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Five Different Formulations of the Finite Strain Perfectly Plastic Equations

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Abstract: The primary objectives of the present exposition focus on five different types of representations of the plastic equations obtained from an elastic-perfectly plastic model by employing different corotational stress rates. They are (a) an affine nonlinear system with a finite-dimensional Lie algebra, (b) a canonical linear system in the Minkowski space, (c) a non-canonical linear system in the Minkowski space, (d) the Lie-Poisson bracket formulation, and (e) a two-generator and two-bracket formulation. For the affine nonlinear system we prove that the Lie algebra of the vector fields is $so(5,1)$, which has dimensions fifteen, and by the Lie theory the superposition principle is available for this system. Although the plastic equations are nonlinear in stress space, we can develop some methods to transform them into the linear systems in the augmented stress spaces with a canonical form and a non-canonical form in the Minkowski space. On the cotangent bundle of yielding manifold, we can introduce the Lie-Poisson bracket to construct an evolutional differential system of plastic equations. The stress trajectory traces a coadjoint orbit in the Poisson manifold under a coadjoint action of the Lie group $SO(5)$. Then, we prove that the plastic equations admit two generators: one conservative and one dissipative, as well as two brackets: the Poisson bracket and dissipative bracket. From a dissipation point of view the yield function is a Casimir function of the dissipative bracket system.

Keyword: Elastoplasticity, Corotational stress rate, Lie algebra, Lie group, Poisson manifold, Generalized Hamiltonian system, Lie-Poisson system, Coadjoint orbit, Two-generator

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1 Introduction

In this paper we present five types of representations of the plastic equations with large deformation by considering the corotational stress rates in the constitutive models. Prager (1960) has proposed a yielding-stationarity criterion, which asserts that for a consistent flow model the vanishing of the stress rate implies the stationarity of yield function. The stationarity of stress invariants for a material undergoing a pure rotation is crucial, since the yield function for isotropic material is defined in terms of these invariants. For any corotational stress rate we have

$$\overset{\circ}{\tau} := \dot{\tau} - \omega\tau + \tau\omega = \mathbf{P}\dot{\tau}_L\mathbf{P}^{\mathrm{T}}, \tag{1}$$

where ω is a certain spin tensor, $\mathbf{P}$ is a corresponding rotation tensor satisfying $\dot{\mathbf{P}}\mathbf{P}^{-1} = \omega$, and $\tau_L = \mathbf{P}^{\mathrm{T}}\tau\mathbf{P}$ is the unrotated Kirchhoff stress. While a superimposed dot denotes the material time derivative, a surmounted circle over the tensor denotes a certain corotational rate under ω. From the last equation it is obvious that the invariants of the unrotated Kirchhoff stress are the same as those of the Kirchhoff stress. Thus, the vanishing of all the corotational stress rates implies the stationary behavior of the stress invariants by Prager's argument [Prager (1960)]. The above discussions indicate that the stress rate must be a corotational type. A further explanation has been made by Lee (1983) for the Jaumann rate, and by Xiao, Bruhns and Meyers (2000) and Bruhns, Xiao and Meyers (2005) for the logarithmic rate.

Through the study by Liu (2004a) it is clear that the employment of the non-corotational stress rates in the constitutive equations leads to the coupling of deviatoric and volumetric parts of the constitutive equations as well as the loss of linearity in an augmented stress space. Conversely, these two-fold advantages of *uncoupling* and *linearity* strongly support the use of corotational stress rates in the constitutive equations. Liu (2004a) has proved that

$$\overset{\circ}{\mathbf{s}} = \dot{\mathbf{s}} - \omega\mathbf{s} + \mathbf{s}\omega, \tag{2}$$

where $\mathbf{s} := \tau - \mathbf{I}_3(\mathrm{tr}\,\tau)/3$ is the deviatoric part of τ, in which tr denotes the trace of the tensor and $\mathbf{I}_3$ is the third-order identity tensor.

Our recent papers demonstrate how powerful of the modern algebraic tools like as the Lie groups can be used to gain a deep insight into the intrinsic structure of the elastoplastic constitutive equations. This paper is originated from a series study of the plastic behavior by viewing plasticity as a dissipatively dynamical system endowed with an on-off switch. As remarked by Liu (2003) the invariant yield condition in stress space renders a natural mathematical frame of plasticity theory from the viewpoints of differentiable manifold and its Lie group transformation, and several results along this way have been published for the Lie symmetry of material models, e.g., Hong and Liu (1999a, 1999b, 2000), Liu and Chang (2004, 2005), Liu and Hong (2001), Liu (2001a, 2003, 2004a, 2004b, 2004c), and Mukherjee and Liu (2003). It is indeed gratifying to notice that a consistent numerical procedure can be established based on the study of Lie symmetry, which can automatically satisfy the consistent condition for the plastic flow models [Liu and Li (2005), Liu (2005, 2006a)].

In addition of the constitutive equations which beyond the small deformations attract much attention as described by Rubinstein and Atluri (1983) and Xiao, Bruhns and Meyers (2006), the special attention has also been paid on the finite rotation effect in the structural mechanics of flexible bodies, including beam, plate, shell, etc. Atluri (1984) has considered finite rotations as direct independent variables in the variational formulations of finitely deformed continua and shells. Han, Rajendran and Atluri (2005) have formulated an effective MLPG approach for solving the nonlinear structural problems of beam with large deformation and rotation. They have shown that the MLPG is more effective than the FEM.

A recent progress to dealing with the finite rotations in beams, plates and shells was also summarized in a special issue of CMES. Lin and Hsiao (2003) have solved the buckling problems of 3-D beams by using the co-rotational formulation. Gotou, Kuwataka, Nishihara and Iwakuma (2003) have introduced the rotational angles associated with the Cartesian coordinates as additional degrees of freedom, where the Euler's angles are used to describe finite rotations. The accuracy of the co-rotational formulation for 3-D Timoshenko's beam is discussed from a theoretical viewpoint by Iura, Suetake and Atluri (2003). Beda (2003) introduced three rotation angles and solved the elastica problem of spatial Euler-Bernoulli beam. Suetake, Iura and Atluri (2003) have derived a symmetric tangent stiffness operator for thick shells undergoing finite rotation. Basar and Kintzel (2003) have developed a finite element model for finite rotation and large strain thin-walled shells. From those papers one can understand that the finite rotations are important in the mechanical analysis of flexible body. Similarly, for the rigid multibody dynamics the finite rotations are also important as shown by Rochinha and Sampaio (2000) and Huston and Liu (2005).

The dissipation of plasticity and the consistency of yielding condition are two main characters of the plastic models during a plastic loading state. How to put these two ingredients, the conservative property of yielding condition and the dissipative nature of plastic deformation, together into a unified framework is still pending in the development of plasticity theory. Even, it is usually asserted that the Hamiltonian formalism can treat only the conservative problem, we are attempting to develop a generalized Hamiltonian formulation of the plastic models to take the dissipative effects into account. In addition a recent paper by Liu (2004d), it seems that there has no attempt in the open literature to study the generalized Hamiltonian structure and the Lie-Poisson bracket formalism of the constitutive models of plasticity. In Section 2 we will give a brief sketch of the generalized Hamiltonian systems for the later requirements.

In this exposition we analyze the constitutive model of perfect elastoplasticity from several different points of view and attempt to achieve a deeper understanding of its underlying structures of Lie algebra properties, Lie group symmetries and dissipative behaviors in different representations.

Although the models of plasticity are nonlinear in stress space, the perfectly elastoplastic model with finite deformation postulated in Section 3 will be proved to admit a superposition principle in Section 4 based on the Lie theory, and to be transformed in Section 5 into the linear systems in the augmented stress spaces with a canonical form in Section 5.1 and a non-canonical form in Section 5.2. In Section 6 we will prove that the plastic equation is a Lie-Poisson system equipped with a structure tensor of the Poisson type and the stress trajectory is a coadjoint orbit under the action of the Lie group $SO(5)$. According to a thermodynamic framework of the general equation for the nonequilibrium reversible-irreversible coupling (GENERIC) we can develop a two-generator formulation of the plastic equation in Section 7, where the dissipation and yielding behaviors are well organized into a single format. Finally, as the conclusions of the present paper we compare the different formulations and address their computational applications in Section 8.

2 Generalized Hamiltonian systems

The classical Hamiltonian mechanics is endowed with an even-dimensional phase space. In practice, there are many mechanical systems whose phase spaces are not canonical. That is, the phase manifold does not admit a cotangent bundle structure on it, but it still has a Poisson bracket equipped with the properties of skew-symmetry, bilinearity, the Leibniz identity and the Jacobi identity. The most famous example is the Euler equations governing the motion of rigid body.

Suppose that P is a manifold. If there is a bracket $\{\bullet,\bullet\}$ defined on the function space $\mathbb{C}(P)$, satisfying the following properties:

$$\text{Skew-symmetry}: \quad \{F,G\} = -\{G,F\}, \tag{3}$$

$$\text{Bilinearity}: \quad \{\lambda F+\mu G,H\} = \lambda\{F,H\} + \mu\{G,H\}, \quad \lambda,\mu \in \mathbb{R}, \tag{4}$$

$$\text{Jacobi identity}: \quad \{F,\{G,H\}\}+\{G,\{H,F\}\} + \{H,\{F,G\}\} = 0, \tag{5}$$

$$\text{Leibniz identity}: \quad \{FG,H\} = F\{G,H\} + \{F,H\}G, \tag{6}$$

then $(P,\{\bullet,\bullet\})$ is a Poisson manifold [Marsden and Ratiu (1994)]. If an observable function $F: P \mapsto \mathbb{R}$ of a dynamical system can be governed by a generalized Hamiltonian function H through

$$\dot{F} = \{F,H\}, \tag{7}$$

then $(P,\{\bullet,\bullet\},H)$ is called a generalized Hamiltonian system.

Let $H: P \mapsto \mathbb{R}$ be a smooth function on P. The generalized Hamiltonian vector field $\mathbf{X}_H$ associated with H is a unique smooth vector field on P, which for every smooth function $F: P \mapsto \mathbb{R}$ satisfies

$$\mathbf{X}_H(F) = \{F,H\}. \tag{8}$$

Instead of the non-degeneracy of the classical Poisson bracket, the bracket defined on the non-canonical Poisson manifold is permitted degenerate.

When P is a finite-dimensional manifold with dimensions n, the local coordinates of P can be assigned as $\mathbf{x} = (x_1,\ldots,x_n)$, and the Poisson bracket on P can be written as

$$\{F,G\} := J_{ij}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}, \tag{9}$$

where $J_{ij}(\mathbf{x})$ is a Poisson tensor. Throughout this paper the Einstein summation convention is adopted for the repeated indices.

Given an $n \times n$ matrix function $J_{ij}(\mathbf{x})$ defined on the open set $P \subset \mathbb{R}^n$, the necessary and sufficient conditions of $J_{ij}(\mathbf{x})$ to be a Poisson tensor are

$$J_{ij} = -J_{ji}, \;\; i,j = 1,\ldots,n, \tag{10}$$

$$J_{i\ell}J_{jk,\ell} + J_{j\ell}J_{ki,\ell} + J_{k\ell}J_{ij,\ell} = 0, \;\; i,j,k = 1,\ldots,n, \tag{11}$$

where $J_{jk,\ell}$ denotes $\partial J_{jk}/\partial x_\ell$.

For all smooth function $H : P \mapsto \mathbb{R}$ defined on P, the bundle mapping $B : T^*P \mapsto TP$ is denoted by $B(dH(\mathbf{x})) = \mathbf{X}_H\,|_{\mathbf{x}}$. TP and T^*P are, respectively, the tangent and cotangent bundles on the Poisson manifold P. The rank of the Poisson bracket at a point $\mathbf{x} \in P$ is defined as the rank of the linear mapping $B\,|_{\mathbf{x}}$: $T^*_{\mathbf{x}}P \mapsto T_{\mathbf{x}}P$. A point $\mathbf{x}$ on the Poisson manifold P is called a regular point, if the ranks for all points in the neighborhood of $\mathbf{x} \in P$ are the same; otherwise, $\mathbf{x}$ is a singular point. The rank of B at $\mathbf{x} \in P$ and the rank of the Poisson tensor $\mathbf{J}$ at point $\mathbf{x}$ are the same. Because of the skew-symmetry of $\mathbf{J}(\mathbf{x})$ the rank is always even.

Suppose that $C : P \mapsto \mathbb{R}$ is a non-constant smooth function on P. If $\{C,F\} = 0$ for all smooth function $F : P \mapsto \mathbb{R}$, then C is a Casimir function on P. If the rank of the Poisson tensor $\mathbf{J}$ at a regular point $\mathbf{x}_0$ is $n-m$, $m > 0$, then there are m functionally independent Casimir functions defined in the neighborhood of the point $\mathbf{x}_0$.

When $J_{ij}(\mathbf{x})$ is a linear function of $\mathbf{x}$, the bracket (9) is called a Lie-Poisson bracket, and correspondingly, Eq. (7) is a Lie-Poisson system written as

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x})\nabla H(\mathbf{x}), \tag{12}$$

where the gradient operator ∇ denotes the derivative with respect to $\mathbf{x}$; and we usually write such $J_{ij}(\mathbf{x})$ to be

$$J_{ij} = C^k_{ij} x_k, \tag{13}$$

where $C^k_{ij} = -C^k_{ji}$ and the Jacobi identity (11) becomes

$$C^\ell_{ij}C^r_{\ell k} + C^\ell_{jk}C^r_{\ell i} + C^\ell_{ki}C^r_{\ell j} = 0. \tag{14}$$

It is known that for this case the underlying space can be given a Lie algebra structure with the structure constants C^k_{ij} in a suitable basis [Marsden and Ratiu (1994)].

The Lie-Poisson system is naturally formulated in a dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$. The solutions of the system are coadjoint orbits of a certain Lie group, constrained on a nonlinear submanifold of $\mathcal{G}^*$ known as a symplectic foliation. In recent years, the applications that fit into the Lie-Poisson formalism and the extensions to dissipation are numermous, for example, Bloch, Krishnaprasad, Marsden and Ratiu (1996) and Pelino and Pasini (2001). Also, for its important applications in some mechanical systems there are numerical integrators developed to preserve the Lie-Poisson structure; see, for example, Ge and Marsden (1988), Channell and Scovel (1991), McLachlan (1993), Austin, Krishnaprasad and Wang (1993), Li and Qin (1995), and Engø and Faltinsen (2001).

3 Constitutive model and switch

3.1 Constitutive model

The constitutive law of elastoplasticity of solid materials proposed by Prandtl (1924) and Reuss (1930) can be re-postulated [Hong and Liu (1999a)] and enlarged to take the large deformation effects into account as in the following system:[1]

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p, \tag{15}$$

$$\overset{\circ}{\mathbf{s}} = 2G\mathbf{D}^e, \tag{16}$$

$$\dot{\lambda}\mathbf{s} = 2\tau_y\mathbf{D}^p, \tag{17}$$

$$\|\mathbf{s}\| \le \sqrt{2}\tau_y, \tag{18}$$

$$\dot{\lambda} \ge 0, \tag{19}$$

$$\|\mathbf{s}\|\dot{\lambda} = \sqrt{2}\tau_y\dot{\lambda}, \tag{20}$$

in which the two material constants, namely the shear modulus G and the shear yield strength τ_y, are determined experimentally and both are assumed to be positive. The bold-faced symbols $\mathbf{D}$, $\mathbf{D}^e$, $\mathbf{D}^p$ and $\mathbf{s}$ stand for the deviatoric parts of the deformation rate, elastic deformation rate, plastic deformation rate, and Kirchhoff's stress, respectively, all being symmetric and traceless tensors,

[1] The volumetric part of the Prandtl-Reuss law is linearly elastic and is thus excluded from the present study in order to focus on the more interesting elastic-plastic behavior of the deviatoric part.

whereas λ is a scalar, called the equivalent shear plastic strain.

As usual, $\|\mathbf{s}\| := \sqrt{\mathbf{s}\cdot\mathbf{s}}$ is the Frobenius norm of a tensor **s** and the dot between two tensors denotes their inner product, and a surmounted circle "$\circ$" on **s** represents a corotational rate of **s** using a certain spin tensor ω as that shown in Eq. (2). Some objective corotational rates usually employed in the constitutive equations are summarized in Table 1. Inserting these ω's into Eq. (2) we thus obtain different corotational stress rates. Simultaneously, employing the different corotational stress rates for the hypoelasticity in Eq. (16) will result in different large deformation models of the elastic-perfect plasticity.

Table 1: Some objective corotational rates

Objective corotational rates	ω
1. Jaumann (J)	$\mathbf{W}$
2. Green-Naghdi (GN)	$\mathbf{\Omega}$
3. Sowerby-Chu (SC)	$\mathbf{\Omega}_E$
4. Xiao-Bruhns-Meyers (XBM)	$\mathbf{\Omega}^{\log}$

The notations used here are defined as follows: $\mathbf{L} := \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the velocity gradient tensor, where **F** is the deformation gradient tensor; **W** is the skew-symmetric part of **L**; $\mathbf{\Omega} := \dot{\mathbf{R}}\mathbf{R}^{-1}$ is the rate of rotation, where **R** is the orthogonal tensor in the polar decomposition of **F**, i.e.,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}; \tag{21}$$

$\mathbf{\Omega}_E := \dot{\mathbf{R}}_E\mathbf{R}_E^{\mathrm{T}}$ is known as the Eulerian spin tensor, where $\mathbf{R}_E$ is the diagonal transformation of **V**, that is,

$$\mathbf{V} = \mathbf{R}_E\boldsymbol{\lambda}\mathbf{R}_E^{\mathrm{T}}, \tag{22}$$

and $\boldsymbol{\lambda}$ is a diagonal tensor containing the eigenvalues of **V**. The logarithmic spin $\mathbf{\Omega}^{\log}$ was introduced by Xiao, Bruhns and Meyers (1997a, 1997b, 2006), which satisfies

$$\overset{\circ}{(\ln\mathbf{V})}{}^{\log} := (\ln\mathbf{V})^{\cdot} + (\ln\mathbf{V})\mathbf{\Omega}^{\log} - \mathbf{\Omega}^{\log}\ln\mathbf{V} = \frac{1}{2}[\mathbf{L}+\mathbf{L}^{\mathrm{T}}]. \tag{23}$$

The hypoelasticity based upon the logarithmic stress rate has been proposed by Xiao, Bruhns and Meyers (1997a), which is exactly integrable [Xiao, Bruhns and Meyers (1999)] and does not exhibit shear oscillation [Bruhns, Xiao and Meyers (2001), Liu and Hong (1999)]. Recently, Zhou and Tamma (2003) further demonstrate that the logarithmic stress rate hypoelasticity model also satisfies the Clausius-Duhem inequality and the models by using other corotational stress rates do not.

3.2 Switching between elastic and plastic phases

From the constitutive model presented in Eqs. (15)-(20), through some analyses we have the following switching criteria:

$$\dot{\lambda} = 0 \ \text{ if } \ \|\mathbf{s}\| < \sqrt{2}\tau_y \ \text{ or } \ \mathbf{s}\cdot\mathbf{D} \le 0, \tag{24}$$

$$\dot{\lambda} = \frac{1}{\tau_y}\mathbf{s}\cdot\mathbf{D} > 0 \ \text{ if } \ \|\mathbf{s}\| = \sqrt{2}\tau_y \ \text{ and } \ \mathbf{s}\cdot\mathbf{D} > 0, \tag{25}$$

as well as a two-phase nonlinear system:

$$\dot{\mathbf{s}} - \omega\mathbf{s} + \mathbf{s}\omega = 2G\mathbf{D} \quad \text{if } \ \|\mathbf{s}\| < \sqrt{2}\tau_y \ \text{ or } \ \mathbf{s}\cdot\mathbf{D} \le 0, \tag{26}$$

$$\dot{\mathbf{s}} - \omega\mathbf{s} + \mathbf{s}\omega = 2G\mathbf{D} - \frac{G\mathbf{s}\cdot\mathbf{D}}{\tau_y^2}\mathbf{s} \quad \text{if } \ \|\mathbf{s}\| = \sqrt{2}\tau_y \ \text{ and } \ \mathbf{s}\cdot\mathbf{D} > 0, \tag{27}$$

of which the former is a linearly elastic equation, and the latter is a nonlinearly plastic equation. From Eq. (26) it can be seen that the elastic equation is rather simple, and therefore we concentrate below on the study of plastic equation (27).

4 Lie algebra of plastic equation

In this section we derive a Lie algebra of the above plastic equation. For this purpose let us consider the problem in a suitable vector space by introducing a five-dimensional stress vector:

$$\mathbf{Q} = \begin{bmatrix} a_1 s^{11} + a_2 s^{22} \\ a_3 s^{11} + a_4 s^{22} \\ s^{23} \\ s^{13} \\ s^{12} \end{bmatrix}, \tag{28}$$

where

$$a_1 := \sin(\theta + \frac{\pi}{3}), \ \ a_2 := \sin\theta,$$

$$a_3 := \cos(\theta + \frac{\pi}{3}), \ \ a_4 := \cos\theta,$$

with θ being any real number,[2] and a five-dimensional deformation rate vector:[3]

$$\dot{\mathbf{q}} := \begin{bmatrix} a_1D_{11} + a_2D_{22} \\ a_3D_{11} + a_4D_{22} \\ D_{23} \\ D_{13} \\ D_{12} \end{bmatrix}. \tag{29}$$

As shown by Liu (2003), upon using

$$\|\mathbf{s}\|^2 = 2\|\mathbf{Q}\|^2, \ \ \mathbf{s}\cdot\mathbf{D} = 2\mathbf{Q}\cdot\dot{\mathbf{q}}, \tag{30}$$

we can rewrite Eq. (27) to the following form:

$$\dot{\mathbf{Q}} = \mathbf{A}_s^s\mathbf{Q} - \frac{k_e}{Q_0^2}\mathbf{Q}\cdot\dot{\mathbf{q}}\mathbf{Q} + k_e\dot{\mathbf{q}}, \tag{31}$$

where

$$\mathbf{A}_s^s := \begin{bmatrix} 0 & 0 & 2a_2\omega_{23} & 2a_1\omega_{13} & 2(a_1-a_2)\omega_{12} \\ 0 & 0 & 2a_4\omega_{23} & 2a_3\omega_{13} & 2(a_3-a_4)\omega_{12} \\ -2a_2\omega_{23} & -2a_4\omega_{23} & 0 & -\omega_{12} & -\omega_{13} \\ -2a_1\omega_{13} & -2a_3\omega_{13} & \omega_{12} & 0 & -\omega_{23} \\ 2(a_2-a_1)\omega_{12} & 2(a_4-a_3)\omega_{12} & \omega_{13} & \omega_{23} & 0 \end{bmatrix} \tag{32}$$

is a skew-symmetric matrix and

$$k_e = 2G, \ \ Q_0 = \tau_y. \tag{33}$$

In order to give a Lie algebra formulation of the plastic equation (31), we write it with the following componential form:

$$\dot{Q}_i = (A_s^s)_{ij}Q_j + k_e\left[\delta_{ij} - \frac{Q_iQ_j}{Q_0^2}\right]\dot{q}_j, \tag{34}$$

where δ_{ij} is the Kronecker delta symbol. This equation can be viewed as an affine nonlinear system with $\dot{q}_j$ and ω_{ij} as inputs and Q_j as outputs, which means that the above equation is linear in both $\dot{q}_j$ and ω_{ij} but nonlinear in Q_j.

[2] See Hong and Liu (1997).

[3] Recall that both **s** and **D** are deviatoric tensors and thus the plastic equation (27) has only five independent components.

For the nonlinear dynamical system:

$$\frac{dx^\mu(t)}{dt} = \eta^\mu(x^1,\ldots,x^n,t), \ \ 1 \le \mu \le n, \tag{35}$$

if the general solution $\mathbf{x}(t) = (x^1(t),\ldots,x^n(t))^{\mathrm{T}}$ can be expressed as a function of m particular solutions $x^1(t),\ldots,x^m(t)$ and n integration constants $c_1,\ldots,c_n$ such that

$$\mathbf{x}(t) = \mathbf{F}(x^1,\ldots,x^m,c_1,\ldots,c_n), \tag{36}$$

then Eq. (35) is said to admit a superposition principle; see, e.g. Hong and Liu (1997) and Cariñena, Grabowski and Ramos (2001).

Lie has proved that Eq. (35) admits a superposition principle iff it can be written as

$$\frac{d\mathbf{x}}{dt} = Z_i(t)\boldsymbol{\xi}_i(\mathbf{x}), \tag{37}$$

and its vector fields

$$Y_i = \xi_i^\mu(\mathbf{x})\frac{\partial}{\partial x^\mu}, \ \ i = 1,\ldots,s, \tag{38}$$

constitute a finite-dimensional Lie algebra, the dimension r of which satisfies $s \le r \le mn$.

Theorem 1. The plastic equation (31) admits a superposition principle.

Proof. The five vector fields of Eq. (34) corresponding to the five inputs $k_e\dot{q}_j,\ j = 1,\ldots,5$ are

$$\mathbf{g}_j = \delta_{ij}\mathbf{e}_i - \frac{Q_iQ_j}{Q_0^2}\mathbf{e}_i, \ \ 1 \le j \le 5, \tag{39}$$

where $\mathbf{e}_i, i = 1,\ldots,5$ are unit bases. The componential forms of $\mathbf{g}_j, j = 1,\ldots,5$ are

$$\mathbf{g}_1 = \begin{bmatrix} 1-\frac{Q_1^2}{Q_0^2} \\ -\frac{Q_1Q_2}{Q_0^2} \\ -\frac{Q_1Q_3}{Q_0^2} \\ -\frac{Q_1Q_4}{Q_0^2} \\ -\frac{Q_1Q_5}{Q_0^2} \end{bmatrix}, \ \ \mathbf{g}_2 = \begin{bmatrix} -\frac{Q_1Q_2}{Q_0^2} \\ 1-\frac{Q_2^2}{Q_0^2} \\ -\frac{Q_2Q_3}{Q_0^2} \\ -\frac{Q_2Q_4}{Q_0^2} \\ -\frac{Q_2Q_5}{Q_0^2} \end{bmatrix}, \tag{40}$$

$$\mathbf{g}_3 = \begin{bmatrix} -\frac{Q_1Q_3}{Q_0^2} \\ -\frac{Q_2Q_3}{Q_0^2} \\ 1-\frac{Q_3^2}{Q_0^2} \\ -\frac{Q_3Q_4}{Q_0^2} \\ -\frac{Q_3Q_5}{Q_0^2} \end{bmatrix}, \quad \mathbf{g}_4 = \begin{bmatrix} -\frac{Q_1Q_4}{Q_0^2} \\ -\frac{Q_2Q_4}{Q_0^2} \\ -\frac{Q_3Q_4}{Q_0^2} \\ 1-\frac{Q_4^2}{Q_0^2} \\ -\frac{Q_4Q_5}{Q_0^2} \end{bmatrix}, \tag{41}$$

$$\mathbf{g}_5 = \begin{bmatrix} -\frac{Q_1Q_5}{Q_0^2} \\ -\frac{Q_2Q_5}{Q_0^2} \\ -\frac{Q_3Q_5}{Q_0^2} \\ -\frac{Q_4Q_5}{Q_0^2} \\ 1-\frac{Q_5^2}{Q_0^2} \end{bmatrix}. \tag{42}$$

Similarly, the three vector fields generated from ω_{12}, ω_{13} and ω_{23} are, respectively,

$$\omega_1 = \begin{bmatrix} 2(a_1-a_2)Q_5 \\ 2(a_3-a_4)Q_5 \\ -Q_4 \\ Q_3 \\ 2(a_2-a_1)Q_1+2(a_4-a_3)Q_2 \end{bmatrix}, \tag{43}$$

$$\omega_2 = \begin{bmatrix} 2a_1Q_4 \\ 2a_3Q_4 \\ -Q_5 \\ -2a_1Q_1-2a_3Q_2 \\ Q_3 \end{bmatrix}, \tag{44}$$

$$\omega_3 = \begin{bmatrix} 2a_2Q_3 \\ 2a_4Q_3 \\ -2a_2Q_1-2a_4Q_2 \\ -Q_5 \\ Q_4 \end{bmatrix}. \tag{45}$$

The Lie bracket of $\mathbf{g}_\alpha$ and $\mathbf{g}_\beta$ is given by

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \frac{\partial \mathbf{g}_\beta}{\partial \mathbf{Q}}\mathbf{g}_\alpha - \frac{\partial \mathbf{g}_\alpha}{\partial \mathbf{Q}}\mathbf{g}_\beta. \tag{46}$$

From Eq. (39) it follows that

$$\frac{\partial g_\alpha^i}{\partial Q_j} = -\frac{Q_i\delta_{\alpha j}}{Q_0^2} - \frac{Q_\alpha\delta_{ij}}{Q_0^2}, \tag{47}$$

where g_α^i is the ith component of $\mathbf{g}_\alpha$. By using the above equation we can prove that

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \frac{Q_\alpha}{Q_0^2}\mathbf{g}_\beta - \frac{Q_\beta}{Q_0^2}\mathbf{g}_\alpha. \tag{48}$$

Inserting Eq. (39) for $\mathbf{g}$, the above right-hand side can be further reduced to

$$\frac{Q_\alpha}{Q_0^2}\mathbf{g}_\beta - \frac{Q_\beta}{Q_0^2}\mathbf{g}_\alpha = \frac{Q_j}{Q_0^2}(\delta_{j\alpha}\delta_{i\beta} - \delta_{j\beta}\delta_{i\alpha})\mathbf{e}_i. \tag{49}$$

Let us consider the 5-dimensional permutation symbol $\varepsilon_{i_1\ldots i_5}$, which is zero if any two indices of $\{i_1,\ldots,i_5\}$ are equal, +1 if $\{i_1,\ldots,i_5\}$ is an even permutation, and -1 if $\{i_1,\ldots,i_5\}$ is an odd permutation. Reminding that

$$\varepsilon_{i_1i_2i_3ij}\varepsilon_{i_1i_2i_3\beta\alpha} = \delta_{j\alpha}\delta_{i\beta} - \delta_{j\beta}\delta_{i\alpha}, \tag{50}$$

from Eqs. (48) and (49) it follows that

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \frac{-1}{Q_0^2}\varepsilon_{i_1i_2i_3\alpha\beta}\varepsilon_{i_1i_2i_3ij}Q_j\mathbf{e}_i. \tag{51}$$

This prompts us to consider the vector fields

$$\mathbf{w}_{i_1i_2i_3} := \varepsilon_{i_1i_2i_3ij}Q_j\mathbf{e}_i, \tag{52}$$

and there are totally ten linearly independent vector fields of $\mathbf{w}$:

$$\mathbf{w}_{123} = \varepsilon_{12345}Q_5\mathbf{e}_4 + \varepsilon_{12354}Q_4\mathbf{e}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Q_5 \\ -Q_4 \end{bmatrix}, \tag{53}$$

$$\mathbf{w}_{124} = \varepsilon_{12435}Q_5\mathbf{e}_3 + \varepsilon_{12453}Q_3\mathbf{e}_5 = \begin{bmatrix} 0 \\ 0 \\ -Q_5 \\ 0 \\ Q_3 \end{bmatrix}, \tag{54}$$

$$\mathbf{w}_{125} = \varepsilon_{12534}Q_4\mathbf{e}_3 + \varepsilon_{12543}Q_3\mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ Q_4 \\ -Q_3 \\ 0 \end{bmatrix},$$

(55)

$$\mathbf{w}_{134} = \varepsilon_{13425}Q_5\mathbf{e}_2 + \varepsilon_{13452}Q_2\mathbf{e}_5 = \begin{bmatrix} 0 \\ Q_5 \\ 0 \\ 0 \\ -Q_2 \end{bmatrix}, \tag{56}$$

$$\mathbf{w}_{135} = \varepsilon_{13524}Q_4\mathbf{e}_2 + \varepsilon_{13542}Q_2\mathbf{e}_4 = \begin{bmatrix} 0 \\ -Q_4 \\ 0 \\ Q_2 \\ 0 \end{bmatrix}, \tag{57}$$

$$\mathbf{w}_{145} = \varepsilon_{14523}Q_3\mathbf{e}_2 + \varepsilon_{14532}Q_2\mathbf{e}_3 = \begin{bmatrix} 0 \\ Q_3 \\ -Q_2 \\ 0 \\ 0 \end{bmatrix}, \tag{58}$$

$$\mathbf{w}_{234} = \varepsilon_{23415}Q_5\mathbf{e}_1 + \varepsilon_{23451}Q_1\mathbf{e}_5 = \begin{bmatrix} -Q_5 \\ 0 \\ 0 \\ 0 \\ Q_1 \end{bmatrix}, \tag{59}$$

$$\mathbf{w}_{235} = \varepsilon_{23514}Q_4\mathbf{e}_1 + \varepsilon_{23541}Q_1\mathbf{e}_4 = \begin{bmatrix} Q_4 \\ 0 \\ 0 \\ -Q_1 \\ 0 \end{bmatrix}, \tag{60}$$

$$\mathbf{w}_{245} = \varepsilon_{24513}Q_3\mathbf{e}_1 + \varepsilon_{24531}Q_1\mathbf{e}_3 = \begin{bmatrix} -Q_3 \\ 0 \\ Q_1 \\ 0 \\ 0 \end{bmatrix}, \tag{61}$$

$$\mathbf{w}_{345} = \varepsilon_{34512}Q_2\mathbf{e}_1 + \varepsilon_{34521}Q_1\mathbf{e}_2 = \begin{bmatrix} Q_2 \\ -Q_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{62}$$

From Eqs. (51) and (52) it follows that

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \frac{-1}{Q_0^2}\varepsilon_{i_1i_2i_3\alpha\beta}\mathbf{w}_{i_1i_2i_3}, \tag{63}$$

and that

$$\frac{\partial w^i_{i_1i_2i_3}}{\partial Q_j} = \varepsilon_{i_1i_2i_3ij}, \tag{64}$$

where $w^i_{i_1i_2i_3}$ is the ith component of $\mathbf{w}_{i_1i_2i_3}$. Hence, by Eqs. (47), (64), (39) and (52) through some calculations we find that

$$[\mathbf{w}_{i_1i_2i_3}, \mathbf{g}_\alpha] = \varepsilon_{i_1i_2i_3\alpha\beta}\mathbf{g}_\beta, \tag{65}$$

$$[\mathbf{w}_{i_1i_2i_3}, \mathbf{w}_{j_1j_2j_3}] = \varepsilon_{j_1j_2j_3im}w^m_{i_1i_2i_3}\mathbf{e}_i - \varepsilon_{i_1i_2i_3im}w^m_{j_1j_2j_3}\mathbf{e}_i. \tag{66}$$

Therefore, the five vector fields of Eq. (39) and the ten vector fields $\mathbf{w}$ in Eq. (52) constitute a finite-dimensional Lie algebra, which is indeed the Lie algebra $so(5,1)$ of the $5+1$-dimensional proper orthochronous Lorentz group $SO_o(5,1)$ with dimensions fifteen. As a consequence, according to Lie's theory we have proved that the plastic equation (31) admits a superposition principle. □

By inspection, the three vector fields defined in Eqs. (43)-(45) can be expanded in terms of the bases defined in Eqs. (53)-(62):

$$\omega_1 = 2(a_2-a_1)\mathbf{w}_{234} - 2(a_4-a_3)\mathbf{w}_{134} - \mathbf{w}_{125}, \tag{67}$$

$$\omega_2 = 2a_1\mathbf{w}_{235} - 2a_3\mathbf{w}_{135} + \mathbf{w}_{124}, \tag{68}$$

$$\omega_3 = 2a_4\mathbf{w}_{145} - 2a_2\mathbf{w}_{245} - \mathbf{w}_{123}. \tag{69}$$

Moreover, from Eqs. (67)-(69) and (66) we can obtain

$$\begin{aligned}[\omega_1, \omega_2] &= [2(a_2-a_1)\mathbf{w}_{234} - 2(a_4-a_3)\mathbf{w}_{134} \\ &\quad - \mathbf{w}_{125}, 2a_1\mathbf{w}_{235} - 2a_3\mathbf{w}_{135} + \mathbf{w}_{124}] \\ &= [4a_1(a_2-a_1) + 4a_3(a_4-a_3) + 1]\mathbf{w}_{123} \\ &\quad + 2a_4\mathbf{w}_{145} - 2a_2\mathbf{w}_{245} \\ &= \omega_3,\end{aligned}$$

(70)

$$
\begin{aligned}
[\omega_1, \omega_3] &= [2(a_2-a_1)\mathbf{w}_{234} - 2(a_4-a_3)\mathbf{w}_{134} \\
&\quad - \mathbf{w}_{125}, 2a_4\mathbf{w}_{145} - 2a_2\mathbf{w}_{245} - \mathbf{w}_{123}] \\
&= [1-4a_2(a_2-a_1) - 4a_4(a_4-a_3)]\mathbf{w}_{124} \\
&\quad + 2a_3\mathbf{w}_{135} - 2a_1\mathbf{w}_{235} \\
&= -\omega_2,
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
[\omega_2, \omega_3] &= [2a_1\mathbf{w}_{235} - 2a_3\mathbf{w}_{135} \\
&\quad + \mathbf{w}_{124}, 2a_4\mathbf{w}_{145} - 2a_2\mathbf{w}_{245} - \mathbf{w}_{123}] \\
&= (1-4a_1a_2 - 4a_3a_4)\mathbf{w}_{125} \\
&\quad + 2(a_3-a_4)\mathbf{w}_{134} + 2(a_2-a_1)\mathbf{w}_{234} \\
&= \omega_1.
\end{aligned} \tag{72}
$$

These show that $[\omega_1, \omega_2], [\omega_1, \omega_3], [\omega_2, \omega_3] \in so(5,1)$, and that $\{\omega_1, \omega_2, \omega_3\}$ forms a closed subalgebra as a rotation Lie algebra in the three-dimensional subspace. However, $\mathbf{g}_i, i = 1, \ldots, 5$ and $\omega_i, i = 1,2,3$ do not constitute a Lie algebra, because the bracket given in Eq. (63) is not an element in the space spanned by $\mathbf{g}_i, i = 1, \ldots, 5$ and $\omega_i, i = 1,2,3$.

Hong and Liu (1997) have derived the superposition formula for Eq. (27) without considering the large deformation. After transforming the plastic equation into a linear system in the next section we can derive a similar superposition formula for $\mathbf{s}$ as that shown by Eq. (68) in the paper by Hong and Liu (1997) for the small deformation perfect elastoplasticity.

5 Two linear systems

The plastic equation (31) admitting a superposition principle gives us a clue to linearize it in this section.

5.1 Canonical form

Let us introduce

$$
\mathbf{X} = \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix} := \frac{X^0}{Q_0} \begin{bmatrix} \mathbf{Q} \\ Q_0 \end{bmatrix}, \tag{73}
$$

and consider the integrating factor

$$
X^0 := \exp\left(\frac{k_e q_0}{Q_0}\right), \tag{74}
$$

where

$$
\dot{q}_0 := \frac{1}{Q_0} \mathbf{Q} \cdot \dot{\mathbf{q}}. \tag{75}
$$

So, multiplying Eq. (31) by X^0/Q_0 we obtain

$$
\dot{\mathbf{X}}^s = \mathbf{A}^s_s \mathbf{X}^s + X^0 \mathbf{A}^s_0, \tag{76}
$$

and at the same time, from Eqs. (74) and (75) it follows that

$$
\dot{X}^0 = \mathbf{A}^s_0 \cdot \mathbf{X}^s, \tag{77}
$$

where

$$
\mathbf{A}^s_0 = \frac{k_e}{Q_0} \dot{\mathbf{q}}. \tag{78}
$$

Then, Eqs. (76) and (77) are combined together into a single linear system:

$$
\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \tag{79}
$$

where

$$
\mathbf{A} = \begin{bmatrix} \mathbf{A}^s_s & \mathbf{A}^s_0 \\ (\mathbf{A}^s_0)^{\mathrm{T}} & 0 \end{bmatrix}. \tag{80}
$$

Eq. (79) is a linear $5+1$-dimensional representation of the constitutive model (15)-(20), in which $\mathbf{X}$ and $\mathbf{A}$ are the augmented state vector and the control input matrix, respectively. $\mathbf{X}$ satisfies the cone condition:

$$
\mathbf{X}^{\mathrm{T}} \mathbf{g} \mathbf{X} = 0 \iff \|\mathbf{Q}\| = Q_0 \iff \|\mathbf{s}\| = \tau_y \tag{81}
$$

with

$$
\mathbf{g} = \begin{bmatrix} \mathbf{I}_5 & \mathbf{0}_{5\times 1} \\ \mathbf{0}_{1\times 5} & -1 \end{bmatrix} \tag{82}
$$

a metric tensor of the Minkowski space $\mathbb{M}^{5+1}$. $\mathbf{I}_5$ is the identity tensor of order 5.

$\mathbf{A}$ satisfies the Lie algebra condition:

$$
\mathbf{A}^{\mathrm{T}} \mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}. \tag{83}
$$

The set of all $(5+1)\times(5+1)$ Lorentzian matrices is denoted by $so(5,1)$. The corresponding solution matrix $\mathbf{G}$ of

$$\dot{\mathbf{G}}(t)=\mathbf{A}(t)\mathbf{G}(t),\ \ \mathbf{G}(0)=\mathbf{I}_{5+1}, \tag{84}$$

satisfying the following group properties [Liu (2001b)]:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G}=\mathbf{g}, \tag{85}$$

$$\det\mathbf{G}=1, \tag{86}$$

$$G_0^0\geq 1, \tag{87}$$

is a proper orthochronous Lorentz group left acting on the Minkowski space $\mathbb{M}^{5+1}$, and is denoted by $SO_o(5,1)$. Here, det is the shorthand of the determinant, and G_0^0 is the 00th component of $\mathbf{G}$.

5.2 *Non-canonical form*

In this section we provide the following results about the nilpotentlization of the time-varying linear system (79).

Theorem 2. Corresponding to the linear system (79) with $\mathbf{A}$ satisfying Eq. (83), there exists a linear system

$$\dot{\mathbf{Y}}=\mathbf{B}\mathbf{Y}, \tag{88}$$

where $\mathbf{B}$ is a zero trace nilpotent matrix function satisfying

$$\operatorname{tr}\mathbf{B}=0,\ \ \mathbf{B}^2=\mathbf{0}. \tag{89}$$

Proof. The integral of Eq. (76) gives

$$\mathbf{X}^s(t)=\mathbf{R}(t)\mathbf{X}^s(0)+\int_0^t\mathbf{R}(t)\mathbf{R}^{\mathrm{T}}(\xi)\mathbf{A}_0^s(\xi)X^0(\xi)d\xi, \tag{90}$$

where $\mathbf{R}$, satisfying

$$\dot{\mathbf{R}}=\mathbf{A}_s^s\mathbf{R},\ \ \mathbf{R}(0)=\mathbf{I}_5, \tag{91}$$

is an element of $SO(5)$, since $\mathbf{A}_s^s$ as shown in Eq. (32) is skew-symmetric.

Substituting Eq. (90) into Eq. (77) we have

$$\dot{X}^0(t)=\mathbf{U}^{\mathrm{T}}(t)\mathbf{X}^s(0)+\int_0^t\mathbf{U}^{\mathrm{T}}(t)\mathbf{U}(\xi)X^0(\xi)d\xi, \tag{92}$$

where

$$\mathbf{U}:=\mathbf{R}^{\mathrm{T}}\mathbf{A}_0^s. \tag{93}$$

Integrating Eq. (92) we obtain

$$X^0(t)=X^0(0)+\mathbf{V}^{\mathrm{T}}(t)\mathbf{X}^s(0)+\int_0^t[\mathbf{V}^{\mathrm{T}}(t)-\mathbf{V}^{\mathrm{T}}(\xi)]\mathbf{U}(\xi)X^0(\xi)d\xi, \tag{94}$$

where

$$\mathbf{V}(t):=\int_0^t\mathbf{U}(\xi)d\xi. \tag{95}$$

Left multiplying Eq. (94) by $[\mathbf{U}^{\mathrm{T}}\ \ \mathbf{V}^{\mathrm{T}}\mathbf{U}]^{\mathrm{T}}$ we obtain a six-dimensional vector integral equation:

$$\begin{bmatrix}\mathbf{U}\\ \mathbf{V}^{\mathrm{T}}\mathbf{U}\end{bmatrix}X^0=\begin{bmatrix}\mathbf{U}\\ \mathbf{V}^{\mathrm{T}}\mathbf{U}\end{bmatrix}\int_0^t\begin{bmatrix}\mathbf{V}^{\mathrm{T}}(t) & -1\end{bmatrix}\begin{bmatrix}\mathbf{U}(\xi)\\ \mathbf{V}^{\mathrm{T}}(\xi)\mathbf{U}(\xi)\end{bmatrix}X^0(\xi)d\xi+\begin{bmatrix}\mathbf{U}X^0(0)+\mathbf{U}\mathbf{V}^{\mathrm{T}}\mathbf{X}^s(0)\\ \mathbf{V}^{\mathrm{T}}\mathbf{U}X^0(0)+\mathbf{V}^{\mathrm{T}}\mathbf{U}\mathbf{V}^{\mathrm{T}}\mathbf{X}^s(0)\end{bmatrix}.$$

Then, introducing the following dyadic tensor:

$$\mathbf{B}:=\begin{bmatrix}\mathbf{U}\\ \mathbf{V}^{\mathrm{T}}\mathbf{U}\end{bmatrix}\begin{bmatrix}\mathbf{V}^{\mathrm{T}} & -1\end{bmatrix}=\begin{bmatrix}\mathbf{U}\mathbf{V}^{\mathrm{T}} & -\mathbf{U}\\ \mathbf{V}^{\mathrm{T}}\mathbf{U}\mathbf{V}^{\mathrm{T}} & -\mathbf{V}^{\mathrm{T}}\mathbf{U}\end{bmatrix} \tag{96}$$

and a new variable:

$$\mathbf{Y}(t)=\begin{bmatrix}\mathbf{Y}^s(t)\\ Y^0(t)\end{bmatrix}:=\int_0^t\begin{bmatrix}\mathbf{U}(\xi)\\ \mathbf{V}^{\mathrm{T}}(\xi)\mathbf{U}(\xi)\end{bmatrix}X^0(\xi)d\xi+\begin{bmatrix}\mathbf{X}^s(0)\\ -X^0(0)\end{bmatrix}, \tag{97}$$

we obtain a linear differential system as that given by Eq. (88), where $\mathbf{B}$ can be proved to satisfy Eq. (89) by direct calculations. □

In terms of $\mathbf{U}$ defined in Eq. (93), $\mathbf{X}^s$ in Eq. (90) can be written as

$$\mathbf{X}^s(t)=\mathbf{R}(t)\left[\mathbf{X}^s(0)+\int_0^t\mathbf{U}(\xi)X^0(\xi)d\xi\right], \tag{98}$$

which upon comparing with $\mathbf{Y}^s$ defined in Eq. (97) readily gives

$$\mathbf{X}^s=\mathbf{R}\mathbf{Y}^s. \tag{99}$$

On the other hand, differentiating Eq. (97) we get, by taking its last row,

$$\dot{Y}^0 = \mathbf{V}^{\mathrm{T}}\mathbf{U}X^0. \tag{100}$$

The term $\dot{Y}^0$ as shown in the last row of Eq. (88) with its $\mathbf{B}$ defined by Eq. (96) is equal to

$$\dot{Y}^0 = \mathbf{V}^{\mathrm{T}}\mathbf{U}\mathbf{V}^{\mathrm{T}}\mathbf{Y}^s - \mathbf{V}^{\mathrm{T}}\mathbf{U}Y^0. \tag{101}$$

From the above two equations we obtain

$$X^0 = \mathbf{V}^{\mathrm{T}}\mathbf{Y}^s - Y^0. \tag{102}$$

In summary, we have the relation between $\mathbf{X}$ and $\mathbf{Y}$ as follows:

$$\begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix} \begin{bmatrix} \mathbf{Y}^s \\ Y^0 \end{bmatrix}, \tag{103}$$

or the inverse relation as follows:

$$\begin{bmatrix} \mathbf{Y}^s \\ Y^0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{\mathrm{T}} & \mathbf{0}_{5\times 1} \\ \mathbf{V}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} & -1 \end{bmatrix} \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix}. \tag{104}$$

In addition Eq. (89), $\mathbf{B}$ possesses the following interesting properties.

Theorem 3. $\mathbf{B}$ defined by Eq. (96) satisfies

$$\mathbf{B}^{\mathrm{T}}\mathbf{h} + \mathbf{h}\mathbf{B} = \begin{bmatrix} \mathbf{V}\mathbf{U}^{\mathrm{T}} + \mathbf{U}\mathbf{V}^{\mathrm{T}} & -\mathbf{U} \\ -\mathbf{U}^{\mathrm{T}} & 0 \end{bmatrix}, \tag{105}$$

where

$$\mathbf{h} := \begin{bmatrix} \mathbf{I}_5 - \mathbf{V}\mathbf{V}^{\mathrm{T}} & \mathbf{V} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix} \tag{106}$$

is an indefinite matrix function. Moreover, we have

$$\mathbf{B}^{\mathrm{T}}\mathbf{h} + \mathbf{h}\mathbf{B} + \dot{\mathbf{h}} = \mathbf{0}. \tag{107}$$

Proof. Substituting Eq. (96) for $\mathbf{B}$ and Eq. (106) for $\mathbf{h}$ into the left-hand side of Eq. (105) and through some calculations we obtain the right-hand side of Eq. (105). For any nonzero $\mathbf{Y} = (\mathbf{Y}^s, Y^0) \in \mathbb{R}^6$ we have

$$\begin{bmatrix} (\mathbf{Y}^s)^{\mathrm{T}} & Y^0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_5 - \mathbf{V}\mathbf{V}^{\mathrm{T}} & \mathbf{V} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix} \begin{bmatrix} \mathbf{Y}^s \\ Y^0 \end{bmatrix} = \|\mathbf{Y}^s\|^2 - (\mathbf{V}^{\mathrm{T}}\mathbf{Y}^s - Y^0)^2, \tag{108}$$

where $\|\mathbf{Y}^s\|^2 := (\mathbf{Y}^s)^{\mathrm{T}}\mathbf{Y}^s$ denotes the squared norm of $\mathbf{Y}^s$. Since the right-hand side may be positive, zero or negative, $\mathbf{h}$ is indefinite. Taking the time derivative of Eq. (106) and noting $\dot{\mathbf{V}} = \mathbf{U}$ by Eq. (95), then substituting the resultant into Eq. (107) which together Eq. (105) lead to Eq. (107). □

Eqs. (103) and (104) render us easily to prove the following result.

Theorem 4. The fundamental matrix $\mathbf{G}$ for Eq. (84) with $\mathbf{A}$ satisfying Eq. (83) has the following representation:

$$\mathbf{G} = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix} \mathbf{H}\mathbf{g}, \tag{109}$$

where $\mathbf{H}$ is the fundamental matrix of Eq. (88), satisfying

$$\dot{\mathbf{H}} = \mathbf{B}\mathbf{H}, \quad \mathbf{H}(0) = \mathbf{I}_6, \tag{110}$$

as well as

$$\mathbf{H}^{\mathrm{T}}\mathbf{h}\mathbf{H} = \mathbf{g}. \tag{111}$$

The quantity

$$\mathbf{Y}^{\mathrm{T}}\mathbf{h}\mathbf{Y} = \mathbf{Y}^{\mathrm{T}}(0)\mathbf{h}(0)\mathbf{Y}(0) \tag{112}$$

is a first integral of the system (88).

Proof. With $\mathbf{H}$ satisfied Eq. (110), the solution of Eq. (88) can be expressed by

$$\mathbf{Y}(t) = \mathbf{H}(t)\mathbf{Y}(0). \tag{113}$$

Substituting it into Eq. (103) and using $\mathbf{Y}(0) = \mathbf{g}\mathbf{X}(0)$ obtained from Eq. (104) by letting $t = 0$ and noting $\mathbf{R}(0) = \mathbf{I}_5$ and $\mathbf{V}(0) = \mathbf{0}$, we get

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix} \mathbf{H}\mathbf{g}\mathbf{X}(0), \tag{114}$$

which upon comparing with the solution of Eq. (79), $\mathbf{X}(t) = \mathbf{G}(t)\mathbf{X}(0)$ with $\mathbf{G}(t)$ satisfying Eq. (84), leads to Eq. (109).

Substituting Eq. (109) for $\mathbf{G}$ into Eq. (85) and noting that

$$\begin{bmatrix} \mathbf{R}^{\mathrm{T}} & \mathbf{V} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \mathbf{g} \begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix} = \mathbf{h}, \tag{115}$$

where $\mathbf{h}$ is defined by Eq. (106), we can prove Eq. (111).

It is known that $\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X}$ is an invariant form of system (79) with its $\mathbf{A}$ satisfying Eq. (83). Substituting Eq. (103) for $\mathbf{X}$ into $\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X}$, we obtain an invariant form $\mathbf{Y}^{\mathrm{T}}\mathbf{h}\mathbf{Y}$ of system (88). Substituting Eq. (113) for $\mathbf{Y}$ into the above quadratic form and using Eq. (111) and $\mathbf{g} = \mathbf{h}(0)$ we can prove that $\mathbf{Y}^{\mathrm{T}}\mathbf{h}\mathbf{Y} = \mathbf{Y}^{\mathrm{T}}(0)\mathbf{h}(0)\mathbf{Y}(0)$ is a first integral of system (88). □

Corollary 1. In addition the property (111) the fundamental matrix $\mathbf{H}$ solved from Eq. (110) satisfies the following properties:

$$\det \mathbf{H} = 1, \tag{116}$$

$$H_0^0 \geq 1 + \mathbf{V}^{\mathrm{T}}\mathbf{H}_0^s. \tag{117}$$

Proof. Since $\operatorname{tr}\mathbf{B} = 0$ as shown in Eq. (89), the property (116) follows from the Abel formula and $\det \mathbf{H}(0) = 1$. It indicates that $\mathbf{H} \in SL(6,\mathbb{R})$, i.e., a six-dimensional real special linear group. We partition $\mathbf{H}$ as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_s^s & \mathbf{H}_0^s \\ \mathbf{H}_s^0 & H_0^0 \end{bmatrix}, \tag{118}$$

where $\mathbf{H}_s^s$, $\mathbf{H}_0^s$ and $\mathbf{H}_s^0$ are of order 5×5, 5×1 and 1×5, respectively, and the scalar H_0^0 is the 00th component of $\mathbf{H}$. Substituting Eq. (118) for $\mathbf{H}$, Eq. (106) for $\mathbf{h}$, and Eq. (82) for $\mathbf{g}$ into Eq. (111), and then comparing the 00th components of both sides, we obtain

$$(H_0^0 - \mathbf{V}^{\mathrm{T}}\mathbf{H}_0^s)^2 = 1 + \|\mathbf{H}_0^s\|^2. \tag{119}$$

The right-hand side is positive, and we may take

$$H_0^0 - \mathbf{V}^{\mathrm{T}}\mathbf{H}_0^s = \pm\sqrt{1 + \|\mathbf{H}_0^s\|^2}.$$

However, at time $t = 0$, $\mathbf{V} = \mathbf{0}$, $\mathbf{H}_0^s = \mathbf{0}$ and $H_0^0 = 1$, and since $\mathbf{H}$ is a continuous matrix function of time t, we should select

$$H_0^0 - \mathbf{V}^{\mathrm{T}}\mathbf{H}_0^s = \sqrt{1 + \|\mathbf{H}_0^s\|^2} \geq 1, \tag{120}$$

and thus the property (117) follows directly. □

Corollary 2. The fundamental matrix $\mathbf{G}$ represented by Eq. (109) satisfies the properties (85)-(87).

Proof. Substituting Eq. (109) for $\mathbf{G}$ into Eq. (85) and using Eq. (111), it follows the property (85). Taking the determinants on both sides of Eq. (109) and noting that

$$\det \begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix} = -\det \mathbf{R} = -1,$$

$$\det \mathbf{H} = 1, \quad \det \mathbf{g} = -1,$$

we obtain $\det \mathbf{G} = 1$, and the property (86) is proved. Substituting Eq. (118) for $\mathbf{H}$ and Eq. (82) for $\mathbf{g}$ into Eq. (109), and comparing the 00th components on both sides we get

$$G_0^0 = H_0^0 - \mathbf{V}^{\mathrm{T}}\mathbf{H}_0^s. \tag{121}$$

By means of Eq. (117) the property (87) follows. □

From the above discussions we know that $\mathbf{h}$ is a metric in the underlying space for $\mathbf{Y}$ and $\mathbf{g}$ is a metric in the underlying space for $\mathbf{X}$. The two metrics are related through a similar transformation as shwon in Eq. (115), which both are indefinite. The three properties (111), (116) and (117) for $\mathbf{H}$ correspond to the three properties (85), (86) and (87) for $\mathbf{G}$. Since $\mathbf{g}$ is a constant metric, Eq. (83) also can be written as

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} + \dot{\mathbf{g}} = \mathbf{0}, \tag{122}$$

which corresponds to the form (107) for $\mathbf{B}$. In Table 2 we compare the Lie groups, Lie algebras and other properties for these two systems about $\mathbf{X}$ and $\mathbf{Y}$.

In Corollary 2 we have proved that the $\mathbf{G}$ in Eq. (109) is an element of $SO_o(5,1)$; however, left and right multiplying Eq. (109) by $\mathbf{g}$, we also obtain another element of $SO_o(5,1)$:

$$\mathbf{g}\mathbf{G}\mathbf{g} = \mathbf{g}\begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix}\mathbf{H}. \tag{123}$$

The element

$$\mathbf{g}\begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ \mathbf{V}^{\mathrm{T}} & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ -\mathbf{V}^{\mathrm{T}} & 1 \end{bmatrix} \tag{124}$$

Table 2: Comparisons of canonical and non-canonical formulations in the Minkowski space

Variable	Equation	Metric	Lie algebra	Lie group	Invariant form
$\mathbf{X}$	$\dot{\mathbf{X}} = \mathbf{AX}$	$\mathbf{g}$	$\mathbf{A}^T\mathbf{g} + \mathbf{gA} = \mathbf{0}$ $\text{tr}\,\mathbf{A} = 0$	$\mathbf{G}^T\mathbf{gG} = \mathbf{g}$ $\det \mathbf{G} = 1$ $G_0^0 > 0$	$\mathbf{X}^T\mathbf{gX}$
$\mathbf{Y}$	$\dot{\mathbf{Y}} = \mathbf{BY}$	$\mathbf{h}$	$\mathbf{B}^T\mathbf{h} + \mathbf{hB} + \dot{\mathbf{h}} = \mathbf{0}$ $\text{tr}\,\mathbf{B} = 0$ $\mathbf{B}^2 = \mathbf{0}$	$\mathbf{H}^T\mathbf{hH} = \mathbf{g}$ $\det \mathbf{H} = 1$ $H_0^0 \geq 1 + \mathbf{V}^T\mathbf{H}_0^s$	$\mathbf{Y}^T\mathbf{hY}$

forms a $SE(5)$ group right-action in the space $\mathbb{R}^5$ via the map

$$(\mathbf{R}, -\mathbf{V}) \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{0}_{5\times 1} \\ -\mathbf{V}^T & 1 \end{bmatrix}. \tag{125}$$

The right-action of $SE(5)$ on $\mathbb{R}^5$ is the rotation $\mathbf{R}$ followed by a translation by the vector $-\mathbf{V}$ and has the expression:

$$\mathbf{x}^T(\mathbf{R}, -\mathbf{V}) = \mathbf{x}^T\mathbf{R} - \mathbf{V}^T \tag{126}$$

for any $\mathbf{x} \in \mathbb{R}^5$.

Corresponding to the linear system (79) representation of the plastic equation in a canonical Minkowski space with metric $\mathbf{g}$, the linear system (88) representation however is a non-canonical one with a non-canonical metric $\mathbf{h}(t)$. The comparisons of the non-canonical one with the canonical Minkowski frame of plasticity as shown in this section are made in Table 2. We should stress that since $\mathbf{B}$ is nilpotent, in the numerical computation of plasticity problem such formulation may have higher accuracy than others [Liu and Tseng (2002), Liu (2006b)].

6 Lie-Poisson bracket formulation

Eq. (31) can be written as

$$\dot{\mathbf{Q}}^R = [\dot{\mathbf{q}}^R(\mathbf{Q}^R)^T - \mathbf{Q}^R(\dot{\mathbf{q}}^R)^T]\nabla H, \tag{127}$$

where

$$H = \frac{k_e \|\mathbf{Q}^R\|^2}{2Q_0^2} \tag{128}$$

is a generalized Hamiltonian function, the yield function $\|\mathbf{Q}^R\|^2$ dividing by $2Q_0^2/k_e$, and $\mathbf{Q}^R = \mathbf{R}^T\mathbf{Q}$ and $\dot{\mathbf{q}}^R = \mathbf{R}^T\dot{\mathbf{q}}$ with $\mathbf{R}$ solved from Eq. (91).

Theorem 5. The plastic equation (31) is a Lie-Poisson system. The solutions of Eq. (31) are the coadjoint orbits of the Lie group $SO(5)$, constrained in the yielding manifold of $\mathfrak{G}^*$ known as a symplectic foliation with the generalized Hamiltonian function H constant on it.

Proof. Define the Poisson tensor $\mathbf{J}$ to be

$$\mathbf{J} = \dot{\mathbf{q}}^R(\mathbf{Q}^R)^T - \mathbf{Q}^R(\dot{\mathbf{q}}^R)^T. \tag{129}$$

We can prove that $\mathbf{J}$ satisfies Eqs. (10) and (11). The first condition of skew-symmetry is obvious. Let us write

$$J_{ij} = \dot{q}_i^R Q_j^R - Q_i^R \dot{q}_j^R, \tag{130}$$

$$J_{ij,\ell} = \dot{q}_i^R \delta_{j\ell} - \delta_{i\ell}\dot{q}_j^R. \tag{131}$$

By using them we can get

$$\begin{aligned}
&J_{i\ell}J_{jk,\ell} + J_{j\ell}J_{ki,\ell} + J_{k\ell}J_{ij,\ell} \\
&= [\dot{q}_i^R Q_\ell^R - Q_i^R \dot{q}_\ell^R][\dot{q}_j^R \delta_{k\ell} - \delta_{j\ell}\dot{q}_k^R] \\
&+ [\dot{q}_j^R Q_\ell^R - Q_j^R \dot{q}_\ell^R][\dot{q}_k^R \delta_{i\ell} - \delta_{k\ell}\dot{q}_i^R] \\
&+ [\dot{q}_k^R Q_\ell^R - Q_k^R \dot{q}_\ell^R][\dot{q}_i^R \delta_{j\ell} - \delta_{i\ell}\dot{q}_j^R] \\
&= \dot{q}_i^R \dot{q}_j^R Q_k^R - \dot{q}_i^R \dot{q}_k^R Q_j^R - \dot{q}_j^R \dot{q}_k^R Q_i^R + \dot{q}_j^R \dot{q}_k^R Q_i^R \\
&+ \dot{q}_j^R \dot{q}_k^R Q_i^R - \dot{q}_i^R \dot{q}_j^R Q_k^R - \dot{q}_i^R \dot{q}_k^R Q_j^R + \dot{q}_i^R \dot{q}_k^R Q_j^R \\
&+ \dot{q}_i^R \dot{q}_k^R Q_j^R - \dot{q}_j^R \dot{q}_k^R Q_i^R - \dot{q}_i^R \dot{q}_j^R Q_k^R + \dot{q}_i^R \dot{q}_j^R Q_k^R \\
&= 0.
\end{aligned} \tag{132}$$

Thus, $\mathbf{J}$ satisfies Eqs. (10) and (11). Moreover, because $\mathbf{J}$ is a linear function of $\mathbf{Q}^R$, the bracket (9) with the above **J** is a Lie-Poisson bracket. Consequently, the plastic equation (31) is a Lie-Poisson system (12) with dimensions $n = 5$.

As that presented in Eq. (13), from Eq. (129) we can identify the structure constants to be

$$C_{ij}^{1} = \begin{bmatrix} 0 & -\dot{q}_2^{\mathrm{R}} & -\dot{q}_3^{\mathrm{R}} & -\dot{q}_4^{\mathrm{R}} & -\dot{q}_5^{\mathrm{R}} \\ \dot{q}_2^{\mathrm{R}} & & & & \\ \dot{q}_3^{\mathrm{R}} & & & & \\ \dot{q}_4^{\mathrm{R}} & & \mathbf{0} & & \\ \dot{q}_5^{\mathrm{R}} & & & & \end{bmatrix}, \tag{133}$$

$$C_{ij}^{2} = \begin{bmatrix} 0 & \dot{q}_1^{\mathrm{R}} & 0 & 0 & 0 \\ -\dot{q}_1^{\mathrm{R}} & 0 & -\dot{q}_3^{\mathrm{R}} & -\dot{q}_4^{\mathrm{R}} & -\dot{q}_5^{\mathrm{R}} \\ 0 & \dot{q}_3^{\mathrm{R}} & & & \\ 0 & \dot{q}_4^{\mathrm{R}} & & \mathbf{0} & \\ 0 & \dot{q}_5^{\mathrm{R}} & & & \end{bmatrix}, \tag{134}$$

$$\vdots = \vdots$$

$$C_{ij}^{5} = \begin{bmatrix} & & & & \dot{q}_1^{\mathrm{R}} \\ & & \mathbf{0} & & \dot{q}_2^{\mathrm{R}} \\ & & & & \dot{q}_3^{\mathrm{R}} \\ & & & & \dot{q}_4^{\mathrm{R}} \\ -\dot{q}_1^{\mathrm{R}} & -\dot{q}_2^{\mathrm{R}} & -\dot{q}_3^{\mathrm{R}} & -\dot{q}_4^{\mathrm{R}} & 0 \end{bmatrix}. \tag{135}$$

Suppose that $\mathbf{Q}^{\mathrm{R}} = Q_k^{\mathrm{R}}\mathbf{e}_k$ and that $\{\mathbf{e}_k, k = 1,\ldots,5\}$ forms a basis of the dual Lie algebra $\mathfrak{G}^*$. The above structure constants can be used to construct a Lie algebra denoted by $\mathfrak{G}$:

$$[\mathbf{f}_i, \mathbf{f}_j] = C_{ij}^k \mathbf{f}_k, \tag{136}$$

where $\{\mathbf{f}_k, k = 1,\ldots,5\}$ forms a basis of the Lie algebra $\mathfrak{G}$ and $[\bullet,\bullet]$ is the Lie commutator; see, e.g., Varadarajan (1984).

Next, we consider the adjoint representation of the Lie algebra $\mathfrak{G}$. For each $\mathbf{f} \in \mathfrak{G}$ the operator ad **f** that maps $\mathbf{g} \in \mathfrak{G}$ into $[\mathbf{f},\mathbf{g}]$ is a linear transformation of $\mathfrak{G}$ onto itself, i.e.,

$$(\mathrm{ad}\,\mathbf{f})\mathbf{g} = [\mathbf{f},\mathbf{g}]. \tag{137}$$

As $\{\mathbf{f}_k, k = 1,\ldots,5\}$ been supposed a basis for the Lie algebra $\mathfrak{G}$, we have

$$(\mathrm{ad}\,\mathbf{f}_i)\mathbf{f}_j = C_{ij}^k \mathbf{f}_k. \tag{138}$$

Therefore the matrix associated with the transformation ad $\mathbf{f}_i$ is

$$(\mathbf{M}_i)_{jk} = C_{ik}^j. \tag{139}$$

Corresponding to the structure constants given in Eqs. (133)-(135), the following $\mathbf{M}_i$ are available:

$$\mathbf{M}_1 = C_{1j}^i = \begin{bmatrix} 0 & -\dot{q}_2^{\mathrm{R}} & -\dot{q}_3^{\mathrm{R}} & -\dot{q}_4^{\mathrm{R}} & -\dot{q}_5^{\mathrm{R}} \\ 0 & \dot{q}_1^{\mathrm{R}} & 0 & 0 & 0 \\ 0 & 0 & \dot{q}_1^{\mathrm{R}} & 0 & 0 \\ 0 & 0 & 0 & \dot{q}_1^{\mathrm{R}} & 0 \\ 0 & 0 & 0 & 0 & \dot{q}_1^{\mathrm{R}} \end{bmatrix}, \tag{140}$$

$$\mathbf{M}_2 = C_{2j}^i = \begin{bmatrix} \dot{q}_2^{\mathrm{R}} & 0 & 0 & 0 & 0 \\ -\dot{q}_1^{\mathrm{R}} & 0 & -\dot{q}_3^{\mathrm{R}} & -\dot{q}_4^{\mathrm{R}} & -\dot{q}_5^{\mathrm{R}} \\ 0 & 0 & \dot{q}_2^{\mathrm{R}} & 0 & 0 \\ 0 & 0 & 0 & \dot{q}_2^{\mathrm{R}} & 0 \\ 0 & 0 & 0 & 0 & \dot{q}_2^{\mathrm{R}} \end{bmatrix}, \tag{141}$$

$$\vdots = \vdots$$

$$\mathbf{M}_5 = C_{5j}^i = \begin{bmatrix} \dot{q}_5^{\mathrm{R}} & 0 & 0 & 0 & 0 \\ 0 & \dot{q}_5^{\mathrm{R}} & 0 & 0 & 0 \\ 0 & 0 & \dot{q}_5^{\mathrm{R}} & 0 & 0 \\ 0 & 0 & 0 & \dot{q}_5^{\mathrm{R}} & 0 \\ -\dot{q}_1^{\mathrm{R}} & -\dot{q}_2^{\mathrm{R}} & -\dot{q}_3^{\mathrm{R}} & -\dot{q}_4^{\mathrm{R}} & 0 \end{bmatrix}. \tag{142}$$

In order to prove that the above $\{\mathbf{M}_k, k = 1,\ldots,5\}$ indeed forms a matrix basis for the Lie algebra $\mathfrak{G}$

satisfying Eq. (136), let us rewrite them to be

$$\mathbf{M}_i = \dot{q}_i^{\mathrm{R}}\mathbf{I}_5 + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -\dot{q}_1^{\mathrm{R}} & -\dot{q}_2^{\mathrm{R}} & -\dot{q}_3^{\mathrm{R}} & \cdots & -\dot{q}_5^{\mathrm{R}} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \leftarrow i\text{th row}, \tag{143}$$

$$\mathbf{M}_j = \dot{q}_j^{\mathrm{R}}\mathbf{I}_5 + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -\dot{q}_1^{\mathrm{R}} & -\dot{q}_2^{\mathrm{R}} & -\dot{q}_3^{\mathrm{R}} & \cdots & -\dot{q}_5^{\mathrm{R}} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \leftarrow j\text{th row}. \tag{144}$$

From these two equations it follows that

$$[\mathbf{M}_i, \mathbf{M}_j] = \mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \dot{q}_j^{\mathrm{R}}\dot{q}_1^{\mathrm{R}} & \dot{q}_j^{\mathrm{R}}\dot{q}_2^{\mathrm{R}} & \dot{q}_j^{\mathrm{R}}\dot{q}_3^{\mathrm{R}} & \cdots & \dot{q}_j^{\mathrm{R}}\dot{q}_5^{\mathrm{R}} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\dot{q}_i^{\mathrm{R}}\dot{q}_1^{\mathrm{R}} & -\dot{q}_i^{\mathrm{R}}\dot{q}_2^{\mathrm{R}} & -\dot{q}_i^{\mathrm{R}}\dot{q}_3^{\mathrm{R}} & \cdots & -\dot{q}_i^{\mathrm{R}}\dot{q}_5^{\mathrm{R}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{145}$$

In above, $(\dot{q}_j^{\mathrm{R}}\dot{q}_1^{\mathrm{R}}, \dot{q}_j^{\mathrm{R}}\dot{q}_2^{\mathrm{R}}, \ldots, \dot{q}_j^{\mathrm{R}}\dot{q}_5^{\mathrm{R}})$ locates at the jth row, and $(-\dot{q}_i^{\mathrm{R}}\dot{q}_1^{\mathrm{R}}, -\dot{q}_i^{\mathrm{R}}\dot{q}_2^{\mathrm{R}}, \ldots, -\dot{q}_i^{\mathrm{R}}\dot{q}_5^{\mathrm{R}})$ locates at the ith row. On the other hand, from Eqs. (133)-(135) we find that the structure constants C_{ij}^k are all zeros except these of $k=i$ or $k=j$, of which we have $C_{ij}^i = -\dot{q}_j^{\mathrm{R}}$ and $C_{ij}^j = \dot{q}_i^{\mathrm{R}}$. Thus, we obtain

$$C_{ij}^k\mathbf{M}_k = C_{ij}^i\mathbf{M}_i + C_{ij}^j\mathbf{M}_j = -\dot{q}_j^{\mathrm{R}}\mathbf{M}_i + \dot{q}_i^{\mathrm{R}}\mathbf{M}_j. \tag{146}$$

Inserting Eqs. (143) and (144) for $\mathbf{M}_i$ and $\mathbf{M}_j$ into the above equation we can get the right-hand side of Eq. (145), that is,

$$[\mathbf{M}_i, \mathbf{M}_j] = C_{ij}^k\mathbf{M}_k. \tag{147}$$

This ends the proof of Eq. (136). Consequently, $\{\mathbf{M}_k, k=1,\ldots,5\}$ forms a matrix basis for the Lie algebra $\mathfrak{G}$.

Let us consider the Lie group $\mathbf{G}_i$ generated from the matrix $\mathbf{M}_i$:

$$\dot{\mathbf{G}}_i = \mathbf{M}_i\mathbf{G}_i, \;\; \mathbf{G}_i(0) = \mathbf{I}_5, \;\; i \text{ not summed}. \tag{148}$$

Since the two matrices on the right-hand side of Eq. (143) commute, we can solve the above $\mathbf{G}_i$ as

$$\mathbf{G}_i = \begin{bmatrix} MAT & MAT \\ I & II \end{bmatrix}, \tag{149}$$

where

$$\begin{bmatrix} MAT \\ I \end{bmatrix} = \begin{bmatrix} e^{q_i^{\mathrm{R}}(t)-q_i^{\mathrm{R}}(0)} & 0 & \cdots & 0 \\ 0 & e^{q_i^{\mathrm{R}}(t)-q_i^{\mathrm{R}}(0)} & \cdots & 0 \\ 0 & 0 & e^{q_i^{\mathrm{R}}(t)-q_i^{\mathrm{R}}(0)} & \cdots \\ v_{i1} & \cdots & v_{i,i-1} & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\begin{bmatrix} MAT \\ II \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ v_{i,i+1} & \cdots & v_{i5} \\ e^{q_i^{\mathrm{R}}(t)-q_i^{\mathrm{R}}(0)} & 0 & 0 \\ 0 & e^{q_i^{\mathrm{R}}(t)-q_i^{\mathrm{R}}(0)} & 0 \\ 0 & 0 & e^{q_i^{\mathrm{R}}(t)-q_i^{\mathrm{R}}(0)} \end{bmatrix},$$

$$v_{ij}(t) = -\int_0^t \dot{q}_j^{\mathrm{R}}(\xi)\exp[q_i^{\mathrm{R}}(\xi) - q_i^{\mathrm{R}}(0)]d\xi. \tag{150}$$

The above $\mathbf{G}_i$ is a dilational translation in the ith plane $Q_i =$constant denoted by $DT_i(4)$. The right-action of $DT_i(4)$ on $\mathbb{R}^4$ is a dilation followed by a translation by the vector $\overline{\mathbf{v}}_i$ and has the expression:

$$(\overline{\mathbf{Q}}_i^{\mathrm{R}})^{\mathrm{T}}(e^{q_i^{\mathrm{R}}(t)-q_i^{\mathrm{R}}(0)}\mathbf{I}_4, \overline{\mathbf{v}}_i) = e^{q_i^{\mathrm{R}}(t)-q_i^{\mathrm{R}}(0)}(\overline{\mathbf{Q}}_i^{\mathrm{R}})^{\mathrm{T}} + \overline{\mathbf{v}}_i^{\mathrm{T}} \tag{151}$$

for any $\overline{\mathbf{Q}}_i^{\mathrm{R}} = (Q_1^{\mathrm{R}}, \ldots, Q_{i-1}^{\mathrm{R}}, Q_{i+1}^{\mathrm{R}}, \ldots, Q_5^{\mathrm{R}})^{\mathrm{T}} \in \mathbb{R}^4$, where $\overline{\mathbf{v}}_i = (v_{i1}, \ldots, v_{i,i-1}, v_{i,i+1}, \ldots, v_{i5})^{\mathrm{T}} \in \mathbb{R}^4$. Note that $DT_i(4)$ embeds into $GL(5,\mathbb{R})$ as that done in Eq. (149); hence, one can operate with $DT_i(4)$ as one would with matrix Lie groups by using the embedding.

Corresponding to the Lie algebra $\mathfrak{G}$ there exists a Lie group denoted by $\mathbf{G}$ which is composed of all $DT_i(4)$, and the adjoint representation of the Lie group is denoted by $\mathrm{Ad}_{\mathbf{g}}$, $\mathbf{g} \in \mathbf{G}$:

$$\mathrm{Ad}_{\mathbf{g}} : \mathfrak{G} \mapsto \mathfrak{G}. \tag{152}$$

$\mathfrak{G}^*$ is foliated by the coadjoint orbits:

$$\mathscr{O}_{\mathbf{Q}} = \{\mathrm{Ad}^*_{\mathbf{g}^{-1}}\mathbf{Q} | \mathbf{g} \in \mathbf{G}\} \subset \mathfrak{G}^*, \tag{153}$$

where the coadjoint action $\mathrm{Ad}^*_{\mathbf{g}^{-1}}$ is defined by

$$< \mathrm{Ad}^*_{\mathbf{g}^{-1}}\mathbf{w}, \mathbf{v} >=< \mathbf{w}, \mathrm{Ad}_{\mathbf{g}^{-1}}\mathbf{v} >, \ \ \mathbf{w} \in \mathcal{G}^*, \ \mathbf{v} \in \mathcal{G}. \tag{154}$$

Here $< \bullet, \bullet >$ denotes a non-degenerate pairing between $\mathcal{G}^*$ and $\mathcal{G}$. For matrices the adjoint action and coadjoint action are, respectively,

$$\mathrm{Ad}_{\mathbf{g}^{-1}}\mathbf{v} = \mathbf{g}^{-1}\mathbf{v}\mathbf{g}, \tag{155}$$

$$\mathrm{Ad}^*_{\mathbf{g}^{-1}}\mathbf{v} = \mathbf{g}\mathbf{v}. \tag{156}$$

Deriving the pair in Eq. (154) with respect to $\mathbf{g}$ then letting $\mathbf{g}$ equal to identity, we obtain

$$< \mathrm{ad}^*_{\mathbf{u}}\mathbf{w}, \mathbf{v} >=< \mathbf{w}, \mathrm{ad}_{\mathbf{u}}\mathbf{v} >, \ \ \mathbf{w} \in \mathcal{G}^*, \ \mathbf{v} \in \mathcal{G}, \tag{157}$$

where $\mathbf{u} = (d/dt)\mathbf{g}(t)|_{t=0}$. ad^* is the coadjoint representation of the Lie algebra $\mathcal{G}$. Then we have [Varadarajan (1984)]

$$\mathrm{ad}^*_{\mathbf{f}_i}\mathbf{w} = -\mathbf{J}(\mathbf{w})\mathbf{f}_i, \ \ \mathbf{w} \in \mathcal{G}^*. \tag{158}$$

Therefore, the matrix associated with the transformation $\mathrm{ad}^*_{\mathbf{f}_i}$ is

$$(\mathbf{M}^*_i)_{jk} = -C^i_{jk}. \tag{159}$$

Since C^i_{jk} is a skew-symmetric matrix for each i, the corresponding coadjoint action is found to be a five-dimensional rotation group denoted by $SO(5)$. Given an initial point $\mathbf{Q}^{\mathrm{R}}(t_i)$ on the yield manifold, a solution of the plastic equation (31) stays on the same coadjoint orbit $\mathscr{O}_{\mathbf{Q}^{\mathrm{R}}(t_i)}$ for all time until unloading happens. Along the coadjoint orbit the generalized Hamiltonian function H defined by Eq. (128) is a constant. □

7 Two-generator formulation

The two-generator bracket formalism is the one that appears first in the original development for the dynamics of dissipative systems by Kaufman (1984), Morrison (1984) and Grmela (1984). Recent progress of this formalism leads to the GENERIC (general equation for the nonequilibrium reversible-irreversible coupling) framework, e.g., Grmela and Öttinger (1997), Öttinger and Grmela (1997), and Beris (2001), in which the time evolution of any isolated thermodynamic system can be written in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{L}(\mathbf{x})\frac{\delta E(\mathbf{x})}{\delta \mathbf{x}} + \mathbf{M}(\mathbf{x})\frac{\delta S(\mathbf{x})}{\delta \mathbf{x}}, \tag{160}$$

where E and S represent, respectively, the total energy and entropy expressed in terms of the state variables $\mathbf{x}$, $\mathbf{L}$ and $\mathbf{M}$ are certain matrices, and $\delta \bullet / \delta \mathbf{x}$ denotes the Frechet derivative when $\bullet$ is a functional of $\mathbf{x}$; otherwise, it is the usual partial derivative. The use of two generators, the energy for the reversible dynamics and the entropy for the irreversible dynamics, is a characteristic feature of GENERIC. According to these concepts we can derive the following result.

Theorem 6. The plastic equation (31), admitting one conservative generator $H = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}$ and one dissipative generator $S = \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{q}}$, can be written as

$$\dot{\mathbf{Q}} = \mathbf{A}^s_s \nabla H + \eta \nabla S, \tag{161}$$

where

$$\eta := k_e \left[\mathbf{I}_5 - \frac{\mathbf{Q}\mathbf{Q}^{\mathrm{T}}}{Q_0^2} \right] \tag{162}$$

is a non-negative metric tensor. Furthermore, there holds one degenerate condition:

$$\eta \nabla H = \mathbf{0}. \tag{163}$$

Proof. We consider a dissipative bracket formulation of the plastic equation (31) by writing it to be

$$\dot{\mathbf{Q}} = \mathbf{A}^s_s \nabla H + k_e \left[\mathbf{I}_5 - \frac{\mathbf{Q}\mathbf{Q}^{\mathrm{T}}}{Q_0^2} \right] \nabla S, \tag{164}$$

where

$$S = \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{q}} = \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{q}}^p \tag{165}$$

is the dissipation power of plastic material, and $H = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}/2$ is the yield function.

As following that done in Eq. (9) we may define the Poisson bracket by

$$\{F, G\} := (A^s_s)_{ij} \frac{\partial F}{\partial Q_i} \frac{\partial G}{\partial Q_j}, \tag{166}$$

where F and G are two differentiable scalar functions of $\mathbf{Q}$. In above the Poisson tensor is defined for $\mathbf{A}_s^s$, which is skew-symmetric as shown in Eq. (32) and also satisfies Eq. (11) since $\mathbf{A}_s^s$ is independent on $\mathbf{Q}$. Introducing the metric tensor (162), and by following that done in Eq. (166) we can define the dissipative bracket by

$$\{\{F,G\}\} := \eta_{ij}\frac{\partial F}{\partial Q_i}\frac{\partial G}{\partial Q_j}, \tag{167}$$

where η_{ij} is a structure matrix of the dissipative bracket system [Liu (2000)]. We can prove that η is non-negative definite. From Eq. (162) the symmetry of η is obvious. Then, by

$$\mathbf{v}^{\mathrm{T}}\left[\mathbf{I}_5 - \frac{\mathbf{Q}\mathbf{Q}^{\mathrm{T}}}{Q_0^2}\right]\mathbf{v} = \|\mathbf{v}\|^2 - \left(\mathbf{v}^{\mathrm{T}}\frac{\mathbf{Q}}{Q_0}\right)^2 \geq 0 \;\forall\, \mathbf{v}\in\mathbb{R}^5, \tag{168}$$

we prove that $\mathbf{I}_5 - \mathbf{Q}\mathbf{Q}^{\mathrm{T}}/Q_0^2 \geq \mathbf{0}$ and that η is non-negative definite.

By using the above defined Poisson bracket and dissipative bracket, Eq. (164) can be represented as

$$\dot{Q}_i = \{Q_i, H\} + \{\{Q_i, S\}\}, \; i = 1,\ldots,5. \tag{169}$$

Obviously, $H = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}/2$ is involutive with all differentiable functions:

$$\{\{H,F\}\} = 0, \tag{170}$$

because of

$$\eta\nabla H = 2k_e\left[\mathbf{I}_5 - \frac{\mathbf{Q}\mathbf{Q}^{\mathrm{T}}}{Q_0^2}\right]\mathbf{Q} = 2k_e[\mathbf{Q}-\mathbf{Q}] = \mathbf{0}$$

in the plastic phase, where $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = Q_0^2$ was used. Thus, the yield function is a Casimir function in the dissipative bracket formulation. □

The dissipative bracket formulation stresses the role of the dissipation power as a potential function. The yielding behavior is guaranteed by preserving the Casimir function invariant. To retain the invariance may benefit from the Lie algebra construction as shown in Eq. (39) for this formulation; however, the Lie algebra is nonlinear in $\mathbf{Q}$ and it may need more effort to construct the corresponding Lie group. Also we find that Eq. (161) does fully fit the format (160) for the two-generator formulation.

8 Conclusions

In this paper we have investigated the plastic behavior from several theoretical aspects, explored five types of representations of the large deformation perfectly plastic equation by considering the corotational stress rates on hypoelasticity. The five representations are compared in Table 3 by displaying the underlying space, metric (structure) tensor, dimension, yielding, linearity, Lie algebra and Lie group. Each has its philosophy as being a different aspect of the same material model of plasticity.

The progress from the flow model in Eqs. (15)-(20) to Eq. (31) reflects the fact that it is an affine nonlinear system from the viewpoint of control theory, which admits the construction of a finite-dimensional Lie algebra with dimensions fifteen. Then taking the Lie group structure into account we may improve its calculation by utilizing the superposition principle.

Comparing the two representations (31) and (79), we are observed that Eq. (31) is nonlinear and of the five orders, but Eq. (79) is linear and of the six orders; hence, the implicit linearity is unfolded at the expense of raising one order up. The representation in the $\mathbf{X}$-space is linear, and moreover, it is easy to retain the Lie group symmetry of the model, thus facilitating the fulfillment of the consistency condition. The representation in the $\mathbf{Y}$-space is also linear; however, it provides a non-canonical Minkowski frame of plasticity, and may be more effective in the numerical computation since the augmented state matrix function $\mathbf{B}$ is nilpotent with index two.

The generalized Hamiltonian formalism nicely highlights the stress yielding behavior as a coadjoint orbit on the symplectic foliation in a dual Lie algebra space. The yield function plays the role as a generalized Hamiltonian function in the Lie-Poisson system. Conversely, in the two-generator and two-bracket formulation the nonlinear behavior of plastic material is reflected by the dissipation power as a potential function. The yielding behavior is guaranteed by preserving the Casimir function invariant. When compared with Eq. (31), Eq. (161) marks a further breakthrough of the

Table 3: Comparisons of five representations of plastic equation (Dim. is Dimension)

Equation(s)	Space	Metric	Dim.	Yielding	Linearity	Lie algebra	Lie group
(31)	$\mathbb{E}^5$	$\mathbf{I}_5$	5	Yield surface	Non-linear	–	–
(79)	$\mathbb{M}^{5+1}$	$\mathbf{g}$	$5+1$	Cone	Linear	$so(5,1)$	$SO_o(5,1)$
(88)	$\mathbb{M}^{5+1}$	$\mathbf{h}$	$5+1$	Cone	Linear	Nilpotent	$SL(6)$
(127)	$\mathcal{G}^*$	$\mathbf{J}$	5	Coadjoint orbit	Linear	$\mathbf{M}_i^*, i=1,...,5$	$SO(5)$
(161)	$\mathbb{E}^5$	η	5	Casimir function	Non-linear	in Eq. (39)	Yes

concept by protruding the dissipation nature of plasticity.

The nonlinear problems of plasticity were usually treated by workers in plasticity with various numerical schemes, which often encounter the tremendous difficulties of plastic nonlinearity and yielding inconsistency. The passage directly from the flow model in Eqs. (15)-(20) to a numerical scheme, if no care is taken of, may alter or destroy the underlying structure of the model, resulting in unstable, inefficient, and inaccurate calculations. Conversely, the passage from our representations of plasticity into numerical schemes may have the merit of the automatic fulfillment of the consistency condition. The extensions of the **X**-space representation in the canonical Minkowski space together with the Lorentz group symmetry to other plasticity models have been carried out by Liu (2001a, 2003, 2004a, 2004b, 2004c, 2005, 2006a) and Liu and Chang (2004). However, more study are required to fit the blank in the extension and numerical realization of these representations to other more complex plasticity models. Fully utilizing the internal symmetries and internal mathematical structures of the plasticity models would facilitate us to a further understanding of the real material behavior and the development of better numerical integrating methods.

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