 CMES, vol.16, no.1, pp.27-39, 2006

Regularized Meshless Method for Solving Acoustic Eigenproblem with Multiply-Connected Domain

K.H. Chen[1], J.T. Chen[2] and J.H. Kao[3]

Abstract: In this paper, we employ the regularized meshless method (RMM) to search for eigenfrequency of two-dimension acoustics with multiply-connected domain. The solution is represented by using the double layer potentials. The source points can be located on the physical boundary not alike method of fundamental solutions (MFS) after using the proposed technique to regularize the singularity and hypersingularity of the kernel functions. The troublesome singularity in the MFS methods is desingularized and the diagonal terms of influence matrices are determined by employing the subtracting and adding-back technique. Spurious eigenvalues are filtered out by using singular value decomposition (SVD) updating term technique. The accuracy and stability of the RMM are verified through the numerical experiments of the Dirichlet and Neumann problems for domains with multiple holes. The method is found to perform pretty well in comparison with analytical solutions and numerical results of boundary element method, finite element method and the point-matching method.

keyword: Regularized meshless method, Hypersingularity, Eigenvalue, Eigenmode, Method of fundamental solutions, Acoustics.

1 Introduction

For a multiply-connected problem, spurious eigensolutions always appear, even when the complex-valued BEM is employed to solve the eigensolutions [Chen J. T.; Lin J. H.; Kuo S. R.; Chyuan S. W. (2001), Kuo S. R.; Yeih W.; Wu Y. C. (2000b)]. In Chen *et al.* [Chen J. T.; Lin J. H.; Kuo S. R.; Chyuan S. W. (2001)], the problem of spurious eigensolutions of the singular and hypersingular BEMs was studied by using circulants for an annular case and treated by using the Burton & Miller approach in a discrete system. Chen *et al.* [Chen J. T.; Liu L. W.; Chyuan S. W. (2004), Chen J. T.; Liu L. W.; Hong H. K. (2003)] studied the spurious and true eigensolutions for a multiply-connected problem by using BIE, BEM and dual BEM. Also, spurious eigensolutions were examined in the MFS for annular eigenproblems [Chen J. T.; Chang M. H.; Chen K. H.; Lin S. R. (2002)]. In this paper, we implement a novel meshless method to solve multiply-connected eigenproblem. Spurious eigenvalues are extracted out by employing SVD updating term technique.

Several meshless methods have also been reported in the literature, for example, the domain-based methods including the reproducing kernel method [Liu W. K.; Jun S.; Li S.; Adee J.; Belytschko T. (1995)], and boundary type methods of collocation approach [Chen J. T.; Chang M. H.; Chen K. H.; Lin S. R. (2002), Kang S. W.; Lee J. M.; Kang Y. J. (1999)], the method of fundamental solution (MFS) approach [Fairweather G.; Karageorghis A. (1998), Kupradze V. D.; Aleksidze M. A. (1964)], the meshless local Petrov-Galerkin approach [Atluri S. N.; Zhu T. L. (2000), Lin H.; Atluri S. N. (2000), Sladek J.; Sladek V.; Atluri S. N. (2004)], the RBF approach [Chen C. S.; Golberg M. A.; Hon Y. C. (1998), Chen J. T.; Chen I. L.; Lee Y. T. (2005), Chen J. T.; Chang M. H.; Chen K. H.; Lin S. R. (2002)] and the boundary knot method (BKM) etc [Chen W.; Hon Y. C. (2003)]. Since neither domain nor boundary meshing is required for the meshless method, it is very attractive for engineers in modeling. Therefore, the meshless method becomes promising in solving engineering problems.

In the MFS [Fairweather G.; Karageorghis A. (1998), Kupradze V. D.; Aleksidze M. A. (1964)], the solution is approximated by a set of fundamental solutions of the governing equations which are expressed in terms of

[1] Assistant Professor, Department of Information Management Toko University, Chia-Yi, Taiwan. Email: khc6177@mail.toko.edu.tw

[2] Distinguished Professor, Department of Harbor and River Engineering, National Taiwan Ocean University, Keeling, Taiwan. Email: jtchen@mail.ntou.edu.tw

[3] Graduate Student, Department of Harbor and River Engineering, National Taiwan Ocean University, Keeling, Taiwan. Email: m93520009@mail.ntou.edu.tw

sources located outside the physical domain. The method is relatively easy to implement. However, the MFS is still not a popular method because of the debatable artificial boundary (fictitious boundary) distance of source location in numerical implementation especially for a complicated geometry. The diagonal coefficients of influence matrices are divergent in conventional case when the fictitious boundary approaches the physical boundary. In spite of its gain of singularity free, the influence matrices become ill-posed when the fictitious boundary is far away from the real boundary. It results in an ill-posed problem since the condition number for the influence matrix becomes very large.

To distribute singularities on the real boundary, imaginary-part kernel method was adopted [Chen J. T.; Chang M. H.; Chen K. H.; Lin S. R. (2002)]. Later, Chen independently employed nonsingular fundamental solution to solve PDE using the similar idea [Chen W.; Hon Y. C. (2003)]. Later, Young *et al.* [Young D. L.; Chen K. H.; Lee C. W. (2005), Young D. L.; Chen K. H.; Lee C. W. (2005)] proposed the novel meshless method, namely regularized meshless method (RMM), to deal with 2-D problems including the Laplace problem and Helmholtz problem of exterior acoustics.

The RMM can be seen as one kind of MFS. The method eliminates the well-known drawback of equivocal artificial boundary. The subtracting and adding-back technique [Chen K. H.; Kao J. H.; Chen J. T.; Young D. L.; Lu M. C. (2006), Young D. L.; Chen K. H.; Lee C. W. (2005), Young D. L.; Chen K. H.; Lee C. W. (2005)] can regularize the singularity and hypersingularity of the kernel functions. This method can simultaneously distribute the observation and source points on the real boundary even using the singular kernels instead of non-singular kernels [Chen W.; Hon Y. C. (2003)]. The diagonal terms of the influence matrices can be extracted out by using the proposed technique. However, previous paper [Young D. L.; Chen K. H.; Lee C. W. (2005)] were limited to the exterior acoustic problem with a single radiator or scatter.

Following the success of previous applications [Young D. L.; Chen K. H.; Lee C. W. (2005)], we investigate the eigenfrequency of interior acoustics with multiply-connected domain by using the RMM in this study. The rationale for choosing double-layer potential as radial basis function (RBF) instead of the single-layer potential in the RMM is to take the advantage of the regularization of the subtracting and adding-back technique. A general-purpose program was developed to solve the multiply-connected eigenproblems of Laplace operator. True and spurious eigenvalues will be examined by using the technique of SVD updating term. Furthermore, the results will be compared with analytical solutions and those of BEM, FEM and PM to show the validity of our method.

2 Formulation

2.1 Governing equation and boundary conditions

Consider an eigenproblem with an acoustic pressure field $u(x)$, which satisfies the Helmholtz equation as follows:

$$(\nabla^2 + k^2)u(x) = 0, \quad x \in D, \tag{1}$$

subject to boundary conditions,

$$u(x) = \overline{u} = 0, \quad x \in B_p^{\overline{u}}, \quad p = 1, \quad 2, \quad 3, \cdots, \quad m \tag{2}$$

$$t(x) = \overline{t} = 0, \quad x \in B_q^{\overline{t}}, \quad q = 1, \quad 2, \quad 3, \cdots, \quad m \tag{3}$$

where ∇^2 is the Laplacian operator, k is the wave number, D is the domain of the problem, $t(x) = \partial u(x)/\partial n_x$, m is the total number of boundaries including m-1 numbers of inner boundaries and one outer boundary (the mth boundary), $B_p^{\overline{u}}$ is the essential boundary (Dirichlet boundary) of the pth boundary in which the potential is prescribed by $\overline{u}$ and $B_q^{\overline{t}}$ is the natural boundary (Neumann boundary) of the qth boundary in which the flux is prescribed by $\overline{t}$. Both $B_p^{\overline{u}}$ and $B_q^{\overline{t}}$ construct the whole boundary of the domain D as shown in Fig. 1 (a).

2.2 Conventional method of fundamental solutions

By employing the RBF technique [Chen J. T.; Chen I. L.; Lee Y. T. (2005), Cheng A. H. D. (2000)], the representation of the eigensolution for multiply-connected problem as shown in Fig. 1 (a) can be approximated in terms of the α_j strengths of the singularities at s_j as

$$\begin{aligned} u(x_i) &= \sum_{j=1}^{N} T(s_j,x_i)\alpha_j \\ &= \sum_{j=1}^{N_1} T(s_j,x_i)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} T(s_j,x_i)\alpha_j + \cdots \\ &+ \sum_{j=N_1+N_2+\cdots+N_{m-1}+1}^{N} T(s_j,x_i)\alpha_j, \end{aligned} \tag{4}$$

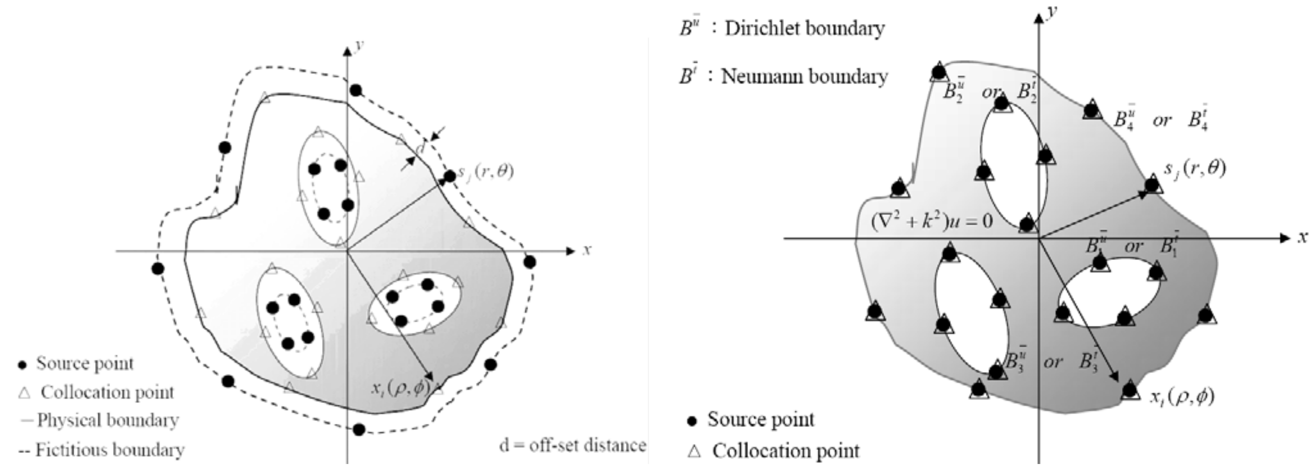

Figure 1 : (a) The distribution of the source points and observation points and definitions of r, θ, ρ, ϕ by using the conventional MFS for the multiply-connected problems; (b) The distribution of the source points and observation points and definitions of r, θ, ρ, ϕ by using the RMM for the multiply-connected problems.

$$
\begin{aligned}
t(x_i) &= \sum_{j=1}^{N} M(s_j, x_i)\alpha_j \\
&= \sum_{j=1}^{N_1} M(s_j, x_i)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} M(s_j, x_i)\alpha_j + \cdots \\
&+ \sum_{j=N_1+N_2+\cdots+N_{m-1}+1}^{N} M(s_j, x_i)\alpha_j,
\end{aligned} \tag{5}
$$

where x_i and s_j represent the ith observation point and the jth source point, respectively, α_j are the jth unknown coefficients (strength of the singularity), $N_1, N_2, \cdots, N_{m-1}$ are the numbers of source points on m-1 numbers of inner boundaries, respectively, N_m is the number of source points on the outer boundary, while N is the total numbers of source points ($N = N_1 + N_2 + \cdots + N_m$) and $M(s_j, x_i) = \partial T(s_j, x_i)/\partial n_{x_i}$. After matching boundary conditions, the coefficients $\{\alpha_j\}_{j=1}^{N}$ are determined. The distributions of source points and observation points are shown in Fig. 1 (a) for the MFS. The chosen bases are the double layer potentials [Young D. L.; Chen K. H.; Lee C. W. (2005)] as

$$
T(s_j, x_i) = -\frac{i\pi k}{2} H_1^{(1)}(kr_{ij}) \frac{((x_i - s_j), n_j)}{r_{ij}}, \tag{6}
$$

$$
\begin{aligned}
M(s_j, x_i) = \frac{i\pi k}{2}\{ & kH_2^{(1)}(kr_{ij}) \frac{((x_i - s_j), n_j)((x_i - s_j), \overline{n_i})}{r_{ij}^2} \\
& - H_1^{(1)}(kr_{ij}) \frac{n_k \overline{n_k}}{r_{ij}} \},
\end{aligned} \tag{7}
$$

where (,) is the inner product of two vectors, $H_2^{(1)}(kr_{ij})$ is the second-order Hankel function of the first kind, $r_{ij} = |s_j - x_i|$, n_j is the normal vector at s_j, and $\overline{n_i}$ is the normal vector at x_i.

It is noted that the double-layer potentials have both singularity and hypersingularity when source and field points coincide, which leads to difficulty in the conventional MFS. The fictitious distance between the fictitious (auxiliary) boundary (B') and the physical boundary (B), defined by d, shown in Fig. 1 (a) needs to be chosen deliberately. To overcome the above mentioned shortcoming, s_j is distributed on the real boundary, as shown in Fig. 1 (b), by using the proposed regularized technique as stated in the following Section 2.3. The rationale for choosing double- layer potential as the form of RBFs instead of the single-layer potential in the RMM is to take the advantage of the regularization of the subtracting and adding-back technique, so that no fictitious distance is needed when evaluating the diagonal coefficients of influence matrices which will be explained in Section 2.4. The single-layer potential can not be chosen because the following Eqs. (13), (16), (19) and (22) in Section 2.3 are not provided. If the single layer potential is used, the regularization of subtracting and adding-back technique can not work [Young D. L.; Chen K. H.; Lee C. W. (2005)].

2.3 *Regularized meshless method*

When the collocation point x_i approaches the source point s_j, the potentials in Eqs. (6) and (7) are approximated by:

$$\lim_{x_i \to s_j} T(s_j, x_i) = \overline{T}(s_j, x_i) = -\frac{n_k y_k}{r_{ij}^2}, \quad (8)$$

$$\lim_{x_i \to s_j} M(s_j, x_i) = \overline{M}(s_j, x_i) + \frac{k^2}{4} i$$
$$= (\frac{2((x_i - s_j), n_j)((x_i - s_j), \overline{n_i})}{r_{ij}^4} - \frac{(n_j, \overline{n_i})}{r_{ij}^2}) + \frac{k^2}{4} i, \quad (9)$$

by using the limiting form for small arguments and the identities form the generalized function as shown below [Abramowitz M.; Stegun I. A. (1972)]

$$\lim_{r_{ij} \to 0} H_1^{(1)}(kr_{ij}) = \frac{kr_{ij}}{2} + \frac{2}{\pi k r_{ij}} i, \quad (10)$$

$$\lim_{r_{ij} \to 0} H_2^{(1)}(kr_{ij}) = \frac{(kr_{ij})^2}{8} + \frac{4}{\pi (kr_{ij})^2} i. \quad (11)$$

The kernels in Eqs. (8) and (9) have the same singularity order as the Laplace equation. Therefore, Eqs. (4) and (5) for the multiply-connected domain problems can be regularized by using the above mentioned regularization of subtracting and adding-back technique [Chen K. H.; Kao J. H.; Chen J. T.; Young D. L.; Lu M. C. (2006), Young D. L.; Chen K. H.; Lee C. W. (2005), Young D. L.; Chen K. H.; Lee C. W. (2005)] as follows:

$$u(x_i^I) = \sum_{j=1}^{N_1} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^I)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^O, x_i^I)\alpha_j - \sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} \overline{T}(s_j^I, x_i^I)\alpha_i,$$
$$x_i^I \in B_p, \quad p = 1, \quad 2, \quad 3, \cdots, \quad m-1, \quad (12)$$

where x_i^I is located on the inner boundary ($p = 1, \quad 2, \quad 3, \cdots, \quad m-1$) and the superscripts I and O denote the inward and outward normal vectors, respectively, and

$$\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} \overline{T}(s_j^I, x_i^I) = 0, \quad x_i^I \in B_p,$$
$$p = 1, \quad 2, \quad 3, \cdots, \quad m-1. \quad (13)$$

Therefore, we can obtain

$$u(x_i^I) = \sum_{j=1}^{N_1} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{i-1} T(s_j^I, x_i^I)\alpha_j + \sum_{j=i+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^I)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^O, x_i^I)\alpha_j - \left[\sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} \overline{T}(s_j^I, x_i^I) - T(s_i^I, x_i^I) \right] \alpha_i,$$
$$x_i^I \in B_p, \quad p = 1, \quad 2, \quad 3, \cdots, \quad m-1. \quad (14)$$

When the observation point x_i^O locates on the outer boundary ($p = m$), Eq. (12) becomes

$$u(x_i^O) = \sum_{j=1}^{N_1} T(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} T(s_j^I, x_i^O)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^O, x_i^O)\alpha_j - \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{T}(s_j^I, x_i^I)\alpha_i,$$
$$x_i^{O \ and \ I} \in B_p, \quad p = m, \quad (15)$$

where

$$\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{T}(s_j^I, x_i^I)\alpha_i = 0, \quad x_i^I \in B_p, \quad p = m. \quad (16)$$

Hence, we obtain

$$
\begin{aligned}
u(x_i^O) = &\sum_{j=1}^{N_1} T(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} T(s_j^I, x_i^O)\alpha_j + \cdots \\
&+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^O)\alpha_j \\
&+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{i-1} T(s_j^O, x_i^O)\alpha_j + \sum_{j=i+1}^{N} T(s_j^O, x_i^O)\alpha_j \\
&- \left[\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{T}(s_j^I, x_i^I) - T(s_i^O, x_i^O) \right] \alpha_i, \\
&x_i^{I \text{ and } O} \in B_p, \quad p = m.
\end{aligned} \tag{17}
$$

Similarly, the boundary flux is obtained as

$$
\begin{aligned}
t(x_i^I) = &\sum_{j=1}^{N_1} M(s_j^I, x_i^I)\alpha_j + \cdots \\
&+ \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_j + \cdots \\
&+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^I)\alpha_j \\
&+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^O, x_i^I)\alpha_j \\
&- \sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} \overline{M}(s_j^I, x_i^I)\alpha_i, \\
&x_i^I \in B_p, \quad p = 1, \quad 2, \quad 3, \cdots, \quad m-1.
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
&\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} \overline{M}(s_j^I, x_i^I) = 0, \quad x_i^I \in B_p, \\
&p = 1, \quad 2, \quad 3, \cdots, \quad m-1.
\end{aligned} \tag{19}
$$

Therefore, we obtain

$$
\begin{aligned}
t(x_i^I) = &\sum_{j=1}^{N_1} M(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{i-1} M(s_j^I, x_i^I)\alpha_j \\
&+ \sum_{j=i+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_j + \cdots \\
&+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^I)\alpha_j \\
&+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^O, x_i^I)\alpha_j \\
&- \left[\sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} \overline{M}(s_j^I, x_i^I) - M(s_i^I, x_i^I) \right] \alpha_i, \\
&x_i^I \in B_p, \quad p = 1, \quad 2, \quad 3, \cdots, \quad m-1.
\end{aligned} \tag{20}
$$

When the observation point locates on the outer boundary ($p = m$), Eq. (18) yields

$$
\begin{aligned}
t(x_i^O) = &\sum_{j=1}^{N_1} M(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} M(s_j^I, x_i^O)\alpha_j + \cdots \\
&+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^O)\alpha_j \\
&+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^O, x_i^O)\alpha_j \\
&- \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{M}(s_j^I, x_i^I)\alpha_i, \\
&x_i^{O \text{ and } I} \in B_p, \quad p = m.
\end{aligned} \tag{21}
$$

where

$$
\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{M}(s_j^I, x_i^I) = 0, \quad x_i^I \in B_p, \quad p = m. \tag{22}
$$

Hence, we obtain

$$t(x_i^O) = \sum_{j=1}^{N_1} M(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} M(s_j^I, x_i^O)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{i-1} M(s_j^O, x_i^O)\alpha_j + \sum_{j=i+1}^{N} M(s_j^O, x_i^O)\alpha_j - \left[\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{M}(s_j^I, x_i^I) - M(s_i^O, x_i^O)\right]\alpha_i,$$
$$x_i^{O\ and\ I} \in B_p, \quad p = m. \tag{23}$$

The detailed derivations of Eqs. (13), (16), (19) and (22) can be found in the reference [Young D. L.; Chen K. H.; Lee C. W. (2005)]. According to the dependence of normal vectors for inner and outer boundaries [Young D. L.; Chen K. H.; Lee C. W. (2005)], their relationships are

$$\begin{cases} \overline{T}(s_j^I, x_i^I) = -\overline{T}(s_j^O, x_i^O), & i \neq j \\ \overline{T}(s_j^I, x_i^I) = \overline{T}(s_j^O, x_i^O), & i = j \end{cases} \tag{24}$$

$$\begin{cases} \overline{M}(s_j^I, x_i^I) = \overline{M}(s_j^O, x_i^O), & i \neq j \\ \overline{M}(s_j^I, x_i^I) = \overline{M}(s_j^O, x_i^O), & i = j \end{cases} \tag{25}$$

where the left and right hand sides of the equal sign in Eqs. (24) and (25) denote the kernels for observation and source point with the inward and outward normal vectors, respectively.

By using the proposed technique, the singular terms in Eqs. (4) and (5) have been transformed into regular terms $\left(-\left[\sum_{j=N_1+N_2+\cdots+N_{P-1}+1}^{N_1+N_2+\cdots+N_p} \overline{T}(s_j^I, x_i^I) - T(s_i^{I\ or\ O}, x_i^{I\ or\ O})\right]\right.$ and $\left.-\left[\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} \overline{M}(s_j^I, x_i^I) - M(s_i^{I\ or\ O}, x_i^{I\ or\ O})\right]\right)$ in Eqs. (14), (17), (20) and (23), respectively, where $p = 1, 2, 3, \cdots, m$. The terms of $\sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} \overline{T}(s_j^I, x_i^I)$ and $\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} \overline{M}(s_j^I, x_i^I)$ are the adding-back terms and the terms of $T(s_i^{I\ or\ O}, x_i^{I\ or\ O})$ and $M(s_i^{I\ or\ O}, x_i^{I\ or\ O})$ are the subtracting terms in the two brackets for regularization. After using the above mentioned method of regularization of subtracting and adding-back technique, we are able to remove the singularity and hypersingularity of the kernel functions.

2.4 *Derivation of influence matrices for arbitrary domain problems*

By collocated N observation points to match with the BCs from Eqs. (14) and (17) for the Dirichlet problem, the linear algebraic equation is obtained

$$\{\overline{u}\} = \{0\} = [T]\{\alpha\} \Leftrightarrow$$
$$\left\{ 0 \right\}_{N\times 1} = \begin{bmatrix} [T_{11}]_{N_1\times N_1} & \cdots & [T_{1m}]_{N_1\times N_m} \\ \vdots & \ddots & \vdots \\ [T_{m1}]_{N_m\times N_1} & \cdots & [T_{mm}]_{N_m\times N_m} \end{bmatrix}_{N\times N} \begin{Bmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{N_1} \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} \alpha_{N_1+N_2+\cdots+N_{m-1}+1} \\ \vdots \\ \alpha_N \end{Bmatrix} \end{Bmatrix}_{N\times 1}, \tag{26}$$

where

$$[T_{11}] = \begin{bmatrix} -\left[\sum_{j=1}^{N_1} \overline{T}(s_j^I, x_1^I) - T(s_1^I, x_1^I)\right] & \cdots & T(s_{N_1}^I, x_1^I) \\ \vdots & \ddots & \vdots \\ T(s_1^I, x_{N_1}^I) & \cdots & -\left[\sum_{j=1}^{N_1} \overline{T}(s_j^I, x_{N_1}^I) - T(s_{N_1}^I, x_{N_1}^I)\right] \end{bmatrix}_{N_1\times N_1}, \tag{27}$$

$$[T_{1m}] = \begin{bmatrix} T(s_{N_1+\cdots+N_{m-1}+1}^O, x_1^I) & \cdots & T(s_N^O, x_1^I) \\ \vdots & \ddots & \vdots \\ T(s_{N_1+\cdots+N_{m-1}+1}^O, x_{N_1}^I) & \cdots & T(s_N^O, x_{N_1}^I) \end{bmatrix}_{N_1\times N_m} \tag{28}$$

$$[T_{m1}] = \begin{bmatrix} T(s_1^I, x_{N_1+\cdots+N_{m-1}+1}^O) & \cdots & T(s_{N_1}^I, x_{N_1+\cdots+N_{m-1}+1}^O) \\ \vdots & \ddots & \vdots \\ T(s_1^I, x_N^O) & \cdots & T(s_{N_1}^I, x_N^O) \end{bmatrix}_{N_m\times N_1}, \tag{29}$$

$$[T_{mm}] = \begin{bmatrix} -\begin{bmatrix} \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{T}(s_j^I, x_{N_1+\cdots+N_{m-1}+1}^I) \\ -T(s_{N_1+\cdots+N_{m-1}+1}^O, x_{N_1+\cdots+N_{m-1}+1}^O) \end{bmatrix} & \cdots & T(s_N^O, x_{N_1+\cdots+N_{m-1}+1}^O) \\ \vdots & \ddots & \vdots \\ T(s_{N_1+\cdots+N_{m-1}+1}^O, x_N^O) & \cdots & -\begin{bmatrix} \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{T}(s_j^I, x_N^I) - T(s_N^O, x_N^O) \end{bmatrix} \end{bmatrix}_{N_m \times N_m}. \quad (30)$$

For the Neumann problem, Eqs. (20) and (23) yield

$$\{\bar{t}\} = \{0\} = [M]\{\alpha\} \Leftrightarrow \begin{Bmatrix} 0 \end{Bmatrix}_{N\times 1} = \begin{bmatrix} [M_{11}]_{N_1\times N_1} & \cdots & [M_{1m}]_{N_1\times N_m} \\ \vdots & \ddots & \vdots \\ [M_{m1}]_{N_m\times N_1} & \cdots & [M_{mm}]_{N_m\times N_m} \end{bmatrix}_{N\times N} \begin{Bmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{N_1} \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} \alpha_{N_1+N_2+\cdots+N_{m-1}+1} \\ \vdots \\ \alpha_N \end{Bmatrix} \end{Bmatrix}_{N\times 1}, \quad (31)$$

in which

$$[M_{11}] = \begin{bmatrix} -\begin{bmatrix} \sum_{j=1}^{N_1} \overline{M}(s_j^I, x_1^I) - T(s_1^I, x_1^I) \end{bmatrix} & \cdots & M(s_{N_1}^I, x_1^I) \\ \vdots & \ddots & \vdots \\ M(s_1^I, x_{N_1}^I) & \cdots & -\begin{bmatrix} \sum_{j=1}^{N_1} \overline{M}(s_j^I, x_{N_1}^I) - M(s_{N_1}^I, x_{N_1}^I) \end{bmatrix} \end{bmatrix}_{N_1\times N_1}, \quad (32)$$

$$[M_{1m}] = \begin{bmatrix} M(s_{N_1+\cdots+N_{m-1}+1}^O, x_1^I) & \cdots & M(s_N^O, x_1^I) \\ \vdots & \ddots & \vdots \\ M(s_{N_1+\cdots+N_{m-1}+1}^O, x_{N_1}^I) & \cdots & M(s_N^O, x_{N_1}^I) \end{bmatrix}_{N_1\times N_m}, \quad (33)$$

$$[M_{m1}] = \begin{bmatrix} M(s_1^I, x_{N_1+\cdots+N_{m-1}+1}^O) & \cdots & M(s_{N_1}^I, x_{N_1+\cdots+N_{m-1}+1}^O) \\ \vdots & \ddots & \vdots \\ M(s_1^I, x_N^O) & \cdots & M(s_{N_1}^I, x_N^O) \end{bmatrix}_{N_m\times N_1}, \quad (34)$$

$$[M_{mm}] = \begin{bmatrix} -\begin{bmatrix} \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{M}(s_j^I, x_{N_1+\cdots+N_{m-1}+1}^I) \\ -M(s_{N_1+\cdots+N_{m-1}+1}^O, x_{N_1+\cdots+N_{m-1}+1}^O) \end{bmatrix} & \cdots & M(s_N^O, x_{N_1+\cdots+N_{m-1}+1}^O) \\ \vdots & \ddots & \vdots \\ M(s_{N_1+\cdots+N_{m-1}+1}^O, x_N^O) & \cdots & -\begin{bmatrix} \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} \overline{M}(s_j^I, x_N^I) - M(s_N^O, x_N^O) \end{bmatrix} \end{bmatrix}_{N_m\times N_m}. \quad (35)$$

For the mixed-type problem, a linear combination of Eqs. (26) and (31) is required to satisfy the mixed-type BCs.

2.5 Extraction of the eigenvalues

In order to sort out the eigenvalues, the SVD technique is utilized [6]. We obtain Eqs. (26) and (31) by using the double-layer potentials approach for the Dirichlet and Neumann problems, respectively. Form Eqs. (26) and (31), we can obtain eigenvalues by using the SVD technique as follows:

$$[T] = [\Phi_T]\,[\Sigma_T]\,[\Psi_T]^H, \quad (36)$$

$$[M] = [\Phi_M]\,[\Sigma_M]\,[\Psi_M]^H, \quad (37)$$

where the superscript H denotes the transpose and conjugate, Σ_T and Σ_M are diagonal matrices with diagonal elements of positive or zero singular values and $[\Phi_T]$, $[\Phi_M]$, $[\Psi_T]$ and $[\Psi_M]$ are the left and right unitary matrices corresponding with $[T]$ and $[M]$, respectively. Thus the minimum singular value of $[T]$ or $[M]$ as a function of k can be utilized to detect the eigenvalue and eigenmodes by using unitary vectors. However, spurious eigenvalues are present for multiply-domain eigenproblem. Spurious eigenvalue can be extracted out by SVD updating term techniques as shown in the next section.

2.6 Treatments of spurious eigenvalues

In order to sort out the spurious eigenvalues, the SVD updating term is utilized [Chen J. T.; Chen I. L.; Lee Y. T. (2005), Chen J. T.; Lin J. H.; Kuo S. R.; Chyuan S. W. (2001)]. We can combine Eqs. (26) and (31) by using the SVD updating term as follows:

$$[P]\{\alpha\} = \begin{bmatrix} [T]_{N\times N} \\ [M]_{N\times N} \end{bmatrix} \{\alpha\} = \{0\}. \tag{38}$$

The rank of the matrix $[P]$ must be smaller than N to have a spurious mode [Chen J. T.; Chen I. L.; Lee Y. T. (2005)]. By using the SVD technique, the matrix in Eq. (38) can be decomposed into

$$[P] = \begin{bmatrix} \Phi_T & 0 \\ 0 & \Phi_M \end{bmatrix} \begin{bmatrix} \Sigma_T & 0 \\ 0 & \Sigma_M \end{bmatrix} \begin{bmatrix} \Psi_T & 0 \\ 0 & \Psi_M \end{bmatrix}^H. \tag{39}$$

Based on the equivalence between the SVD technique and the Least-squares method, we extract out the spurious eigenvalue by detecting zero singular values for $[P]$ matrix.

2.7 Flowchart of solution procedures

Following the section 2.3 to section 2.6, the flowchart of solution procedures by using the RMM is shown in Fig. 2.

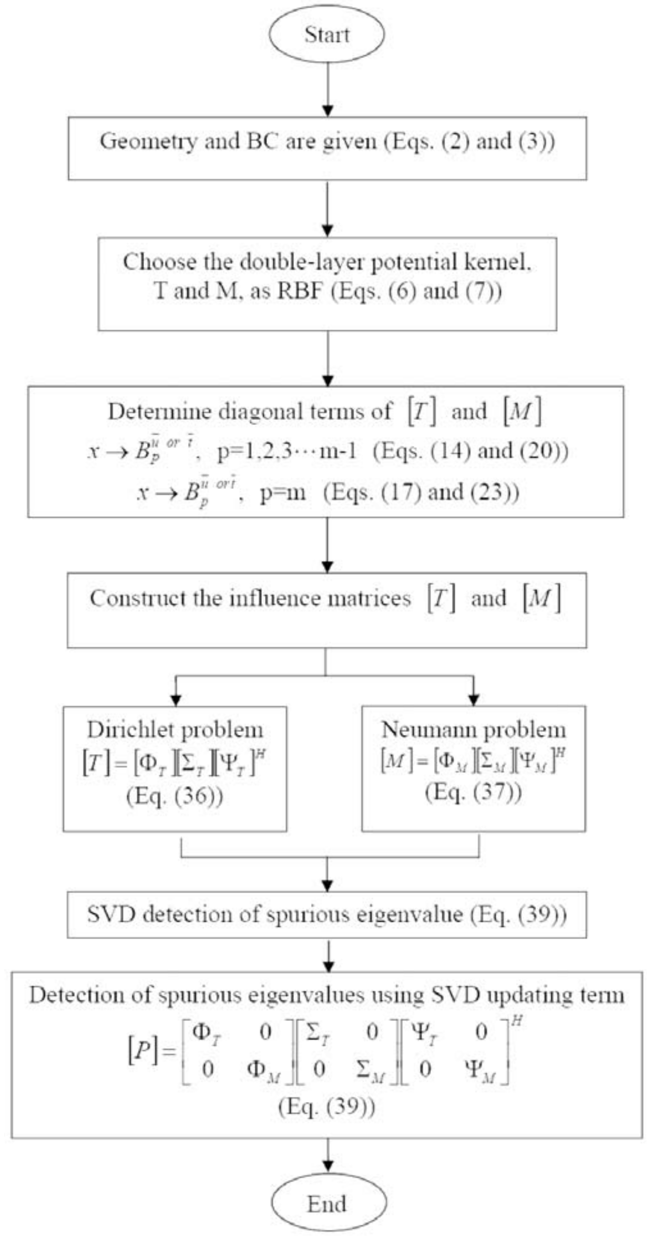

Figure 2 : Flowchart of solution procedures.

3 Numerical examples

In order to show the accuracy and validity of the proposed method, four cases with simply-connected and multiply-connected domain subjected to the Dirichlet and Neumann BCs are considered.

Case 1: Square problem (simply-connected case)

The length of the square domain is $L = 1.0$. All the boundary conditions are the Dirichlet type $(u = 0)$ as shown in Fig. 3. The analytical solution of true eigenequations [Chen J. T.; Chang M. H.; Chen K. H.; Lin S. R. (2002)] for this case is shown below:

$$k_{mn} = \pi\sqrt{(m/L)^2 + (n/L)^2}, \quad m,n = 1,2,3\cdots, \tag{40}$$

The former five eigenvalues for the Dirichlet BC by using our proposed method is shown in Fig. 4. Good agreement is obtained after comparing with analytical solutions. Since the domain is simply connected, no spuri-

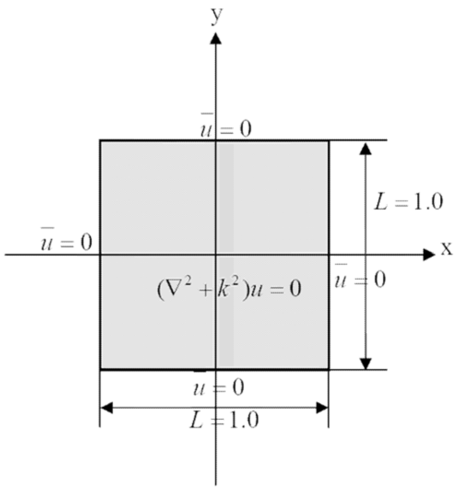

Figure 3 : Problem sketch for the case 1.

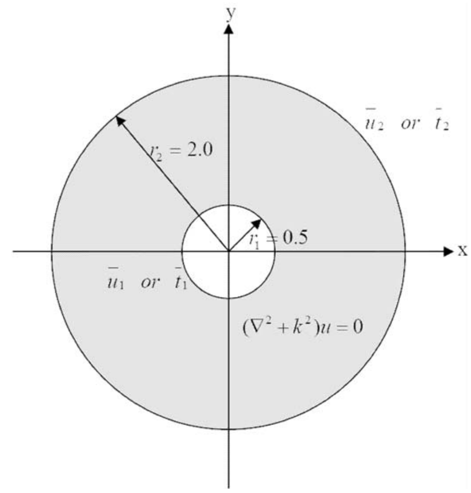

Figure 5 : Problem sketch for case 2.

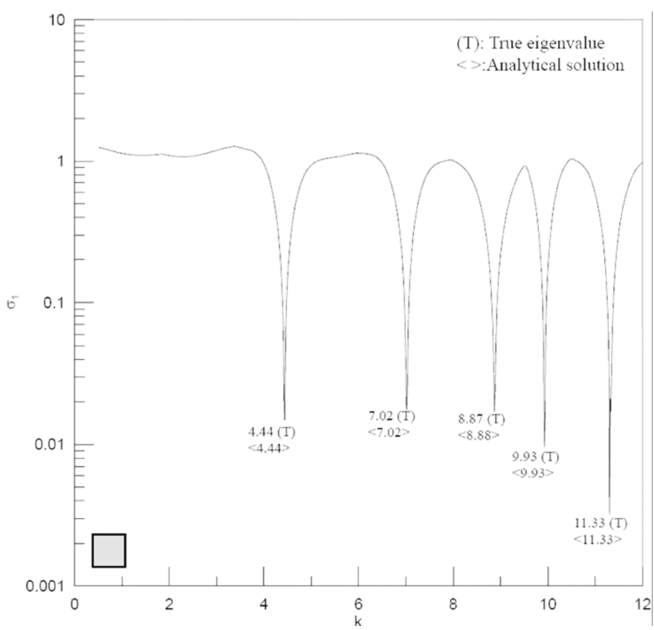

Figure 4 : The first minimum singular value versus wave number.

ous eigenvalue is found as expected in simply-connected case.

Case 2: Annular problem

The inner and outer radii of domain are $r_1 = 0.5$ and $r_2 = 2.0$, respectively. All the boundary conditions are the Dirichlet type $(u = 0)$ and Neumann type $(t = 0)$ as shown in Fig. 5. The analytical solution of true eigenequations [Chen J. T.; Liu L. W.; Hong H. K. (2003)] for Dirichlet and Neumann types, respectively, is shown below:

$$J_n(kr_1)Y_n(kr_2) - J_n(kr_2)Y_n(kr_1) = 0, \text{ (Dirichlet)} \tag{41}$$

$$J_n'(kr_1)Y_n'(kr_2) - J_n'(kr_2)Y_n'(kr_1) = 0, \text{ (Neumann)}. \tag{42}$$

The analytical solutions of spurious eigenequations [Chen J. T.; Liu L. W.; Hong H. K. (2003)] for both types are the same as:

$$J_n'(kr_1) = 0. \tag{43}$$

The minimum singular value versus wave number by using our proposed method for the Dirichlet and Neumann BCs are shown in Figs. 6 (a) and (b), respectively.

Good agreement is obtained after comparing with analytical solutions. The spurious eigenvalues for the Dirichlet and Neumann problems are found out by employing SVD updating term as shown in Fig. 6 (c).

From Fig. 6 (c), we find that one spurious eigenvalue appear at $k_s = 3.68$ (J'^1_1) in the range of $0 < k \leq 5$. This spurious eigenvalue (J'^1_1) is found to be the true eigenvalue of Neumann eigenproblem of interior circular with radius 0.5.

Case 3: A circular domain with two equal holes

In this case, the eigenvalues were obtained by Chen and his coworkers [Chen J. T.; Liu L. W.; Chyuan S. W. (2004)]. The radius R of the outer boundary is 1.0 and the eccentricity e and radius c of the inner circular boundaries are 0.5 and 0.3, respectively, as shown in Fig. 7.

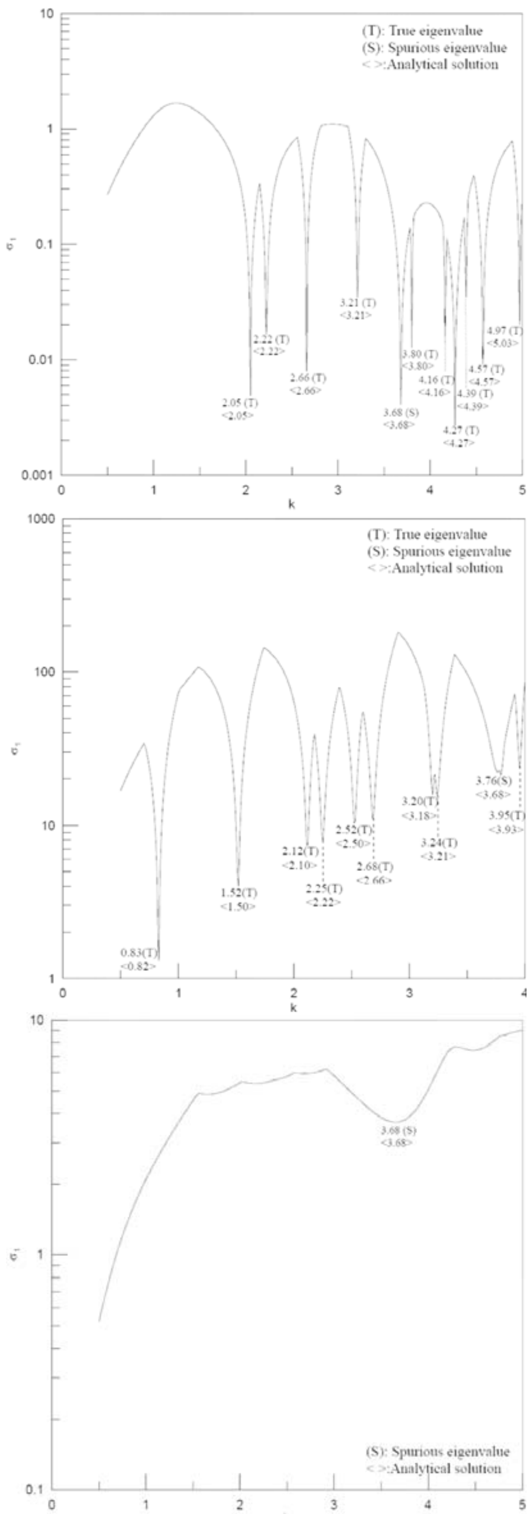

Figure 6 : (a) The result of RMM and analytical solution for the Dirichlet BC; (b) The result of RMM and analytical solution for the Neumann BC.

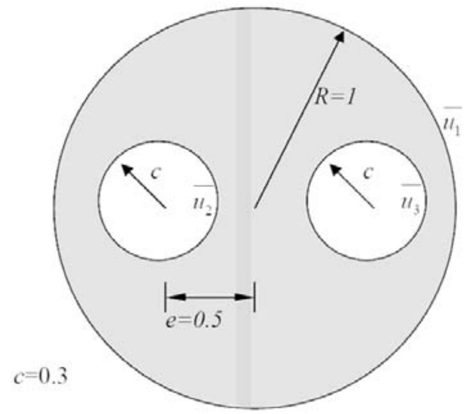

Figure 7 : Problem sketch for case 3.

Table 1 : The former five eigenvalues for a circular domain with two equal holes by using different approaches.

Method / Eigenvalue	RMM	BEM	FEM	PM
k_1	4.50(SS)	4.50(SS)	4.453	4.548(SS)
k_2	4.56(AS)	4.50(AS)	4.512	4.588(AS)
k_3	6.40(AA)	6.37(AA)	6.267	6.457(AA)
k_4	6.40(SA)	6.37(SA)	6.269	6.472(SA)
k_5	7.10(SS)	7.16(SS)	6.930	7.083(SS)

All the boundary conditions are the Dirichlet type $(u = 0)$. Numerical data of eigenvalues for RMM, BEM, FEM and point-matching method (PM), are shown in Table 1. In Table 1, the (S) and (A) symbols denote the symmetric and antisymmetric with respect to the x and y axis, respectively [Chen J. T.; Liu L. W.; Chyuan S. W. (2004)]. It is easy to find that the mode shapes of RMM, BEM and PM approach match well. In this case, the first spurious eigenvalue $k_s = 6.14$ is found by comparing with the analytical solution J'^{1}_{1}. From Table 1, it is found that the former five eigenvalues match well with those of RMM and BEM. On the other hand, the former five eigenmodes are shown in Fig. 8, respectively, by using the RMM and the BEM approach. Good agreement is made.

Case 4: A circular domain with four equal holes

In this case, the eigenvalues were obtained by Chen and his coworkers [Chen J. T.; Liu L. W.; Chyuan S. W. (2004)]. The radius R of outer boundary is 1.0 and the eccentricity e and radius c of the inner circular boundaries are 0.5 and 0.1, respectively. Dirichlet problem is considered as shown in Fig. 9.

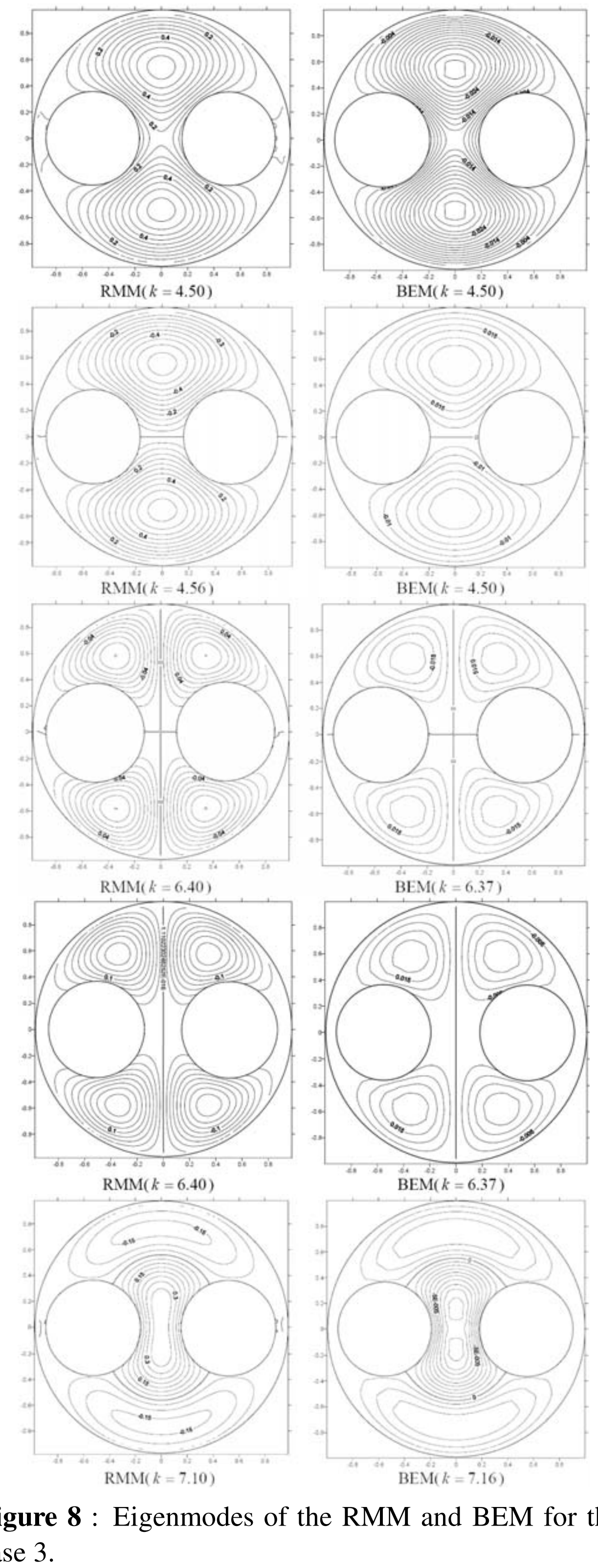

Figure 8 : Eigenmodes of the RMM and BEM for the case 3.

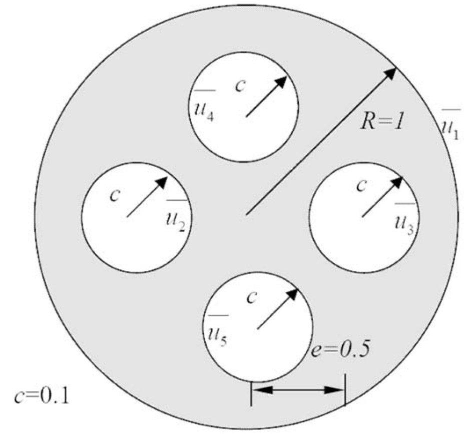

Figure 9 : Problem sketch for case 4.

Table 2 : The former five eigenvalues for a circular domain with four equal holes by using different approaches.

Eigenvalue \ Method	RMM	BEM	FEM	PM
k_1	4.50(SS)	4.47(SS)	4.443	4.655(SS)
k_2	5.38(AS)	5.37(AS)	5.316	N/A
k_3	5.38(SA)	5.37(SA)	5.320	N/A
k_4	5.55(AA)	5.54(AA)	5.486	5.561(SA)
k_5	5.95(SS)	5.95(SS)	5.884	5.868(SS)

The former five eigenvalues by using the RMM, BEM, FEM and PM are listed in Table 2, where the results of PM miss the eigenvalues of k_2 and k_3.

In this case, no spurious eigenvalue is found in the range of $0 < k < 6$ sine the first spurious eigenvalue is 18.412 (J'^{1}_{1}). The eigenvalues of k_2 and k_3 are roots of multiplicity two by finding the second successive zero singular value in SVD when using RMM and BEM. Besides, the symmetry of the fourth mode shape by using the PM is quite different with the results of RMM and BEM. The former five eigenmodes of the RMM and the BEM are shown in Fig. 10. Agreeable results of the RMM are obtained by comparing with the BEM data.

4 Conclusions

In this study, we used the RMM to solve the acoustic eigenproblems with multiply-connected domain subjected to the Dirichlet and Neumann BCs, respectively. Only the boundary nodes on the physical boundary are required. The perplexing fictitious boundary in the MFS

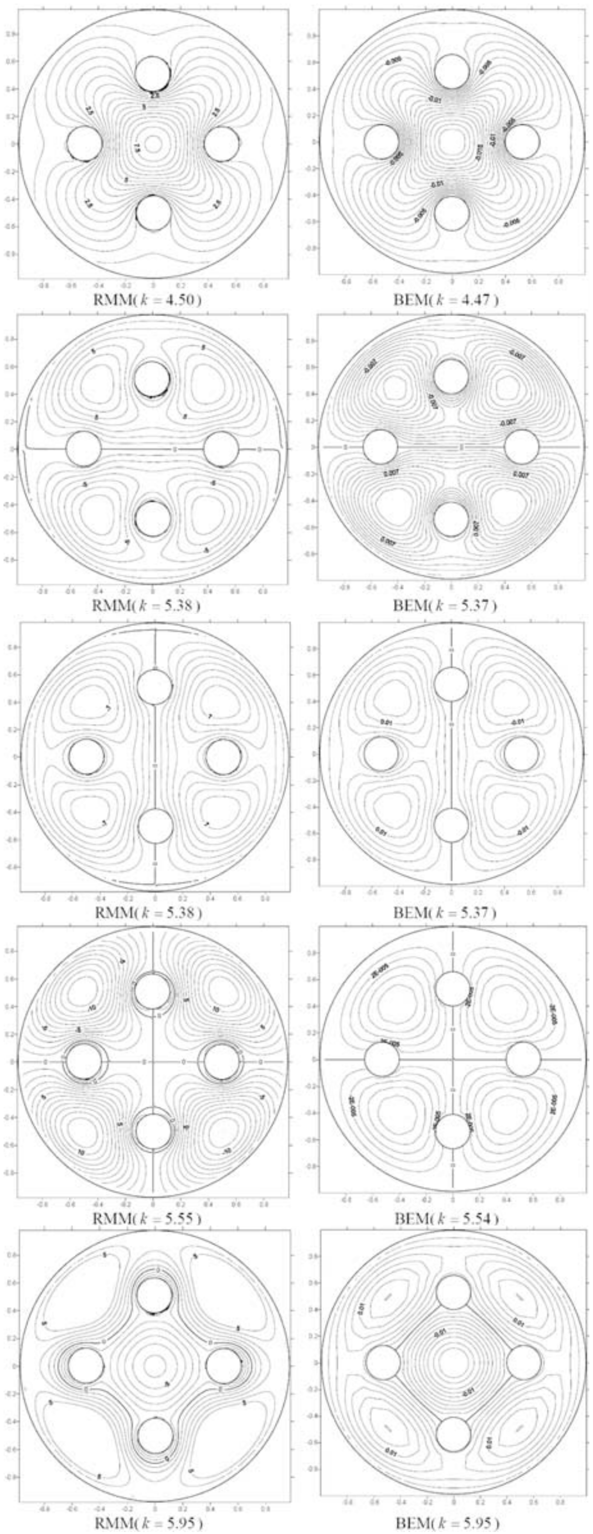

Figure 10 : Eigenmodes of the RMM and BEM for the case 4.

is then circumvented. Despite the presence of singularity and hypersingularity of double-layer potentials, the finite values of the diagonal terms of the influence matrix can be extracted out by employing subtracting and add-back techniques. Four numerical experiments were performed to demonstrate not only the occurring mechanism of spurious eigenvalue due to inner boundaries but also the suppression of the spurious eigenvalue by using SVD technique of updating term. The numerical results were obtained by applying the developed program to problems of simply-connected and multiply-connected domain subjected to Dirichlet and Neumann BCs. Numerical results agreed very well with analytical solutions and those of BEM, FEM and the PM.

Acknowledgement: Financial support from the National Science Council under Grant No. NSC-94-2211-E-464-004 to the first author for Toko University is gratefully acknowledged.

References

Abramowitz M.; Stegun I. A. (1972): "Handbook of mathematical functions with formulation graphs and mathematical tables", *New York. Dover.*.

Atluri S. N.; Han Z. D.; Shen S. (2003): "Meshless Local Petrov-Galerkin (MLPG) Approaches for Solving the Weakly-Singular Traction & Displacement Boundary Integral Equations", *CMES: Computer Modeling in Engineering & Sciences*, 4, 507-518.

Chen C. S.; Golberg M. A.; Hon Y. C. (1998): "The method of fundamental solutions and quasi-Monte-Carlo method for diffusion equations", *International Journal for Numerical Methods in Engineering*, 43, 1421-1435.

Chen J. T.; Chen I. L.; Lee Y. T. (2005): "Eigensolutions of multiply connected membranes using the method of fundamental solutions", *Engineering Analysis with Boundary Elements*, 29, 166-174.

Chen J. T.; Chang M. H.; Chen K. H.; Lin S. R. (2002): "The boundary collocation method with meshless concept for acoustic eigenanalysis of two-dimensional cavities using radial basis function", *Journal of Sound and Vibration*, 257, 667-711.

Chen J. T.; Lin J. H.; Kuo S. R.; Chyuan S. W. (2001): "Boundary element analysis for the Helmholtz eigenvalue problems with a multiply connected domain", *Proceedings of the Royal Society of London. Series A.*, 457, 2521-2546.

Chen J. T.; Liu L. W.; Chyuan S. W. (2004): "Acoustic eigenanalysis for multiply-connected problems using dual BEM", *Communications in Numerical Methods in Engineering*, 20, 419-440.

Chen J. T.; Liu L. W.; Hong H. K. (2003): "Spurious and true eigensolutions of Helmholtz BIEs and BEMs for a multiply connected problem", *Proceedings of the Royal Society of London. Series A.*, 459, 1891-1924.

Chen K. H.; Kao J. H.; Chen J. T.; Young D. L.; Lu M. C. (2006): "Regularized meshless method for multiply-connected-domain Laplace problems", *Engineering Analysis with Boundary Elements*, vol. 30, 882-896.

Chen W.; Hon Y. C. (2003): "Numerical investigation on convergence of boundary knot method in the analysis of homogeneous Helmholtz, modified Helmholtz and convection-diffusion problems", *Computer Methods in Applied Mechanics and Engineering*, 192, 1859-1875.

Cheng A. H. D. (2000): "Particular solutions of Laplacian, Helmholtz-type, and polyharmonic operators involving higher order radial basis functions", *Engineering Analysis with Boundary Elements*, 24, 531-538.

Fairweather G.; Karageorghis A. (1998): "The method of fundamental solutions for elliptic boundary value problems", *Advances in Computational Mathematics*, 9, 69-95.

Kang S. W.; Lee J. M.; Kang Y. J. (1999): "Vibration analysis of arbitrary shaped membranes using non-dimensional dynamic influence function", *Journal of Sound and Vibration*, 221, 117–132.

Kuo S. R.; Yeih W.; Wu Y. C. (2000b): "Applications of the generalized singular-value decomposition method on the eigenproblem using the incomplete boundary element formulation", *Journal of Sound and Vibration*, 235, 813-845.

Kupradze V. D.; Aleksidze M. A. (1964): "The method of functional equations for the approximate solution of certain boundary value problems", *U.S.S.R. Computational Mathematics and Mathematical Physics*, 4, 199-205.

Lin H.; Atluri S. N. (2000): "Meshless Local Petrov-Galerkin (MLPG) Method for Convection-Diffusion Problems", *CMES: Computer Modeling in Engineering & Sciences*, 1, 45-60.

Liu W. K.; Jun S.; Li S.; Adee J.; Belytschko T. (1995): "Reproducing kernel particle methods for structural dynamics", *International Journal for Numerical Methods in Engineering*, 38, 1655-1679.

Sladek J.; Sladek V.; Atluri S. N. (2004): "Meshless Local Petrov-Galerkin Method for Heat Conduction Problem in an Anisotropic Medium", *CMES: Computer Modeling in Engineering & Sciences*, 6, 309-318.

Young D. L.; Chen K. H.; Lee C. W. (2005): "Novel meshless method for solving the potential problems with arbitrary domain", *Journal of Computational Physics*, 209, 290-321.

Young D. L.; Chen K. H.; Lee C. W. (2005): "Singular meshless method using double layer potentials for exterior acoustics", *Journal of the Acoustical Society of America*, 119, 96-107.