of the possibilities of triangulation in the SPR operation can be rejected rapidly during recursive procedure. Some strategies discussed in section 4 can also speed up the SPR process greatly.

3.3 Application in boundary recovery

In mesh generation, boundary conformity is a basic requirement, but it is not easy to satisfy. If, after a meshing process for a geometric model, some boundary edges or faces are not existed in the mesh in a meaningful form, specific post-process must be performed to recover them. Several algorithms are designed to fulfill this target [George, Hecht and Saltel (1991)], which depend heavily on adding extra points namely Steiner nodes that is prone to produce degenerated elements. A better idea is to recover missed entities by the SPR operation. It has the advantages of creating none or fewer Steiner points and producing better elements in the local region where boundary recovery takes place. In this situation, the goal of a SPR operation is to seek the best local triangulation of a small polyhedron with some extra geometric restrictions. This is the reason that we define the second form of SPR operation.

4 Framework of the SPR approach

4.1 The SPR approach

The SPR approach based on the SPR operation can be applied to improve the quality of worst element of a mesh in a step by step manner. First construct a small polyhedron that includes the worst element and its neighbors, and then do SPR operation to find out the best tetrahedralization of this polyhedron. Each time when a SPR operation is performed, the mesh quality of local region adjacent to the worst element can be improved. Next, find another worst element according to its quality measurement and repeat above procedure. The cycle continues until the improvement reaches its limit, that is no better tetrahedralization existed for the small polyhedron most recently constructed.

Same as the previous local transformations, the presented SPR approach can also be combined with smoothing process. Numerical test in section 5 will show that the SPR approach is more suitable for combining with smoothing approach, and combination of the SPR approach and smoothing may achieve substantial improvement in mesh quality.

By the way, we also notice recent work of Moore and Saigal (2005) to eliminate sliver shaped elements in 3-dimensional finite element models, which first merges the slivers with neighboring elements to create a polyhedron, and then subdivides the polyhedron into well-shaped tetrahedral elements by adding a temporary centroidal node rather than searching for the best triangulation of the polyhedron without extra node added.

4.2 The recursive enumerative searching algorithm

The most important issue for the SPR approach is the efficiency of searching algorithm for optimal tetrahedralization of the polyhedron. Here a recursive enumerative searching algorithm is presented as follows.

First choose a triangle F on the boundary of the polyhedron P, and construct an element (denoted by ELE) by F and one of the other vertexes of the polyhedron. Thus the original polyhedron is divided into the element ELE and a new smaller polyhedron (denoted by Q). Next solve the smaller problem for the new smaller polyhedron Q by

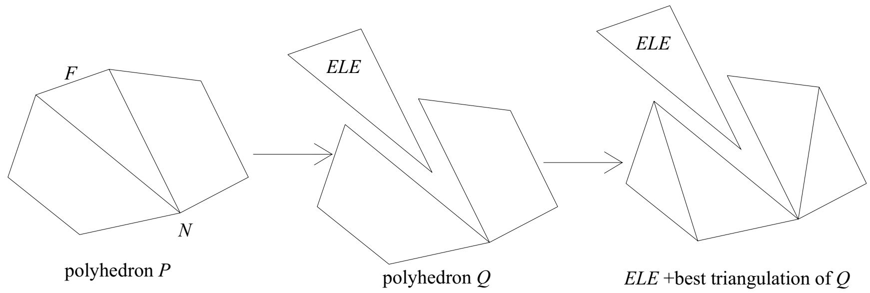

Figure 6 : Recursive procedure for the SPR approach illustrated in form of two-dimensional case (*ELE*+ the best triangulation of $Q \Longrightarrow$ a triangulation of P)

```
int OptimalTetMeshForSmallPolyhedron (q_0, P, T)

input:    q_0, quality of the initial mesh;
          P, the small polyhedron.
output:   T, the best triangulation.
          If there is no triangulation with quality better than q_0, T will be NULL.
return value: "succeed" or "fail".
temporary variables: T_c, the best triangulation among these already tested;
                     q_c, quality of T_c;
                     r_t, return value of the recursive call.

1    q_c=q_0, T_c=NULL
2    select a triangle F on the polyhedron
3    for each vertex N on the polyhedron, do
     {
4      if ( F and N can construct a valid tetrahedron ELE
5           and quality of ELE is better than q_c)
       {
6        remove ELE from polyhedron P, construct a new smaller polyhedron Q
7        r_t= OptimalTetMeshForSmallPolyhedron (q_c, Q, T_Q)
8        if ( r_t is "succeed")
         {
9            merge T_Q and ELE to create a new triangulation of P
10           update T_c and q_c
         }
       }
     }
11   if (a better mesh found) { T=T_c, return "succeed"}
12   else return "fail"
```

Figure 7 : Pseudocode of the recursive algorithm for the SPR approach

the same algorithm recursively, and then merge its result with the element *ELE* to get a feasible solution for the original polyhedron *P*. Here, the so-called feasible solution is in some sense optimal, since it includes the optimal solution of the smaller polyhedron *Q*. Such process is repeated for all rest vertexes. Finally choose the best tetrahedralization from all feasible solutions, thus the final solution is exactly the optimal solution for the polyhedron *P*. The recursive procedure is illustrated in form of two-dimensional case in Figure 6, and the pseudocode for the algorithm is listed in Figure 7.

4.3 Consideration of efficiency

Enumerating all possible tetrahedralization will be a very time-consuming work. The following is a rough estimation. Actually, every sub-polyhedron produced in line 6 of the algorithm (Figure 7) will result in many possibilities. Suppose that the whole problem will be discrete into *N* elements. Its first level sub-problems have *N*-1 elements, the second level sub-problems have *N*-2 elements, and so on. So there are an estimation of O(*N*!) possible ways to subdivide the problem. The computational cost of enumerating all possibilities is far beyond the capability of current personal computer when *N* is large, say, greater than 100.

Fortunately, most of triangulation ways will be aborted and rejected earlier when a bad element (in sense of quality) is generated, or invalid situations such as overlapping and gap occur.

The algorithm given here, while appearing plain and naive, is able to obtain the optimal triangulation of a small polyhedron with size *S* less than 15 with acceptable time cost. A more efficient searching algorithm will make the SPR approach a more powerful tool to improve mesh quality.

Additionally, there are a few further strategies to speed up above searching algorithm, such as avoiding solving the same sub-polyhedrons repeatedly, choosing smartly digging face on the small polyhedron where new elements are to be created, selecting the optimal digging directions, subdividing the polyhedron into several sub-polyhedrons as earlier as possible, etc. Among them the most effective one is being aware of polyhedron subdivision illustrated in Figure 8. If the polyhedron can be broken into 2 or 3 smaller polyhedrons or blocks after a dig, then each block will be treated separately and the result is obtained by merging the result of all separate blocks. The details of these strategies will be discussed in subsequent papers.

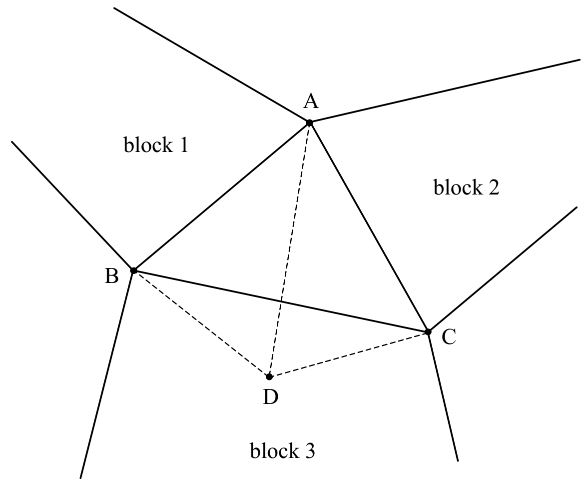

Figure 8 : Polyhedron subdivision (here the polyhedron is broken into 3 blocks after a dig of tetrahedron ABCD)

5 Examples and discussions

Some artificial small polyhedrons with so-called spherical structure and nonspherical structure are designed for testing performance of the SPR operation, and several examples of finite element mesh are given to demonstrate the effectiveness of the presented SPR approach. The finite element meshes in examples are generated by tetrahedral mesh generation package **AutoMesh3D** [Liu (2001)] through the ball-packing method [Liu (1991, 2003)]. The presented SPR procedure is already embedded in **AutoMesh3D**. The γ coefficient in Equation (1) is adopted as the quality measurement for tetrahedral element. All tests are performed on the following platform:

A Pentium IV PC (2.4 GHz CPU and 256 MB RAM) with compiler of Visual C++ 6.0.

5.1 Tests for artificial small polyhedrons with different structures

Figure 9 and Figure 10 illustrate some artificial small polyhedrons with different structures which are generated randomly. The SPR operation is performed to these polyhedrons respectively. The results listed in Table 1 indicate that the time costs are much different from case to case and depend more on the structure of the polyhedron than its size. The SPR operation for polyhedrons

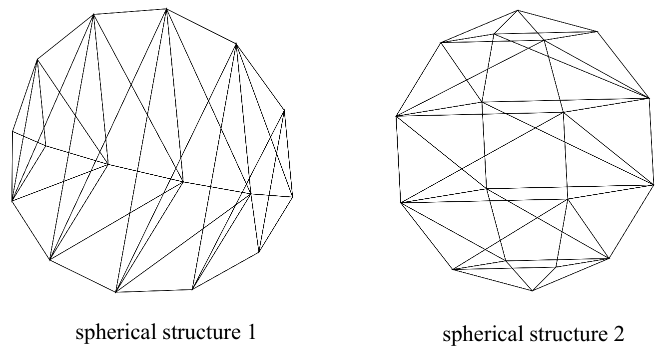

Figure 9 : Two artificial small polyhedrons with spherical structure

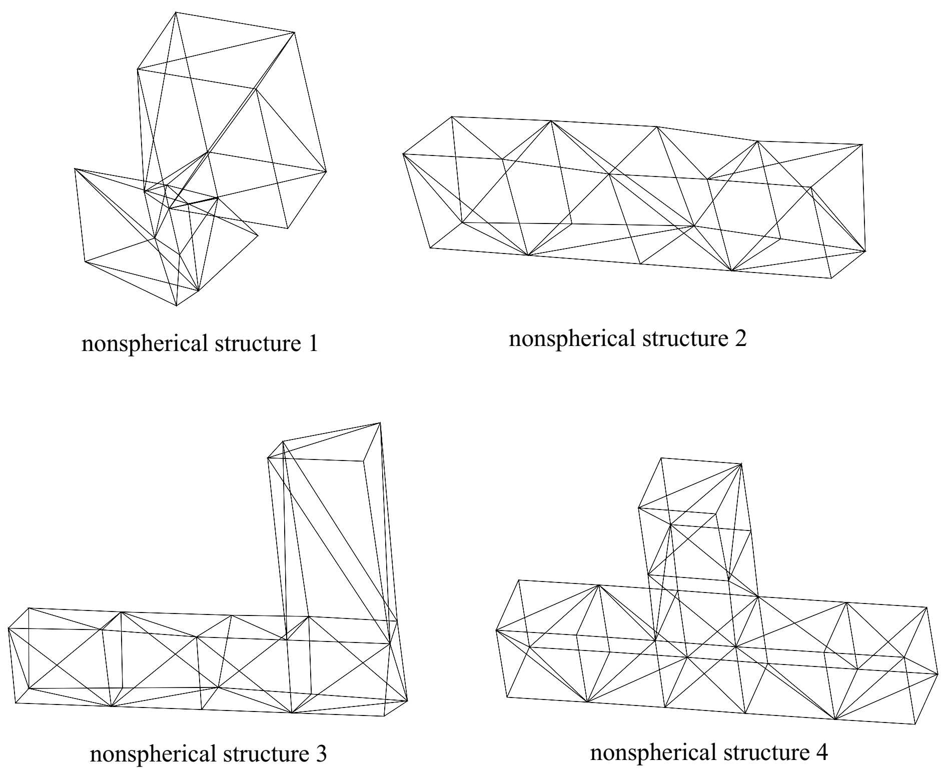

Figure 10 : Some artificial small polyhedrons with nonspherical structure

Table 1 : Time costs of the SPR operation for different polyhedrons

	Number of nodes	Number of triangles on boundary	Running time (second)
spherical structure 1	16	28	0.781
spherical structure 2	18	32	5.078
nonspherical structure 1	19	34	0.016
nonspherical structure 2	20	36	0.047
nonspherical structure 3	24	44	0.344
nonspherical structure 4	32	60	0.063

Table 2 : Quality statistics of coefficient γ for the first finite element mesh

	Ranges of γ	0.00∼0.03	0.03∼0.12	0.12∼0.30	0.30∼0.66	> 0.66
Number of elements	Initial mesh (113975 elements, min. γ 0.00118)	34	423	1763	13526	98229
	After SPR only (111470 elements, min. γ 0.321)	0	0	0	10495	100975

with spherical structure is much more time-consuming than that with nonspherical structure; however, it is not suggested to employ nonspherical structure excessively when constructing small polyhedron, since the final optimal result for the polyhedrons with spherical or near spherical structure is usually better than those with nonspherical structure. When the efficiency of the algorithm is tolerable the polyhedron with spherical or near spherical structure is recommended.

5.2 Tests for finite element meshes

The SPR approach can be applied to improve the quality of mesh step by step. First find the worst element according to a specific quality measure. Then construct a small polyhedron that includes the worst element and its neighbors, and perform SPR operation to find out the best tetrahedralization of this polyhedron to improve the quality of local region adjacent to the worst element. Next, find another worst element and repeat above procedure. The procedure will stop until the tetrahedralization of the polyhedron that includes current worst element can not be improved. The SPR operations are usually performed in limited times in practice and the payment for time cost is reasonable.

Three examples of finite element mesh are tested to demonstrate the effectiveness of the presented SPR procedure. The size of the small polyhedron, *S*, defined by the number of triangles, is set to 25 in following tests. The first finite element mesh shown in Figure 11 consists of 22392 nodes and 113975 tetrahedral elements initially. Its quality is not good. There are 34 elements with the quality value below 0.03, and the lowest value is 0.00118. The statistics of initial quality and quality after the presented SPR approach are listed in Table 2, which shows remarkable improvement of mesh quality by the SPR approach. The minimum value of γ increases to 0.321. The substantial improvement in quality of large number of elements indicates that, as a new local transformation procedure, the SPR approach works effectively on optimizing mesh topology around the worst element, and hence improves the quality of whole mesh. In this example, the SPR operations with total number of 5754 are performed, and the running time (about 260 seconds) is acceptable considering substantial improvement in mesh quality.

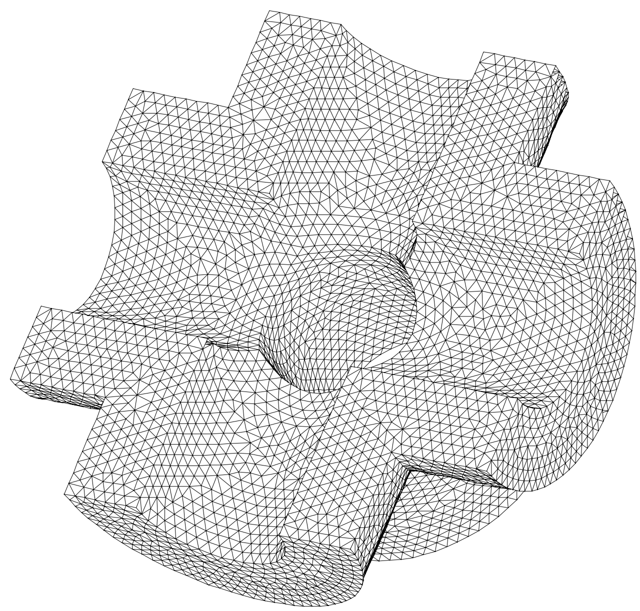

Figure 11 : The first finite element mesh

The second finite element mesh includes 2726 nodes and 8359 tetrahedral elements initially (Figure 12). Its quality is also not good enough. There are 13 elements with the quality value below 0.03, and the lowest value is 0.0036. The elementary local transformations (or ELT for abbreviating) and the presented SPR approach are applied to the initial mesh, respectively. Table 3 shows the

Table 3 : Quality statistics of coefficient γ for the second finite element mesh

	Ranges of γ	0.00∼0.03	0.03∼0.12	0.12∼0.30	0.30∼0.66	>0.66
Number of elements	Initial mesh (8359 elements, min. γ 0.0036)	13	33	188	2500	5625
	After ELT only (8063 elements, min. γ 0.181)	0	0	33	2304	5726
	After SPR only (7960 elements, min. γ 0.275)	0	0	4	2358	5598

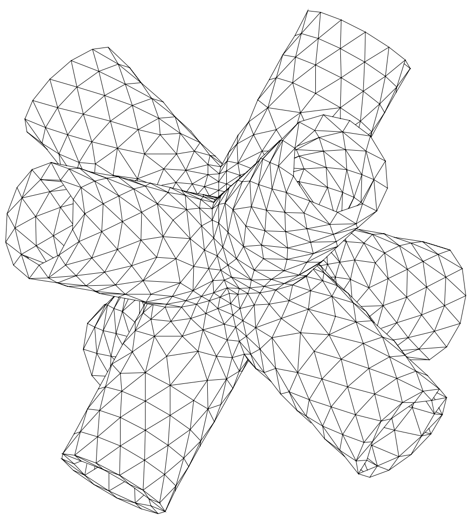

Figure 12 : The second finite element mesh

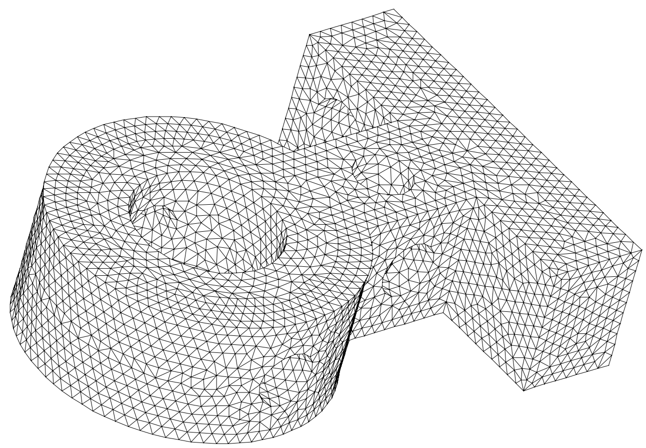

Figure 13 : The third finite element mesh

statistics of initial quality and quality after optimization. Both ELT and SPR procedures improve the mesh quality; however, as expected, the SPR approach gives better result. The minimum value of γ increases from 0.0036 to 0.275 and there are only 4 elements with quality value lower than 0.30. The running time for SPR is about 7.5 seconds. We believe that the superiority in effectiveness makes the SPR approach more useful and become a potential replacement for previous local transformations in mesh topological optimization.

The results of above examples indicate that the proposed SPR procedure is able to significantly improve the quality of tetrahedral mesh. In practice, the topological modification and node reposition should be combined together to get more effective results. In next example, it can be seen that the combination of the proposed SPR procedure and smoothing will achieve substantial improvement in mesh quality.

The third finite element mesh illustrated in Figure 13 consists of 11007 nodes and 53710 tetrahedral elements, and the minimum value of γ is 0.0110 initially. First, ELT and SPR procedures are applied to the initial mesh, respectively. The result listed in Table 4 indicates that the mesh quality has only limited improvement after ELT or SPR procedure. Almost same results are obtained for the two approaches. The minimum value of γ increases from 0.0110 to 0.0195. It is found that, by monitoring the optimization procedure, the processes for both approaches are quickly blocked by the same worst element, since no further improvement can be made by topological modification alone to the local small polyhedron that includes current worst element.

In order to obtain further improvement in mesh quality, smoothing or node reposition is applied to combine with topological optimization. Here, an efficient smoothing approach based on chaos searching algorithm [Sun and

Table 4 : Quality statistics of coefficient γ for the third finite element mesh

	Ranges of γ	0.00∼0.03	0.03∼0.12	0.12∼0.30	0.30∼0.66	> 0.66
Number of elements	Initial mesh (53710 elements, min. γ 0.0110)	26	159	794	7605	45126
	After ELT only (53695 elements, min. γ 0.0195)	11	159	794	7601	45130
	After SPR only (53695 elements, min. γ 0.0195)	11	159	794	7600	45131
	After ELT + smoothing + ELT (53694 elements, min. γ 0.0990)	0	1	494	11211	41988
	After SPR + smoothing + SPR (52546 elements, min. γ 0.332)	0	0	0	9948	42598

Liu (2003)] is adopted. The running time for smoothing procedure is 76 seconds. After smoothing procedure, ELT and SPR procedures are performed respectively again. The direct effect on quality improvement by smoothing is not very distinct; however, the smoothing procedure has optimized node distribution or configuration around the worst element, and such an improvement provides favorable conditions for topological optimization and makes topological optimization work more effectively. It can be seen from Table 4 that both ELT and SPR procedures do actually take effect after the smoothing procedure. Similarly, the SPR procedure gives much better result while the running time of 120 seconds is acceptable. The minimum value of γ increases to 0.332.

Compared with ELT, the presented SPR approach is obviously more suitable for combining with smoothing approach, and combination of SPR and smoothing approach is a better choice for mesh improvement. The time cost of SPR approach is reasonable and worthy to be paid.

It can also be observed in above examples that the number of elements generally decreases by several percentages after topological optimization, since most of the bad elements which usually occupy small volumes are removed.

By the way, same quality measure should be adopted in smoothing and topological transformation procedures. Otherwise the optimization process may probably suffer "zigzag" problem since some quality measures are found to induce inconsistent evaluation for quality change of element in some circumstances [Sun and Liu (2003); Nie, Liu and Sun (2003)].

6 Conclusion and future work

The small polyhedron re-connection is a new and very effective way to improve tetrahedral meshes. Although further speedup is expected for the searching algorithm, examples show that the presented SPR approach can be applied to practical mesh improvement with acceptable payment of time cost and is able to give much better results than the most commonly used local transformations. In addition, the presented SPR approach is more suitable for combining with smoothing approach. We believe that the superiority in effectiveness makes the SPR approach more useful with the further speedup of its efficiency and become a potential replacement for previous local transformations in mesh topological optimization.

While large size of the small polyhedron in the SPR approach may conduce to better results, the time cost may not be afforded if the size is too large. However, experimental investigation indicates the time cost of the SPR operation depends much more on the structure rather than the number of vertexes or surface triangles of the small polyhedron. Therefore one clear direction for further research is how to evaluate and utilize structural characteristics of a polyhedron. We think some geometric essentials should be found first in order to develop more efficient algorithm. Moreover, the initial mesh, if exists, also influences heavily on the efficiency of the SPR approach. The better the initial quality is, the lower the time cost will be.

The superior performance of the SPR approach makes it worthy to be further studied. Some works are in progress, including how to construct more appropriate polyhedron, development of data structure for supporting searching algorithm to avoid solving the same sub-polyhedrons repeatedly, choosing smartly digging face on the small polyhedron where new elements are to be created, selecting the optimal digging directions, subdividing the polyhedron into several sub-polyhedrons as earlier as possible, etc.

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