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Boundary Element Method for Magneto Electro Elastic Laminates

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Abstract: A boundary integral formulation and its numerical implementation are presented for the analysis of magneto electro elastic media. The problem is formulated by using a suitable set of generalized variables, namely the generalized displacements, which are comprised of mechanical displacements and electric and magnetic scalar potentials, and generalized tractions, that is mechanical tractions, electric displacement and magnetic induction. The governing boundary integral equation is obtained by generalizing the reciprocity theorem to the magneto electro elasticity. The fundamental solutions are calculated through a modified Lekhnitskii's approach, reformulated in terms of generalized magneto-electro-elastic displacements. To assess the reliability and effectiveness of the formulation, some numerical analyses have been carried out and the convergence of the method has been studied. The multidomain approach has been developed for the analysis of multilayered structures. Numerical results obtained show good agreement with those found in the literature.

keyword: Magneto-electro-elastic materials; Boundary element method; Laminates modeling.

1 Introduction

The new class of magneto-electro-elastic materials has recently emerged in the field of smart structures and materials by virtue of their ability to convert energy into three different forms, i.e. magnetic, electric and mechanical. This characteristic distinguishes them from the extensively investigated classical piezoelectric materials – for recent studies see for example Han, Ding and Liu (2005) or Han, Pan, Roy and Yue (2006) – and makes them particularly suitable for the construction of smart devices, such as sensors, actuators or transducers. Magneto-electro-elastic media can fundamentally be of two kinds: a) particulate composites having a 0-3 connectivity; b) composites having a 2-2 connectivity, see Ryu, Priya, Uchino and Kim (2002). The first type is constituted by a piezoelectric matrix with piezomagnetic inclusions, or *vice versa*, see Buchanan (2004), while the second one is a real laminate made by piezoelectric and piezomagnetic layers. The optimal exploitation of these materials relies upon the correct analysis of their coupled response to external stimul. While the inherent coupling of the three fields makes them particularly attractive in the framework of intelligent systems, on the other hand it makes more complex the mathematical modeling of their behaviour. Indeed, analytical solutions to the governing differential equations are rather rare and either devoted to the treatment of infinite domain problems or limited to special configurations. Wang and Shen (2002) obtained the general solution for three-dimensional transversely isotropic magneto-electro-elastic media by using five harmonic potential functions and they applied it to determine the fundamental solution for a generalized dislocation and derive Green's functions for a half-space. Huo, Leung and Ding (2003) used the general harmonic potential solution to obtain extended Boussinesq and Cerruti solutions for magneto-electro-elastic half-space and applied them to the treatment of elliptical Hertzian contact. Ding and Jiang (2004) derived a general solution for plane magneto-electro-elastic problems, expressing it in terms of four harmonic potentials and then derived 2D fundamental solutions for an infinite plane. Guan and He (2005) used the same general solution to study the problem of an infinite half plane loaded by a point force lying on its free surface. Hou, Ding and Chen (2005) derived Green's functions for transversely isotropic magneto-electro-elastic media for the case of distinct and multiple eigenvalues. Some exact solutions have been devised for interesting structural configurations. Pan (2001) derived the exact solutions for three-dimensional magneto-electro-elastic simply supported multilayered rectangular plates under static generalized loads by using the propagator matrix method and then extended it to the analysis of free vibrations, (see Pan and Heyliger (2002) Pan and Heyliger (2003), and Heyliger, Ramirez and Pan

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(2004) analysed simply supported multilayered plates in cylindrical bending by using the propagator matrix and discrete layer approximations methods. Pan and Han (2005) obtained exact solutions for rectangular plates made up of functionally graded layers. Wang, Chen and Fang (2003) developed a state vector approach for three-dimensional multilayered magneto-electro-elastic plates and used it to determine a solution in terms of infinite series expansion. Chen, Lee and Ding (2005) studied the free vibration problem of functionally graded transversely isotropic magneto-electro-elastic plates. Actually the analysis of configurations with general boundary conditions requires the use of numerical methods. FEM models are indeed rather rare, as also pointed out by Bhangale and Ganesan (2005), who derived a hybrid formulation to study the free vibrations of functionally graded magneto-electro-elastic cylindrical shells. They used series expansions in the circumferential and axial directions and finite elements in the radial one. In Buchanan (2003) and Buchanan (2004) finite elements were used to compare multilayer and multiphase approaches to magneto-electro-elastic materials. A mixed finite element approach to magneto-electro-elastic plates has been developed on the basis of the mixed Reissner variational principle by Lage, Mota Soares C.M., Mota Soares C.A. and Reddy (2004). A BEM model for plane magneto-electro-elasticity has been developed by Ding and Jiang (2004), who found the 2D fundamental solutions starting from the general harmonic potential solution. Ding, Jiang, Hou and Chen (2005) also found 3D Green's functions for transversely isotropic magneto-electro-elasticity and used it in the 3D BEM analysis of an annular plate.

In the present paper a multidomain boundary element model for 2D magneto-electro-elastic laminates is developed. The formulation is expressed in terms of suitably defined generalized variables, namely generalized displacements and tractions. By using these variables the magneto–electro- elastic governing equations can be recast in a form that resembles the governing equations of classical elasticity and allows the straightforward extension of classical methods to the magneto-electro-elastic analysis. The boundary integral representation is deduced by generalizing the reciprocity theorem. The fundamental solutions are determined by using a generalized displacement based modified Lekhnitskii's approach, see Lekhnitskii (1963). Finally, the numerical solution of the formulation is obtained by the boundary element method. A multidomain approach, obtained by enforcing suitable continuity and equilibrium conditions between adjacent layers, has been used to model laminate configurations. Some numerical results are presented to evaluate the effectiveness and the reliability of the proposed model.

2 Basic equations

The formulation will be developed for two-dimensional magneto-electro-elastic domains Ω with boundary $\partial\Omega$ lying in the x_1x_2 plane. It is assumed that the magneto-electro-elastic response does not vary along the x_3 direction, so that the analysis leads to a generalized plane strain elasticity problem and an in-plane magneto-electrostatic problem. To maintain a compact and efficient matrix notation the strain component γ_{33}, the electric field component E_3 and the magnetic field component H_3, which are trivially zero due to the assumption of generalized plane strain and in-plane electrostatics and magnetostatics, are kept in the formulation and the generalized in-plane behaviour will be expressed by suitable differential operators. It is worthwhile to note that, due to the anisotropy of the material and the inherent coupling between the different fields, the presence of out of plane shear strains cannot be trivially excluded even in the case of generalized plane strain. For the sake of generality the formulation will be therefore developed considering the possibility of their presence. The elastic state of the body is described in terms of displacements $\mathbf{u}^T = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$, elastic strains $\boldsymbol{\gamma}^T = \begin{bmatrix} \gamma_{11} & \gamma_{22} & \gamma_{12} & \gamma_{13} & \gamma_{23} & \gamma_{33} \end{bmatrix}$ and elastic stresses $\boldsymbol{\sigma}^T = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{12} & \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$. The electric state is defined by the electric potential φ, the electric field $\mathbf{E}^T = \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}$ and the electric displacement field $\mathbf{D}^T = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix}$. Assuming that there is no external current density in the domain, the magnetic field variables are the scalar magnetic potential ψ, the magnetic field $\mathbf{H}^T = \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$ and the magnetic induction field $\mathbf{B}^T = \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}$.

The above quantities are involved in the following relationships

$$\boldsymbol{\gamma} = \mathbf{C}\mathbf{u}\,,\ \mathbf{E} = -\mathbf{L}\varphi\,,\ \mathbf{H} = -\mathbf{L}\psi \tag{1}$$

$$\mathbf{C}^T\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}, \mathbf{L}^T\mathbf{D} - \rho = 0, \mathbf{L}^T\mathbf{B} = 0 \tag{2}$$

The eqs. (1) express the strain-displacement relations and the irrotationality conditions of electric and magnetic fields. It should be noted that the equation linking the magnetic field and the scalar magnetic field holds for non conducting materials only. On the other hand, the eqs. (2) represent the classical elastic indefinite equilibrium equations and the stationary Maxwell equations for the electric displacement and the magnetic induction fields respectively. In the above equations $\mathbf{f}$ is the body forces vector, ρ is the free electric charge density and the differential operators are defined as

$$\mathbf{C}=\begin{bmatrix} \partial/\partial x_1 & 0 & \partial/\partial x_2 & 0 & 0 & 0 \\ 0 & \partial/\partial x_2 & \partial/\partial x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial/\partial x_1 & \partial/\partial x_2 & 0 \end{bmatrix}^T,$$
$$\mathbf{L}=\begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ 0 \end{bmatrix} \tag{3}$$

Moreover, the following constitutive equations hold for magneto-electro-elastic materials

$$\begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{D} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{e}^T & \mathbf{d}^T \\ \mathbf{e} & -\boldsymbol{\varepsilon} & -\mathbf{g} \\ \mathbf{d} & -\mathbf{g} & -\boldsymbol{\mu} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\gamma} \\ -\mathbf{E} \\ -\mathbf{H} \end{bmatrix} = \mathbf{R} \cdot \begin{bmatrix} \boldsymbol{\gamma} \\ -\mathbf{E} \\ -\mathbf{H} \end{bmatrix} \tag{4}$$

where $\mathbf{C}$ is the elasticity matrix, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are the matrices of dielectric constants and magnetic permeability respectively, $\mathbf{e}$ and $\mathbf{d}$ are the matrices of piezoelectric and piezomagnetic constants and $\mathbf{g}$ is the matrix describing the direct magneto-electric coupling. In order to maintain a compact notation, the generalized constitutive matrix will be denoted as $\mathbf{R}$ in the following, as shown in the last member of eq. (4).

The eqs. (1), (2) and (4) should be completed by considering the suitable essential and natural boundary conditions, which can be expressed in the form

$$\mathbf{u}=\overline{\mathbf{u}} \text{ on } \partial\Omega_{u_1}, \varphi=\overline{\varphi} \text{ on } \partial\Omega_{\varphi_1}, \psi=\overline{\psi} \text{ on } \partial\Omega_{\psi_1} \tag{5}$$

$$\mathbf{t}=\overline{\mathbf{t}} \text{ on } \partial\Omega_{u_2}, D_n=\overline{D}_n \text{ on } \partial\Omega_{\varphi_2}, B_n=\overline{B}_n \text{ on } \partial\Omega_{\psi_2} \tag{6}$$

where $\mathbf{t}=\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}^T$ are the elastic boundary tractions as defined in the classical elasticity theory. The form of the eqs. (1), (2) and (4) suggests the extension to the general magneto-electro-elastic problem of the Barnett and Lothe's formalism for piezoelectrics, see Barnett and Lothe (1975). More specifically, the analysis can be expressed in terms of generalized quantities, namely generalized displacements $\mathbf{U}$, generalized body forces $\mathbf{F}$, generalized strains $\boldsymbol{\Gamma}$ and generalized stresses $\boldsymbol{\Sigma}$, defined as

$$\mathbf{U}^T=[\mathbf{u} \quad \varphi \quad \psi] \tag{7}$$
$$\mathbf{F}^T=\begin{bmatrix} \mathbf{f}^T & -\omega & 0 \end{bmatrix} \tag{8}$$
$$\boldsymbol{\Gamma}^T=\begin{bmatrix} \boldsymbol{\gamma}^T & -\mathbf{E}^T & -\mathbf{H}^T \end{bmatrix} \tag{9}$$
$$\boldsymbol{\Sigma}^T=\begin{bmatrix} \boldsymbol{\sigma}^T & \mathbf{D}^T & \mathbf{B}^T \end{bmatrix} \tag{10}$$

By introducing the generalized compatibility operator

$$\mathbf{D}=\begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} \end{bmatrix}, \tag{11}$$

eqs. (1) can be written in a compact matrix form as

$$\boldsymbol{\Gamma}=\mathbf{D}\,\mathbf{U} \tag{12}$$

The generalized constitutive equations (4) can be rewritten in compact notation as

$$\boldsymbol{\Sigma}=\mathrm{R}\boldsymbol{\Gamma} \tag{13}$$

and the generalized equilibrium equations, that is eqs (2), are expressed as

$$\mathbf{D}^T\boldsymbol{\Sigma}+\mathbf{F}=\mathbf{0} \tag{14}$$

Finally, by combining the eqs. (12), (13) and (14), the generalized governing equations are obtained

$$\mathbf{D}^T\mathbf{R}\mathbf{D}\,\mathbf{U}+\mathbf{F}=\mathbf{0} \tag{15}$$

The boundary conditions associated with eq (15) can be then expressed by

$$\begin{aligned} &\mathbf{U}=\overline{\mathbf{U}} \text{ on } \partial\Omega_1 \\ &\mathbf{T}=\mathbf{D}_n^T\mathbf{R}\mathbf{D}\,\mathbf{U}=\overline{\mathbf{T}} \text{ on } \partial\Omega_2 \end{aligned} \tag{16}$$

where $\mathbf{T}^T=\begin{bmatrix} t_1 & t_2 & t_3 & D_n & B_n \end{bmatrix}$ is the generalized tractions vector and $\mathbf{D}_n$ is the generalized traction operator, obtained by substituting the derivatives with the corresponding boundary outer normal direction cosines in the generalized compatibility operator $\mathbf{D}$, Davì and Milazzo (1997). There is no contradiction between eq. (16) and eqs. (5) and (6), since absolutely general boundary conditions can be imposed in the generalized notation. It is worth noting that the generalized variables make it possible to retrieve a form of the magneto-electro-elastic governing equations formally analogous to that of the anisotropic elasticity.

3 Boundary integral representation

Let $\mathbf{U}_j$ and $\mathbf{F}_j$ be a system of generalized displacements and forces which satisfies eq. (15), and let $\mathbf{T}_j$ be the corresponding generalized tractions. Applying the reciprocity theorem to the generalized magneto-electro-elastic problem, the following equation can be written, see Aliabadi (2002),

$$\int_{\partial\Omega} \left(\mathbf{U}_j^T\mathbf{T} - \mathbf{T}_j^T\mathbf{U}\right) d\partial\Omega = \int_{\Omega} \left(\mathbf{F}_j^T\mathbf{U} - \mathbf{U}_j^T\mathbf{F}\right) d\Omega \qquad (17)$$

If $\mathbf{F}_j = \mathbf{c}_j\delta(P-P_0)$, where the $\mathbf{c}_j$ is the load intensity and $\delta(P-P_0)$ is the Dirac's delta function, $\mathbf{U}_j$ and $\mathbf{T}_j$ represent the problem fundamental solution and the following equation holds

$$\mathbf{c}_j^T\mathbf{U}(P_0) + \int_{\partial\Omega} \left(\mathbf{T}_j^T\mathbf{U} - \mathbf{U}_j^T\mathbf{T}\right) d\partial\Omega = \int_{\Omega} \mathbf{U}_j^T\mathbf{F}\, d\Omega \quad (18)$$

This equation is the analogous of the classical Somigliana's identity of elasticity and constitutes the boundary integral representation of the magneto-electro-mechanical problem. By using five independent fundamental solutions associated with the concentrated point load directed along the three axes, with a concentrated charge and with a concentrated current, the three displacement components and the two potential at the generic point P_0 can be expressed in terms of generalized displacements and generalized tractions on the boundary. In compact matrix notation one writes

$$\mathbf{c}^*\mathbf{U}(P_0) + \int_{\partial\Omega} (\mathbf{T}^*\mathbf{U} - \mathbf{U}^*\mathbf{T})\, d\partial\Omega = \mathbf{0} \qquad (19)$$

where the hypothesis of null body forces has been assumed.

The kernel terms U_{ij}^* and T_{ij}^* are the *j-th* components of the generalized displacement and generalized tractions at the point P due to a concentrated generalized point load acting along the *i-th direction* at the point P_0. According to Davì (1989) and Davì and Milazzo (2001), the matrix $\mathbf{c}^*$ can be calculated by the following integration

$$\mathbf{c}^* = -\int_{\partial\Omega} \mathbf{T}^* d\partial\Omega \qquad (20)$$

When collocated at the boundary, eq (19) provides the boundary integral equations which, coupled with the essential and natural conditions (16), allow the determination of the unknowns on the boundary. Once the boundary solution is determined, the boundary integral representation gives the generalized displacements at the generic point P_0 of the domain in terms of the boundary variables. The following boundary integral representation for the generalized strain field holds, see Banerjiee and Butterfield (1981)

$$\mathbf{\Gamma}(P_0) = \int_{\partial\Omega} (\mathbf{\Xi}^*\mathbf{T} - \mathbf{\Theta}^*\mathbf{U})\, d\partial\Omega \qquad (21)$$

where

$$\mathbf{\Theta}^* = \mathsf{DC}^{*-1}\mathbf{T}^*\,,\ \mathbf{\Xi}^* = \mathsf{DC}^{*-1}\mathbf{U}^* \qquad (22)$$

Finally, the boundary integral representation for the generalized stress can be simply obtained by pre-multiplying eq (21) for the generalized stiffness matrix **R**

$$\mathbf{\Sigma}(P_0) = \int_{\partial\Omega} (\mathbf{R}\mathbf{\Xi}^*\mathbf{T} - \mathbf{R}\mathbf{\Theta}^*\mathbf{U})\, d\partial\Omega \qquad (23)$$

The integral eqs (21) and (23) hold for the internal points only. Their collocation on the boundary would require special consideration for the singularities arising from the presence of the derivatives of the fundamental solutions in the kernel. In this paper, however, the generalized stresses at the boundary not directly obtained as tractions have been approximated by suitably using the constitutive equations and the derivatives of the boundary variables with respect to the local tangential coordinate.

4 Fundamental solutions

The formulation of the boundary integral equations relies on the magneto-electro-elastic fundamental solution, which is governed by the following equation

$$\mathsf{D}^T\mathbf{R}\mathsf{D}\mathbf{U}_j + \mathbf{c}_j\delta(P-P_0) = \mathbf{0} \qquad (24)$$

in the infinite domain Ω_∞. Ding and Jiang (2004) derived the fundamental solution from a general solution found by means of the strict differential operator theory. In their derivation they distinguished between two different problems, dealing first with the case of point electric charge, point electric current and point force along the anisotropic direction, which are treated in the same way, and then solving separately for the case of point force acting in the plane of transverse isotropy. In the present paper the fundamental solution is deduced by extending to magneto-electro-elastic materials a modified Lekhnitskii's approach previously used for piezoelectric materials, see Davì and Milazzo (2001). In this framework, the solution of eq (24) is sought of the form

$$\mathbf{U} = \lambda\mathbf{a}\ln(X_1 + \alpha X_2) \qquad (25)$$

where λ, $\mathbf{a}$ and α are complex constants to be determined and

$$X_i = x_i(P) - x_i(P_0) \;\; i = 1,2 \tag{26}$$

Feeding the expression (25) into eq (24) produces the following generalized eigenvalue problem

$$\left[\mathbf{I}_1^T\mathbf{R}\mathbf{I}_1 + \alpha\left(\mathbf{I}_1^T\mathbf{R}\mathbf{I}_2 + \mathbf{I}_2^T\mathbf{R}\mathbf{I}_1\right) + \alpha^2\mathbf{I}_2^T\mathbf{R}\mathbf{I}_2\right]\mathbf{a} = \mathbf{0} \tag{27}$$

where the matrices $\mathbf{I}_i$ $(i = 1,2)$ are obtained from the generalized compatibility operator eq (11) by setting the derivatives with respect to x_i equal to one and replacing all the other terms with zeros. The solution of eq. (27) produces ten eigenvalues α_k with the associated eigenvectors $\mathbf{a}_k$, which form conjugate pairs for non degenerate materials. In the case of distinct eigenvalues the fundamental solution is then expressed as superposition of functions of the form (25), associated with the calculated eigenvalues α_k. By selecting the eigenvalues with positive imaginary parts, $Im(\alpha_k) > 0$, the generalized fundamental solution can be expressed as

$$\mathbf{U}_j = 2\sum_{k=1}^{5} Re\left[\lambda_{jk}\mathbf{a}_k \ln(X_1 + \alpha_k X_2)\right] \tag{28}$$

The corresponding tractions are given by

$$\mathbf{T}_j = 2\sum_{k=1}^{5} Re\left[\lambda_{jk}\mathsf{D}_n^T\mathbf{R}\mathsf{D}_{\alpha_k}\mathbf{a}_k\frac{1}{X_1 + \alpha_k X_2}\right] \tag{29}$$

where the matrix D_{α_k} is obtained from the differential operator D by replacing the derivative with respect to x_1 with one and the derivative with respect to x_2 with α_k. The constants λ_{jk} are determined by enforcing the compatibility and equilibrium conditions on the Gauss plane. The vector $\boldsymbol{\lambda}_j = \begin{bmatrix} \lambda_{j1} & \lambda_{j2} & \lambda_{j3} & \lambda_{j4} & \lambda_{j5} \end{bmatrix}^T$, corresponding to the *j-th* generalized point load (point load, concentrated electric charge or current) is then computed by

$$\boldsymbol{\lambda}_j = \left(\mathbf{B} + \tilde{\mathbf{B}}\tilde{\mathbf{A}}^{-1}\mathbf{A}\right)^{-1}\mathbf{c}_j \tag{30}$$

where the tilde denotes the complex conjugate, $\mathbf{A}$ is the matrix containing as columns the computed eigenvectors $\mathbf{a}_k$, and the columns $\mathbf{b}_k$ of the matrix $\mathbf{B}$ are defined as

$$\mathbf{b}_k = \overline{\mathsf{D}}_k\mathbf{R}\mathsf{D}_{\alpha_k}\mathbf{a}_k \tag{31}$$

where

$$\overline{\mathsf{D}}_k = C(\alpha_k)\begin{bmatrix} -1 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & i & 0 \end{bmatrix} \tag{32}$$

with $C(\alpha_k) = 2\pi(1 + i\alpha_k)/\left(1 + \alpha_k^2\right)$

It is worth noting that, by virtue of the generalized formalism used, the present fundamental solution has been derived in compact matrix notation, which has proved to be very practical and advantageous for computer implementation.

5 Numerical model and solution

The boundary integral formulation has been numerically implemented by using the BEM, Banerjiee and Butterfield (1981). The boundary $\partial\Omega$ of the considered domain is subdivided into M elements and the governing integral equations are therefore discretized by expressing the generalized boundary variables $\mathbf{U}$ and $\mathbf{T}$ in terms of their nodal values $\boldsymbol{\Delta}$ and $\mathbf{P}$ as

$$\begin{aligned} \mathbf{U} &= \mathbf{N}_U(\xi)\boldsymbol{\Delta} \text{ on } \partial\Omega \\ \mathbf{T} &= \mathbf{N}_T(\xi)\mathbf{P} \text{ on } \partial\Omega \end{aligned} \tag{33}$$

where $\mathbf{N}_U(\xi)$ and $\mathbf{N}_T(\xi)$ are matrices of standard shape function, expressed in terms of the isoparametric local coordinate ξ. Feeding the boundary approximation (33) into eq (19) produces

$$\mathbf{c}_i^*\boldsymbol{\Delta}_i + \sum_{j=1}^{M}\mathbf{H}_{ij}\,\boldsymbol{\Delta}_j + \sum_{j=1}^{M}\mathbf{G}_{ij}\,\mathbf{P}_j = \mathbf{0} \tag{34}$$

where

$$\begin{aligned} \mathbf{H}_{ij} &= \int_{\partial\Omega_j}\mathbf{T}^*(\xi)\,\mathbf{N}_U(\xi)\,J(\xi)\,d\xi \\ \mathbf{G}_{ij} &= -\int_{\partial\Omega_j}\mathbf{U}^*(\xi)\,\mathbf{N}_T(\xi)\,J(\xi)\,d\xi \end{aligned} \tag{35}$$

In the preceding equation $J(\xi)$ is the Jacobian involved in the transformation from the global to the local curvilinear coordinate. By collocating the point P_i at the boundary nodes using the collocation technique and absorbing the $\mathbf{c}_i^*$ matrix with the corresponding block $\boldsymbol{H}_{ii}$, we obtain a linear algebraic system which can be written as

$$\mathbf{H}\boldsymbol{\Delta}+\mathbf{G}\mathbf{P}=\mathbf{0} \tag{36}$$

where $\boldsymbol{\Delta}$ and $\mathbf{P}$ are the vectors of the nodal generalized displacements and tractions respectively, while $\mathbf{H}$ and $\mathbf{G}$ are the square influence matrix. Eq (36), coupled with the magneto-electro-mechanical boundary conditions, provides the solution of the problem for a single domain. To address problems involving laminated structural configurations, a multidomain approach is proposed, see Davì and Milazzo (2001) and Banerjiee and Butterfield (1981). In this scheme the original domain is subdivided into N subdomains, one for each ply of the laminate, and the boundary of each individual ply is discretized into M_k boundary elements. Repeating the procedure previously illustrated for each subdomain, the following set of subsystems is obtained

$$\mathbf{H}^{(k)}\boldsymbol{\Delta}^{(k)}+\mathbf{G}^{(k)}\mathbf{P}^{(k)}=\mathbf{0} \quad k=1,2...,N \tag{37}$$

where the superscript (k) indicate quantities associated with the *k-th* subdomain. To retrieve the domain unity, the suitable continuity and equilibrium conditions at the interface between contiguous plies are enforced. To do this let us introduce a partition of the linear algebraic system (37) in such a way that the generic vector $\mathbf{y}^{(k)}$ can be written as

$$\mathbf{y}^{(k)}=\begin{bmatrix}\mathbf{y}^{(k)}_{\partial\Omega_{i1}}\\ \vdots \\ \mathbf{y}^{(k)}_{\partial\Omega_{iN}}\end{bmatrix} \tag{38}$$

where the vector $\mathbf{y}^{(k)}_{\partial\Omega_{ij}}$ collects the components of $\mathbf{y}^{(k)}$ associated with the nodes belonging to the interfaces $\partial\Omega_{ij}$ between the *i-th* and *j-th* subdomains, with the convention that $\partial\Omega_{ii}$ denotes the external boundary of the ith subdomain (see Fig. 1)

By so doing the interface compatibility and equilibrium conditions, that is the interface continuity conditions, are given by

$$\begin{aligned}\boldsymbol{\Delta}^{(i)}_{\partial\Omega_{ij}}&=\boldsymbol{\Delta}^{(j)}_{\partial\Omega_{ij}} \quad i=1,...,M-1; \quad j=i+1,...,M\\ \mathbf{P}^{(i)}_{\partial\Omega_{ij}}&=-\mathbf{P}^{(j)}_{\partial\Omega_{ij}} \quad i=1,...,M-1; \quad j=i+1,...,M\end{aligned} \tag{39}$$

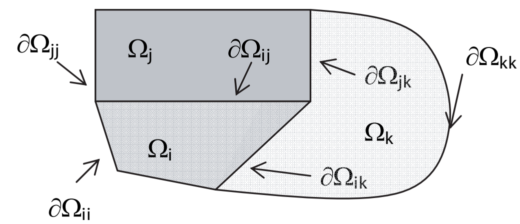

Figure 1 : Multidomain configuration.

It should be noted that, if the *i-th* and *j-th* subdomains have no common boundary, $\mathbf{y}^{(k)}_{\partial\Omega_{ij}}$ is a zero-order vector and eqs. (39) are no longer valid. The system (37) and the interface continuity conditions (39) provide a set of relationships which, together with the boundary conditions on the external boundaries $\partial\Omega_{ii}$, allow the solution of the analyzed magneto-electro-mechanical problem.

6 Numerical results

Some numerical results are presented to assess the effectiveness and the reliability of the proposed model. In the first application, used as benchmark for the numerical implemented model, the magneto-electro-elastic behaviour of a column having dimension $b = 0.6\,m$ and $h = 0.02\,m$ is analysed. The analysis is carried out under generalized plane strain hypothesis.

The material properties of the magneto-electro-elastic medium have been extracted from Ding and Jiang (2004) and are listed in Tab. 1. The generalized boundary conditions for the three analysed load cases are summarized in Tab. 2.

In Tab. 3 the mechanical displacements and the electric and magnetic scalar potentials, computed for the column centre point by using 32 linear elements, are listed for the three considered load cases. The values in parentheses represent the exact solution as extracted from Ding and Jiang (2004).

In the second application a rectangular simply-supported beam made of the same material as that of the previous application is considered (see Fig. 2).

The problem is treated under plane stress hypothesis and the boundary conditions are listed in Tab. 4. The beam dimensions are $b = 0.2\,m$ and $h = 0.02\,m$. Due to the side length ratio, the response of the 2D model will approach that of the 1D model.

Tab. 5 and Tab. 6 show the results obtained us-

Table 1 : Magneto-electro-elastic constitutive coefficients.

C_{ij} [GPa]	ε_{ij} [nF/m]	μ_{ij} [$\mu N/A^2$]	e_{ij} [C/m^2]	d_{ij} [$N/A\ m$]	g_{ij} [$N\ sec/V\ C$]
$C_{11} = 166$	$\varepsilon_{11} = 11.2$	$\mu_{11} = 5$	$e_{21} = -4.4$	$d_{21} = 580.3$	$g_{11} = 5\text{x}10^{-12}$
$C_{22} = 162$	$\varepsilon_{22} = 12.6$	$\mu_{22} = 10$	$e_{22} = 18.6$	$d_{22} = 699.7$	$g_{22} = 3\text{x}10^{-12}$
$C_{12} = 78$			$e_{14} = 11.6$	$d_{14} = 550$	
$C_{13} = 77$					
$C_{44} = 43$					
$C_{55} = 44.5$					

Table 2 : Boundary conditions for the magneto-electro-elastic column.

Boundary	**Load Case**	**Boundary conditions**			
$x_1 = 0, b$	**1, 2, 3**	$t_1 = 0$	$t_2 = 0$	$D_1 = 0$	$B_1 = 0$
$x_2 = 0, h$	**1**	$t_1 = 0$	$t_2 = 10\ Pa$	$D_2 = 0$	$B_2 = 0$
	2	$t_1 = 0$	$t_2 = 0$	$D_2 = 10^{-10}\ C/m^2$	$B_2 = 0$
	3	$t_1 = 0$	$t_2 = 0$	$D_2 = 0$	$B_2 = 10^{-8}\ N/Am$

Table 3 : Displacements and potentials at the column centre point in comparison with the exact values (in parentheses) furnished by Ding and Jiang (2004).

Load case	$u_1\ [m]$	$u_2\ [m]$	$\varphi\ [V]$	$\psi\ [A]$
1	-9.649e-12 (-9.500e-12)	5.6734e-13 (5.683e-13)	9.510e-4 (9.495e-4)	2.1058e-5 (2.139e-5)
2	-2.146e-13 (-2.108e-13)	9.4508e-15 (9.495e-15)	-6.2416e-5 (-6.289e-5)	2.5956e-7 (2.567e-7)
3	5.130e-13 (5.077e-13)	2.1217e-14 (2.139e-14)	2.5374e-5 (2.567e-5)	-7.4864e-6 (-7.521e-6)

Table 4 : Boundary conditions for the magneto-electro-elastic beam.

Boundary	**Boundary conditions**			
$x_1 = 0, b$	$t_1 = 0$	$u_2 = 0$	$\varphi = 0$	$\psi = 0$
$x_2 = 0$	$t_1 = 0$	$t_2 = 0$	$D_2 = 0$	$B_2 = 0$
$x_2 = h$	$t_1 = 0$	$t_2 = -10\ sin(\pi x_1/b)$	$D_2 = 0$	$B_2 = 0$

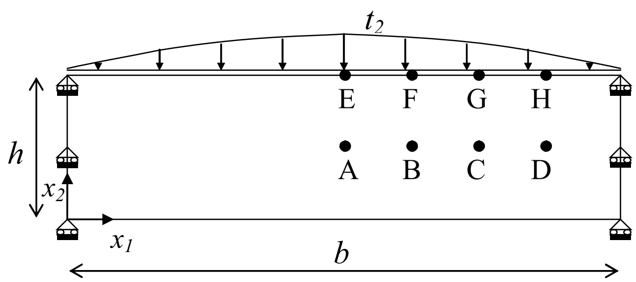

Figure 2 : Magneto-electro-elastic beam scheme.

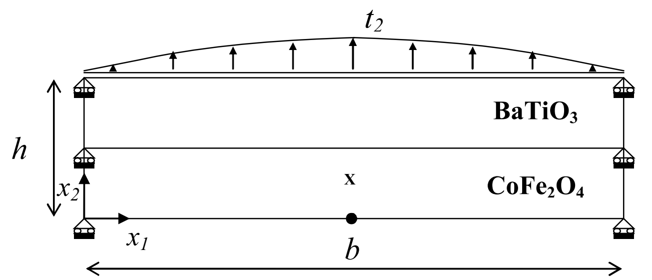

Figure 4 : Magneto-electro-elastic laminate scheme.

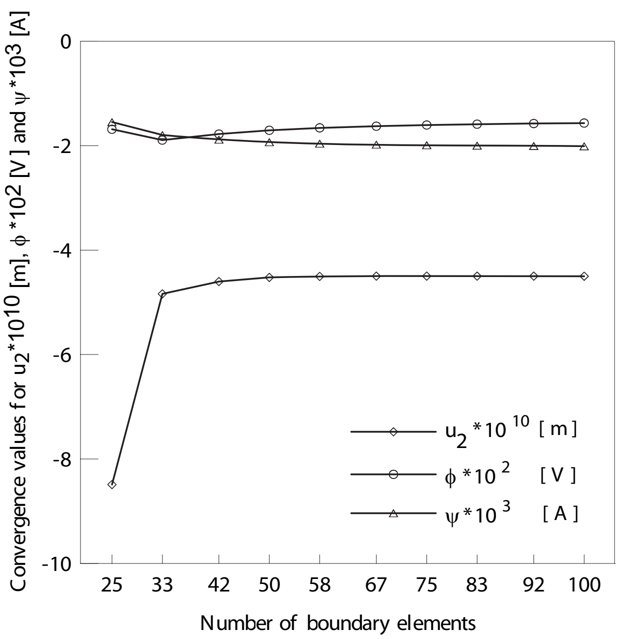

Figure 3 : Convergence analysis for the point E of the magneto-electro-elastic clamped beam.

ing totally 180 linear elements with the exact values given in Ding and Jiang (2004) for the internal points $A \equiv$(0.100,0.010), $B \equiv$(0.125,0.010), $C \equiv$(0.150,0.010), $D \equiv$(0.175,0.010) and for the boundary points $E \equiv$(0.100,0.020), $F \equiv$(0.125,0.020), $G \equiv$(0.150,0.020), $H \equiv$(0.175,0.020) respectively (see Fig. 2).

The same beam with clamped ends has also been analysed. Boundary conditions are the same as the precedent case except for the u_1 displacement which is set to zero for $x_1 = 0, b$. Fig. 3 shows the convergence analysis carried out for the transverse displacement u_2 and for the electric and magnetic potential at point E. The convergence of the electric and magnetic potentials appears slower than that of the elastic displacement and, for a numerical evaluation of the beam behaviour, at least 100 linear elements have been used.

Tab. 7 and Tab. 8 compare the results obtained using the present approach with that numerically evaluated by Ding and Jiang (2004).

As last application, the magneto-electro-elastic behavior of the two layer composite laminate shown in Fig. 4 has been studied. The analysis is carried out under cylindrical bending conditions and generalized plain strain hypothesis. The plate dimensions are b = 0.01 m and $h = 0.001$ m.

The material properties of the piezoelectric and piezomagnetic layers have been extracted from Heyliger, Ramirez and Pan (2004) and are listed in Tab. 9 and Tab. 10 respectively. The generalized boundary conditions are summarized in Tab. 11. In Tab 12 the transverse displacement, the electric potential and the magnetic scalar potential, computed by using 427 linear elements, are listed for three selected points. The values in parentheses are the exact solutions as extracted from Heyliger, Ramirez and Pan (2004).

Fig. 5 shows the convergence analysis for the transverse displacement and the electric and magnetic potential at the point x$\equiv$ (b/2,0) (see Fig. 4); it appears that the magnetic scalar potential exhibits the slowest convergence.

In Fig. 6 to Fig. 13 the components of the generalized traction vector are shown as functions of the plate thickness.

Fig. 6 and Fig. 7 show the presence of a small discontinuity, both in terms of intensity and inclination, in the distribution of the stress component σ_{11} and σ_{33} at the interface between the two layers, due to the different material properties, as also pointed out by Heyliger, Ramirez and Pan (2004), while the stress components σ_{22} and σ_{12}, shown in Fig. 8 and Fig. 9 respectively, are continuous

Table 5 : Values at the four internal points of the magneto electro elastic simply supported beam. Values in parentheses are taken from Ding and Jiang (2004).

Point	u_2 [m]	φ[V]	ψ[A]	σ_1 [Pa]	σ_2 [Pa]	D_2 [C/m^2]	B_2 [N/Am]
	-1.975e-9	-2.313e-2	-1.796e-3	2.795e-2	-4.999	-1.114e-11	-2.283e-10
A	(-2.000e-9)	(-2.314e-2)	(-1.808e-3)	(2.818e-2)	(-5.000)	(-1.114e-11)	(-2.282e-10)
	-1.825e-9	-2.137e-2	-1.659e-3	2.577e-2	-4.619	-1.027e-11	-2.105e-10
B	(-1.847e-9)	(-2.138e-2)	(-1.671e-3)	(2.604e-2)	(-4.619)	(-1.029e-11)	(-2.109e-10)
	-1.397e-9	-1.6364e-2	-1.270e-3	1.958e-2	-3.535	-7.872e-12	-1.614e-10
C	(-1.414e-9)	(-1.6372-2)	(-1.279e-3)	(1993e-2)	(-3.5351)	(-7.875e-12)	(-1.614e-10)
	-7.562e-10	-8.859e-3	-6.878e-4	1.031e-2	-1.913	-4.282e-12	-8.734e-11
D	(-7.652e-10)	(-8.857e-3)	(-6.921e-4)	(1.078e-2)	(-1.913)	(-4.262e-12)	(-8.734e-11)

Table 6 : Values at the four boundary points of the magneto electro elastic simply supported beam. Values in parentheses are taken from Ding and Jiang (2004).

Point	u_2 [m]	φ [V]	ψ [A]
	-1.9693e-9	-8.9121e-3	-2.1676e-3
E	(-1.9930e-9)	(-8.7280e-3)	(-2.1840e-3)
	-1.8195e-9	-8.2311e-3	-2.0028e-3
F	(-1.8420e-9)	(-8.0630e-3)	(-2.0180e-3)
	-1.3927e-9	-6.2932e-3	-1.5334e-3
G	(-1.4090e-9)	(-6.1720e-3)	(-1.5440e-3)
	-7.5389e-10	-3.3972e-3	-8.3023e-4
H	(-7.6280e-10)	(-3.3400e-3)	(-8.3570e-4)

Table 7 : Transverse displacement, electric and magnetic potential at four boundary points of the clamped beam. Values in parenthess are extracted from Ding and Jiang (2004).

Point	u_2 [m]	φ [V]	ψ [A]
E	-4.4988e-10	-1.5755e-2	-2.0052e-3
	(-4.2140e-10)	(-1.6430e-2)	(-1.9630e-3)
F	-3.9603e-10	-1.5037e-2	-1.8434e-3
	(-3.6990e-10)	(-1.5660e-2)	(-1.8030e-3)
G	-2.5651e-10	-1.3000e-2	-1.3817e-3
	(-2.3710e-10)	(-1.3450e-2)	(-1.3520e-3)
H	-9.4191e-11	-0.9979e-2	-6.8681e-4
	(-8.3910e-11)	(-1.0170e-2)	(-6.7180e-4)

through the interface.

The electric displacement components D_1 and D_2 (see Fig. 10 and Fig. 11) and the magnetic induction components B_1 and B_2, shown in Fig. 12 and Fig. 13, present a more relevant discontinuity crossing the interface, both in the intensity (B_1 and D_1) and in the slope (B_2 and D_2), due to the transition from a piezoelectric to a piezomagnetic layer; moreover they show a characteristic behaviour, also depicted by Heyliger, Ramirez and Pan (2004): the through thickness electric displacement distribution is linear in the magnetostrictive layer and non-linear in the piezoelectric one, while the through thickness magnetic induction distribution is linear in the piezoelectric medium and non-linear in the magnetostrictive one.

In all the studied cases, the agreement between previous results and those calculated through the developed formulation appears fully satisfying.

7 Conclusions

In this work a multidomain boundary element approach for the analysis of general magneto-electro-elastic laminates has been developed. The model is entirely expressed in terms of generalized magneto-electro-elastic variables, which allow the extension of anisotropic elasticity techniques to the more general magneto-electro-elastic problem. In particular, the Somigliana's identity is rewritten in the extended notation and the boundary integral representation is then directly deduced. The fun-

Table 8 : Transverse displacement, electric and magnetic potential at four internal points of the clamped beam. Values in parentheses are numerically evaluated by Ding and Jiang (2004).

Point	u_2 [m]	φ[V]	ψ[A]	σ_1 [Pa]	σ_2 [Pa]	D_2 [C/m^2]	B_2 [N/Am]
A	-4.5180e-10	-2.0292e-2	-1.8712e-3	-8.6756e-1	-4.9985	-1.1147e-11	-2.2826e-10
	(-4.2310e-10)	(-2.0860e-2)	(-1.8270e-3)	(-8.6780e-1)	(-4.9969)	(-1.1130e-11)	(-2.2540e-10)
B	-3.9713e-10	-1.8531e-2	-1.7359e-3	-8.6970e-1	-4.6180	-1.0299e-11	-2.1085e-10
	(-3.7110e-10)	(-1.9060e-2)	(-1.6940e-3)	(-8.7020e-1)	(-4.6167)	(-1.0320e-11)	(-2.1080e-10)
C	-2.5551e-10	-1.3511e-2	-1.3499e-3	-8.7582e-1	-3.5344	-7.8473e-12	-1.6004e-10
	(-2.3720e-10)	(-1.3980e-2)	(-1.3190e-3)	(-8.7600e-1)	(-3.5334)	(-7.8670e-12)	(-1.5960e-10)
D	-9.0907e-11	-5.9896e-3	-7.7025e-4	-8.8483e-1	-1.9107	-2.6208e-12	-1.2924e-11
	(-8.2220e-11)	(-6.3170e-3)	(-7.5400e-4)	(-8.8410e-1)	(-1.9096)	(-2.6250e-12)	(-6.6430e-12)

Table 9 : $BaTiO_3$ constitutive coefficients.

C_{ij} [GPa]	ε_{ij} [nF/m]	μ_{ij} [$\mu N/A^2$]	e_{ij} [C/m^2]	d_{ij} [N/Am]	g_{ij} [$Nsec/VC$]
$C_{11} = 166$	$\varepsilon_{11} = 11.2$	$\mu_{11} = 5$	$e_{21} = -4.4$	$d_{21} = 0$	$g_{11} = 0$
$C_{22} = 162$	$\varepsilon_{22} = 12.6$	$\mu_{22} = 10$	$e_{22} = 18.6$	$d_{22} = 0$	$g_{22} = 0$
$C_{12} = 78$			$e_{14} = 11.6$	$d_{14} = 0$	
$C_{13} = 77$					
$C_{44} = 43$					
$C_{55} = 44.5$					

Table 10 : $CoFe_2O_4$ constitutive coefficients.

C_{ij} [GPa]	ε_{ij} [nF/m]	μ_{ij} [$\mu N/A^2$]	e_{ij} [C/m^2]	d_{ij} [N/Am]	g_{ij} [$Nsec/VC$]
$C_{11} = 286$	$\varepsilon_{11} = 0.08$	$\mu_{11} = -590$	$e_{21} = 0$	$d_{21} = 580.3$	$g_{11} = 0$
$C_{22} = 269.5$	$\varepsilon_{22} = 0.093$	$\mu_{22} = 157$	$e_{22} = 0$	$d_{22} = 699.7$	$g_{22} = 0$
$C_{12} = 170.5$			$e_{14} = 0$	$d_{14} = 550$	
$C_{13} = 173$					
$C_{44} = 45.3$					
$C_{55} = 56.5$					

damental solutions are calculated generalizing a modified Lekhnitskii's approach and are expressed in a very compact matrix notation.

The analysis of some configurations has been carried out and some characteristic features of magneto-electro-elastic laminates have been pointed out. In particular, it has been found that the in-plane normal stress components show a linear distribution in the through the thick-

Table 11 : Boundary conditions for the magneto-electro-elastic simply supported laminate.

Boundary	Boundary conditions			
$x_1 = 0, b$	$t_1 = 0$	$u_2 = 0$	$\varphi = 0$	$\psi = 0$
$x_2 = 0$	$t_1 = 0$	$t_2 = 0$	$D_2 = 0$	$B_2 = 0$
$x_2 = h$	$t_1 = 0$	$t_2 = sin(\pi x_1/b)$	$D_2 = 0$	$B_2 = 0$

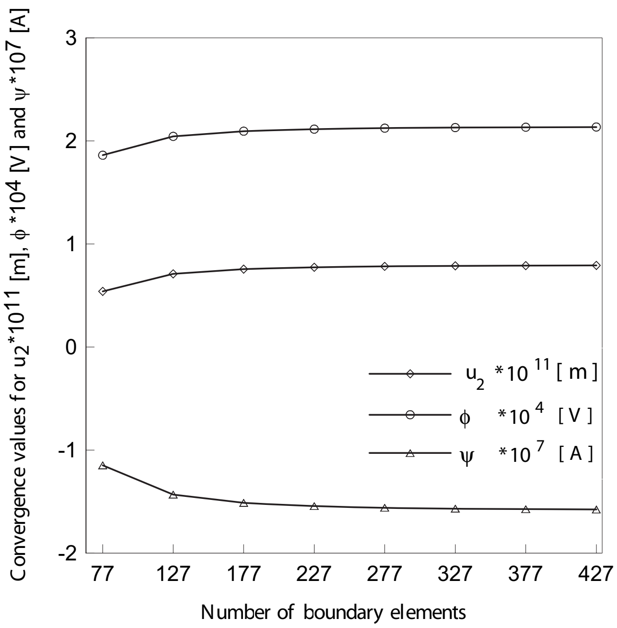

Figure 5 : Convergence analysis for the point X≡(b/2,0) of **t**he magneto electro elastic simply supported laminate.

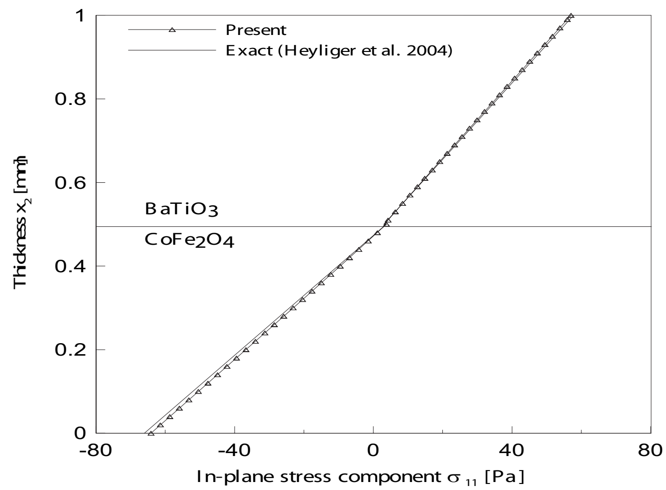

Figure 6 : Through thickness distribution of the in plane normal stress σ_{11}.

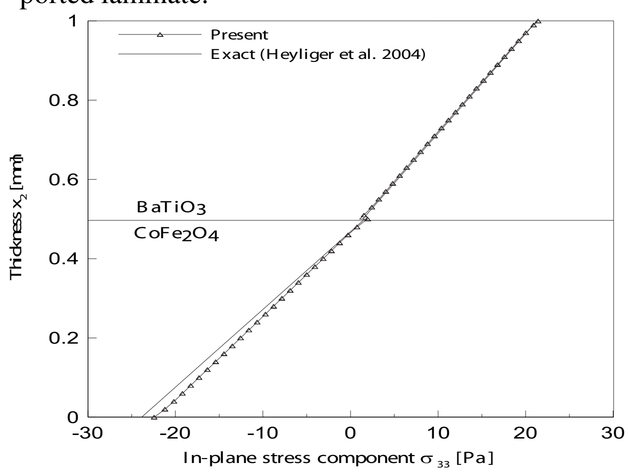

Figure 7 : Through thickness distribution of the in plane normal stress σ_{33}.

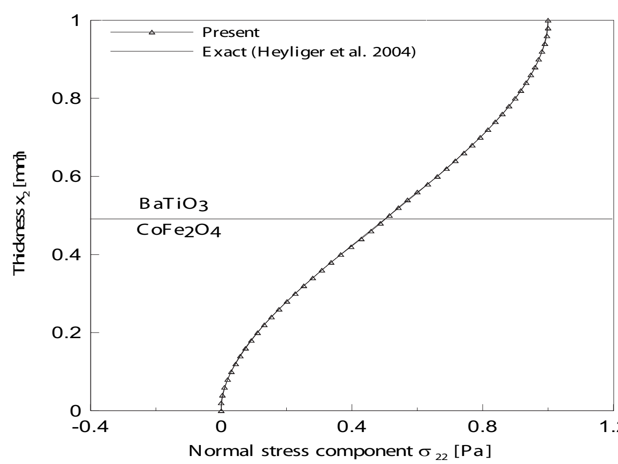

Figure 8 : Through thickness behavior of σ_{22}.stress component.

ness direction with a small discontinuity in slope and intensity at the interface; on the other hand the transverse normal component and the shear component are continuous and vary non-linearly. More interesting are the through the thickness distributions of the electric and magnetic fields. Both the components of the elec-

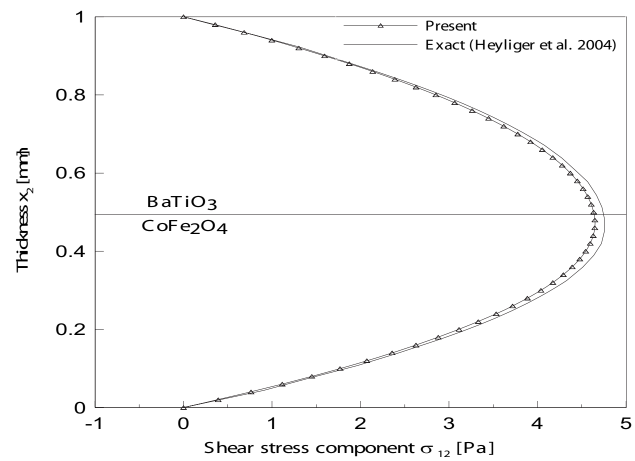

Figure 9 : Through thickness distribution of the transverse shear stress σ_{12}.

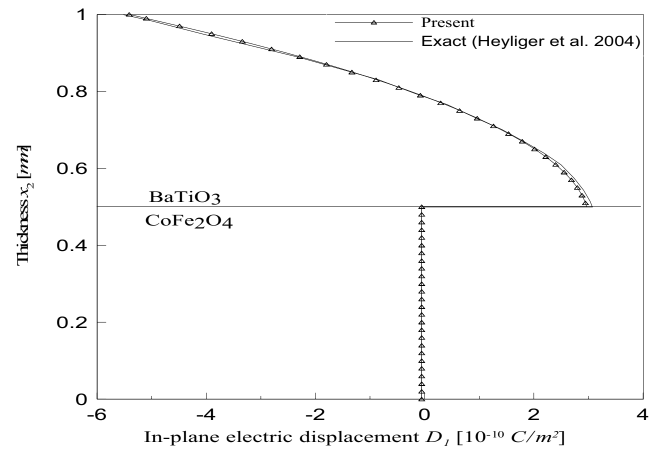

Figure 10 : Through thickness distribution of the in-plane electric displacement component D_1.

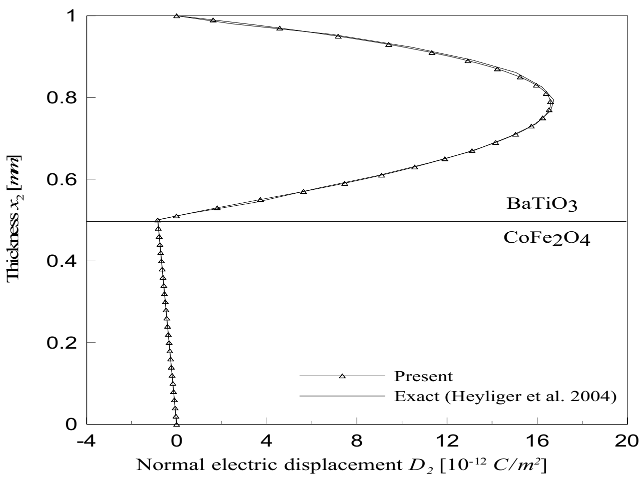

Figure 11 : Through thickness distribution of the normal electric displacement component D_2.

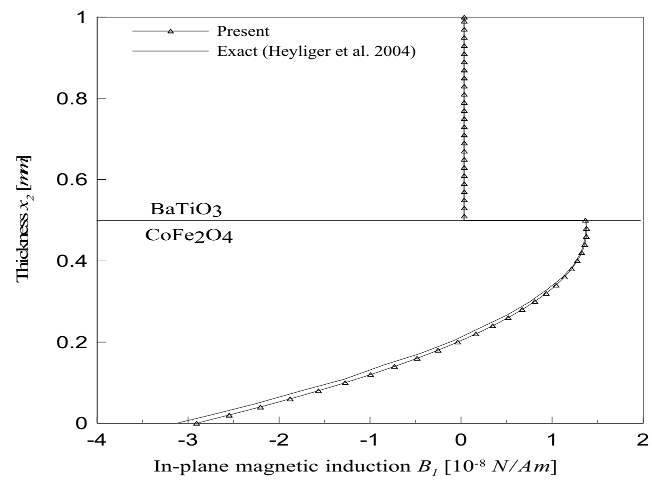

Figure 12 : Through thickness distribution of the in-plane magnetic induction component B_1.

tric field vary linearly in the piezomagnetic lamina and non-linearly in the piezoelectric one; in the same way, both the magnetic field components behave linearly in the piezoelectric medium and non-linearly in piezomagnetic one. Moreover, the in-plane components of the electric and magnetic vectors show a strong discontinuity in magnitude crossing the interface, while the transverse components show a discontinuity only in slope. These results have shown a good agreement with those reported in the literature, confirming therefore the accuracy of the proposed numerical scheme.

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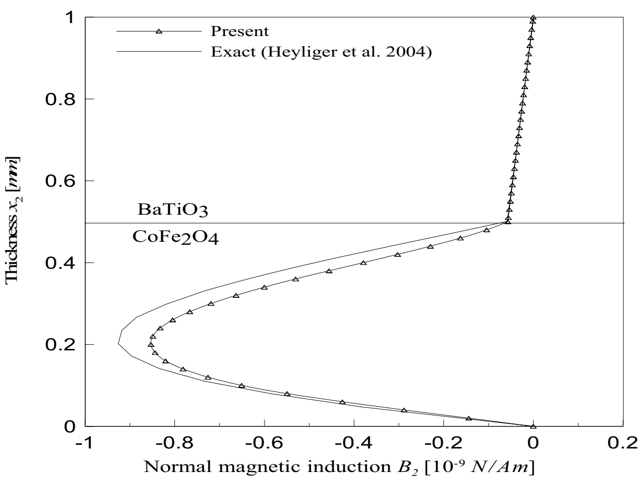

Figure 13 : Through thickness distribution of the normal magnetic induction component B_2.

Table 12 : Transverse displacement and electric and magnetic scalar potentials at three points for the $CoFe_2O_4/BaTiO_3$ laminate.

x_1	x_2	u_2 [m]	φ [V]	ψ [A]
		7.9190e-12	2.1345e-4	-1.5767e-7
	0	(7.9832e-12)	(2.141e-4)	(-1.6977e-7)
		7.9730e-12	2.1572e-4	-2.3101e-7
b/2	h/2	(8.0377e-12)	(2.1637e-4)	(-2.4008e-7)
		7.9353e-12	1.228e-4	-2.296e-7
	h	(7.9997e-12)	(1.2268e-4)	(-2.386e-7)

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