 CMES, vol.17, no.1, pp.55-62, 2007

General Corotational Rate Tensor and Replacement of Material-time Derivative to Corotational Derivative of Yield Function

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Abstract: Constitutive equation describing the mechanical properties of material has to be formulated in an identical form independent of coordinate systems by which it is described even if there exist any mutual configuration and/or mutual rotation between the material and coordinate systems. This mechanical requirement is attained by describing rate variables as corotational rate tensors with objectivity in constitutive equations in rate form. Besides, in order to use the material-time derivative of yield condition as a consistency condition it has to be replaced to the corotational derivative. In this note a general corotational rate for tensors in arbitrary order having the objectivity is shown first, and further it is verified that the material-derivative of yield condition can be replaced generally to the corotational derivative, i.e. the consistency condition.

Keyword: constitutive equation, corotational rate, elastoplasticity, objectivity.

1 Introduction

Mechanical property of material is observed to be identical independent of states of observers. Then, it should be described by a unique equation independent of mutual configuration and/or rotation between material and observers. This fact is advocated and called the *principle of material-frame indifference* by Oldroyd (1950). On account of this principle, all of physical quantities used in constitutive equations have to be described by the tensor quantities obeying the common translation rule, called the *objective transformation*, between coordinate systems. In particular, constitutive equation of inelastic deformation has to be formulated as the relation between rate variables through the stress and internal variables since there does not exist one-to-one correspondence between stress and strain. Whilst all state quantities obey the objective transformation, pertinent tensors obeying the objective transformation independent of material rotation have to be adopted for their rate variables. In addition, they have to be physical quantities capable of describing rates of mechanical state appropriately evaluating a rotation of material. They can be given by the *corotational rate tensor*s which have components obtained by the objective inverse-transformation from the components observed by the coordinate system rotating with material to the fixed coordinate system describing the constitutive relation.

In the formulation of plastic constitutive equation the consistency condition is obtained first by material-time differentiation of yield condition. In order to use it as a constitutive relation one has to translate the stress rate and rates of internal variables to their corotational rate. The fact that a rate variable involved in the material-time derivative of yield function can be directly replaced with the corotational rate has been verified for isotropic hardening models by Hashiguchi et al. (2002), for isotropic-kinematic hardening models [Edelman and Drucker (1951), Ishlinsky (1954), Prager (1955)] by Papamichos and Vardoulakis (1995) and Bruhns et al. (2003) and for isotropic-rotational hardening models [Sekiguchi and Ohta (1977), Hashiguchi (1977, 1994, 2001), Hashiguchi and Chen (1998)] by Asaoka et al. (2002). It was also verified for a general yield function of a single tensor by Dafalias (1985, 1998) without a detailed discussion.

Material elements are often subjected to rotation

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independent of the occurrence of deformation as seen in gears, wheels, etc. Therefore, the adoption of corotational rate and the replacement of consistency condition to the corotational derivative are of importance for the mechanical design in the field of engineering. Needless to say, these would be inevitable step for the steady development of mechanics without a logical jump.

In this note a general corotational rate for tensors in arbitrary order having the objectivity is shown first. Further, it is verified that the material-derivative of yield condition involving arbitrary tensors can be replaced to the corotational derivative, i.e. the consistency condition which can be used as a constitutive relation.

2 Preliminary: Objective transformation

Consider the normalized-orthogonal coordinate systems $\{\mathrm{O}-x_i\}$ (i=1, 2, 3) with the base $\{\mathbf{e}_i\}$ and $\{\mathrm{O}^*-x_i^*(t)\}$ (t: time) with the arbitrary base $\{\mathbf{e}_i^*(t)\}$. Here, let $\{\mathbf{e}_i\}$ be the fixed standard base and $\{\mathbf{e}_i^*(t)\}$ the movable base, provided that the latter has coincided with the former in the initial state ($t=0$). Let it be assumed that the material particle P which had the position vector $\mathbf{X}$ at $t=0$ has the components $x_i(\mathbf{X},t)$ and $x_i^*(\mathbf{X},t)$ in the coordinate systems $\{\mathrm{O}-x_i\}$ and $\{\mathrm{O}^*-x_i^*(t)\}$, respectively. Further, let the position vector of the origin O^* of the coordinate system $\{\mathrm{O}-x_i'\}$ have the components $c_i(t)$ in the coordinate system $\{\mathrm{O}-x_i\}$. Then, the following relations holds between these components.

$$\left.\begin{aligned} x_i^*(\mathbf{X},t) &= Q_{ir}(t)\,(x_r(\mathbf{X},t)-c_r(t)) \\ x_i(\mathbf{X},t) &= Q_{ri}(t)x_r^*(\mathbf{X},t)+c_i(t) \end{aligned}\right\} \tag{1}$$

where the Einstein's summation convention is used throughout this note. Eq. (1) is rewritten in the symbolic notation as

$$\left.\begin{aligned} \mathbf{x}^*(\mathbf{X},t) &= \mathbf{Q}(t)(\mathbf{x}(\mathbf{X},t)-\mathbf{c}(t)) \\ \mathbf{x}(\mathbf{X},t) &= \mathbf{Q}^T(t)\mathbf{x}^*(\mathbf{X},t)+\mathbf{c}(t) \end{aligned}\right\} \tag{2}$$

The notation $(\ )^T$ stands for the transpose, and hereafter the superscript $*$ is added to the components for the movable base $\{\mathbf{e}_i^*(t)\}$. $\mathbf{Q}(t)$ is the orthogonal tensor of the base $\{\mathbf{e}_i^*(t)\}$ with respect to the standard base $\{\mathbf{e}_i\}$ and has the components

$$Q_{ij}(t) \equiv \mathbf{e}_i^*(t)\bullet\mathbf{e}_j\,(=\cos(\mathbf{e}_i^*(t),\mathbf{e}_j)) \tag{3}$$

where the dot $\bullet$ denotes the scalar product. The symbol (t) describing the time dependence is omitted hereafter. Q_{ij} is rewritten from Eq. (1) as follows:

$$Q_{ij} = \frac{\partial x_i^*}{\partial x_j} = \frac{\partial x_j}{\partial x_i^*} \tag{4}$$

From Eq. (3) one has the relation

$$\left.\begin{aligned} \mathbf{e}_i(&=(\mathbf{e}_i\bullet\mathbf{e}_r^*)\mathbf{e}_r^*) = Q_{ri}\mathbf{e}_r^* \\ \mathbf{e}_i^*(&=(\mathbf{e}_i^*\bullet\mathbf{e}_r)\mathbf{e}_r) = Q_{ir}\mathbf{e}_r \end{aligned}\right\} \tag{5}$$

between these bases and the equation

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{6}$$

where $\mathbf{I}$ is the second-order identity tensor having the components of Kronecker's delta $\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$ and thus $\mathbf{I}\equiv\mathbf{e}_i\otimes\mathbf{e}_i=\delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ ($\otimes$: tensor product).

The tensor $\mathbf{Q}$ is described in the following form with the bases.

$$\mathbf{Q} = Q_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = Q_{ij}\mathbf{e}_i^*\otimes\mathbf{e}_j^* \tag{7}$$

or

$$\mathbf{Q} = \mathbf{e}_r\otimes\mathbf{e}_r^* \tag{8}$$

due to Eq. (5). The relation between these bases is also described from Eq. (8) as follows:

$$\left.\begin{aligned} \mathbf{e}_i(&=\mathbf{e}_r\otimes\mathbf{e}_r^*\mathbf{e}_i^*) = \mathbf{Q}\mathbf{e}_i^* \\ \mathbf{e}_i^*(&=\mathbf{e}_r^*\otimes\mathbf{e}_r\mathbf{e}_i) = \mathbf{Q}^T\mathbf{e}_i \end{aligned}\right\} \tag{9}$$

Introduce the second-order tensor

$$\mathbf{\Omega} \equiv \dot{\mathbf{e}}_r^*\otimes\mathbf{e}_r^* \tag{10}$$

where $(\dot{\ })$ denotes the material-time derivative. Eq. (10) is rewritten as

$$\mathbf{\Omega} \equiv \dot{Q}_{ri}Q_{rj}\mathbf{e}_i\otimes\mathbf{e}_j, \quad \mathbf{\Omega} \equiv \dot{\mathbf{Q}}^T\mathbf{Q} \tag{11}$$

due to Eq. (5). It is known that $\mathbf{\Omega}$ is the skew-symmetric tensor from Eq. (11) and means the spin of the base $\{\mathbf{e}_i^*\}$ from the equation

$$\dot{\mathbf{e}}_i^* = \mathbf{\Omega}\mathbf{e}_i^* \tag{12}$$

obtained from Eq. (10).

The following equation for the velocity $\mathbf{v}$ of material particle P is obtained from Eq. (2).

$$\begin{aligned}\mathbf{v}^* = \dot{\mathbf{x}}^* &= \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} - Q\dot{\mathbf{c}} - \dot{\mathbf{Q}}\mathbf{c} \\ &= \mathbf{Q}\mathbf{v} + \bar{\mathbf{\Omega}}\mathbf{x}^* - \mathbf{Q}\dot{\mathbf{c}} - \dot{\mathbf{Q}}\mathbf{c} \qquad (13)\end{aligned}$$

where

$$\bar{\mathbf{\Omega}} \equiv \dot{\mathbf{Q}}\mathbf{Q}^T = -\mathbf{Q}\mathbf{\Omega}\mathbf{Q}^T \qquad (14)$$

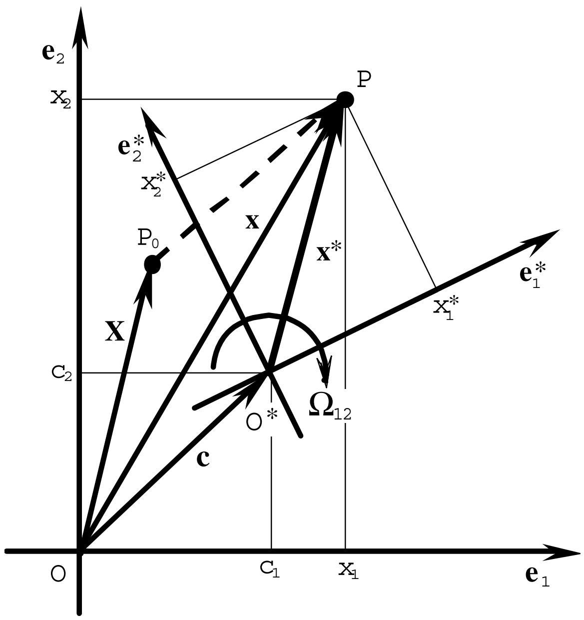

Figure 1: Description for position of material particle in the fixed standard coordinate system and the movable coordinate system

The transformation rule of m-th order tensor $\mathbf{T}$ describing the mechanical state of material is given by

$$\left.\begin{aligned} T^*_{p_1p_2\cdots p_m} &= Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m} \\ T_{p_1p_2\cdots p_m} &= Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}T^*_{q_1q_2\cdots q_m}\end{aligned}\right\} \qquad (15)$$

which is called the *objective transformation rule*. Eq. (15) is written for a vector and a second-order tensor by the symbolic notations as follows:

$$\left.\begin{aligned}\mathbf{T}^* &= \mathbf{Q}\mathbf{T} \\ \mathbf{T} &= \mathbf{Q}^T\mathbf{T}^*\end{aligned}\right\} \quad \left.\begin{aligned}\mathbf{T}^* &= \mathbf{Q}\mathbf{T}\mathbf{Q}^T \\ \mathbf{T} &= \mathbf{Q}^T\mathbf{T}^*\mathbf{Q}\end{aligned}\right\} \qquad (16)$$

by the symbolic notation.

3 General corotational rate tensor

Let the general corotational tensor in an arbitrary order be considered below.

The material-time derivative of Eq. (15) is given as

$$\begin{aligned}\dot{T}^*_{p_1p_2\cdots p_m} =& \dot{Q}_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m} \\ &+ Q_{p_1q_1}\dot{Q}_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m} \\ &+\cdots \\ &+ Q_{p_1q_1}Q_{p_2q_2}\cdots \dot{Q}_{p_mq_m}T_{q_1q_2\cdots q_m} \\ &+ Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}\dot{T}_{q_1q_2\cdots q_m} \qquad (17)\end{aligned}$$

or inversely

$$\begin{aligned}\dot{T}_{p_1p_2\cdots p_m} =& \dot{Q}_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}T^*_{q_1q_2\cdots q_m} \\ &+ Q_{q_1p_1}\dot{Q}_{q_2p_2}\cdots Q_{q_mp_m}T^*_{q_1q_2\cdots q_m} \\ &+\cdots \\ &+ Q_{q_1p_1}Q_{q_2p_2}\cdots \dot{Q}_{q_mp_m}T^*_{q_1q_2\cdots q_m} \\ &+ Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}\dot{T}^*_{q_1q_2\cdots q_m} \qquad (18)\end{aligned}$$

which are rewritten as

$$\begin{aligned}\dot{T}^*_{p_1p_2\cdots p_m} =& Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}\Big(\dot{T}_{q_1q_2\cdots q_m} \\ &- \Omega_{q_1r_1}T_{r_1q_2\cdots q_m} - \Omega_{q_2r_2}T_{q_1r_2\cdots q_m} \\ &-\cdots \\ &- \Omega_{q_mr_m}T_{q_1q_2\cdots r_m}\Big) \qquad (19)\end{aligned}$$

or inversely

$$\begin{aligned}\dot{T}_{p_1p_2\cdots p_m} =& Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}\Big(\dot{T}^*_{q_1q_2\cdots q_m} \\ &- \bar{\Omega}_{q_1r_1}T^*_{r_1q_2\cdots q_m} - \bar{\Omega}_{q_2r_2}T^*_{q_1r_2\cdots q_m} \\ &-\cdots \\ &- \bar{\Omega}_{q_mr_m}T^*_{q_1q_2\cdots r_m}\Big) \qquad (20)\end{aligned}$$

Eqs. (19) and (20) are written for a vector and a second-order tensor by the symbolic notations as follows:

$$\left.\begin{aligned}\dot{\mathbf{T}}^* &= \mathbf{Q}(\dot{\mathbf{T}} - \mathbf{\Omega}\mathbf{T}) \\ \dot{\mathbf{T}} &= \mathbf{Q}^T(\dot{\mathbf{T}}^* - \bar{\mathbf{\Omega}}\mathbf{T}^*)\end{aligned}\right\}$$
$$\left.\begin{aligned}\dot{\mathbf{T}}^* &= \mathbf{Q}(\dot{\mathbf{T}} - \mathbf{\Omega}\mathbf{T} + \mathbf{T}\mathbf{\Omega})\mathbf{Q}^T \\ \dot{\mathbf{T}} &= \mathbf{Q}^T(\dot{\mathbf{T}}^* - \bar{\mathbf{\Omega}}\mathbf{T}^* - \mathbf{T}^*\bar{\mathbf{\Omega}})\mathbf{Q}\end{aligned}\right\} \qquad (21)$$

The rate of tensor quantity used for constitutive equations in rate forms has to fulfill the following conditions.

1. It obeys the objective transformation since the material properties are independent of the observers.
2. The components in the standard fixed coordinate system describing constitutive equation changes only when the components observed in the coordinate system moving with material changes.

As known from Eqs. (17)-(21), the material-time derivative does not obey the objective transformation and it changes even if the components observed by the coordinate system rotating with material does not change, provided that the coordinate system $\{\mathbf{e}_i^*\}$ rotates with material, selecting the spin $\mathbf{\Omega}$ as the spin of material. In other words, the material-time derivative violates both conditions 1 and 2.

Then, consider the tensor $\overset{\triangledown}{\mathbf{T}}$ having the components in the coordinate system $\{\mathrm{O}-x_i\}$, which are obtained from the components observed in the movable coordinate system $\{\mathrm{O}^*-x_i^*(t)\}$ by the objective inverse transformation rule, i.e.

$$\begin{aligned}\overset{\triangledown}{T}_{p_1p_2\cdots p_m} &= Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}\dot{T}^*_{q_1q_2\cdots q_m}\\ &= \dot{T}_{p_1p_2\cdots p_m} - \Omega_{p_1r_1}T_{r_1p_2\cdots p_m}\\ &\quad - \Omega_{p_2r_2}T_{p_1r_2\cdots p_m}\\ &\quad - \cdots\\ &\quad - \Omega_{p_mr_m}T_{p_1p_2\cdots r_m}\end{aligned} \tag{22}$$

In order to verify that the tensor $\overset{\triangledown}{\mathbf{T}}$ obeys the objective transformation, introduce the other coordinate system $\{\mathrm{O}'-x_i'(t)\}$ with the base $\{\mathbf{e}_i'\}$ rotating with the spin

$$\tilde{\mathbf{\Omega}} \equiv \dot{\mathbf{e}}_r' \otimes \mathbf{e}_r' = \dot{\tilde{\mathbf{Q}}}^T\tilde{\mathbf{Q}} \tag{23}$$

where

$$\tilde{Q}_{ij}(t) \equiv \mathbf{e}_i'(t) \bullet \mathbf{e}_j (= \cos(\mathbf{e}_i'(t), \mathbf{e}_j)) \tag{24}$$

The objective transformation

$$\mathbf{\Omega}' = \tilde{\mathbf{Q}}(\mathbf{\Omega}-\tilde{\mathbf{\Omega}})\tilde{\mathbf{Q}}^T \tag{25}$$

holds between the spin $\mathbf{\Omega}'$ of the base $\{\mathbf{e}_i^*\}$ observed by the base $\{\mathbf{e}_i'\}$ and the spin $\mathbf{\Omega}-\tilde{\mathbf{\Omega}}$ of the base $\{\mathbf{e}_i^*\}$ from the base $\{\mathbf{e}_i'\}$ observed by the base $\{\mathbf{e}_i\}$, where the component of the coordinate system with $\{\mathbf{e}_i'\}$ is denoted by the superscript $(\ )'$. It holds from Eqs. (19) and (25) that

$$\begin{aligned}\dot{T}'_{p_1p_2\cdots p_m} =& \tilde{Q}_{p_1q_1}\tilde{Q}_{p_2q_2}\cdots\tilde{Q}_{p_mq_m}\Big(\dot{T}_{p_1p_2\cdots p_m}\\ &- \tilde{\Omega}_{p_1r_1}T_{r_1p_2\cdots p_m} - \tilde{\Omega}_{p_2r_2}T_{p_1r_2\cdots p_m}\\ &- \cdots\\ &- \tilde{\Omega}_{p_mr_m}T_{p_1p_2\cdots r_m}\Big)\end{aligned} \tag{26}$$

$$\begin{aligned}&\Omega'_{p_rr_r}T'_{p_1p_2\cdots r_r\cdots p_m}\\ &= \tilde{Q}_{p_rs}(\Omega_{st}-\tilde{\Omega}_{st})\tilde{Q}_{r_rt}\tilde{Q}_{p_1q_1}\tilde{Q}_{p_2q_2}\cdots\\ &\quad \tilde{Q}_{p_{r-1}q_{r-1}}\tilde{Q}_{r_rq_r}\tilde{Q}_{p_{r+1}q_{r+1}}\cdots\tilde{Q}_{p_mq_m}T_{q_1q_2\cdots q_r\cdots q_m}\\ &= \tilde{Q}_{p_rs}(\Omega_{st}-\tilde{\Omega}_{st})\delta_{tq_r}\tilde{Q}_{p_1q_1}\tilde{Q}_{p_2q_2}\cdots\\ &\quad \tilde{Q}_{p_{r-1}q_{r-1}}\tilde{Q}_{p_{r+1}q_{r+1}}\cdots\tilde{Q}_{p_mq_m}T_{q_1q_2\cdots q_r\cdots q_m}\\ &= \tilde{Q}_{p_rq_r}(\Omega_{q_rr_r}-\tilde{\Omega}_{q_rr_r})\tilde{Q}_{p_1q_1}\tilde{Q}_{p_2q_2}\cdots\\ &\quad \tilde{Q}_{p_{r-1}q_{r-1}}\tilde{Q}_{p_{r+1}q_{r+1}}\cdots\tilde{Q}_{p_mq_m}T_{q_1q_2\cdots r_r\cdots q_m}\\ &= \tilde{Q}_{p_1q_1}\tilde{Q}_{p_2q_2}\cdots\tilde{Q}_{p_mq_m}(\Omega_{q_rr_r}-\tilde{\Omega}_{q_rr_r})T_{q_1q_2\cdots r_r\cdots q_m}\end{aligned} \tag{27}$$

and then

$$\begin{aligned}\overset{\triangledown}{T}'_{p_1p_2\cdots p_m} &= \dot{T}'_{p_1p_2\cdots p_m} - \Omega'_{p_1r_1}T'_{r_1p_2\cdots p_m}\\ &\quad - \Omega'_{p_2r_2}T'_{p_1r_2\cdots p_m} - \cdots - \Omega'_{p_mr_m}T'_{p_1p_2\cdots r_m}\\ &= \tilde{Q}_{p_1q_1}\tilde{Q}_{p_2q_2}\cdots\tilde{Q}_{p_mq_m}\Big\{\dot{T}_{q_1q_2\cdots q_m}\\ &\quad - \tilde{\Omega}_{q_1r_1}T_{r_1q_2\cdots q_m} - \tilde{\Omega}_{q_2r_2}T_{q_1r_2\cdots q_m}\\ &\quad - \cdots - \tilde{\Omega}_{q_mr_m}T_{q_1q_2\cdots r_m}\\ &\quad - (\Omega_{q_1r_1}-\tilde{\Omega}_{q_1r_1})T_{r_1q_2\cdots q_m}\\ &\quad - (\Omega_{q_2r_2}-\tilde{\Omega}_{q_2r_2})T_{q_1r_2\cdots q_m}\\ &\quad - \cdots - (\Omega_{q_mr_m}-\tilde{\Omega}_{q_mr_m})T_{q_1q_2\cdots r_m}\Big\}\\ &= \tilde{Q}_{p_1q_1}\tilde{Q}_{p_2q_2}\cdots\tilde{Q}_{p_mq_m}\Big(\dot{T}_{q_1q_2\cdots q_m}\\ &\quad - \Omega_{q_1r_1}T_{r_1q_2\cdots q_m} - \Omega_{p_2r_2}T_{q_1r_2\cdots q_m}\\ &\quad - \cdots - \Omega_{q_mr_m}T_{q_1q_2\cdots r_m}\Big)\\ &= \tilde{Q}_{p_1q_1}\tilde{Q}_{p_2q_2}\cdots\tilde{Q}_{p_mq_m}\overset{\triangledown}{T}_{q_1q_2\cdots q_m}\end{aligned} \tag{28}$$

which means that $\overset{\triangledown}{\mathbf{T}}$ obeys the objective transformation.

The rate tensor $\overset{\triangledown}{\mathbf{T}}$ is called the *corotational rate* and denoted by $\overset{\circ}{\mathbf{T}}$ when $\mathbf{\Omega}$ is selected to be the spin of material, $\boldsymbol{\omega}$, i.e.

$$\overset{\circ}{T}_{p_1p_2\cdots p_m} = \dot{T}_{p_1p_2\cdots p_m} - \omega_{p_1r_1}T_{r_1p_2\cdots p_m} - \omega_{p_2r_2}T_{p_1r_2\cdots p_m} - \cdots - \omega_{p_mr_m}T_{p_1p_2\cdots r_m} \tag{29}$$

The following equations holds for the corotational rate of a vector and a second-order tensor from Eqs. (28) and (29) as follows:

$$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \boldsymbol{\omega}\mathbf{T}, \quad \overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \boldsymbol{\omega}\mathbf{T} + \mathbf{T}\boldsymbol{\omega} \tag{30}$$

$$\left.\begin{array}{l} \overset{\circ}{\mathbf{T}}{}^* = \mathbf{Q}\overset{\circ}{\mathbf{T}} \\ \overset{\circ}{\mathbf{T}} = \mathbf{Q}^T\overset{\circ}{\mathbf{T}}{}^* \end{array}\right\} \quad \left.\begin{array}{l} \overset{\circ}{\mathbf{T}}{}^* = \mathbf{Q}\overset{\circ}{\mathbf{T}}\mathbf{Q}^T \\ \overset{\circ}{\mathbf{T}} = \mathbf{Q}^T\overset{\circ}{\mathbf{T}}{}^*\mathbf{Q} \end{array}\right\} \tag{31}$$

Introduce the following notation for the coordinate transformation.

$$\left.\begin{array}{l} (\mathbf{Q}\,[\![\mathbf{T}]\!])_{p_1p_2\cdots p_m} \equiv Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m} \\ (\mathbf{Q}^T\,[\![\mathbf{T}]\!])_{p_1p_2\cdots p_m} \equiv Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}T_{q_1q_2\cdots q_m} \end{array}\right\} \tag{32}$$

By use of this notation the following expressions for an arbitrsry tensor $\mathbf{T}$ hold from Eqs. (15), (22) and (28).

$$\mathbf{T}^* = \mathbf{Q}[\![\,\mathbf{T}\,]\!], \quad \mathbf{T} = \mathbf{Q}^T[\![\,\mathbf{T}^*\,]\!] \tag{33}$$

and

$$\overset{\circ}{\mathbf{T}}{}^* = \mathbf{Q}\left[\!\!\left[\overset{\circ}{\mathbf{T}}\right]\!\!\right], \quad \overset{\circ}{\mathbf{T}} = \mathbf{Q}^T\left[\!\!\left[\overset{\circ}{\mathbf{T}}{}^*\right]\!\!\right] (\dot{\mathbf{T}} = \mathbf{Q}^T\left[\!\!\left[\dot{\mathbf{T}}{}^*\right]\!\!\right]) \tag{34}$$

4 Explicit corotational rate tensors

While $\boldsymbol{\omega}$ has been defined merely to be the spin of material above, the corotational rate tensor will be formulated below by adopting the well-known spins of material.

The velocity gradient $\mathbf{L}$, the strain rate $\mathbf{D}$ and the continuum spin $\mathbf{W}$ are given by

$$\mathbf{L} \equiv \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \tag{35}$$

$$\mathbf{D} \equiv \frac{1}{2}(\mathbf{L}+\mathbf{L}^T), \quad \mathbf{W} \equiv \frac{1}{2}(\mathbf{L}-\mathbf{L}^T) \tag{36}$$

Regarding the component $(\ )^*$ as $(\ )'$ and noting $\partial/\partial x'_i = \partial/\partial x_r Q_{ir}$ in Eq. (13), the transformations of $\mathbf{L}$, $\mathbf{D}$ and $\mathbf{W}$ are given as

$$\mathbf{L}' = \tilde{\mathbf{Q}}(\mathbf{L}-\tilde{\mathbf{\Omega}})\tilde{\mathbf{Q}}^T \tag{37}$$

$$\mathbf{D}' = \tilde{\mathbf{Q}}\mathbf{D}\tilde{\mathbf{Q}}^T, \quad \mathbf{W}' = \tilde{\mathbf{Q}}(\mathbf{W}-\tilde{\mathbf{\Omega}})\tilde{\mathbf{Q}}^T \tag{38}$$

Obviously, the continuum spin $\mathbf{W}$ obeys the translation rule of Eq. (25) for $\mathbf{\Omega}$.

The following corotational rate with the continuum spin $\mathbf{W}$ is regarded as the generalization of *Zaremba-Jaumann rate* [Zaremba (1903a,b), Jaumann (1911)].

$$T_{p_1p_2\cdots p_m} = \dot{T}_{p_1p_2\cdots p_m} - W_{p_1r_1}T_{r_1p_2\cdots p_m} - W_{p_2r_2}T_{p_1r_2\cdots p_m} - \cdots - W_{p_mr_m}T_{p_1p_2\cdots r_m} \tag{39}$$

Jaumann rate is determined merely geometrically by an external appearance of body independent of material properties and deformation history.

While the corotational spin would have to reflect the rotation of substructure in material, it is not so large as calculated by the continuum spin $\mathbf{W}$ when a plastic deformation is induced as pointed out by Mandel (1971) and Kratochvil (1971). Based on this concept, Dafalias (1983) and Loret (1983) proposed the relation between the corotational rate $\overset{*}{\boldsymbol{\sigma}}$ of stress $\boldsymbol{\sigma}$ and the elastic strain rate D^e

$$\overset{*}{\boldsymbol{\sigma}}(\equiv \dot{\boldsymbol{\sigma}} - \mathbf{W}^e\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W}^e) = \mathbf{E}\mathbf{D}^e \tag{40}$$

where $\mathbf{W}^e$ is the elastic spin given by

$$\mathbf{W}^e \equiv \mathbf{W} - \mathbf{W}^p \tag{41}$$

$\mathbf{W}^p$ is called the *plastic spin* and is formulated pertinently by Zbib and Aifantis (1989) as follows:

$$\mathbf{W}^p = \mu(\boldsymbol{\sigma}, \mathbf{H}, \mathbf{H})(\mathbf{D}^p\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{D}^p) \tag{42}$$

where $\mathbf{D}^p$ is the plastic strain rare, and μ is the material function of stress $\boldsymbol{\sigma}$ and internal variables of scalar quantity H for isotropic hardening/softening and tensor-valued quantity $\mathbf{H}$ for inherent and/or induced anisotropy. The elastic spin

$\mathbf{W}_e$ also obeys the translation rule of Eq. (25) for $\mathbf{\Omega}$, i.e.

$$\mathbf{W}^{e\prime} = \tilde{\mathbf{Q}}(\mathbf{W}^e - \tilde{\mathbf{\Omega}})\tilde{\mathbf{Q}}^T \tag{43}$$

noting $\mathbf{W}^{p\prime} = \tilde{\mathbf{Q}}\mathbf{W}^p\tilde{\mathbf{Q}}^T$.

The generalized corotational rate $\overset{\circ}{\mathbf{T}}$ based on the plastic spin is given as follows.

$$\overset{\circ}{T}_{p_1p_2\cdots p_m} = \overset{\bullet}{T}_{p_1p_2\cdots p_m} - W^e_{p_1r_1}T_{r_1p_2\cdots p_m} - W^e_{p_2r_2}T_{p_1r_2\cdots p_m} - \cdots - W^e_{p_mr_m}T_{p_1p_2\cdots r_m} \tag{44}$$

The corotational rate has to be adopted for rates of tensor valued state variables. Thus, it is used for stress and tensor-valued internal variables for describing anisotropy including the kinematic hardening variable for metals and rotational hardening variable for soils [Sekiguchi and Ohta (1977), Hashiguchi and Chen (1988)].

5 Transformation to Corotational Tensor in Consistency Condition

In order to obtain the consistency condition from the material-time derivative of yield condition, which is pertinent as a constitutive relation, one has to translate the stress rate and rates of internal variables to their corotational rate.

Yield condition is described generally as

$$f(\mathbf{A},\mathbf{B},\cdots) = 0 \tag{45}$$

where $\mathbf{A},\mathbf{B},\cdots$ are the arbitrary tensors. The material-time derivative of Eq. (45) is described as

$$\begin{aligned}&\overset{\bullet}{f}(\mathbf{A},\mathbf{B},\cdots)\\ &= \mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial \mathbf{A}}\overset{\bullet}{A}^T\right) + \mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial \mathbf{B}}\overset{\bullet}{B}^T\right)\\ &\quad + \cdots\\ &= 0\end{aligned} \tag{46}$$

Here, the yield condition is the scalar quantity independent of the state of observer and then it holds that

$$\left.\begin{aligned}f(\mathbf{A},\mathbf{B},\cdots) &= f(\mathbf{A}^*,\mathbf{B}^*,\cdots)\\ \overset{\bullet}{f}(\mathbf{A},\mathbf{B},\cdots) &= \overset{\bullet}{f}(\mathbf{A}^*,\mathbf{B}^*,\cdots)\end{aligned}\right\} \tag{47}$$

where

$$\begin{aligned}&\overset{\bullet}{f}(\mathbf{A}^*,\mathbf{B}^*,\cdots)\\ &= \mathrm{tr}\left(\frac{\partial f(\mathbf{A}^*,\mathbf{B}^*,\cdots)}{\partial \mathbf{A}^*}\overset{\bullet}{\mathbf{A}}^{*T}\right)\\ &\quad + \mathrm{tr}\left(\frac{\partial f(\mathbf{A}^*,\mathbf{B}^*,\cdots)}{\partial \mathbf{B}^*}\overset{\bullet}{\mathbf{B}}^{*T}\right)\\ &\quad + \cdots\\ &= \mathrm{tr}\left(\mathbf{Q}\left[\!\left[\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial \mathbf{A}}\right]\!\right]\overset{\bullet}{\mathbf{A}}^{*T}\right)\\ &\quad + \mathrm{tr}\left(\mathbf{Q}\left[\!\left[\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial \mathbf{B}}\right]\!\right]\overset{\bullet}{\mathbf{B}}^{*T}\right)\\ &\quad + \cdots\\ &= \mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial \mathbf{A}}\overset{\circ}{\mathbf{A}}^{T}\right)\\ &\quad + \mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial \mathbf{B}}\overset{\circ}{\mathbf{A}}^{T}\right)\\ &\quad + \cdots\end{aligned} \tag{48}$$

using Eq. (34) and the following relation for two arbitrary tensors $\mathbf{T}$ and $\mathbf{S}$.

$$\begin{aligned}&\mathrm{tr}\left(\mathbf{Q}\left[\!\left[\mathbf{T}\right]\!\right]\mathbf{S}\right)\\ &= \left(\mathrm{Q}_{p_1q_1}\mathrm{Q}_{p_2q_2}\cdots\mathrm{Q}_{p_mq_m}\mathrm{T}_{q_1q_2\cdots q_m}\right)\mathrm{S}_{p_1p_2\cdots p_m}\\ &= \mathrm{T}_{q_1q_2\cdots q_m}\left(\mathrm{Q}_{p_1q_1}\mathrm{Q}_{p_2q_2}\cdots\mathrm{Q}_{p_mq_m}\mathrm{S}_{p_1p_2\cdots p_m}\right)\\ &= \mathrm{tr}\left(\mathbf{T}\left(\mathbf{Q}^T\left[\!\left[\mathbf{S}\right]\!\right]\right)\right)\end{aligned} \tag{49}$$

From Eqs. (46), (47) and (48) we have

$$\mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial \mathbf{A}}\overset{\circ}{\mathbf{A}}^{T}\right) + \mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial \mathbf{B}}\overset{\circ}{\mathbf{B}}^{T}\right) + \cdots = 0 \tag{50}$$

Then, it is concluded that the material-time derivative of the yield function can be directly replaced to the corotational derivative, while the notation of transpose can be omitted for stress and anisotropic hardening variables which are symmetric tensors. Here, note that only a part of derivative terms cannot be replaced to the corotational derivative in the case that the yield function f involves plural non-scalar variables. In the case of a single non-scalar variable the transformation to the corotational derivative can also be proved by

$$\mathrm{tr}\left\{\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}}(\mathbf{A}\boldsymbol{\omega}-\boldsymbol{\omega}\mathbf{A})\right\} = \mathrm{tr}\left\{(a_0\mathbf{I}-a_1\mathbf{A}-a_2\mathbf{A}^2)(\mathbf{A}\boldsymbol{\omega}-\boldsymbol{\omega}\mathbf{A})\right\} = 0 \tag{51}$$

due to the Cayley-Hamilton theorem, noting $\mathrm{tr}(\mathbf{A}^n\mathbf{A}\boldsymbol{\omega}) = \mathrm{tr}(\mathbf{A}^n\boldsymbol{\omega}\mathbf{A})$, etc.

6 Concluding remarks

The general form of corotational rate tensor with objectivity is shown and it is verified that the rate variables involved in the material-time derivative of yield function can be directly replaced to the objective corotational rate tensors. While various corotational rate have been proposed, the concept of plastic spin would be physically most pertinent. However, it requires the formulation of rotation in addition to deformation as constitutive equations. On the other hand, the plastic spin is negligible up to one hundred and several ten percents of shear strain even in the simple shear in which the plastic spin is induced most remarkably [Dafalias (1983, 1985, 1988)]. One could use Zaremba-Jaumann rate in a usual elastoplastic deformation process [cf. Han et al. (2005), Himple et al. (2005), Liu (2006)].

Acknowledgement: The author expresses the sincere gratitude to Prof. B. Raniecki, Inst. Technological Research, Polish Academy of Sciences, for valuable discussions and advices on the present problem.

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