Topology Optimization of 2D Potential Problems Using Boundary Elements

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Abstract: Topological Optimization provides a powerful framework to obtain the optimal domain topology for several engineering problems. The Topological Derivative is a function which characterizes the sensitivity of a given problem to the change of its topology, like opening a small hole in a continuum or changing the connectivity of rods in a truss.

A numerical approach for the topological optimization of 2D potential problems using Boundary Elements is presented in this work. The formulation of the problem is based on recent results which allow computing the topological derivative from potential and flux results. The Boundary Element analysis is done using a standard direct formulation. Models are discretized using linear elements and a periodic distribution of internal points over the domain. The total potential energy is selected as cost function. The evaluation of the topological derivative at internal points is performed as a post-processing procedure. Afterwards, material is removed from the model by deleting the internal points with the lowest (or highest depending the nature of the problem) values of the topological derivate. The new geometry is then remeshed using a weighted Delaunay triangularization algorithm capable of detecting "holes" at those positions where internal points have been removed. The procedure is repeated until a given stopping criteria is satisfied.

The proposed strategy proved to be flexible and robust. A number of examples are solved and results are compared to those available in the literature.

keyword: Topology optimization, Boundary elements, Potential problems.

Abstract: Introduction

A classical problem in engineering design consists in finding the optimum geometric configuration of a body that maximizes or minimizes a given cost function while it satisfies the problem boundary conditions. The most general approach to tackle these problems is by means of topological optimization tools, which allow not only to change the shape of the body but its topology via the creation of internal holes. Topological optimization tools are capable of deliver optimal designs with a priori poor information on the optimal shape of the body.

Homogenization methods are possibly the most used approach for topology optimization [Bensoe and Kikuchi, 1988]. In these methods a material model with microscale voids is introduced and the topology optimization problem is defined by seeking the optimal porosity of such a porous medium using one of the optimality criteria. In this way, the homogenization technique is capable of producing internal holes without prior knowledge of their existence. However, the homogenization method often produces designs with infinitesimal pores that make the structure not manufacturable. A number of variations of the homogenization method have been investigated to deal with these issues, such as penalization of intermediate densities and filtering procedures [Sigmund and Peterson, 1998]. On the other hand, there exist the so-called level set methods which are based on the moving of free boundaries [Wang and Wang, 2004; Wang and Wang, 2006]. The main drawback of level set methods is that they require of pre-existent holes within the model domain in order to conduct a topological optimization.

Alternative approaches are the Topological Derivative (D_T) methods [Novotny, et al., 2003; Ceá, et al., 2000]. The basic idea behind the D_T is the evaluation of cost function sensitivity to the creation of a hole. Wherever this sensitivity is low enough (or high enough depending on the nature of the problem) the material can be progressively eliminated. Topological derivative methods aim to solve the aforementioned limitations of the homogenization methods.

A numerical approach for the topological optimization of 2D potential problems using Boundary Elements is presented in this work. The formulation of the problem is based on some recent results by Novotny et al. [Novotny, et al., 2003] which allow computing the topological derivative using potential and flux results. The

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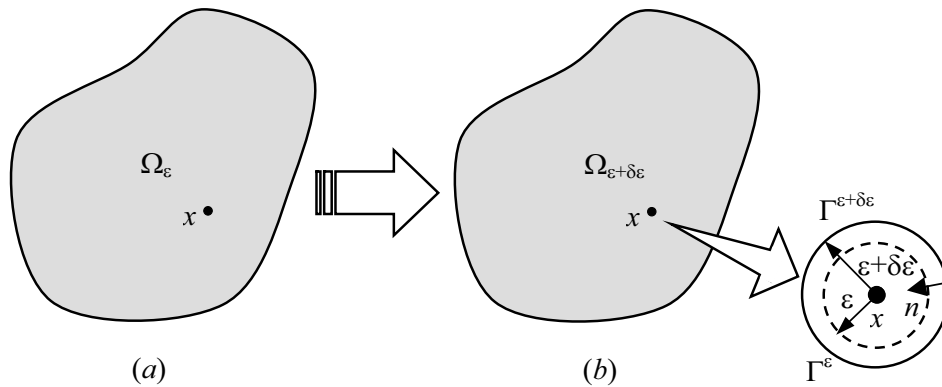
Figure 1 : New definition of the D_T as proponed by Novotny et al [Novotny et al., 2003]: (a) Original domain with pre-existing hole $\psi(\Omega_\varepsilon)$, (b) Perturbed domain $\psi(\Omega_{\varepsilon+\delta\varepsilon})$

Boundary Element analysis is done using a standard direct formulation. Models are discretized using linear elements and a periodic distribution of internal points over the domain. The total potential energy is selected as cost function. Afterwards, material is removed from the model by deleting the internal points with the lowest (or highest) values of the topological derivate. The new geometry is remeshed using an Extended Delaunay Tessellation algorithm capable of detecting "holes" at those positions where internal points and nodes have been removed. The procedure is repeated until a given stopping criteria is satisfied. The performance of the proposed strategy is illustrated for a number of examples.

To the author's knowledge, the only antecedent in the implementation of the Novotny et al. [Novotny, et al., 2003] approach for the computation of the D_T using BEM is a recent work by Marckzak [Marckzak, 2005]. Both implementations, that due to Marckzak [Marckzak, 2005] and the one presented in this work use similar procedures for the computation of the D_T at internal points. However, they differ in the strategy proposed for the creation of the holes and the model remeshing.

1 Topological Sensitivity Analysis

The original definition of the D_T relates the sensitivity of a cost function $\psi(\Omega)$ when the topology of the optimization domain Ω is altered by creating a small hole. However, the direct application and implementation of this concept is not straightforward, as it is not possible to establish a homeomorphism between the domains with different topologies (domains with and without the hole).

Novotny et al [Novotny, et al., 2003] proposed an alter-native definition of the D_T that overcomes the problem. They assimilated the creation of a hole to the perturbation of a pre-existing hole whose radius tends to zero (see Figure 1). Therefore, both topologies of the optimization domain Ω are now similar and it is possible to establish a homeomorphism between them. According to this new definition, the expression for the D_T is

$$D_T(x) = \lim_{\substack{\varepsilon \to 0 \\ \delta\varepsilon \to 0}} \frac{\psi(\Omega_{\varepsilon+\delta\varepsilon}) - \psi(\Omega_\varepsilon)}{f(\varepsilon+\delta\varepsilon) - f(\varepsilon)} \tag{1}$$

where $\psi(\Omega_\varepsilon)$ and $\psi(\Omega_{\varepsilon+\delta\varepsilon})$ are the cost function evaluated for the original and perturbed domain, ε is the initial radius of the hole, $\delta\varepsilon$ is a small perturbation of the hole radius and f is a regularization function. The function f is problem dependent and $f(\varepsilon) \to 0$ when $\varepsilon \to 0$.

The new definition of the D_T in equation (1) merely provides the sensitivity of the problem when the size of the hole is perturbed and not when it is effectively created (as one has in the original definition of the topological derivative). However, it is understood that to expand a hole of radius ε, when $\varepsilon \to 0$, is nothing more than creating it (a complete mathematical proof that establishes the relation between both definitions of the D_T is given in [Novotny, et al., 2003]). Moreover, the relationship between the two definitions constitutes the formal relation between the D_T and the shape sensitivity analysis. The advantage of the novel definition for the topological derivative given by Eq. (1) is that the whole mathematical framework developed for the shape sensitivity analysis can now be used to compute the D_T.
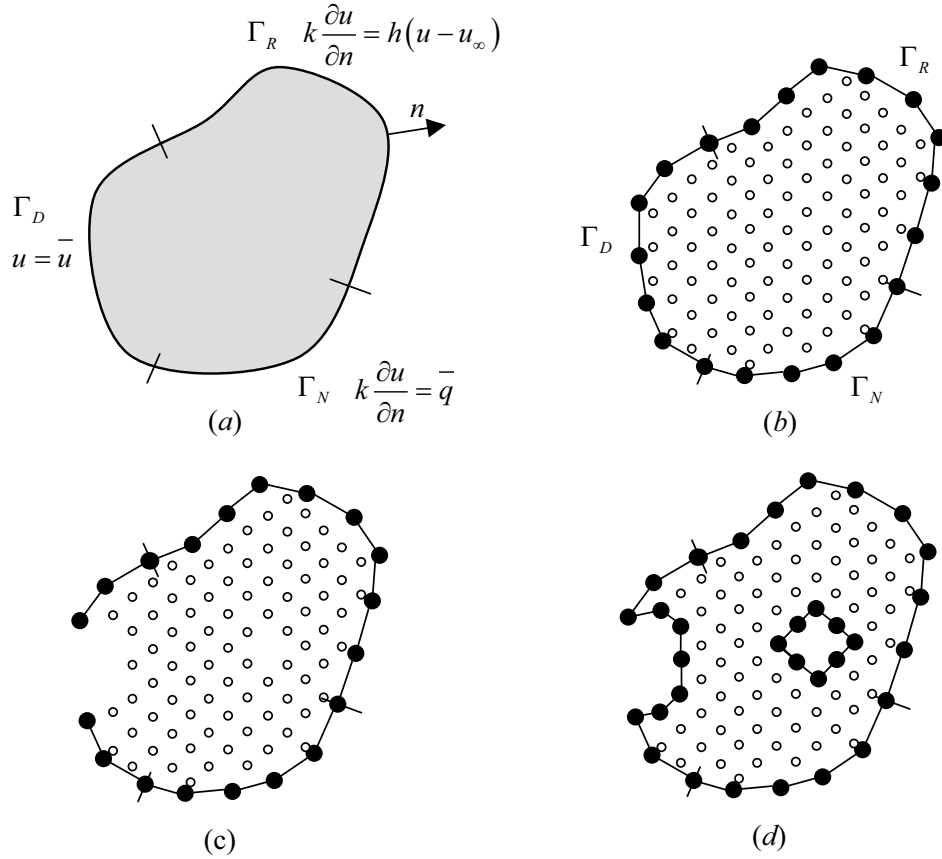
Figure 2 : BEM implementation: (a) Problem definition and boundary conditions, (b) Initial BEM model, (c) Elimination internal points, (d) BEM model remeshing.

2 The Topological Derivative for Linear Potential Problems

In the present work the D_T is applied to the optimization of two dimensional linear potential problems. With this purpose the solution of equation (1) for different boundary conditions is needed. These results are briefly presented next following Novotny et al [Novotny, et al., 2003].

The general potential problem for the temperature field u governed by Laplace equation $k\nabla^2 u = 0$ is defined on the domain Ω_ε and subjected to Dirichlet $u = \overline{u}$, Newman $k\frac{\partial u}{\partial n} = \overline{q}$ and Robin $k\frac{\partial u}{\partial n} = h(u - u_\infty)$ boundary conditions on complementary portions of its frontier Γ_D, Γ_N and Γ_R respectively (see Figure 2a). The symbol k stands for the thermal conductivity while n is the outward normal to the boundary. Similarly, boundary conditions will be imposed on the boundary of the holes Γ^ε either in the potential (Dirichlet), in the flux (Neumann) or even in both variables (Robin). Thus, the function

$$g(\alpha,\beta,\gamma) = \alpha(u - \overline{u}^\varepsilon) + \beta\left(k\frac{\partial u}{\partial n} - \overline{q}^\varepsilon\right)$$
$$+\gamma\left(k\frac{\partial u}{\partial n} + h^\varepsilon(u - u_\infty^\varepsilon)\right) = 0 \quad (2)$$

with $0 \le \alpha, \beta, \gamma \le 1$ and $\alpha + \beta + \gamma = 1$ takes into account the prescribed boundary conditions on the holes. Thus by doing $\alpha = 1$ and $\beta = \gamma = 0$ the temperature $\overline{u}^\varepsilon$ is prescribed on the holes, while for $\beta = 1$ and $\alpha = \gamma = 0$ the flux $\overline{q}^\varepsilon$ is prescribed. The symbols u_∞^ε and h^ε are the temperature and the transfer coefficient in the interior of the holes, respectively, when $\gamma = 1$ and $\alpha = \beta = 0$ are specified.

The cost function $\psi(\Omega_\varepsilon)$ is, in a certain way, arbitrary. For the case of heat conduction problems the total potential energy can be adopted. The expression of the total

potential energy in absence of body loads is

$$\psi(\Omega_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} k\nabla u \cdot \nabla u \, d\Omega_\varepsilon + \frac{1}{2}\int_{\Gamma_R} hu^2 d\Gamma$$

$$+ \frac{1}{2}\gamma\int_{\Gamma^\varepsilon} h^\varepsilon u^2 d\Gamma^\varepsilon + \int_{\Gamma_N}\overline{q}u d\Gamma - \int_{\Gamma_R} hu_\infty u \, d\Gamma$$

$$+ \beta\int_{\Gamma^\varepsilon}\overline{q}^\varepsilon u d\Gamma^\varepsilon - \gamma\int_{\Gamma^\varepsilon} h^\varepsilon u_\infty^\varepsilon u d\Gamma^\varepsilon \qquad (3)$$

The cost function (3) can be used to derive the expression for the D_T using equation (1) for the three types of boundary conditions on the holes. These results are reported in Table 1 following Novotny et al [Novotny, et al., 2003]. Note that the total potential energy serves as a measure of the energy in transit within a solid, and as it will be illustrated in Section 5, it can be used to find the optimum geometric configuration of heat conductors.

3 Bem Implementation

The idea behind the numerical implementation is to compute the D_T using BEM analysis results and the formulas in Table 1. Next, the topology of the problem domain is perturbed by creating holes at the positions with extreme values of D_T (the selection of maximum or minimum values will depend on the nature of the problem). The process is repeated until a given stopping criterion is satisfied.

The evaluation of the D_T using the expressions in Table 1 only requires the potential and fluxes to be known at internal points. These can be easily solved using a standard BEM formulation (see for example Brebbia et al. [Brebbia, et al, 1984]). In the present work BEM models are discretized using linear elements.

The optimization algorithm can be summarized as follows (the index j stands for iteration number):

1. Provide an initial domain $\Omega^{j=0}$ and the stopping criterion.

2. Solve the BEM model for the Ω^j domain (Figure 2b). Compute the potential u^j and flux q^j fields at internal points.

3. Compute the $D_T(x)$ using the formulas in Table 1.

4. Select the points with the extreme values of D_T (a few percent of the total number of points)

**Table 1** : Topological Derivatives for 2D potential problems.

Boundary Condition	D_T	$f(\varepsilon)$
Newman with $\overline{q}^\varepsilon = 0$	$k\nabla u \cdot \nabla u$	$-\pi\varepsilon^2$
Newman with $\overline{q}^\varepsilon \neq 0$	$\overline{q}^\varepsilon u$	$-2\pi\varepsilon$
Robin	$-\frac{1}{2}h^\varepsilon u\left(u - 2u_\infty^\varepsilon\right)$	$-2\pi\varepsilon$
Dirichlet	$-\frac{1}{2}k\left(u - \overline{u}^\varepsilon\right)$	$\dfrac{2\pi}{\log(\varepsilon)}$

5. Create holes by removing the points selected in step 4 (Figure 2c).

6. Check stopping criterion. If necessary, make $j = j + 1$, define a new domain Ω^j, remesh the BEM model (Figure 2d) and go to step 2.

7. At this stage the desired final topology is obtained.

The remeshing of the BEM model after the removal of internal points is a key issue for the performance of the optimization algorithm. With this purpose an α-shapes algorithm is employed [Calvo, et al, 2003]. Alpha shapes can be viewed as Delaunay triangularization of a point set weighted by the parameter α. Alpha shapes formalize the intuitive notion of shape, and for varying parameter α, it ranges from crude to fine shapes. The most crude shape is the convex hull itself, which is obtained for very large values of α. As α decreases, the shape shrinks and develops cavities that may join to form holes. In this work the parameter α is selected as the average distance between the boundary nodes and the internal points of the BEM model. This is the reason why internal points are distributed on the model domain using a regular array (see Figure 2b).

4 Examples

Results for three examples are presented in this section. In order to assess the performance of the BEM algorithm, the examples are taken from Novotny et al and the results compared to those obtained in that work using finite element models.
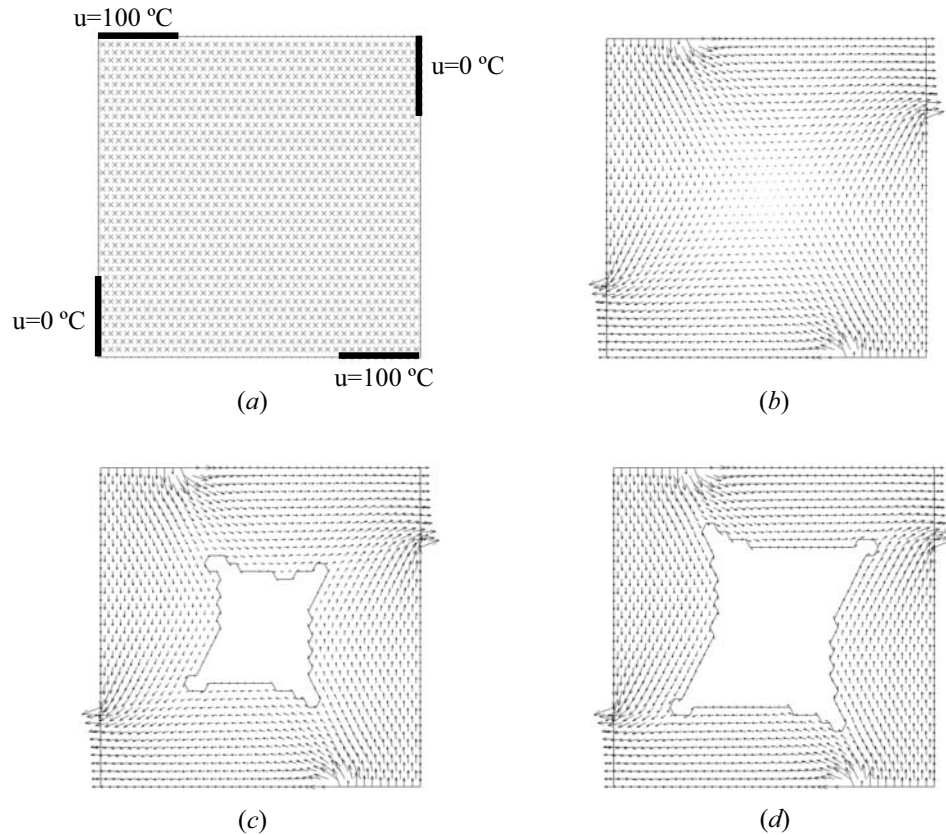
Figure 3 : Non-symmetric heat conductor: (a) Problem definition and initial BEM discretization, (b) Initial flux solution, (c) Optimized geometry and flux solution after 20 iterations, (d) Final model geometry and flux solution (iteration 40)

4.1 Non-symmetric heat conductor

The first example consists in an initial squared domain with dimensions 10 m ×10 m and prescribed temperatures in the corners as illustrated in Figure 3a. The remaining boundary is isolated. The model was discretized using 160 boundary elements and 1600 internal points. The goal of the optimization is to diminish the total potential energy (which can be interpreted in this case as a measure of the heat flux, or even, as a measure of energy in transit in the body under analysis) in order to improve the geometry of the heat conductor. In this way, holes are created with Newman boundary conditions ($\overline{q}^{\varepsilon} = 0$) where the D_T assumes smallest values (first case in Table 1) in order to . Following the reference, 0.5% of surface of the initial domain was eliminated in each iteration (8 points per iteration) until 20% of the initial domain was removed (40 iterations). The evolution of the model geometry and the flux solution are presented in Figures 3b, c and d. Obtained results are in perfect agreement with

those reported by Novotny et al. Similar results were obtained using a coarser mesh consisting in 80 boundary nodes and 400 internal points.

4.2 Heat conductor with a initial hole

In this example the topology of a squared heat conductor containing a initial circular hole of $R = 2,5$ m is improved following the same optimization criterion of the previous example. The initial external dimensions of the conductor are 10 m ×10 m with temperatures prescribed along the central portion of the lateral sides (see Figure 4a). The remaining boundary is isolated. The symmetry of the problem allows only half of its geometry to be discretized using 130 boundary elements and 688 internal points. In this case 1% of the initial model surface was eliminated in each iteration until 60% of the initial domain was removed. The evolution of the model geometry together with the flux and D_T solutions are presented in Figures 4b, c and d. It can be observed that the regions
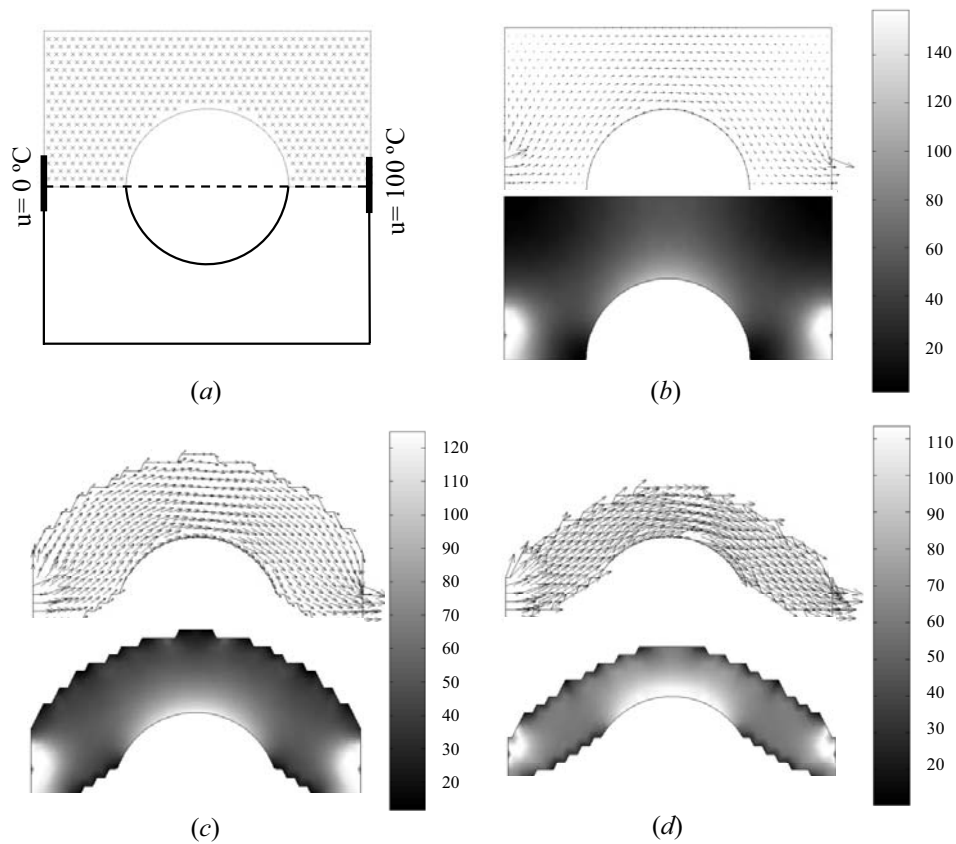
Figure 4 : Heat conductor with an initial hole: (a) Problem definition and initial BEM discretization, (b) Initial flux and D_T solutions, (c) Optimized geometry, flux and D_T solutions after 30 iterations, (d) Final geometry, flux and D_T solutions (iteration 60)

of the lowest flux always correspond to the lowest values of D_T and therefore they are progressively removed from the model.

4.3 Design of a heat exchanger

This example is devoted to the design of the heat exchanger illustrated in Figure 5. The cooling surface is in the top, with a Robin boundary condition. A periodic prescribed flux is specified to the bottom surface. The initial model was discretized using 120 boundary elements and 800 internal points. The optimization process consists in creating holes (cooling channels) in the central portion of the wall with prescribed Robin boundary conditions $u_\infty^\varepsilon = 25^o$C and $h^\varepsilon = 200$ W/(m^2 ^{o}C). The goal of the optimization is to diminish the temperature of the wall, and so the channels are open at the points with the highest values of D_T.

The BEM model was constructed considering the periodic boundary conditions of the problem. Two points
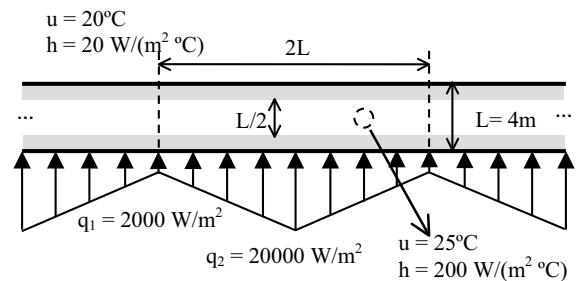
Figure 5 : Geometry and boundary conditions of the heat exchanger example.

(approximately 0,125% of the initial model surface) were eliminated in each iteration. The process was completed in 7 steps.

The evolution of the problem topology and the temperature solution are presented in Figure 6 for the initial model and after 3 and 7 iterations. It can be observed that the temperature of the wall effectively diminishes as

(*a*)

(*b*)

(*c*)
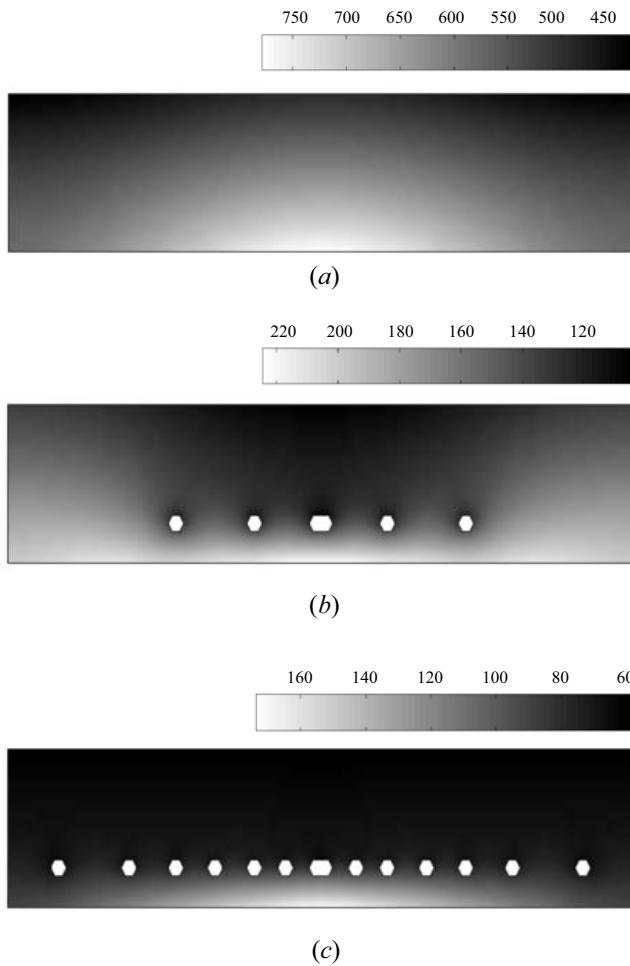
Figure 6 : Design of a heat exchanger: (a) Initial geometry and temperature solution, (b) Optimized geometry and temperature solution after 3 iterations (c) Final geometry and temperature solution (7 iterations)

the optimization progresses. Figure 7 illustrates the evolution of the maximum temperature in the wall, which achieves the limit value $T_{\max} \approx 173\,^{\circ}\mathrm{C}$ after 5 iterations. As with the previous examples, the computed results are in complete agreement with those of the reference [Novotny, et al., 2003].

5 Conclusions

An effective BEM implementation for the topological optimization of 2D potential problems was presented in this work. The problem formulation is based on some recent results by Novotny et al. [Novotny, et al., 2003] which allow computing the topological derivative us-
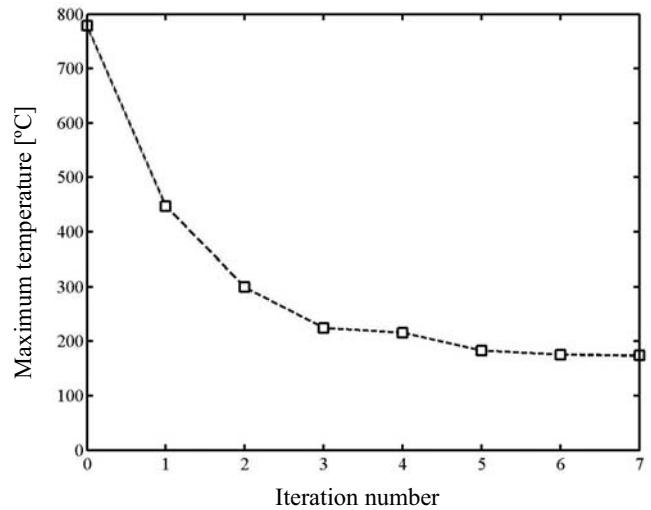
Figure 7 : Evolution of the maximum temperature in the wall.

ing potential and flux results. The optimization process consists in the progressive creation of holes within the model domain until a given stopping criterion is satisfied. The BEM model discretization is updated using a weighted Delaunay triangularization algorithm. The proposed method proves to be efficient and robust. Its performance is illustrated for a number of examples from the bibliography.

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