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Dynamic Analysis of Piezoelectric Structures by the Dual Reciprocity Boundary Element Method

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Abstract: The aim of the present work is to show the formulation and application of the dual reciprocity boundary element method (BEM) to free vibrations of two-dimensional piezoelectric structures. The piezoelectric materials are modelled as homogenous, linear – elastic, transversal isotropic and dielectric. Displacements and electric potentials are treated as generalized displacements and tractions and electric charge flux densities are treated as generalized tractions. The static fundamental solutions, which are required in the proposed approach, are derived using the Stroh formalism. The domain inertial integral is transformed to the equivalent boundary integral using the dual reciprocity method (DRM). The boundary quantities are interpolated using constant elements. The developed method is used to compute frequencies and mode shapes of natural vibrations of two-dimensional piezoelectric structures. The boundary conditions are imposed using the condensation method. In this method, the degrees of freedom, which correspond to the prescribed generalized displacements are eliminated. The eigenvalue problem is solved using the Lanczos method. The numerical results computed by the present method and finite element method are compared with the available analytical solutions given in the literature.

Keyword: piezoelectric material, coupled fields, eigenvalue problem, dynamics, boundary element method, dual reciprocity method.

1 Introduction

In piezoelectric materials, mechanical and electric fields are coupled, namely, they produce an electric field when deformed or conversely, they deform when subjected to an electric field. Piezoelectric materials are widely used as sensors and actuators in smart structures and micro-electro-mechanical systems (MEMS). The piezoelectric phenomenon is also applied in ultrasonic transducers, electromechanical filters and microphones. Dynamic analysis of piezoelectric materials requires the solution of coupled electric and mechanical partial differential equations of motion [Tiersten (1969)]. These equations, particularly for piezoelectric structures with arbitrary geometries and boundary conditions, are usually solved by numerical methods. One of the versatile computer methods, which is intensively applied in piezoelectricity in the last decade, is the boundary element method (BEM). In this paper the BEM [Brebbia and Dominguez (1992)] is implemented to solve a free vibration problem of linear piezoelectricity.

Recently the use of various methods has increased in the area of the piezoelectric material analysis. Heyliger and Brooks (1995) investigated the free vibration problem of piezoelectric laminates in cylindrical bending using analytical methods. Analytical methods were also used by Benjeddou and Deu (2002) and Vel, Mever and Batra (2004) to solve the free vibrations problem of 2D piezoelectric plates. Saravanos, Heyliger and Hopkins (1997) used layerwise mechanics and the FEM to the dynamic analysis of piezoelectric composite plates. Wang, Yong and Imai (1999) used the FEM to analysis of the piezoelectric vibrations of quartz plate resonators. Denda, Araki and Yong (2004) derived the fundamental generalized displacement solution for 2D piezoelectric solids by the Radon transform. Dynamic fundamental solutions for piezoelectricity in time-domain, frequency-domain and Laplace-transform domain were also presented in Wang

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and Zhang (2005). The direct formulation of the time – harmonic BEM was applied to the determination of the eigenfrequencies of piezoelectric and general anisotropic solids. Recently Kogl and Gaul (2000b, 2003) solved a free vibration problem of 3D anisotropic solids by the dual reciprocity BEM.

The most popular piezoelectric materials are ceramics. These piezoelectrics are solids, which belong to the hexagonal symmetry class of the crystals [Tiersten (1969)]. Piezoceramics have anisotropic physical properties (both mechanical and electrical), therefore in the present work homogeneous, transversal isotropic, linear elastic and dielectric model of the piezoelectric material is considered. The anisotropy of the material increases the number of the material constants, and derivation of the fundamental solutions becomes difficult [Wang and Zhang (2005)].

The Stroh formalism [Ting (1996), Ting and Wang (1997), Pan (1999)] is a powerful and elegant analytic technique for the anisotropic elasticity, which is expanded to the linear piezoelectricity in this work. The formalism requires the solution of the special eigenvalue problem with respect to the material constants of the piezoelectric. Derivation of the fundamental solutions in frequency or time-domain is much more complicated [Denda, Araki and Yong (2004)]. The dual reciprocity boundary element method allows the use of the static fundamental solutions for dynamic problems [Brebbia and Dominguez (1992), Kogl and Gaul (2000a, 2000b), Kogl and Gaul (2003)]. The dual reciprocity formulation is derived from the reciprocal relation between a static state and a dynamic state in which the inertia forces are treated as body forces. This method is used to transform the domain integral, which depends on inertia, into the boundary integral. After discretization the system matrices are independent of time.

The advantages of the dual reciprocity BEM are: application of the static fundamental solutions, which have simpler form than the corresponding fundamental solutions in time or frequency – domain; the dual reciprocity method is a universal approach - it allows to analyze structures subjected to body forces, to solve eigenvalue and transient problems, using procedures, which are similar to the finite element algorithms; this method is also faster in comparison with frequency and time-domain BEM approach, especially for the transient analysis.

The main drawback of the dual reciprocity BEM is an additional interpolation of the generalized displacements. To obtain more accurate results the application of additional internal points is necessary.

In this paper, the method proposed by Kogl and Gaul (2000a) is applied to 2D linear piezoelectricity, but the fundamental solutions are obtained by the different method – the Stroh formalism. Also the influence of electric boundary conditions and the polarization direction is considered.

The Stroh formalism applied in the dual reciprocity BEM gives the new, efficient and effective numerical tool for solution of dynamic problems of 2D linear piezoelectricity. The influence of the polarization direction is also taken into account using the Stroh formalism.

A numerical example is presented and it shows that the dual reciprocity boundary element method allows to analyze efficiently the free vibration problem of linear piezoelectricity.

This paper is organized as follows. In section 2, the dual reciprocity BEM for linear piezoelectricity is briefly reviewed. In section 3, the Stroh formalism is introduced and applied to obtain the fundamental solutions. The Stroh formalism is also used to investigate the influence of the polarization direction. In section 4, the particular solutions of the linear piezoelectricity operator are proposed. In section 5, the eigenvalue problem formulation is presented. In section 6, the numerical examples are given to show accuracy and applications of the method. Finally, concluding remarks are given in section 7.

2 The dual reciprocity BEM formulation of linear piezoelectricity

The coupled field equations of piezoelectricity are given by the following system of partial differential equations [Tiersten (1969), Kogl and Gaul

(2000a, 2000b)]:

$$\begin{aligned} C_{ijkl}u_{k,li} + e_{lij}\phi_{,li} &= \rho \ddot{u}_j - b_j \\ e_{ikl}u_{k,li} - \varepsilon_{il}\phi_{,li} &= 0 \end{aligned} \tag{1}$$

The tensors C_{ijkl}, e_{lij}, ε_{il} denote elastic moduli, measured in a constant electric field, piezoelectric constants and dielectric constants, measured at constant strains, respectively; u_k is the displacement vector, ϕ denotes the electric potential; b_j is the body force vector per unit volume and ρ denotes the density. Double dots denote the second derivative with respect to time. In equations (1) the intrinsic electric charge is neglected.

To get the classical boundary-initial value problem formulation, equations (1) must be completed with the boundary conditions. The mechanical and electric boundary conditions are:

$$\begin{aligned} &\Gamma_u : u_i = \bar{u}_i; \quad \Gamma_t : t_i = \bar{t}_i \\ &\Gamma_\phi : \phi = \bar{\phi}; \quad \Gamma_q : q = \bar{q} \end{aligned} \tag{2}$$

In equations (2) t_i denotes the tractions, q is the charge flux density; Γ_u, Γ_t, Γ_ϕ and Γ_q denote parts of the boundary Γ where displacements, tractions, potentials and charge flux densities are prescribed. The overbar denotes the prescribed boundary conditions. These parts of the boundaries fulfill the following relations:

$$\begin{aligned} &\Gamma = \Gamma_t \cup \Gamma_u = \Gamma_\phi \cup \Gamma_q \\ &\Gamma_t \cap \Gamma_u = \emptyset \\ &\Gamma_\phi \cap \Gamma_q = \emptyset \end{aligned} \tag{3}$$

The initial conditions have a form:

$$\begin{aligned} &u_k(\tau = 0) = u_k^0, \quad \phi(\tau = 0) = \phi^0 \\ &\dot{u}_k(\tau = 0) = \dot{u}_k^0, \quad \dot{\phi}(\tau = 0) = \dot{\phi}^0 \end{aligned} \tag{4}$$

The superscript ^{0} denotes the prescribed initial condition and τ is time. The coupled field equations with boundary and initial conditions formulate the direct problem of linear piezoelectricity.

The boundary-initial value problem of linear piezoelectricity can be formulated in a much more convenient form using generalized quantities. The following vectors are introduced:

$$U_K = \begin{Bmatrix} u_k \\ \phi \end{Bmatrix}; \quad T_J = \begin{Bmatrix} t_j \\ q \end{Bmatrix}; \quad B_J = \begin{Bmatrix} b_j \\ 0 \end{Bmatrix} \tag{5}$$

where U_K, T_J and B_J are the generalized displacement, traction and body force vector, respectively. Then, the coupled field equations are given by the operator equation:

$$L_{JK}U_K = D_{JK}U_K - B_J \tag{6}$$

where L_{JK} is the 2D elliptic operator of static piezoelectricity and D_{JK} is a differential operator, which has a form:

$$D_{JK} = \begin{bmatrix} \rho\partial_t^2 & 0 & 0 \\ 0 & \rho\partial_t^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

In the above equation the operator ∂_t^2 is a differential operator, which denotes double differentiation with respect to time.

In the present work homogeneous, transversal isotropic, linear elastic and dielectric model of the piezoelectric material is chosen. For this model, the operator L_{JK} , for the two - dimensional case, has a form [Kogl and Gaul (2000a)]:

$$L_{JK} = \begin{bmatrix} c_{11}\partial_{11} + c_{44}\partial_{22} & (c_{13}+c_{44})\partial_{12} & (e_{31}+e_{15})\partial_{12} \\ & c_{44}\partial_{11} + c_{33}\partial_{22} & e_{15}\partial_{11} + e_{33}\partial_{22} \\ sym & & -\varepsilon_{11}\partial_{11} - \varepsilon_{33}\partial_{22} \end{bmatrix} \tag{8}$$

where ∂_{ij} is a differential operator, which denotes differentiation with respect to the spatial coordinates. The coefficients c_{ij}, e_{ij} and ε_{ij} are the values of the elastic, piezoelectric and dielectric constants, respectively.

The next step in the BEM formulation is application of a reciprocity relation. This process is well known [Brebbia nad Dominguez (1992), Kogl and Gaul (2000a)]. The system of equations (6) is weighted with a test function and integrated by parts. Then, the piezoelectric reciprocity relation has a form:

$$\int_\Omega (L_{JK}U_K U_{MJ} - L_{JK}U_{MK}U_J)\, d\Omega = \int_\Gamma (U_{MJ}T_J - T_{MJ}U_J) d\Gamma \tag{9}$$

where U_{MJ} is the test function and T_{MJ} depends on the derivative of the test function and Ω denotes the region which is occupied by the piezoelectric body. In this equation the generalized body force vector is neglected.

The fundamental solution U_{MK} is given by the relation:

$$L_{JK}U_{MK} = -\delta_{JM}\delta \tag{10}$$

where δ_{JM} is the Kronecker delta and δ denotes the Dirac distribution, T_{KJ} depends on the derivative of the fundamental solution:

$$T_{MJ} = C_{iJKl}U_{MK,l}n_i \tag{11}$$

where C_{iJKl} is the piezoelectric constitutive tensor, which contains elastic, piezoelectric and dielectric constants and n_i denotes the unit outward normal vector.

When the test function is chosen as a fundamental solution of the static piezoelectric operator L_{JK} and a limiting process to the boundary is done, the boundary integral formula is given by:

$$c_{KJ}U_J + \int_\Gamma T_{KJ}U_J d\Gamma = \int_\Gamma U_{KJ}T_J d\Gamma - \int_\Omega U_{KJ}D_{JL}U_L d\Omega \tag{12}$$

where c_{KJ} denotes a free term coefficient at the source point.

To obtain the fundamental solutions, the Stroh formalism is used [Ting and Wang (1997), Pan (1999), Dziatkiewicz and Fedelinski (2005a, 2005b)]. The derivation of the fundamental solution will be presented in detail in the next section.

The domain integral in equation (12), which describes the inertia effect, will be transformed into the boundary integral using the reciprocal theorem between two static states [Brebbia and Dominguez (1992), Kogl and Gaul (2000a, 2000b)].

Let the generalized displacements be approximated using a sum of functions multiplied by unknown coefficients:

$$D_{JL}U_L \approx \sum_{m=1}^{M} F_{JN}^m \alpha_N^m \tag{13}$$

In the above equation the unknown coefficients α_N^m depend on time, but functions F_{JN}^m are time independent. The functions F_{JN}^m are related to the inhomogeneous differential equation of static piezoelectricity:

$$L_{JK}U_{KN}^m = F_{JN}^m \tag{14}$$

Weighting the equation above with the static fundamental solution U_{KJ} , one can obtain:

$$\int_\Omega U_{KJ}F_{JN}^m d\Omega = -c_{KJ}U_{JN}^m - \int_\Gamma T_{KJ}U_{JN}^m d\Gamma + \int_\Gamma U_{KJ}T_{JN}^m d\Gamma \tag{15}$$

Using this new reciprocal relation and approximation of the generalized acceleration field leads to the dual reciprocity formulation of the dynamic piezoelectricity [Kogl and Gaul (2000)]:

$$c_{KJ}U_J + \int_\Gamma T_{KJ}U_J d\Gamma = \int_\Gamma U_{KJ}T_J d\Gamma + \sum_{m=1}^{M} \left(c_{KJ}U_{JN}^m + \int_\Gamma T_{KJ}U_{JN}^m d\Gamma - \int_\Gamma U_{KJ}T_{JN}^m d\Gamma \right) \alpha_N^m \tag{16}$$

To solve approximately the boundary integral equation, the boundary element method is applied. The boundary Γ is divided into boundary elements. The boundary generalized displacements and tractions are approximated using shape functions. In the present method constant boundary elements are used. The coefficients α_N^m are calculated using collocation at boundary nodes and internal nodes.

The operator D_{JK} shows that the electric potential has no influence on the inertia term, hence the approximation, using collocation, of the generalized acceleration field has a special form:

$$D_{JK}U_K = \begin{bmatrix} \rho \ddot{u}_m \\ 0 \end{bmatrix} = \begin{bmatrix} F_{mm} & F_{me} \\ F_{em} & F_{ee} \end{bmatrix} \begin{bmatrix} \alpha_m \\ \alpha_e \end{bmatrix} \tag{17}$$

where indices m and e denote the mechanical and electric field respectively and $\ddot{u}_m$ denotes the nodal values of the mechanical acceleration. The vector coefficient α_e can be eliminated from equation (17) and the vector coefficient α can be expressed using the generalized acceleration vector

$\ddot{U}$, in the following matrix form:

$$\alpha = \rho F_*^{-1}\ddot{U} \tag{18}$$

where:

$$F_*^{-1} = \begin{bmatrix} F_{*mm}^{-1} & 0 \\ -F_{ee}^{-1}F_{em}F_{*mm}^{-1} & 0 \end{bmatrix} \tag{19}$$

The submatrix F_{*mm} is equal to:

$$F_{*mm} = F_{mm} - F_{me}F_{ee}^{-1}F_{em} \tag{20}$$

The discretized equation (16) is applied to all boundary nodes, and it leads to the following linear system of equations:

$$HU - GT = (H\hat{U} - G\hat{T})\alpha \tag{21}$$

where $\hat{U}$ and $\hat{T}$ denote the nodal values of the particular solutions of equation (14), H denotes a matrix which depends on derivatives of the fundamental solution, G is a matrix which depends on the fundamental solution. The vectors U and T contain discretized values of the boundary generalized displacements and tractions. Using expression (18) one can obtain the following system of linear ordinary differential equations:

$$M\ddot{U} + HU = GT \tag{22}$$

The mass matrix M is equal to:

$$M = \rho(G\hat{T} - H\hat{U})F_*^{-1} \tag{23}$$

In the DRM the system of ordinary differential equation in time – domain (22) is similar to equations of the finite element method (FEM). The present system of equations, with the boundary and initial conditions is an approximated discrete form of the boundary-initial value problem of linear piezoelectricity.

3 The Stroh formalism

Since piezoelectric materials are anisotropic, the fundamental solutions are rather complicated, even for the transversal isotropic model of the material [Kogl and Gaul (2000a, 2000b)]. To obtain the fundamental solutions, the Stroh formalism is used. The Stroh formalism is a powerful and elegant analytic technique for the anisotropic elasticity [Ting (1996), Ting and Wang (1997)], which is expanded to the linear piezoelectricity in this case [Pan (1999), Dziatkiewicz and Fedelinski (2005a, 2005b)]. The orientation of the polarization direction is also taken into account using this formalism. The formalism requires the solution of the special eigenvalue problem with respect to the material constants of the piezoelectric material. The eigenvalues and eigenvectors, related to these constants, are specially transformed according to the polarization direction [Ting (1996)].

3.1 The fundamental solutions

In the Stroh formalism it is assumed that the field of the generalized displacements has a form [Ting and Wang (1997), Pan (1999), Dziatkiewicz and Fedelinski (2005a, 2005b)]:

$$U = af(z) \tag{24}$$

where a is the unknown vector and $f(z)$ is an analytic complex function and z is a complex variable:

$$z = x_1 + px_2 \tag{25}$$

where x_1 and x_2 are the coordinates, p denotes the unknown complex constant. Introducing equation (24) into the coupled field equations (1) for a static case, the quadratic eigenvalue problem is obtained:

$$\left\{Q + p(R + R^T) + p^2T\right\}a = 0 \tag{26}$$

where the matrices Q, R and T depend only on the material constants. The above equation can be transformed into the standard eigenvalue problem:

$$N\xi = p\xi \tag{27}$$

where:

$$N = \begin{bmatrix} -T^{-1}R^T & T^{-1} \\ RT^{-1}R^T - Q & (-T^{-1}R^T)^T \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix} \tag{28}$$

where the vector b is equal to:

$$b = -\frac{1}{p}(Q + pR)\,a \tag{29}$$

It is known, that the eigenvalue problem (26) or (27), in a two – dimensional case, gives three pairs of complex conjugate eigenvalues and corresponding eigenvectors. If p_J (J=1,2,3) are eigenvalues with a positive imaginary part, then:

$$p_{J+3} = \bar{p}_J, \quad a_{J+3} = \bar{a}_J, \quad b_{J+3} = \bar{b}_J \tag{30}$$

where the overbar denotes the complex conjugate. In this formulation only a quasi-isotropic material can be analyzed, but the difference between the solution based on a quasi-isotropic and a pure isotropic model is negligible [Pan (1999)]. For piezoelectric materials eigenvalues are distinct, so it can be written that the solution (24) is equal to:

$$U = \sum_{J=1}^{3} \{a_J f_J(z_J) + \bar{a}_J f_{J+3}(\bar{z}_J)\} \tag{31}$$

where functions $f_1, \ldots, f_6$ have arguments in the form:

$$z_J = x_1 + p_J x_2 \tag{32}$$

Most often functions f have the same form, but with different complex coefficients vector:

$$\begin{aligned} &f_J(z_J) = q_J f(z_J), \\ &f_{J+3}(\bar{z}_J) = \bar{q}_J \bar{f}(\bar{z}_J), \end{aligned} \tag{33}$$

where the vector q must be determined. Let:

$$\begin{aligned} &A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \\ &B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}, \\ &[f(z_*)] = diag\begin{bmatrix} f(z_1) & f(z_2) & f(z_3) \end{bmatrix}, \\ &q = \begin{Bmatrix} q_1 & q_2 & q_3 \end{Bmatrix}. \end{aligned} \tag{34}$$

Then equation (31) can be written in the form [Pan (1999)]:

$$U = 2Re\{A[f(z_*)]\,q\} \tag{35}$$

Now, the fundamental solution can be obtained in the following way. Let a generalized dislocation be $U^* = (U^+ - U^-)$ at the source point (x_1^0, x_2^0) and the generalized force is $T^* = (T^- - T^+)$. Then the complex function f_J will have a form [Pan (1999)]:

$$f(z_J) = \ln(z_J - s_J) \tag{36}$$

where:

$$s_J = x_1^0 + p_J x_2^0 \tag{37}$$

Let the jump of the generalized quantities be equal to U^* and T^* and if $U^* = 0$ and the generalized force T^* is a unit point force then q in equation (35) is equal to [Pan (1999)]:

$$q = \frac{1}{2\pi}\Big[B^{-1}\left(Y + \bar{Y}\right)^{-1} U^* - A^{-1}\left(Y^{-1} + \bar{Y}^{-1}\right)^{-1} T^*\Big] \tag{38}$$

where the matrix Y is equal to:

$$Y = iAB^{-1} \tag{39}$$

where $i = \sqrt{-1}$.

The Green functions (the fundamental solutions) of piezoelectric elliptic operator L_{JK} (8) are then given by:

$$\begin{aligned} U_{KL} &= \frac{-1}{\pi} Re\left[\sum_{J=1}^{3} A_{LJ} V_{JK} \ln(z_J - s_J)\right], \\ T_{KL} &= \frac{1}{\pi} Re\left[\sum_{J=1}^{3} B_{LJ} \frac{p_J n_1 - n_2}{z_J - s_J} V_{JK}\right], \end{aligned} \tag{40}$$

where:

$$V = A^{-1}(Y^{-1} + \bar{Y}^{-1})^{-1} \tag{41}$$

In equations (40) z_J and s_J are given by (32) and (37), respectively.

3.2 The influence of the polarization direction

The last stage of the manufacturing process of the piezoceramics is polarization [Tiersten (1969)]. The polarization direction has a great influence on the behaviour of piezoelectric materials. For example, when the piezoelectric linear actuators are designed, relatively small changes of the polarization direction vary the designed displacement fields. For this reason, determining the relationship between the polarization direction and the response of the structure is an important task. The orientation of the polarization direction is taken into account using the Stroh formalism. The eigenvalues and the eigenvectors of the eigenvalue problem (27) are specially transformed according to the polarization direction [Ting (1996)].

The properties of the piezoceramics are measured in the coordinate system, which is parallel to the polarization axis. When the polarization direction is not parallel to the axis of the global coordinate system, a rotational transformation is necessary.

The Stroh formalism requires the transformation of the eigenvalues and the eigenvectors only. It takes a form [Ting (1996)]:

$$A = \Omega^T A^* \qquad B = \Omega^T B^* \tag{42}$$

for the eigenvectors, and:

$$p = \frac{\sin\Theta + p^* \cos\Theta}{\cos\Theta - p^* \sin\Theta} \tag{43}$$

for the eigenvalues, where Θ denotes the angle between the coordinate system where the properties are measured and the coordinate system where the boundary – value problem is solved, as shown in Figure 1.

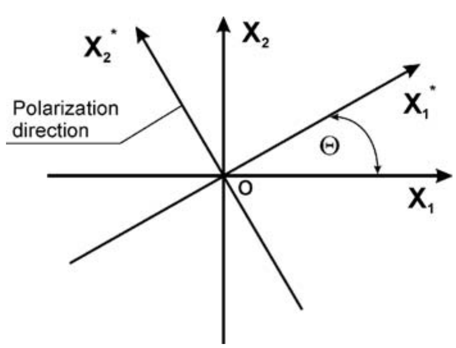

Figure 1: The coordinate system transformation

The transformation matrix Ω is given by:

$$\Omega = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{44}$$

4 Particular solutions

The dual reciprocity method requires the solution of the inhomogeneous partial differential equation of the static piezoelectricity (14). The procedure, which leads to the solution, is quite difficult for piezoelectric materials because the governing equations for an anisotropic material have complicated forms. An alternative is to assume a particular solution and find the corresponding functions.

In the present fomulation the particluar solution is assumed as a radial basis function with a constant term [Kogl and Gaul (2000a)]:

$$U^m_{KN} = \delta_{KN}(r^3 + r^2 + C) \tag{45}$$

where δ_{KN} is a Kronecker isotropic tensor, a C is a constant. To obtain the corresponding traction field and an “artificial" body force term, the derivatives of the assumed particular solution are calculated:

$$\begin{aligned} U^m_{KN,l} &= \delta_{KN}(3r^2 + 2r)r_{,l} \\ U^m_{KN,li} &= \delta_{KN}[(3r+2)\delta_{li} + 3rr_{,i}r_{,l}] \end{aligned} \tag{46}$$

Then the traction field and body force field are described by the following equations:

$$\begin{aligned} T^m_{JN} &= C_{iJKl}U^m_{KN,l}n_i \\ F^m_{JN} &= C_{iJKl}U^m_{KN,li} \end{aligned} \tag{47}$$

5 The eigenvalue problem formulation

When the boundary conditions are known, the matrix equation of motion can be written in the form [Kogl and Gaul (2003)]:

$$\begin{bmatrix} M_{uu} & M_{ut} \\ M_{tu} & M_{tt} \end{bmatrix} \begin{bmatrix} \ddot{U}_u \\ \ddot{U}_t \end{bmatrix} + \begin{bmatrix} H_{uu} & H_{ut} \\ H_{tu} & H_{tt} \end{bmatrix} \begin{bmatrix} U_u \\ U_t \end{bmatrix} = \begin{bmatrix} G_{uu} & G_{ut} \\ G_{tu} & G_{tt} \end{bmatrix} \begin{bmatrix} T_u \\ T_t \end{bmatrix} \tag{48}$$

where the subscript u is related to the part of the boundary, where the generalized displacements are prescribed and the generalized tractions are unknown; t corresponds to the part of the boundary, where the generalized tractions are known and the generalized displacements are unknown. Eliminating the unknown generalized tractions and assuming that the known generalized tractions are equal to zero, the following system of equations can be obtained:

$$M_{red_1}\ddot{U}_t + H_{red_1}U_t = 0 \quad (49)$$

where:

$$\begin{aligned} M_{red_1} &= M_{tt} - G_{tu}G_{uu}^{-1}M_{ut} \\ H_{red_1} &= H_{tt} - G_{tu}G_{uu}^{-1}H_{ut} \end{aligned} \quad (50)$$

In the free vibration problem it is assumed that the solution of the equation of motion are harmonic functions. In this case:

$$\ddot{U}_t = -\omega^2 U_t \quad (51)$$

where ω is an angular frequency. Then the generalized eigenvalue problem is described by the equation:

$$H_{red_1}X = \omega^2 M_{red_1}X \quad (52)$$

where X is an amplitude. The above formulation is equivalent to :

$$(H_{red_1} - \lambda M_{red_1})X = 0 \quad (53)$$

where $\lambda = \omega^2$. From the formulation of the dynamic problem of linear piezoelectricity it is known that the electric potential does not influence the inertia term. The eigenvalue problem can be formulated as follows:

$$\left(\begin{bmatrix} H_{mm} & H_{me} \\ H_{em} & H_{ee} \end{bmatrix} - \lambda \begin{bmatrix} M_{mm} & M_{me} \\ M_{em} & M_{ee} \end{bmatrix}\right)\begin{bmatrix} X_m \\ X_e \end{bmatrix} = 0 \quad (54)$$

where the index m denotes the mechanical part of the quantity and e – the electric part.

After reduction, the eigenvalue problem is represented by the equation:

$$(H_{red_2} - \lambda M_{red_2})X_m = 0 \quad (55)$$

where :

$$\begin{aligned} H_{red_2} &= H_{mm} - H_{me}H_{ee}^{-1}H_{em} \\ M_{red_2} &= M_{mm} - H_{me}H_{ee}^{-1}M_{em} \end{aligned} \quad (56)$$

The two - step condensation process reduces number of degrees of freedom. The above equation can be transformed into the standard eigenvalue problem:

$$\lambda_{red}X = AX \quad (57)$$

where:

$$\begin{aligned} \lambda_{red} &= \frac{1}{\lambda} \\ A &= H_{red_2}^{-1}M_{red_2} \end{aligned} \quad (58)$$

The present eigenvalue problem is solved using the Lanczos method.

6 Numerical examples

The eigenvalue problem for the rectangular piezoelectric plate made of PZT-4 ceramic is considered [Vel, Mever and Batra (2004)]. The length of the plate is equal to L=0.04 m, and height is equal to H=0.01 m, as shown in Figure 2.

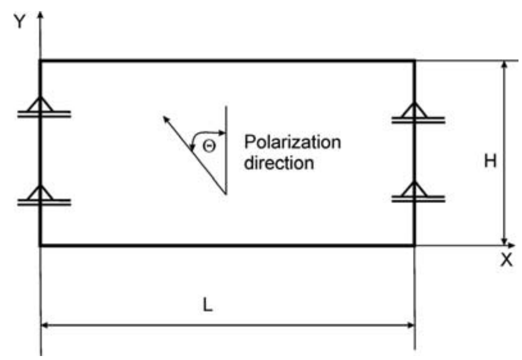

Figure 2: Piezoelectric strip

Table 1: Material properties of the PZT-4

constant	value
c_{11} [Pa]	$1.39\cdot10^{11}$
c_{33}	$1.14\cdot10^{11}$
c_{55}	$25.6\cdot10^{10}$
c_{12}	$74.3\cdot10^{10}$
e_{15} [C/m^2]	12.72
e_{31}	-5.20
e_{33}	12.72
ε_{11} [C/Vm]	$1.31\cdot10^{-8}$
ε_{33}	$1.12\cdot10^{-8}$
ρ [kg/m^3]	7600

The material properties of PZT-4 are given in Table 1.

The piezoelectric plate is simply supported – the nodes on the vertical edges can only move in the horizontal direction. The edges are electrically grounded to zero potential – this boundary condition is called the closed condition. To discretize the boundary of the plate 100 constant boundary elements and 114 internal nodes are applied.

The problem is also solved using the finite element method. For the FEM model 100 quadrilateral finite elements with quadratic shape functions are used. The solutions of the eigenvalue problem for the smallest 5 thickness modes are given in Table 2.

The eigenfrequencies are scaled using the relation:

$$\bar{\omega} = \omega \left(\frac{L^2}{H} \right) \sqrt{\frac{\rho}{c_{11}}} \tag{59}$$

where $\bar{\omega}$ denotes the scaled dimensionless eigenfrequency and ω is the computed eigenfrequency.

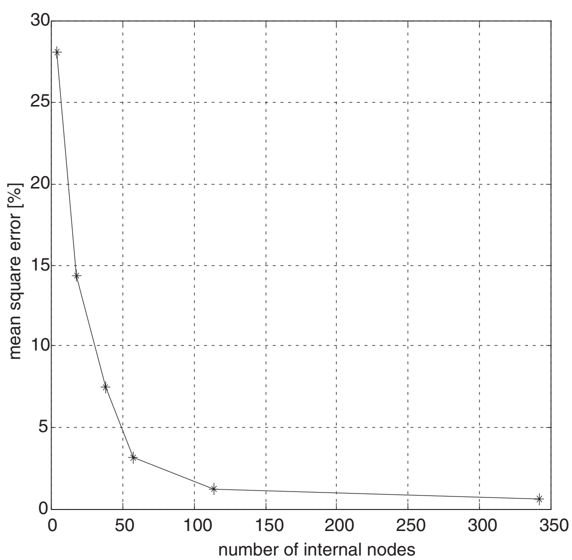

Figure 3: Mean square error of smallest 5 thickness eigenfrequencies vs. number of internal nodes

In [Vel, Mever and Batra (2004)] is shown, that these eigenfrequencies are thickness modes corresponding to large axial displacements, which are generated by the sinusoidal potential applied to the top surface of the plate. In Table 2, the fact that the smallest 5 thickness eigenfrequencies are not equivalent to the smallest eigenfrequencies is presented.

When only the boundary is discretized, the inertia is modelled inaccurately – for higher modes, the complex eigenvalues are found in the spectrum. [Kogl and Gaul (2003)]. The internal nodes are necessary to improve the accuracy of the solution and remove complex eigenfrequencies. All eigenfrequencies presented in Table 2 are real.

In this table a comparison between the analytical [Vel, Mever and Batra (2004)], the BEM and FEM results with the relative percentage error are shown. A good agreement between the analytical, FEM and BEM solutions can be observed. The FEM results are better than BEM results for the first and second thickness eigenfrequencies and worse for higher modes.

Figure 3 shows the influence of internal nodes on the mean square error of smallest 5 thickness eigenfrequencies. When the internal nodes are used, the accuracy of the solution is better and in the spectrum there are not many complex eigenfrequencies. An application of 57 internal nodes gives satisfactory accuracy.

The mode shapes shown in Figures 4-8 agree well with the results in [Vel, Mever and Batra (2004)].

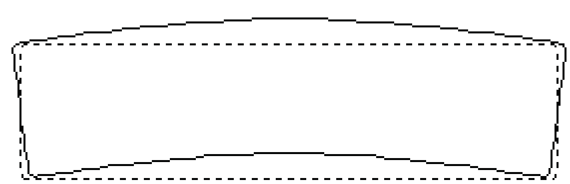

Figure 4: 1^{st} mode shape of the plate

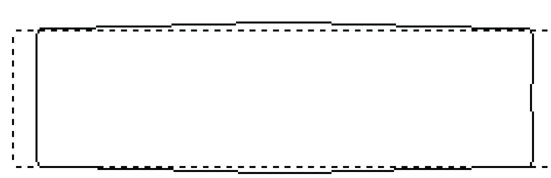

Figure 5: 2^{nd} mode shape of the plate

Table 2: Comparison between analytical, BEM and FEM solution

No. of thickness mode	No. of mode	$\bar{\omega}_0$ analytical	$\bar{\omega}_0$ BEM	$\bar{\omega}_0$ FEM	relative error FEM [%]	relative error BEM [%]
1	2	2.26	2.31	2.25	0.44	2.21
2	4	10.09	9.88	9.98	1.09	2.08
3	8	24.09	24.10	23.84	1.04	0.04
4	15	41.66	41.83	39.83	4.15	0.41
5	20	49.51	50.00	47.94	3.17	0.99

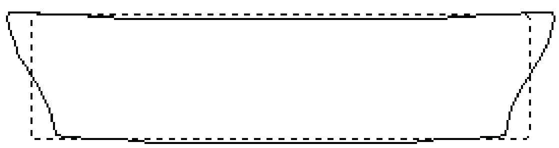

Figure 6: 3^{rd} mode shape of the plate

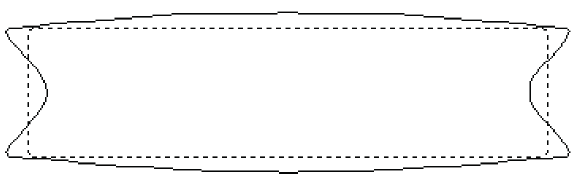

Figure 7: 4^{th} mode shape of the plate

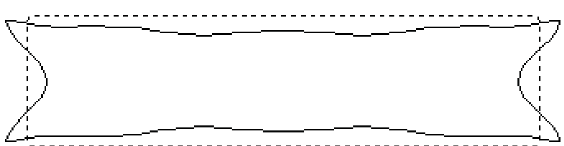

Figure 8: 5^{th} mode shape of the plate

When on the bottom and top edges of the plate the charge flux density is equal to zero – the boundary conditions are called open [Heyliger nad Brooks (1995)]. The influence of the boundary conditions on normalized eigenfrequencies is given in Table 3. It is easy to see, that for the same mode, the eigenfrequencies for the open circuit are bigger.

The influence of the polarization direction is also explored. The same plate is considered. In Figure 9 the change of the smallest 2 thickness eigenfreqeuncies vs. polarization direction is presented.

Table 3: Comparison between open and closed boundary condition

mode	$\bar{\omega}_c$ closed (ϕ=0)	$\bar{\omega}_0$ open (q=0)
1	2.31	2.33
2	9.88	10.84
3	24.10	28.53
4	41.83	42.50
5	50.00	52.32

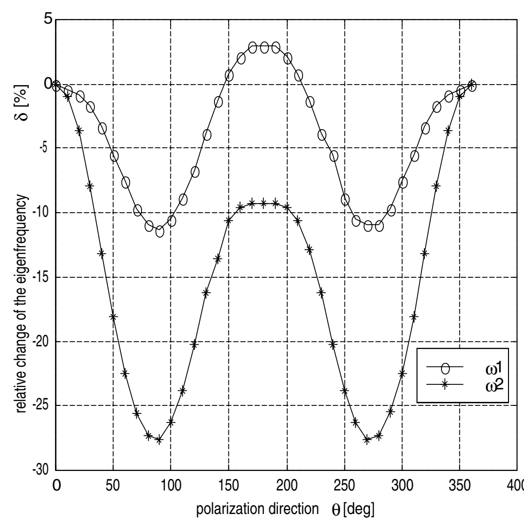

Figure 9: Relative change of the smallest 2 thickness eigenfreqeuncies vs. polarization direction

7 Conclusions

In this paper, the dual reciprocity BEM for the free vibration problem of linear piezoelectricity is developed. The results confirm that the present formulation allows to solve the eigenvalue problem of linear piezoelectricity with arbitrary boundary conditions – especially with different electric boundary conditions. The two step condensation method reduces degrees of freedom. This is an important property of the present method, because an accurate analysis requires many internal degrees of freedom [Kogl and Gaul (2003)], which allows to avoid complex eigenfrequencies. The utilized radial basis functions, as the particular solutions, give good approximation of the inertia term in the dual reciprocity BEM. The influence of the polarization direction is also taken into account using the Stroh formalism.

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