

Thermo-Poro-Elastostatic Green's Functions for Unsaturated Soils

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Abstract: In this paper after a discussion about the evolution of the unsaturated soils' governing differential equations and a brief history of the Green's functions for porous media, the governing equations, i.e., the mathematical model in the presence of heat effects are presented and simplified so as the derivation of the associated Green's functions be in the realm of possibility. The thermal two- and three-dimensional, full- and half-space Green's functions for unsaturated porous media, although in a relatively simplified form, are being introduced for the first time, following the previous works of the authors. The derived Green's functions have been demonstrated graphically and verified mathematically by comparing with the previously introduced corresponding Green's functions. The resulted solutions may be used in a BEM or other meshless numerical models.

Keyword: Boundary element method; thermal Green's function; fundamental solution; porous media; unsaturated soils.

1 Introduction

The study of coupled heat and moisture transfer in a deformable partly saturated porous medium is an area of research receiving considerable current attention. One reason for this interest is that the problem is of importance in the strategic, international issue of the safe disposal of high-level nuclear waste [Schrefler (2002)]. As a consequence, a significant number of papers in this field have appeared in the recent literature [de Boer (1996)]. Two main distinguished strategies are generally

used to arrive at the description of the behavior of the porous continua: one starts from macro-mechanics and the other from micro-mechanics. Phenomenological approaches and the mixture theories belong to the first strategy while averaging theories belong to the second one.

Historically, Woltman (1974) introduced the concept of volume fractions and Delesse (1848) dealt with the concept of surface fractions. Fick (1855) studied the problem of diffusion of mixtures, Darcy (1856) the motion of a liquid in a porous solid and Stefan (1871) the diffusion of gas through a porous diaphragm. Fillunger (1913) introduced the concept of effective stress, i.e., the stress which controls stress-strain, volume change and strength behavior in a porous medium and studied the problems of uplift, friction and capillarity in porous solids. Terzaghi (1923) investigated saturated deformable porous solids and also made use of the effective stress principle. Biot (1941, 1956) developed the phenomenological approach of Terzaghi further and extended it to the three-dimensional case. A macroscopic thermodynamic approach to Biot's theory was used by Coussy (1995).

Modern mixture theories were developed by Morland (1972), Goodman and Cowin (1972), Sampaio and Williams (1979), Bowen (1980, 1982), Passman, Nunziato and Walsh (1984), Svendsen and Hutter (1955), and de Boer (2000). Averaging theories were developed by Hassanizadeh and Gray (1979, 1979) and Whitaker (1980) and the theory of homogenization was used by Auriault (1987, 1991). For more detailed review see Schrefler (2002) and for an extensive review of the history of porous media theories see de Boer and Ehlers (1988).

The Boundary Element Method (BEM), as the most efficient one among the numerical methods

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for solving the boundary value problems governing the various physical phenomena, is going to be employed for more complicated and coupled ones regarding the behavior and consequently the governing differential equations. Main advantages of the method include the reduction of the problem dimension by one and the implicit fulfillment of the radiation condition for unbounded domains.

As in this method, during formulating boundary integral equations, the applied mathematics concept of the Green's functions has been employed; this type of fundamental solutions for the governing partial differential equations should be first derived and therein lies the reason that the development of the BEM is restricted by the enclosed difficulties. The Green's functions or in more general term the fundamental solutions of the governing partial differential equations have been presented and explained in relatively complete forms for isothermal saturated soils phenomenon.

The first fundamental solutions for governing differential equations of saturated soils have been introduced by Cleary (1977) for quasi-static problem following the earlier work of Nowacki (1966). Closed form Laplace transform domain quasistatic poroelastic fundamental solutions were obtained by Cheng and Liggett (1984, 1984). The first fundamental solutions for dynamic poroelasticity seem to be presented by Burridge and Vargas (1979) who presented a general solution procedure similar to that of Deresiewicz (1960-1967). Norris (1985) derived time harmonic Green's functions for a point force in the solid and a point force in the fluid. Kaynia and Banerjee (1992) used the same solution scheme and derived the fundamental solution in the Laplace transform domain as well as transient short-time solution.

The time harmonic poroelastic fundamental solutions were introduced by Bonnet (1987) and Boutin, Bonnet and Bard (1987). Boutin worked on the equations that are based upon the homogenization theory for periodic structures but his solution is in symmetrical form, while the Green's functions for this problem should not be symmetric. Weibe and Antes (1991) obtained a time do-

main fundamental solution for the Biot type dynamic poroelasticity for the first time, neglecting the viscous coupling and without numerical evaluation of the kernel functions. Chen (1994, 1994) provided nearly the most complete analytical time domain Green's functions for both two and three-dimensional full dynamic poroelasticity. Thereupon, Gatmiri and Kamalian (2002) modified his two-dimensional solutions and boundary integral formulation to lead to more accurate results. Schanz and Pryl (2004) derived dynamic fundamental solutions for deformable soil's solid skeleton with compressible and incompressible fluid in Laplace transform domain. Simultaneously, Gatmiri and Nguyen (2005) derived closed form Green's functions for two-dimensional saturated soil with incompressible fluid. Both of the papers have concluded that an incompressible model can only be used in wave propagation problems if not the short time behavior is considered and also if the ratios of the compression moduli are very insignificant. More recently, Seyrafiyan, Gatmiri and Noorzad (2006) have presented analytical Green's functions for a continuously non-homogeneous half-space saturated media under a time-harmonic vertical point load without employing any potential function.

For unsaturated soils, the authors have derived the first Green's functions for the nonlinear governing differential equations for static [Gatmiri and Jabbari (2004, 2004)] and quasi-static [Gatmiri and Jabbari (2005, 2005)] poroelastic media for both two and three-dimensional problems and half-space domains [Jabbari and Gatmiri (2006)].

The present research is an effort for deriving thermo-poro-elastic full- and half-space Green's functions for two- and three-dimensional unsaturated media, for the first time, using a few necessary and sophisticated simplifications which certainly may be picked gradually off in the future by a few more analytic endeavors to improve the validity of the results.

2 Governing equations

For an unsaturated material influenced by heat effects, the governing partial differential equations consisted of four main groups: equilibrium equa-

tions, air transfer equations, fluid transfer equations [Gatmiri, Delage and Cerrolaza (1998)] and heat diffusion equations.

2.1 Equilibrium and constitutive equations of solid skeleton

Quasi-static equilibrium equations based on the two independent parameters $(\sigma - p_a)$ and $(p_a - p_w)$, with linear elastic behavior are

$$(\sigma_{ij} - \delta_{ij} p_a)_{,j} + p_{a,i} + b_i = 0 \quad (1)$$

and stress-strain relations including heat effects (in finite deformations form)

$$d(\sigma_{ij} - \delta_{ij} p_a) = D_{ijkl} d\epsilon_{kl} + \delta_{ij} D_s (dp_a - dp_w) - \delta_{ij} C_T dT \quad (2)$$

or

$$(\sigma_{ij} - \delta_{ij} p_a) = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} + \delta_{ij} D_s (p_a - p_w) - \delta_{ij} C_T T \quad (3)$$

and (linear, i.e., assuming small deformation gradients) strain-displacement relationships, describing the kinematics of deformation are

$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}). \quad (4)$$

Substituting the constitutive relations into the equilibrium equations yields the final governing (Navier's) equations, stating the equilibrium of solid skeleton, in the form of

$$(\lambda + \mu) u_{j,ij} + \mu u_{i,jj} + (D_s - 1) p_{a,i} - D_s p_{w,i} + C_T T_{,i} + b_i = 0. \quad (5)$$

In Eqs.1-5 λ and μ are Lamé's coefficients, D_{ijkl} are the coefficients of soil elasticity (constitutive tensor) and D_s is the coefficient of deformations due to suction effect. In addition, σ , ϵ , u and b stand for stress, strain, displacement of soil's solid skeleton and the body forces, respectively. Also, p_a , p_w and T denote air and water pressures and temperature and δ_{ij} represents the Kronecker delta operator.

In Eq.5 C_T is the specific heat capacity of unsaturated mixture and is defined as

$$C_T = (1 - n) \rho_s C_{ps} + n(1 - Sr) \rho_a C_{pa} + n Sr \rho_w C_{pw} + n(1 - Sr) \rho_v C_{pv}, \quad (6)$$

but will be considered an average constant in the medium. In Eq.6 C_{ps} , C_{pa} , C_{pw} and C_{pv} are specific heat capacities of soil particles, air, water and vapor, respectively. Furthermore, ρ_s , ρ_a , ρ_w and ρ_v are soil particles, air, water and vapor mass densities, respectively. n and S_r stand for porosity and degree of saturation.

2.2 Continuity and transfer equations for air

According to generalized Darcy's law (1856) for air transfer the air velocity, u_a , is defined as

$$u_a = -K_a \nabla \left(\frac{p_a}{\gamma_a} + Z \right). \quad (7)$$

Considering that p_a is a function of temperature, i.e. $p_a = p_a(T)$, Eq.7 can be written as

$$u_a = -\frac{K_a}{\gamma_a} \frac{\partial p_a}{\partial T} \nabla T - K_a \nabla \left(\frac{p_a}{\gamma_a} + Z \right). \quad (8)$$

Using the thermodynamic state equations for gases, the $\frac{1}{\gamma_a} \frac{\partial p_a}{\partial T}$ term can be replaced by

$$\frac{1}{\gamma_a} \frac{\partial p_a}{\partial T} = \frac{p_a + p_{atm}}{(T + 273) \gamma_a} = \beta_{pa} \quad (9)$$

this yields

$$u_a = -K_a \beta_{pa} \nabla T - K_a \nabla \left(\frac{p_a}{\gamma_a} + Z \right) \quad (10)$$

where γ_a and Z are air unit weight and the element's height from an arbitrary level, respectively. p_{atm} is atmosphere pressure and β_{pa} is considered a constant. The air coefficient of permeability, K_a , is defined as

$$K_a = D_K \frac{\gamma_a}{\mu_a} [e(1 - S_r)]^{E_K} \quad (11)$$

where μ_a and e are air dynamic viscosity and void ratio, respectively and D_K and E_K are constants [Lambe and Whitman (1969)].

Again a generalized Darcy's law for liquid transfer states that

$$u_w = -K_w \nabla \left(\frac{p_w}{\gamma_w} + Z \right) \quad (12)$$

in which γ_w is water unit weight. The water permeability, K_w , is defined as [Kovacs (1981)]

$$K_w = K_{wz0} \left(\frac{S_r - S_{ru}}{1 - S_{ru}} \right)^{3.5} \quad (13)$$

where S_{ru} is residual degree of saturation and K_{wz0} is the intrinsic water permeability defined as

$$K_{wz0} = a_{Kw} 10^{\alpha_{Kw} e} \quad (14)$$

where a_{Kw} and α_{Kw} are constant coefficients.

Again like the air transfer, as the water transfer phenomenon (water pressure) is temperature dependent, the water velocity, u_w , should be rewritten as

$$u_w = -K_{Tw} \nabla T - K_w \nabla \left(\frac{p_w}{\gamma_w} + Z \right) \quad (15)$$

in which the thermal water diffusivity, K_{Tw} , is considered herein a constant coefficient, see Appendix A.

Now, the mass conservation law for air unit volume should be written as [Alonso, Battle, Gens and Llorca (1988)]

$$\frac{\partial}{\partial t} \{ \rho_a n [1 - S_r (1 - H)] \} + \text{div} [\rho_a (u_a + H u_w)] = 0 \quad (16)$$

in which H is Henry's coefficient which denotes the amount of dissolved air in water, t is time variable and n stands for porosity.

With the assumption of constant ρ_a and K_a in space, dispensing with time variations, and remembering that the Laplacian of Z is zero, we have

$$\frac{\rho_a K_a}{\gamma_a} \nabla^2 p_a + \frac{H \rho_a K_w}{\gamma_w} \nabla^2 p_w + (K_a \beta_{pa} \rho_a + H K_{Tw}) \nabla^2 T = 0 \quad (17)$$

where div , ∇ and ∇^2 stand for divergence, gradient and Laplacian operators, respectively.

Indeed, assuming K_a , K_w and K_{Tw} as constants in the considered domain enables us to suppose the governing differential equation in the linear form although some key features of the nonlinearity are still reserved, using variable coefficient K_a , K_w and K_{Tw} in the derived Green's functions [Gatmiri and Jabbari (2005)]. In other words, keeping above parameters as a variable, makes the differential equation nonlinear (or with variable coefficients) so that deriving the considered Green's functions will become too difficult, at least with common methods. Therefore as a first step of deriving the Green's functions, it is reasonable to keep the effects of mentioned conductivity coefficients after deriving the Green's functions, using the linear form of the governing differential equations.

2.3 Continuity and transfer equations for water

The moisture phase transfer can be normally divided to liquid and vapor transfer components. The water velocity is represented in Eq.15 while the equation of vapor diffusion in porous media according to Philip and de Vries (1957) theory is given as

$$q_{vap} = -D_v \nabla \rho_v \quad (18)$$

where q_{vap} is the vector of vapor flux density and D_v depends partially on the degree of saturation and porosity. Since the procedure is both temperature and pressure dependent, the vapor velocity can be represented as

$$u_v = \frac{q_{vap}}{\rho_w} = -D_{pv} \nabla p_a - D_{Tv} \nabla T \quad (19)$$

in which D_{pv} and D_{Tv} are considered constant coefficients, see Appendix A.

The conservation law for total moisture transfer, dispensing with time variations, may be represented as

$$-\text{div} [\rho_w (u_w + u_v)] = \frac{\partial (\rho_m)}{\partial t} = 0 \quad (20)$$

in which ρ_m is moisture density.

Considering Eq.20 and again assuming constant ρ_w , D_{pv} , D_{Tv} and K_{Tw} , we have

$$\rho_w D_{pv} \nabla^2 p_a + \frac{\rho_w K_w}{\gamma_w} \nabla^2 p_w + \rho_w (D_{Tv} + K_{Tw}) \nabla^2 T = 0. \quad (21)$$

A discussion similar to that brought for K_a , K_w and K_{Tw} shows that it is inevitable to dispense with variations of D_{Tv} and D_{pv} in order to arrive at a linearized form of the governing differential equations, although the approach still reserves some nonlinearity features.

2.4 Heat equations

Heat equations are consisted of heat diffusion equation and energy conservation equation. Based on the Philipe and de Vries theory (1957) the flow of heat, Q , in an unsaturated porous medium is

$$Q = -\lambda_m \nabla T + [C_{pa} \rho_a u_a + C_{pw} \rho_w u_w + C_{pv} \rho_w u_v] \cdot (T - T_0) + \rho_w h_{fa} u_v + \rho_v h_{fa} u_a \quad (22)$$

in which λ_m accounts for Fourier heat diffusion coefficient

$$\lambda_m = (1 - n) \lambda_s + n Sr \lambda_w + n (1 - Sr) \lambda_v. \quad (23)$$

In Eq.23 the first term denotes the conductive heat flow, the second term is related to the convective heat flow in liquid, vapor and air and two last terms denote the latent heat related to evaporation. In addition, λ_s , λ_w and λ_v stand for soil particles, water and vapor thermal conductivities, respectively. h_{fa} is the latent heat of vaporization of soil water and T_0 is initial temperature, see Appendix A.

Energy conservation equation in a porous medium can be expressed as [Philipe and de Vries (1957)]

$$\frac{\partial \Phi}{\partial t} \lambda_m + \text{div} Q = 0 \quad (24)$$

in which Φ is the volumetric buck heat content of the medium [Gatmiri, Seyed, Delage and Fry (1997)] and is defined by

$$\Phi = C_T (T - T_0) + n(1 - Sr) \rho_v h_{fa}. \quad (25)$$

Eq.25 is valid if the local thermal equilibrium among the soil particles, the air and the water can be justified.

Combining above equations and neglecting time variations yield

$$\text{div} \left\{ -\lambda_m \nabla T + [C_{pa} \rho_a u_a + C_{pw} \rho_w u_w + C_{pv} \rho_w u_v] (T - T_0) + \rho_w h_{fa} u_v + \rho_v h_{fa} u_a \right\} = 0 \quad (26)$$

which may be simplified as

$$\left\{ (C_{pa} \rho_a K_a \beta_{pa} + C_{pv} \rho_w D_{pv}) \overline{(T - T_0)} + \rho_a h_{fa} \frac{K_a}{\gamma_a} + \rho_w h_{fa} D_{pv} \right\} \nabla^2 p_a + \left\{ C_{pw} \rho_w \frac{K_w}{\gamma_w} \overline{(T - T_0)} \right\} \nabla^2 p_w + \left\{ \lambda_m + (C_{pa} \rho_a K_a \beta_{pa} + C_{pw} \rho_w K_{Tw} + C_{pv} \rho_w D_{Tv}) \overline{(T - T_0)} + \rho_a h_{fa} K_a \beta_{pa} + \rho_w h_{fa} D_{Tv} \right\} \nabla^2 T = 0. \quad (27)$$

In Eq.27 we have again dispensed with the variations of K_a , K_w , K_{Tw} , D_{pv} and D_{Tv} to use the linear form of the equations and $\overline{(T - T_0)}$ means the mean value of $(T - T_0)$ in the computational domain.

2.5 Simplified mathematical model

The governing partial differential equations based on the considered linearization assumptions may be summarized and simplified as

$$c_{11} u_{j,i} + c_{12} u_{i,j} + c_{13} p_{a,i} + c_{14} p_{w,i} + c_{15} T_{,i} + c_{16} = 0$$

$$c_{21} \nabla^2 p_a + c_{22} \nabla^2 p_w + c_{23} \nabla^2 T = 0$$

$$c_{31} \nabla^2 p_a + c_{32} \nabla^2 p_w + c_{33} \nabla^2 T = 0$$

$$c_{41} \nabla^2 p_a + c_{42} \nabla^2 p_w + c_{43} \nabla^2 T = 0$$

$$(28)$$

where $i, j = \overline{1, 2}$ for two- and $i, j = \overline{1, 3}$ for three-dimensional problems and the c_{ij} coefficients are

$$\begin{aligned} c_{11} &= \lambda + \mu, & c_{12} &= \mu, \\ c_{13} &= -1 + D_s, & c_{14} &= -D_s, \\ c_{15} &= C_T, & c_{16} &= b_i, \\ c_{21} &= \frac{\rho_a K_a}{\gamma_a}, & c_{22} &= \frac{H \rho_a K_w}{\gamma_w}, \\ c_{23} &= K_a \beta_{pa} \rho_a + H K_{T_w}, \\ c_{31} &= \rho_w D_{pv}, & c_{32} &= \rho_w \frac{K_w}{\gamma_w}, \\ c_{33} &= \rho_w (D_{T_v} + K_{T_w}), \\ c_{41} &= \left(C_{pa} \rho_a \frac{K_a}{\gamma_a} + C_{pv} \rho_w D_{pv} \right) \overline{(T - T_0)} \\ &\quad + \rho_a h_{fa} \frac{K_a}{\gamma_a} + \rho_w h_{fa} D_{pv}, \\ c_{42} &= C_{pw} \rho_w \frac{K_w}{\gamma_w} \overline{(T - T_0)}, \\ c_{43} &= \lambda_m + \\ &\quad (C_{pa} \rho_a K_a \beta_{pa} + C_{pw} \rho_w D_{T_w} + C_{pv} \rho_w D_{T_v}) \overline{(T - T_0)} \\ &\quad + \rho_a h_{fa} K_a \beta_{pa} + \rho_w h_{fa} D_{T_v}. \end{aligned} \quad (29)$$

The differential equations system (28) may be written in the following matrix form

$$[C_{ij}] \times \vec{\omega} = \vec{f} \quad (30)$$

where $C_{ij} = c_{ij} \times d_{ij}$ and d_{ij} are the differential operators. Also, for two- and three-dimensional cases we have

Two-dimensional:

$$\begin{aligned} \omega_i &= u_i, & f_i &= -b_i, & i, j &= \overline{1, 2}, \\ \omega_3 &= p_a, & \omega_4 &= p_w, & \omega_5 &= T. \end{aligned} \quad (31)$$

Three-dimensional:

$$\begin{aligned} \omega_i &= u_i, & f_i &= -b_i, & i, j &= \overline{1, 3}, \\ \omega_4 &= p_a, & \omega_5 &= p_w, & \omega_6 &= T. \end{aligned} \quad (32)$$

Now, considering the above linearized differential equations system we will be able to derive the Green's functions using the classical methods as the first approximations of the unsaturated soils' thermal Green's functions.

3 Green's functions

Based on the method of Kupradze (1979) or Hörmander (1963) [also see Manolis and Pavlou (2002)], the Green's functions of a set of differential equations with linear differential operators are the cofactors of C_{ij} :

$$[g_{ij}] = [C_{ij}^*] \varphi \quad (33)$$

in which φ is a potential function and satisfies the equation

$$\det(C_{ij}) \varphi + \delta(x) = 0 \quad (34)$$

where $\delta(x)$ is the Dirac delta function of dimension five or six for two- or three-dimensional problems, respectively. By definition of the potential function φ , a set of fundamental solutions will be achieved [Melnikov and Melnikov (2001)]. This leads to such equation

$$\begin{aligned} \Pi_i \nabla^n \varphi_i + \delta(x) &= 0, \\ \Pi_i &= -c_{12}^{i-1} (c_{11} + c_{12}) \left(c_{23} (c_{32} c_{41} - c_{31} c_{42}) \right. \\ &\quad \left. + c_{22} (c_{31} c_{43} - c_{33} c_{41}) + c_{21} (c_{33} c_{42} - c_{32} c_{43}) \right) \end{aligned} \quad (35)$$

where

$$\begin{cases} \text{two-dimensional problem:} & n = 10 & i = 2, \\ \text{three-dimensional problem:} & n = 12 & i = 3 \end{cases} \quad (36)$$

in which $\nabla^{2m} = (\nabla^2)^m$ is m occurrence(s) of the Laplacian operator. The solutions of Eq.35 in symmetric (only r -dependent) domains are

$$\varphi_2 = \frac{r^8 [25 - 12 \ln(r)]}{3,538,944 D_2 \pi} \quad (37)$$

for two-dimensional problem and

$$\varphi_3 = \frac{r^9}{14,515,200 D_3 \pi} \quad (38)$$

for three-dimensional problem.

3.1 Full-space Green's functions

Based on the resulted φ functions, g_{ij} Green's functions or $[C_{ij}^*]$ φ matrix's elements can be derived for two and three-dimensional cases as follows

Two-dimensional problem:

$$\begin{aligned}
 g_{ij} &= F_{11}^2 \nabla^8 \delta_{ij} \varphi + F_{12}^2 \nabla^6 \partial_i \partial_j \varphi \quad i, j = \overline{1, 2} \\
 g_{i3} &= F_{13}^2 \nabla^6 \delta_i \varphi, \quad g_{i4} = F_{14}^2 \nabla^6 \delta_i \varphi, \\
 g_{i5} &= F_{15}^2 \nabla^6 \delta_i \varphi, \\
 g_{ij} &= 0 \quad i = \overline{3, 5} \quad j = \overline{1, 2}, \\
 g_{33} &= F_{21}^2 \nabla^8 \varphi, \quad g_{34} = F_{22}^2 \nabla^8 \varphi, \\
 g_{35} &= F_{23}^2 \nabla^8 \varphi, \\
 g_{43} &= F_{31}^2 \nabla^8 \varphi, \quad g_{44} = F_{32}^2 \nabla^8 \varphi, \\
 g_{45} &= F_{33}^2 \nabla^8 \varphi, \\
 g_{53} &= F_{41}^2 \nabla^8 \varphi, \quad g_{54} = F_{42}^2 \nabla^8 \varphi, \\
 g_{55} &= F_{43}^2 \nabla^8 \varphi.
 \end{aligned} \tag{39}$$

Three-dimensional problem:

$$\begin{aligned}
 g_{ij} &= F_{11}^3 \nabla^{10} \delta_{ij} \varphi + F_{12}^3 \nabla^8 \partial_i \partial_j \varphi \quad i, j = \overline{1, 3}, \\
 g_{i4} &= F_{13}^3 \nabla^8 \delta_i \varphi, \quad g_{i5} = F_{14}^3 \nabla^8 \delta_i \varphi, \\
 g_{i6} &= F_{15}^3 \nabla^8 \delta_i \varphi, \\
 g_{ij} &= 0 \quad i = \overline{4, 6} \quad j = \overline{1, 3}, \\
 g_{44} &= F_{21}^3 \nabla^{10} \varphi, \quad g_{45} = F_{22}^3 \nabla^{10} \varphi, \\
 g_{46} &= F_{23}^3 \nabla^{10} \varphi, \\
 g_{54} &= F_{31}^3 \nabla^{10} \varphi, \quad g_{55} = F_{32}^3 \nabla^{10} \varphi, \\
 g_{56} &= F_{33}^3 \nabla^{10} \varphi, \\
 g_{64} &= F_{41}^3 \nabla^{10} \varphi, \quad g_{65} = F_{42}^3 \nabla^{10} \varphi, \\
 g_{66} &= F_{43}^3 \nabla^{10} \varphi.
 \end{aligned} \tag{40}$$

The F_{ij}^2 coefficients are presented in Appendix B while F_{ij}^3 coefficients are related to F_{ij}^2 coefficients with

$$F_{ij}^3 = c_{12} F_{ij}^2. \tag{41}$$

3.2 Half-space Green's functions

When the domain is half-space, as the most of the engineering problems, the method of reflection

[McOwen (2003)] should be used to compute the Green's functions and other corresponding kernels. Based on this method, the full-space Green's function should be superposed with its reflection respect to the considered axis or plane in two- and three-dimensional problems, respectively, i.e.

$$\begin{aligned}
 G_{half-space}(x - \xi) \\
 = G_{full-space}(x - \xi) - G_{full-space}(x - \xi^*)
 \end{aligned} \tag{42}$$

in which G is the Green's function, x is the location of the potential function or receiver point, ξ is the location of the excitation point and ξ^* is its reflection respect to the considered axis or plane. The resulted half-space Green's functions are illustrated in the following figures.

4 Verification

Evidently, the final and exact verification of the resulted Green's functions should be performed by implementing them in a proper BEM model, but as a preliminary check, one may observe that by approaching K_{T_w} , D_{T_v} and D_{p_v} , as the main characteristics of the three phase model, to zero, F_{11}^i , F_{12}^i and Π_i will approach to

$$\begin{aligned}
 F_{11}^2 &= \frac{\lambda_m (\lambda + 2\mu) K_a K_w \rho_a \rho_w}{\gamma_a \gamma_w}, \\
 F_{12}^2 &= \frac{\lambda_m (\lambda + \mu) K_a K_w \rho_a \rho_w}{\gamma_a \gamma_w}, \\
 \Pi_2 &= \frac{\lambda_m \mu (\lambda + 2\mu) K_a K_w \rho_a \rho_w}{\gamma_a \gamma_w},
 \end{aligned} \tag{43}$$

for the two-dimensional problem, while for the three-dimensional problem the corresponding values are

$$\begin{aligned}
 F_{11}^3 &= \frac{\lambda_m \mu (\lambda + 2\mu) K_a K_w \rho_a \rho_w}{\gamma_a \gamma_w}, \\
 F_{12}^3 &= \frac{\lambda_m \mu (\lambda + \mu) K_a K_w \rho_a \rho_w}{\gamma_a \gamma_w}, \\
 \Pi_3 &= \frac{\lambda_m \mu^2 (\lambda + 2\mu) K_a K_w \rho_a \rho_w}{\gamma_a \gamma_w},
 \end{aligned} \tag{44}$$

and consequently, implementing them in the g_{ij} Green's functions results in

Two-dimensional problem:

$$g_{ij} = \frac{[(\lambda + \mu) - 2(\lambda + 3\mu) \ln(r)] r^2 \delta_{ij} + 2(\lambda + \mu) x_i x_j}{8\pi r^2 \mu (\lambda + 2\mu)}$$

Three-dimensional problem:

$$g_{ij} = \frac{(\lambda + 3\mu) r^2 \delta_{ij} + (\lambda + \mu) x_i x_j}{8\pi r^3 \mu (\lambda + 2\mu)} \quad (45)$$

those are exactly the well-known two- and three-dimensional elastostatic Green's functions [Gatmiri and Jabbari (2005)].

5 Illustration

For instance, some of the derived Green's functions for two- and three-dimensional and full- and half-space cases are shown through Figs. 1-8 with the following initial values. The consequent parameters are provided in Appendix A.

$$\begin{aligned} E &= 3 \times 10^4 \text{ kPa} & \nu &= 0.35 \\ H &= 0.02 & D_s &= 2 \\ \rho_a &= 1.293 \text{ kg/m}^3 \\ \mu_a &= 1.85 \times 10^{-5} \text{ kg/ms} \\ \rho_w &= 1000 \text{ kg/m}^3 & g &= 9.806 \text{ m/s}^2 \\ a_{Kw} &= 1.2 \times 10^{-9} \text{ m/s} & \alpha_{Kw} &= 5 \\ S_r &= 0.5 & S_{ru} &= 0.05 \\ e_0 &= 0.75 \\ D_{Ka} &= 1 \times 10^{-4} \text{ m}^2 & E_{Ka} &= 2.6 \\ C_{ps} &= 800 \text{ J/Kg}^\circ\text{K} & C_{pw} &= 4180 \text{ J/Kg}^\circ\text{K} \\ C_{pv} &= 1870 \text{ J/Kg}^\circ\text{K} & C_{pa} &= 1000 \text{ J/Kg}^\circ\text{K} \\ h_{fa} &= 2.4 \times 10^6 \text{ J/Kg} \\ \lambda_s &= 0.9 \text{ W/m}^\circ\text{K} \\ \lambda_a &= 0.0258 \text{ W/m}^\circ\text{K} \\ \lambda_w &= 0.6 \text{ W/m}^\circ\text{K} \\ \hat{p}_a &= 11000 \text{ Kg/m}^2 & \hat{p}_w &= 1200 \text{ Kg/m}^2 \\ T_0 &= 20^\circ\text{C} & T &= 25^\circ\text{C} \end{aligned} \quad (46)$$

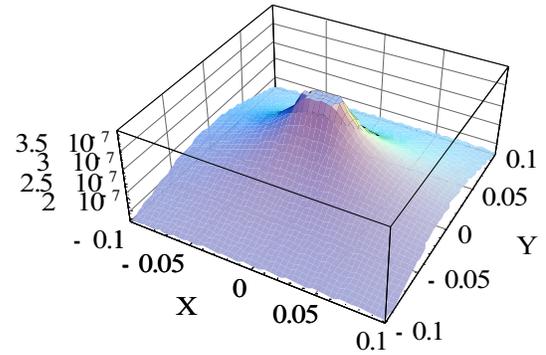


Figure 1: 2D Full-space Green's function g_{11} , solid skeleton displacement in direction one due to unit point load in direction one.

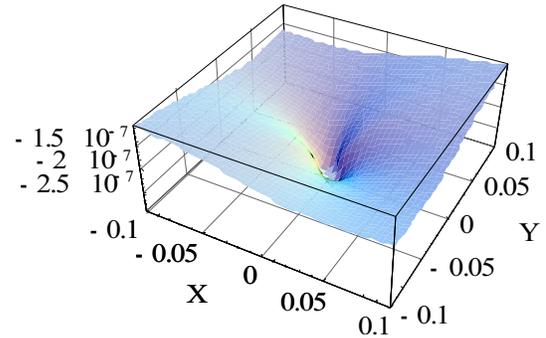


Figure 2: 2D Full-space Green's function g_{12} , solid skeleton displacement in direction one due to unit point load in direction two.

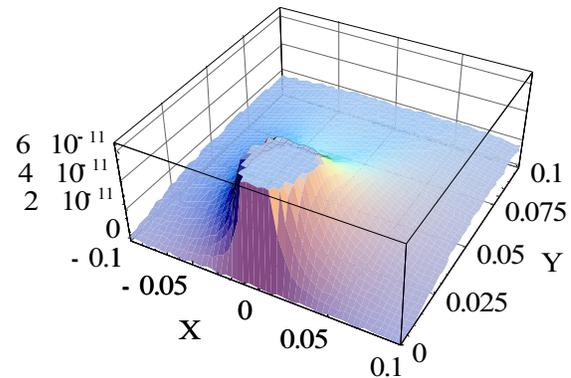


Figure 3: 2D Half-space Green's function g_{11} , solid skeleton displacement in direction one due to unit point load in direction one.

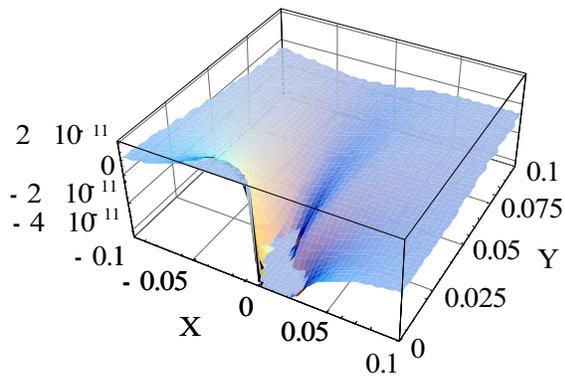


Figure 4: 2D Half-space Green's function g_{12} , solid skeleton displacement in direction one due to unit point load in direction two.

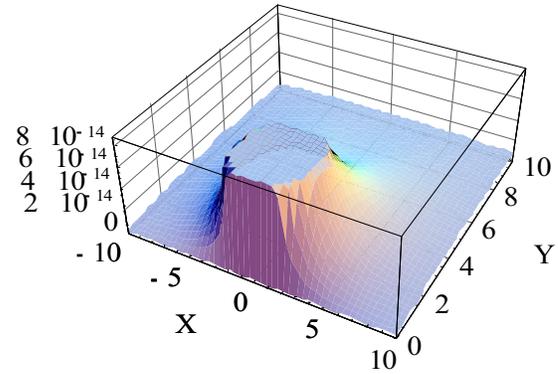


Figure 7: 3D Half-space Green's function g_{11} , solid skeleton displacement in direction one due to unit point load in direction one at $z = 0.10m$.

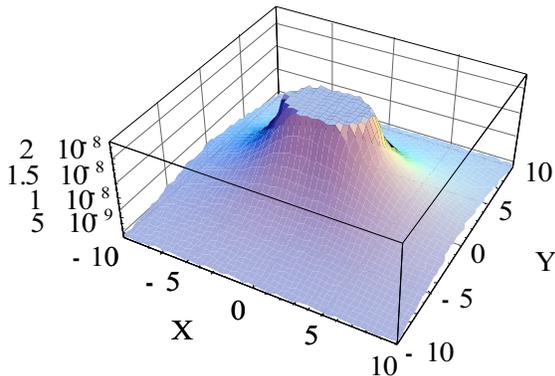


Figure 5: 3D Full-space Green's function g_{11} , solid skeleton displacement in direction one due to unit point load in direction one at $z = 0.10m$.

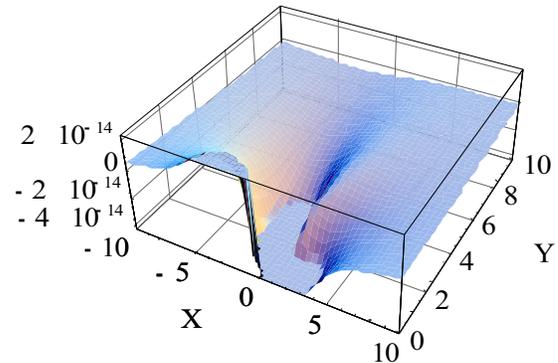


Figure 8: 3D Half-space Green's function g_{12} , solid skeleton displacement in direction one due to unit point load in direction two at $z = 0.10m$.

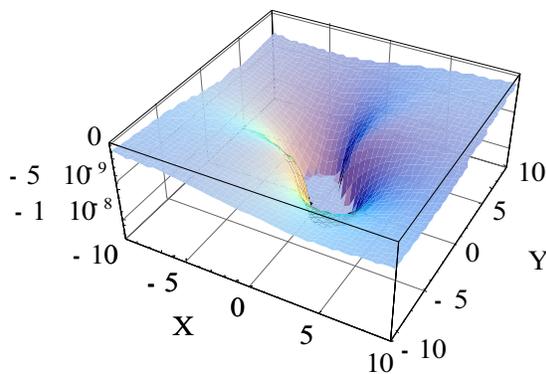


Figure 6: 3D Full-space Green's function g_{12} , solid skeleton displacement in direction one due to unit point load in direction two at $z = 0.10m$.

6 Conclusion

In this paper we introduced two- and three-dimensional, full- and half-space thermo-poro-elastostatic Green's functions for the three phase phenomenon of unsaturated soils governing differential equations, for the first time. The presented Green's functions have been derived using a few simplifications to linearize the governing partial differential equations and therefore rationalize the use of common straightforward methods for Green's functions derivation. As a primary verification, it has been demonstrated that the presented Green's functions approach to their corresponding elastostatic ones when the principal characteristics of the thermal three phase phenomenon will vanish. The resulted Green's func-

tions are also demonstrated through figures.

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Appendix A: Coefficients estimation

D_{pv} has been defined as

$$D_{pv} = -\frac{D_0}{\rho_w^2} \omega n h \frac{\rho_0}{R T} \frac{\sigma(T)}{\sigma_r} \quad (\text{A-1})$$

where ω is the mass flow factor close to unity at normal temperature [Gatmiri, Seyedi, Delage and Fry (1997)], $R = 4.615 \times 10^6$ is the ideal gas constant and h , the relative humidity, is defined as [Geraminegad and Saxena (1986)]

$$h = \left[1 + \left(\frac{n S r}{0.04 \rho_0} \right)^{-4.27} \right]^{-0.42} \quad (\text{A-2})$$

D_0 is the molecular diffusivity of water in air and is defined as [Kirsher and Rohnalter (1940)]

$$D_0 = \frac{244 \times 10^{-7}}{p_a} \left(\frac{T}{273} \right)^{2.3} \quad (\text{A-3})$$

ρ_0 is the mass density of saturated water vapor and is defined as [Geraminegad and Saxena (1986)]

$$\rho_0 = \frac{1}{194.4} \exp(0.06374 T - 0.0001634 T^2) \quad (\text{A-4})$$

σ_r is the surface energy of soil water at reference temperature while $\sigma(T)$ is at temperature T (Celsius) and is defined as [Gatmiri, Seyedi, Delage and Fry (1997)]

$$\sigma(T) = -75.882 + 0.165T. \quad (\text{A-5})$$

D_{Tv} has been expressed as

$$D_{Tv} = \frac{D_0}{\rho_w} \omega n h D_l \frac{d\rho_0}{dT} \quad (\text{A-6})$$

in which D_l is the ratio of microscopic temperature gradient in pore space to macroscopic temperature gradient as [Gatmiri, Seyedi, Delage and Fry (1997)]

$$D_l = \frac{1}{3} \left[\frac{2}{1+B} + \frac{1}{1+B(1-2G)} \right] \quad (\text{A-7})$$

where B is bulk modulus defined as

$$B = \frac{\lambda_a + \lambda_v}{\lambda_w} - 1 \quad (\text{A-8})$$

and the definition of G is

$$0.3333 - 0.325(1 - Sr) \quad (\text{A-9})$$

for $0.09 < n Sr < n$ and

$$0.0033 + 11.11 \left(0.33 - 0.325 \frac{(n-0.09)}{n} \right) n Sr \quad (\text{A-10})$$

for $0 < n Sr < 0.09$.

λ_v or vapor thermal conductivity is given as

$$\lambda_v = D_0 \omega h h_{fa} \frac{d\rho_0}{dT}. \quad (\text{A-11})$$

K_{Tw} has been introduced as [Gatmiri, Seyedi, Delage and Fry (1997)]

$$K_{Tw} = K_w \left(\frac{v_r}{v_T} \right) \frac{\hat{p}_w - \hat{p}_a}{\gamma_w \sigma_r} \frac{d\sigma(T)}{dT} \quad (\text{A-12})$$

in which \hat{p}_a and \hat{p}_w are the mean values of air and water pressures over the computational domain. v_r is the dynamic viscosity of water at a reference

temperature and v_T is at temperature T defined as [White (1999)]

$$v_T = \frac{1.788 \times 10^{-3}}{\rho} \cdot \exp(-1.704 - 5.306 \xi + 7.003 \xi^2) \quad (\text{A-13})$$

in which

$$\xi = \frac{273}{T+273}, \quad (\text{A-14})$$

$$\rho = 1000 - 0.0178 |T - 4|^{1.7}.$$

Appendix B: F_{ij}^2 coefficients

$$F_{11}^2 = -(c_{11} + c_{12}) \left(c_{23} (c_{32}c_{41} - c_{31}c_{42}) \right. \\ \left. + c_{22} (c_{31}c_{43} - c_{33}c_{41}) + c_{21} (c_{33}c_{42} - c_{32}c_{43}) \right)$$

$$F_{12}^2 = c_{11} \left(c_{23} (c_{32}c_{41} - c_{31}c_{42}) \right. \\ \left. + c_{22} (c_{31}c_{43} - c_{33}c_{41}) + c_{21} (c_{33}c_{42} - c_{32}c_{43}) \right)$$

$$F_{13}^2 = c_{12} \left(c_{15} (c_{32}c_{41} - c_{31}c_{42}) \right. \\ \left. + c_{14} (c_{31}c_{43} - c_{33}c_{41}) + c_{13} (c_{33}c_{42} - c_{32}c_{43}) \right)$$

$$F_{14}^2 = c_{12} \left(c_{15} (c_{21}c_{42} - c_{22}c_{41}) \right. \\ \left. + c_{14} (c_{23}c_{41} - c_{21}c_{43}) + c_{13} (c_{22}c_{43} - c_{23}c_{42}) \right)$$

$$F_{15}^2 = c_{12} \left(c_{15} (c_{22}c_{31} - c_{21}c_{32}) \right. \\ \left. + c_{14} (c_{21}c_{33} - c_{23}c_{31}) + c_{13} (c_{23}c_{32} - c_{22}c_{33}) \right)$$

$$\kappa = -c_{12} (c_{11} + c_{12})$$

$$F_{21}^2 = \kappa (c_{33}c_{42} - c_{32}c_{43}),$$

$$F_{22}^2 = \kappa (c_{22}c_{43} - c_{23}c_{42}),$$

$$F_{23}^2 = \kappa (c_{23}c_{32} - c_{22}c_{33}),$$

$$F_{31}^2 = \kappa (c_{31}c_{43} - c_{33}c_{41}),$$

$$F_{32}^2 = \kappa (c_{23}c_{41} - c_{21}c_{43}),$$

$$F_{33}^2 = \kappa (c_{21}c_{33} - c_{23}c_{31}),$$

$$F_{41}^2 = \kappa (c_{32}c_{41} - c_{31}c_{42}),$$

$$F_{42}^2 = \kappa (c_{21}c_{42} - c_{22}c_{41}),$$

$$F_{43}^2 = \kappa (c_{22}c_{31} - c_{21}c_{32}).$$

(B-1)

