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Adaptive Random Field Mesh Refinements in Stochastic Finite Element Reliability Analysis of Structures

M. Manjuprasad[1] **and C. S. Manohar**[2]

Abstract: A technique for adaptive random field refinement for stochastic finite element reliability analysis of structures is presented in this paper. Refinement indicator based on global importance measures are proposed and used for carrying out adaptive random field mesh refinements. Reliability index based error indicator is proposed and used for assessing the percentage error in the estimation of notional failure probability. Adaptive mesh refinement is carried out using hierarchical graded mesh obtained through bisection of elements. Spatially varying stochastic system parameters (such as Young's modulus and mass density) and load parameters are modeled in general as non-Gaussian random fields with prescribed marginal distributions and covariance functions in conjunction with Nataf's models. Expansion optimum linear estimation method is used for random field discretisation. A framework is developed for spatial discretisation of random fields for system/load parameters considering Gaussian/non-Gaussian nature of random fields, multidimensional random fields, and multiple-random fields. Structural reliability analysis is carried out using first order reliability method with a few refined features, such as, treatment of multiple design points and/or multiple regions of comparable importance. The gradients of the performance function are computed using direct differentiation method. Problems of multiple performance functions, either in series, or in parallel, are handled using the method based on product of conditional marginals. The efficacy of the proposed adaptive technique is illustrated by carrying out numerical studies on a set of examples covering linear static, free vibration and forced vibration problems.

[1] Scientist, Structural Engineering Research Centre, Taramani, Chennai, India

[2] Professor, Department of Civil Engineering, Indian Institute of Science, Bangalore, India

1 Introduction

One of the major developments in the recent years has been the extension of deterministic finite element methods to cover problems involving uncertainties in structural properties and in specifying loads which has resulted in the development of stochastic finite element methods (SFEMs). These developments include treatment of spatially varying structural properties that are modeled as random fields, loads that are modeled as space-time random processes, structural behavior covering static and dynamic regimes, linear and nonlinear structural mechanical behavior, and characterization of response variability and determination of structural reliability measures. Various mathematical tools, including, discretisation of random fields, random eigenvalue analyses, solution of stochastic boundary value problems, inversion of random matrix and differential operators, computation of reliability measures via nonlinear optimization tools, Monte Carlo simulation methods including variance reduction strategies, and applications of response surface modeling have been developed (for example see Ghanem and Spanos, 1991; Kleiber and Hien, 1992; Schueller, 1997; Manohar and Ibrahim, 1999; Haldar and Mahadevan, 2000; Jefferson Stroud, Krishnamurthy and Steven Smith, 2002; Manohar and Gupta, 2005).

The finite element method, even when applied to deterministic problems, is already approximate in nature, and, when the inputs to the finite element models are additionally treated as being probabilistic in nature, the character of approximations involved gets further compounded by inaccuracies realized in treatment of probability measure and uncertainty propagation through the system mechanics. Consequently, questions on errors involved in response predictions, *vis-à-vis* the choices made in handling of uncertainties, lead to challenging research problems. In a problem involving reliability analysis of structures analyzed using SFEM, there could be several sources of numerical errors. These errors can be grouped as follows:

Group I:

- errors of discretisation of displacement fields,
- errors of time stepping in direct integration,
- errors in computation of stress fields,
- errors in determination of natural frequencies and mode shapes, and
- errors due to truncation of modal expansion of response.

Group II:

- errors associated with discretisation of random fields,
- truncation errors in series representations used for random field discretisation,
- treatment of non-Gaussian nature of random fields via transformation to normal space,
- errors associated with characterization of reliability measures via linearization and subsequent solution of constrained optimization problems,
- errors associated with computation of gradients needed in calculation of reliability indices,
- errors associated with treatment of multiple design points and multiple regions of comparable importance in reliability calculations,
- errors associated with treatment of multiple failure modes,
- treatment of time-variant reliability using asymptotic theories of extremes of random processes, and
- treatment of sampling fluctuations if Monte Carlo simulation methods are employed in reliability assessment.

In any given problem, all the above sources of errors are present simultaneously and interact in a complicated way contributing to the overall error in reliability estimates. A unified treatment of all these errors, with a view to select parameters of numerical analysis so as to control the errors in reliability analysis, leads to complicated questions. In the present study, we restrict our aim to address issues related to impact of decisions made on discretisation of random fields on assessment of reliability of structure on hand.

In the context of SFEM based reliability analysis of dynamic structures with spatially distributed random properties, there could be three different meshes that need to be formulated; these are: (a) mesh for discretizing the displacement fields, (b) mesh for discretizing the random fields, and (c) time steps for integrating the equilibrium equations. The accuracy of random response characterization depends upon the choices made in formulating these meshes. In an adaptive process, attempts are made to estimate the errors without excessively increasing the numerical effort and to leave the decision of suitably refining the mesh/time steps more or less to a computer program. Considerable research has been carried out on adaptive mesh/time step refinements, particularly in the context of deterministic finite elements (for example see Zienkiewicz and Taylor, 2000; Ainsworth and Oden, 2000; Stein, Ruter and Ohnimus, 2004; Jin Ma, Hongbing Lu, and Ranga Komanduri, 2006; Wacher; and Givoli, 2006). However, literature available on adaptive methods for stochastic finite element analysis is limited. Liu, Belytschko and Mani (1986) have studied the error induced by discretising a random field and demonstrated that the number of random variables required to represent a random field is controlled by the correlation length of the random field. Li and Der Kiureghian (1993) have carried out studies on the error in different random field discretisation methods as a function of element size for selected correlation functions. Liu and Liu (1993) studied the influence of random field mesh on the performance of stochastic finite element reliability analysis in static problems. They presented rules for the selection and refinement of random field meshes. The error was computed using reliability index as the error norm and the refinement of random field mesh was carried using the gradients of Limit-state function as error indicators. Deb, Babuska and Oden (2001) have developed a framework for the construction of Galerkin approximations of elliptic boundary-value problems with stochastic input data. A theory of posteriori error estimation and corresponding adaptive approaches based on practical experience can be utilized. Pellissetti (2003) has discussed issues related to adaptive data refinement in the spectral stochastic finite element method.

The present paper addresses some of the questions related to the issue of random field mesh refinements. Specifically, we consider linear systems having non-Gaussian spatially distributed properties subject to static or dynamic loads. The problem of refinement of the mesh used to discretize the random fields, in order to achieve an acceptable estimate of the reliability of the system, is considered. The study proposes a general framework to achieve this objective. The reliability analysis procedure used is based on concepts from first order reliability methods involving several improvements to handle possible existence of multiple design points or multiple regions of comparable importance and subsequent use of methods of system reliability analysis, and, to take into account random dynamic loads. The refinement procedure itself is based on a global importance measure (GIM) assigned primarily to the basic random variables in standard normal space. These measures are subsequently assigned to the original non-Gaussian random variables using a sequence of transformations. Adaptive mesh refinement is carried out using hierarchical graded mesh obtained through bisection of elements. The study aims at refining the reliability index/failure probability estimates obtained through adaptive refinement of finite element mesh and random field mesh (RF mesh). Illustrative examples involving static/dynamic problems of structures with one-dimensional Euler beam elements and plane stress elements are presented.

It should be noted that exact solutions to problems of reliability analysis of structure modeled using SFEM are seldom available. In the proposed method for adaptive refinement of random field mesh, the first step consists of solving the problem with a coarse uniform mesh configured based on user's judgment. The proposed GIM is derived based on this coarse mesh and these measures indicate spatial regions where the RF meshes need to be refined. Reliability analysis with the refined mesh is subsequently performed, which, in turn, leads to the improved plots of GIM. This process is continued till a satisfactory convergence is obtained on the reliability measure being estimated. In the present study, to demonstrate the feasibility of the proposed mesh refinement procedures, we first solve the reliability analysis problem using a fairly fine displacement field mesh (DF mesh) and RF mesh configurations. The result from this analysis is treated as a benchmark against which the merit of the mesh configuration at a given stage of refinement is evaluated. It is to be noted, however, that the application of refinement procedure developed herein, does not depend upon the availability of a benchmark solution. It is also important to note that the refinement strategy here is goal oriented. That is, specifically, we focus on one or more specified performance functions with respect to which the structural reliability is analysed. The performance functions are typically in terms of response at a set of specified points. In the present paper, we focus on performance functions involving displacements at specified locations or natural frequencies of specified modes of the structure.

2 Problem Formulation

Consider the problem of reliability analysis of a structure in which uncertainties enter through one or more of the following routes:

1. spatially varying structural properties (such as, material properties, strength characteristics, joint characteristics or boundary conditions on an edge), which are modeled as a set of non-Gaussian random fields; these are de-

noted collectively by the vector random field $\boldsymbol{w}(\boldsymbol{x})$ with $\boldsymbol{x}$ denoting the vector of spatial co-ordinates;

2. loads that act on the structure that need to be modeled as random processes evolving is space and/or time; these are denoted collectively by $\boldsymbol{f}(\boldsymbol{x},t)$ with $\boldsymbol{x}$ denoting the vector of spatial co-ordinates and t denoting the time; and,

3. those properties of structure or the applied loads for which a random variable model would suffice; these are denoted collectively by the vector of mutually dependent non-Gaussian random variables Θ.

This list could also include uncertainties in modeling; this, however, has not been included in the present study. It is assumed that the random fields $\boldsymbol{w}(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x},t)$ would be discretised suitably leading to a vector of random variables Ξ, which, in general, would be non-Gaussain and mutually dependent in nature. Furthermore, the random variable vectors Θ and Ξ are collectively denoted by the vector $\Psi = [\Theta\,\Xi]^t$. We define a set of performance functions $g_i(\Psi)$ $(i = 1,2\ldots,n)$ such that the regions in the Ψ-space for which $g_i(\Psi) < 0$ and $g_i(\Psi) > 0$, respectively, denote the unsafe and safe regions with the surface $g_i(\Psi) = 0$ representing the limit surface with respect to the i^{th} performance function. We denote by P_{fi} the estimate of the probability of failure with respect to i^{th} performance function and P_f as the estimate of the probability of failure from structural system reliability analysis. The problem on hand consists of selecting and modifying the details of random field discretisation mesh so that the accuracy of estimation of P_f is improved.

3 Overview of the Proposed Adaptive Solution Strategy

In this section we present the broad features of the proposed adaptive solution strategy. The detailed description of the various steps involved is discussed in subsequent sections. In the present study, an adaptive technique is proposed for carrying out stochastic finite element reliability analysis of structures. It aims at refining the reliability index/failure probability estimates obtained through adaptive refinement of random field mesh. The following are the broad features of strategy adopted to tackle the problem stated in the previous section.

A. Random field discretisation: We model one or more of the structural properties as a vector of mutually dependent non-Gaussian random fields. These fields are taken to be differentiable in the mean square sense and are assumed to be spatially homogeneous and isotropic. We follow the expansion optimum linear estimation (EOLE) method of discretisation combined with Nataf's transformation technique to represent uncertainties through an equivalent vector of standard normal random variables.

B. Discretisation of load fields: We handle the spatially distributed surface load as a Gaussian random field. This field is taken to be differentiable in the mean square sense and is assumed to be spatially homogeneous and isotropic. We again follow the expansion optimum linear estimation (EOLE) method of discretisation to represent uncertainties through an equivalent vector of standard normal random variables. We model the time varying stochastic excitations due to nodal loads as zero-mean stationary Gaussian random process with a specified power spectral density (PSD) function.

C. Mesh refinement and error indicator: We have proposed the use of GIM with appropriate transformations as mesh refinement indicators and this is one of the original ideas developed for carrying out adaptive random field mesh refinements proposed in this paper. A refinement norm based on a notional reliability index is also proposed and used as an error indicator for estimating the percentage error in the estimation of P_f.

D. Mesh refinement strategy: We focus attention mainly on placement of nodes for random field meshes. The method developed does not impose any restriction on relative sizes of different RF meshes as well as the DF mesh. All these meshes could be refined independent of each other. The method has provisions for assembly of various structural matrices taking into account differences in the associated RF meshes. The mesh refinement itself is carried on by using the method of successive bisection. The study also examines the possibility of derefinement of meshes, wherein, we reduce the number of elements in spatial regions considered less important by the computed GIMs. The question of using GIM also to refine DF mesh and time steps of direct integration problem are also addressed albeit very briefly.

E. Reliability estimation: Structural reliability analysis is carried out using FORM with the following improvements: (a) treatment of multiple design points or multiple regions of comparable importance in reliability calculations, (b) system reliability problems involving multiple performance functions, and (c) dynamic problems involving random excitations. The performance function is computed using finite element method and the gradients needed in reliability calculations are evaluated numerically using direct differentiation method.

4 Random Field Discretisation

In this study, we consider the random fields that model the structural properties as a vector of non-Gaussian mutually dependent random fields. We also take that these random fields are only partially specified with knowledge on their properties limited to the first order marginal probability density function (pdf), and the set of auto-covariance and cross-covariance functions. We discretise these random fields using the EOLE method (Li and Der Kiureghian, 1993) in conjunction with the Nataf's transformation technique (Der Kiureghian and Liu, 1986), which leads to the definition of a vector of mutually dependent non-Gaussian random variables.

In order to clarify the steps involved in discretisation of random field, and its relation to discretisation of displacement field we consider a one-dimensional problem involving a displacement field $u(x)$ and a non-Gaussian random field $w(x)$ with $0 < x < L$. Let $F_w[w;x]$ and $\rho_{ww}(x,x')$ denote, respectively, the first order pdf and auto-correlation function associated with $w(x)$. Let the displacement field be discretised into N^* elements of length $\Delta' x$ and (N^*+1) nodes such that $N^*\Delta' x = L$. Let $\{x_i'\}_{i=1}^{N^*}$ denote the centroidal value of the i^{th} element. In formulating the structural matrix for the i^{th} element we need the value of $w(x_i')$.

To arrive at the value of $w(x_i')$, we begin by first transforming $w(x)$ into an equivalent Gaussian random field $v(x)$ using the rule $v(x) = \Phi^{-1}[F_w(w;x)]$ (Der Kiureghian and Liu, 1986). The Gaussian random field $v(x)$ is discretised using another mesh with N nodes and $(N-1)$ elements of length Δx such that $(N-1)\Delta x = L$. Here, $\{x_i\}_{i=1}^{N}$ represents the nodal coordinates and $v(x_i)$, $i = 0,1\ldots,N$ constitutes a set of correlated Gaussian random variables. We represent $v(x)$ using the EOLE expansion such that for any point $x = x_i$,

$$\hat{v}(x_i) = c(x_i) + \sum_{j=1}^{N} b_j(x_i)\left(\mu(x_j) + \sum_{k=1}^{r} \Gamma_k' \sqrt{\theta_k}\phi_{kj}\right), \quad i = 0,1\ldots,\hat{N} \tag{1a}$$

where, $\mu(x)$ represents the mean of $v(x)$, $\{\theta_k\}_{k=1}^{r}$ and $\{\varphi_k\}_{k=1}^{r}$ are the r highest eigenvalues and eigenvectors associated with auto-covariance matrix of $v(x)$ evaluated at $x = x_1, x_2\ldots,x_N$; and, $\{\Gamma_k'\}_{k=1}^{r}$ are a set of standard normal random variables. The functions $c(x)$ and $\{b_j(x)\}_{j=1}^{N}$ are the interpolation functions used in discretizing $v(x)$, selected such that the variance of error given by $\left\langle \{v(x) - \hat{v}(x)\}^2 \right\rangle$ is minimized for every x in $(0,L)$ (Li and Der Kiureghian, 1993). Here, $\langle\rangle$ is the expectation operator. When $c(x)$ and $\{b_j(x)\}_{j=1}^{N}$ are so selected, the optimal error is obtained at any point $x = x_i$ as,

$$\left\langle \varepsilon_i^2 \right\rangle = \left\langle \left\{ v(x_i) - \sum_{j=1}^{N} b_j(x_i) \left(\mu(x_j) + \sum_{k=1}^{r} \Gamma_k' \sqrt{\theta_k} \phi_{kj} \right) \right\}^2 \right\rangle, \quad i = 0, 1 \ldots, \hat{N} \quad (2a)$$

After substituting for $c(x)$ and $\{b_j(x)\}_{j=1}^{N}$ and further simplifying, Eq. (1a) and Eq. (2a) take the following forms:

$$\hat{v}(x_i) = \mu(x_i) + \sum_{k=1}^{r} \frac{\Gamma_k'}{\sqrt{\theta_k}} \sum_{j=1}^{N} \phi_{kj} S_{ji}, \quad i = 0, 1 \ldots, \hat{N} \quad (1b)$$

$$\left\langle \varepsilon_i^2 \right\rangle = \sigma^2(x_i) - \sum_{k=1}^{r} \frac{1}{\theta_k} \left(\sum_{j=1}^{N} \phi_{kj} S_{ji} \right)^2, \quad i = 0, 1 \ldots, \hat{N} \quad (2b)$$

where, $\boldsymbol{S}_{N\hat{N}}$ is a $N \times \hat{N}$ matrix containing the expectations $\langle (v(x_j) - \mu(x_j))(\hat{v}(x_i) - \mu(x_i)) \rangle$ ($j = 0, 1 \ldots, N$ and $i = 0, 1 \ldots, \hat{N}$) indicating the covariance of $v(x_j)$ with $\hat{v}(x_{\mathbf{i}})$.

Once the details of discretisation are obtained, one can obtain $w(x_i')$ needed for formulation of structural matrices for the i^{th} element by noting that $w(x_i') = F_w^{-1}[\boldsymbol{\Phi}(\hat{v}(x_i'))]$ with,

$$\hat{v}(x_i') = \mu(x_i') + \sum_{k=1}^{r} \frac{\Gamma_k'}{\sqrt{\theta_k}} \sum_{j=1}^{N} \phi_{kj} S_{ji}, \quad i = 0, 1 \ldots, N^* \quad (1c)$$

For the purpose of illustration of some of the features of the EOLE, we consider a Gaussian random field with unit mean, unit variance and correlation function of the form,

$$\rho(x, x') = exp\left(-\frac{(x - x')^2}{a^2} \right) \quad (3)$$

where, a is the correlation length factor.

In the following studies we fix $N^* = 32$. The influence of varying r on variance of error for $a = 0.25L, 0.5L, 1.0L, 2.0L$ and $N = 9, 17, 33$ is shown in Figs. 1a to 1c. In Fig. 2 we fix $a = 0.25L$, r=5 to show the variation of the variance of error of discretisation as a function of x/L for $N = 9, 17, 33$. The influence of varying a on the variance of error for r=5, N=33 is studied in Fig. 3. Fig. 4 shows the plot of correlation function for different values of a. If we define the length over which correlation drops by a given factor (say, 0.5) as the correlation length of the random field, then it can be observed from Fig. 4 that for $a = 0.25L, 0.5L, 1.0L$ the correlation lengths are respectively $0.22L$, $0.42L$, and $0.82L$. For $a = 2L$, the correlation length is much greater than L. Clearly, shorter the correlation length with respect to L, greater is the need for accounting for the random spatial variability in the analysis.

In summary, for the given form of $\rho(x, x')$ and value of a it can be noted that the error of discretisation of the random field depends upon the choice of N and r. In the present studies we take r=5 and $N = 8, 16, 32$, $a = 0.25L, 0.5L, 1.0L$ and $2.0L$. For the range values of r and N considered, the standard deviation of the error of discretisation is found to be within 0.04 (See Fig. 2). In the adaptive refinement procedures discussed subsequently, we hold $r = 5$ and vary N.

It is important however to note that the error of discretisation that has been discussed in this section has so far been in the context of random field discretisation alone and these errors are not necessarily the ones which we wish to eventually control. Indeed, it is the error in the estimation of structural reliability that we wish to eventually control.

We would also note here that in the context of problems involving two-dimensional random fields we assume random fields to be homogeneous and isotropic with correlation function of the form,

$$\rho(x, x') = exp\left(-\frac{l^2}{a^2} \right) \quad (4)$$

where, l represents the distance between two points in the random field. Consequently, the discussion presented above in the context of one-dimensional random fields remains broadly valid

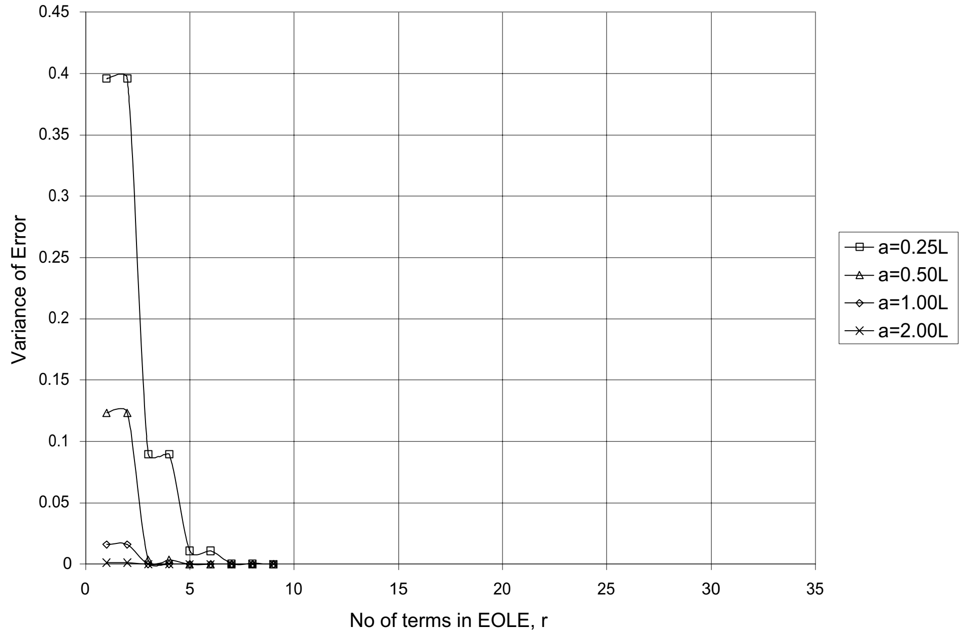


Figure 1a: Influence of varying the number of terms of expansion (r) on variance error of discretisation using EOLE; with correlation length factor, $a = 0.25L, 0.5L, 1.0L, 2.0L$; number of random field elements, $N = 9; x/L = 0.5; N^{*} = 32$.

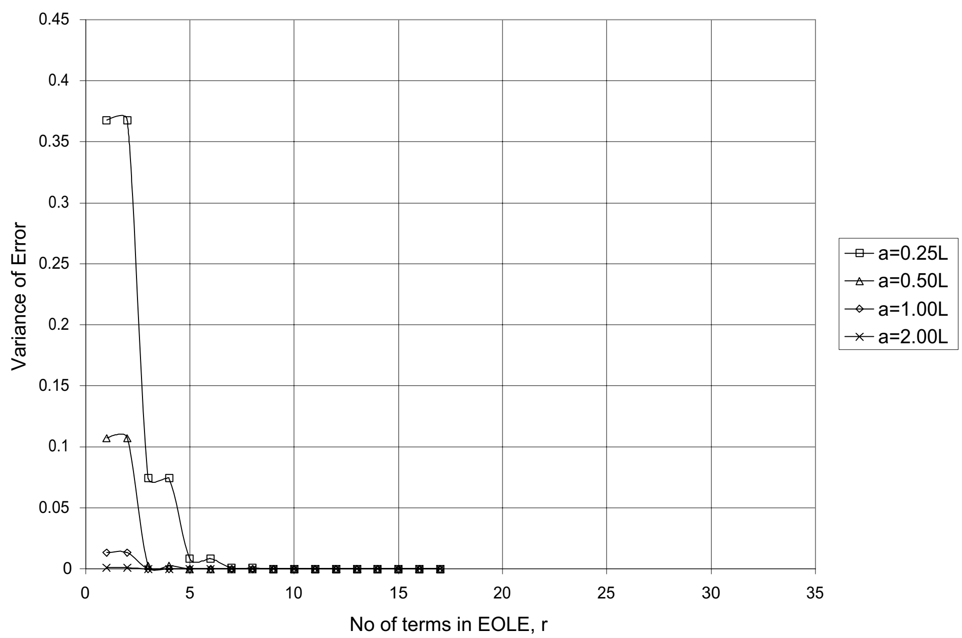


Figure 1b: Influence of varying the number of terms of expansion (r) on variance error of discretisation using EOLE; with correlation length factor, $a = 0.25L, 0.5L, 1.0L, 2.0L$; number of random field elements, $N = 17; x/L = 0.5; N^{*} = 32$.

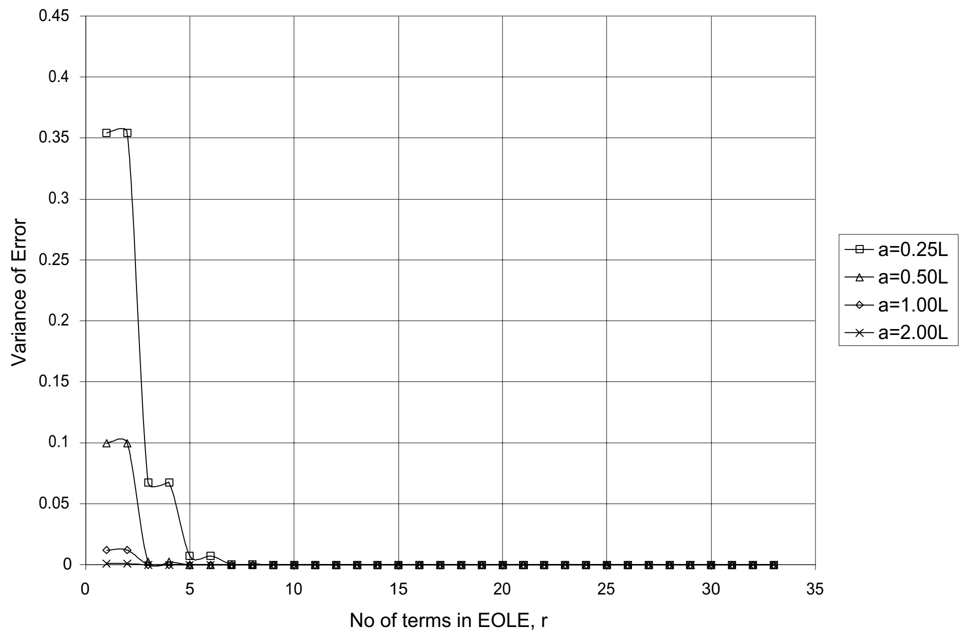


Figure 1c: Influence of varying the number of terms of expansion (r) on variance error of discretisation using EOLE; with correlation length factor, $a = 0.25L, 0.5L, 1.0L, 2.0L$; number of random field elements, $N = 33; x/L = 0.5; N^* = 32$.

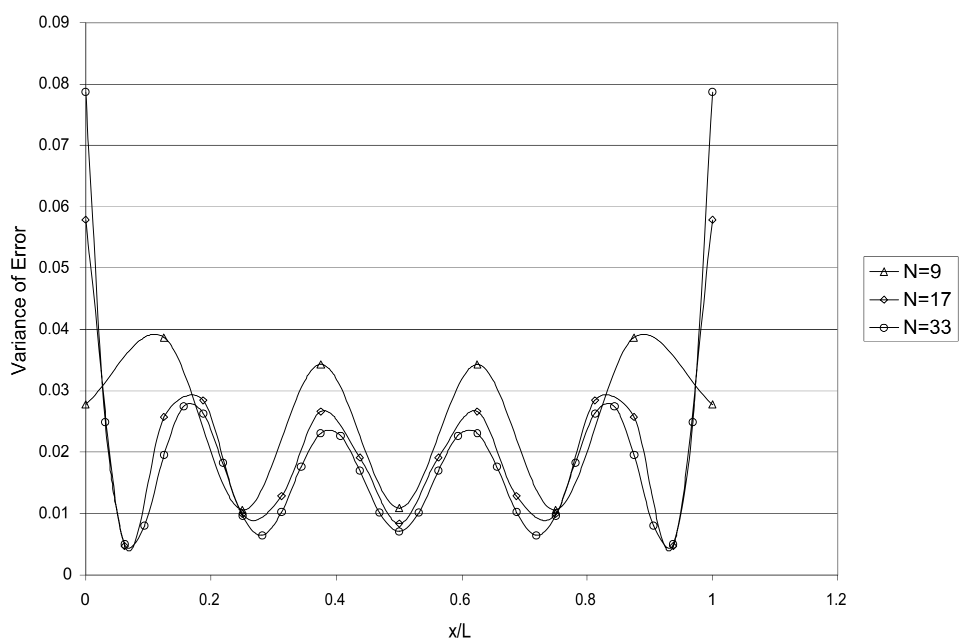


Figure 2: Variation of the variance of error of discretisation using EOLE as a function of x/L for number of terms of expansion, $r = 5$, correlation length factor $a = 0.25L$ and number of random field elements $N = 8, 16, 33; N^* = 32$.

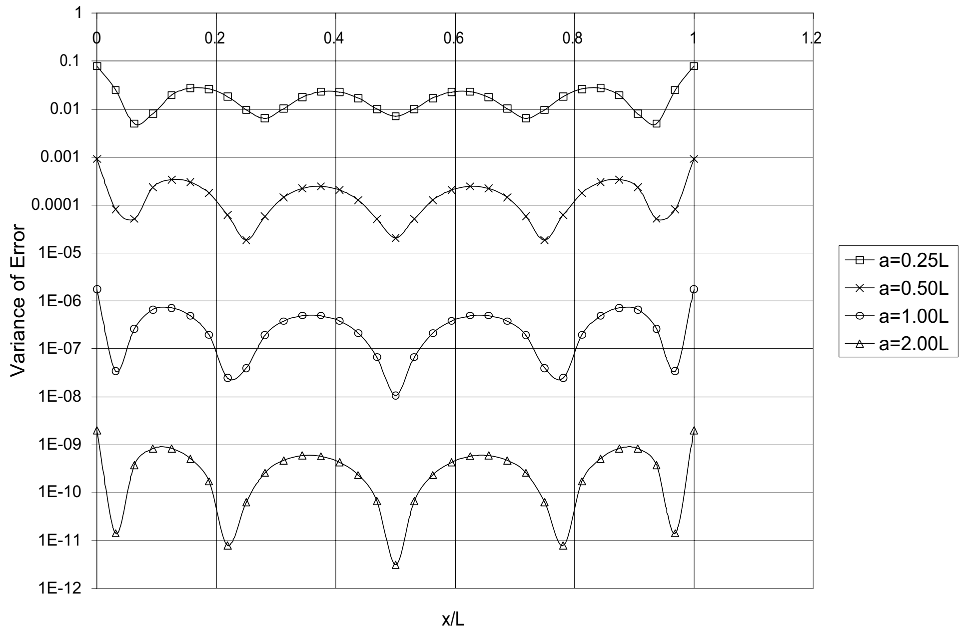


Figure 3: Variation of the variance of error of discretisation using EOLE as a function of $x/L$for number of terms of expansion, $r = 5$, number of random field elements $N = 33$ and correlation length factor $a = 0.25L, 0.5L, 1.0L, 2.0L; N^* = 32$.

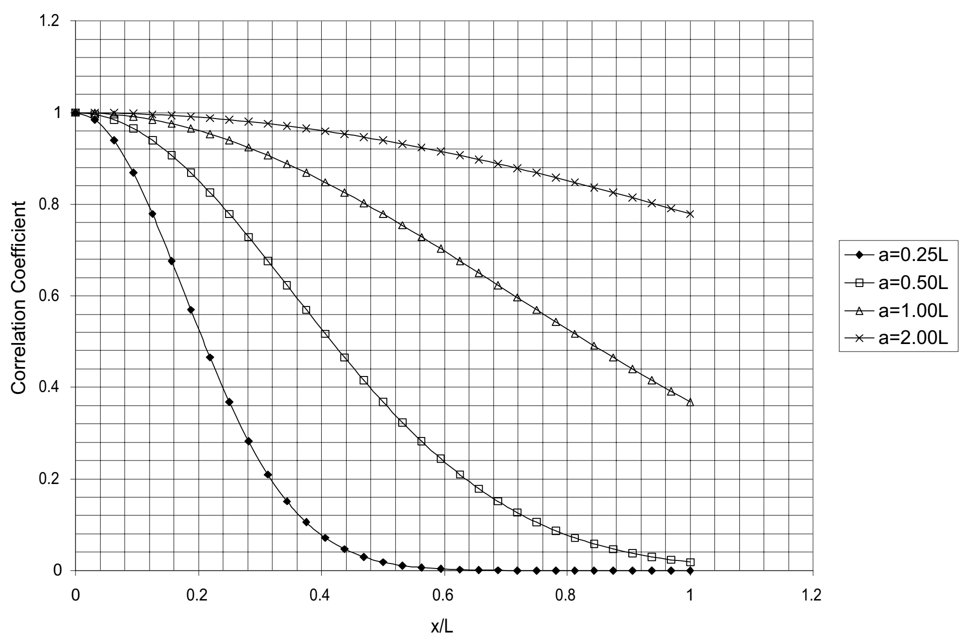


Figure 4: Variation of the correlation function for correlation length factor, $a = 0.25L, 0.5L, 1.0L, 2.0L; N^* = 32$.

in this case also.

5 Adaptive Mesh Refinement

Our objective is to propose a method to adaptively refine the random field mesh, without excessively increasing the number of random field elements. It is expected that the adaptively refined mesh will result in improved estimation of reliability of the problem considered. In an adaptive refinement process, the following issues will be of interest:

- identification of a refinement indicator which determines regions where the mesh refinements are most effective,
- identification of an error indicator which serves as a figure of merit for the current solution, and
- implementation of effective computational algorithms for new improved solutions.

In the following sections we discuss the proposed refinement indicator, error indicator and computational algorithm for adaptive random field mesh refinement in the context of stochastic finite element analysis.

5.1 Proposed Refinement Indicator

It is noted from literature that the importance measures for ranking the random variables have been used successfully in conjunction with sampling/simulation methods to reduce the computational effort by eliminating the lowly ranked random variables from the analysis (Brenner and Bucher, 1995; Gupta and Manohar, 2004). However, we realize that, the importance measures in general, and the global importance measures in particular, possess potential, which could be utilized in the context of stochastic finite element method. With appropriate transformations, these importance measures offer an effective means to identify the spatial regions in the stochastic finite element model where refinement of random field mesh is required. In other words, the global importance measures can be used to serve the purpose of indicators for carrying out adaptive refinement of random field mesh. This is analogous to the energy based error indicators used in conventional deterministic adaptive finite element methods. Furthermore, this approach works with multiple random fields and performance functions with multiple regions of comparable importance. We have used global importance measures with appropriate transformations as mesh refinement indicators.

In the present work, we assume that the performance function could consist of multiple regions of comparable importance. Fig. 5 shows a few performance functions where there are multiple regions contributing to the failure probability. The limit state surfaces in Fig 5(a) and (b), respectively, has two and four points, which have the same minimal distance from the origin O. These are called multiple design points, each of which makes equally important contributions to the failure probability. Fig. 5(c) shows a performance function, where points a and b are situated at distances β_1 and β_2 respectively from the origin, where $\beta_2 - \beta_1 < \varepsilon$ with ε being a very small number. In this case, even though there exists only a single design point a, it is obvious that point $b$also makes significant contribution to the failure probability. Fig. 5(d) shows two performance functions, one that is a perfect circle (full line) and the other an ellipse obtained by a minor modification to the circle (dotted line). For the performance function that is a circle, all points on the limit surface lie at the same distance from the origin and hence, the number of design points is infinite in number. On the other hand, for the ellipse, there exist two design points, c and d, but infinitely many points that are not strictly the design points, nevertheless, make significant contributions to the failure probability.

We use the global importance measure proposed by Gupta and Manohar (2004) for performance functions with multiple design points for ranking the discretised random variables in the order of their relative importance, and, also, to investigate if such ranking could be used for adaptive mesh refinement in stochastic finite element reliability analysis. If R distinct points are identified in the n-dimensional standard normal space, the global importance measure for the i^{th} element in the vector of random variables Ψ in non-Gaussian space

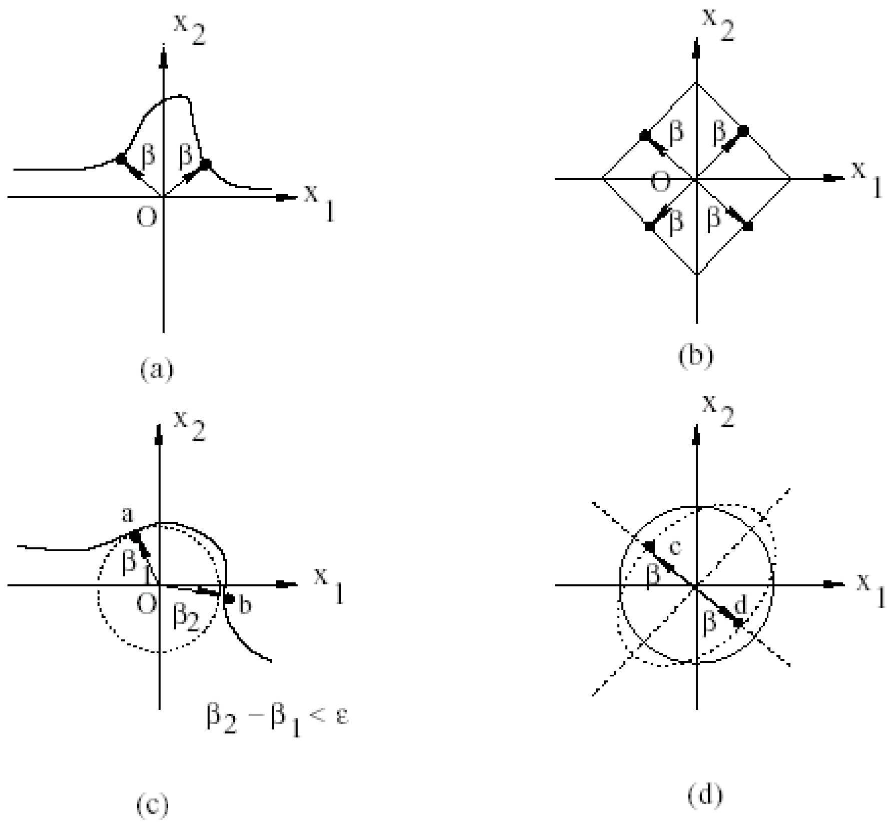


Figure 5: Performance functions in standard normal space with multiple design points (Gupta, 2004)

is computed using the expression,

$$\gamma_i = \sum_{l=1}^{R} w_l \lambda_l^{i^2} \Big/ \sum_{i=1}^{n}\sum_{l=1}^{R} w_l \lambda_l^{i^2} \tag{5}$$

where, the parameters w_l and w_l are defined later. The vector of correlated non-Gaussian random variables $\boldsymbol{\Psi}$ is related to the vector of normal variables $\boldsymbol{\Gamma}$, through Nataf's transformation, given by $\boldsymbol{\Gamma} = \boldsymbol{\Phi}^{-1}\left[F_{\Psi}(\boldsymbol{\Psi})\right]$. Here, F_{Ψ} and $\boldsymbol{\Phi}(\cdot)$ are, respectively, the marginal probability distribution functions of $\boldsymbol{\Psi}$and $\boldsymbol{\Gamma}$. The vector of correlated normal random variables $\boldsymbol{\Gamma}$are related to the standard normal variables $\boldsymbol{\Gamma'}$ through the expression $\boldsymbol{\Gamma} = \boldsymbol{L\Gamma'}$, where $\boldsymbol{L}$ is the lower triangular matrix, such that $\boldsymbol{C_\Gamma} = \boldsymbol{LL}^T$. Here, $\boldsymbol{C_\Gamma}$ is the correlation coefficient matrix of $\boldsymbol{\Gamma}$ expressed in terms of the known correlation coefficients $\boldsymbol{C_\Psi}$ of $\boldsymbol{\Psi}$ (Der Kiureghian and Liu, 1986). The matrix $\boldsymbol{L}$ is determined by Cholesky decomposition of $\boldsymbol{C_\Gamma}$. Since the proposed reliability analysis algorithm is carried out in the standard normal space, the direction cosines for the identified design points $\left(\frac{\partial G(\boldsymbol{\Gamma'})}{\partial \boldsymbol{\Gamma'}}\right)$are obtained in the $\boldsymbol{\Gamma'}$-space. The corresponding direction cosines in the $\boldsymbol{\Psi}$-space are expressed as, $\left(\frac{\partial g(\boldsymbol{\Psi})}{\partial \boldsymbol{\Psi}}\right) = \boldsymbol{J}\left(\frac{\partial G(\boldsymbol{\Gamma'})}{\partial \boldsymbol{\Gamma'}}\right)$, where, $\boldsymbol{J}$ is the Jacobian matrix, $g(\boldsymbol{\Psi})$ is the performance function in non-Gaussian space and $G(\boldsymbol{\Gamma'})$ is the performance function in standard normal space. The parameter λ_l^i in Eq. (5) is now defined as,

$$\lambda_l^i = \sum_{j=1}^{R} w_l \left\{ \left(\frac{\partial g(\Psi)}{\partial \Psi_j} \right)^i \right\}^2 \tag{6}$$

Here w_l are weighting functions. The details of computation of gradients $\frac{\partial g(\boldsymbol{\Psi})}{\partial \boldsymbol{\Psi}}$ of the performance function are discussed in the work by Manjuprasad (2005). To give a greater weightage to the points closer to the origin in the $\boldsymbol{\Gamma'}$-space, the weights w_l are defined in terms of probability content associated with each point in the $\boldsymbol{\Gamma}'$-space, and are expressed as,

$$w_l = \Phi(-\beta_l) \Big/ \sum_{j=1}^{R} \Phi(-\beta_j) \tag{7}$$

where, β_l is the Hasofer-Lind reliability index of the lth point in the $\boldsymbol{\Gamma}'$-space. Thus, points in the failure domain more likely to fail are given greater weightage.

While carrying out SFEM based reliability analysis, the vector $\boldsymbol{\Psi}$ also contains a subset of discretised random field variables Ξ. Hence, the set of global importance measures computed using the procedure discussed above would also contain a subset of GIM values corresponding to the discretised random field variables. In this study, it is proposed to use these global importance measures with respect to discretised random field variables as refinement indicators for carrying out adaptive refinement of random field element meshes.

5.2 *Proposed Error Indicator*

In the conventional adaptive mesh refinements it is common to use energy-based norms for estimation of errors and to arrive at global error indicators. In the present method we propose to use the reliability index based norm for estimation of error. We begin by describing the random field mesh using a coarse mesh and subsequently refine the mesh. Let P_f denote the estimate of failure probability with respect to one or more performance functions using a specific mesh configuration. We treat a uniformly refined 'fine' mesh as the 'reference mesh'. Since it is not practically possible to estimate the exact value of failure probability in advance, it is assumed that the estimate obtained using the 'reference mesh' as the best solution. Let P_f^* denote the estimate of failure probability using this reference mesh. The relative error percentage, η_P, in estimation of failure probability obtained using any specified mesh can therefore be defined as,

$$\eta_P(\%) = \frac{\left|P_f - P_f^*\right|}{P_f^*} \times 100 \tag{8}$$

The probabilities P_f and P_f^* could also be expressed in terms of notional reliability indices $\beta = -\Phi^{-1}(P_f)$ and $\beta^* = -\Phi^{-1}\left(P_f^*\right)$ leading to an alternative definition of relative error index, η_β, given by

$$\eta_\beta(\%) = \frac{|\beta - \beta^*|}{|\beta^*|} \times 100 \tag{9}$$

Fig. 6 shows the relationship between P_f and β. It is of interest to note that β is zero when P_f is 0.5, β is negative for $P_f > 0.5$, and, β is positive for $P_f < 0.5$. The presence of modulus in the denominator of Eq. 9, thus takes into account the probability of β^* being negative. Also, given the fact that β varies monotonically with respect to P_f, it is clear that either η_P or η_β could equally well serve the role of error indicator. In the present study we have preferred to use η_β for discussion.

6 Mesh Refinement Strategy

As has been already pointed out, there exist separate meshes for displacement field and each of the random fields that represent different structural properties. In the present study, the configuration of all these meshes could be different from each other. Each of the structural property random fields is represented through its own EOLE. While the displacement field mesh could be coarser or finer than the random field meshes, in the formulation of the structural matrices however, the structural properties, such as, elastic modulus and mass density, within a displacement field element is taken to be a constant. These constants themselves are deduced from the respective EOLE of the relevant random fields. In the implementation of mesh refinements, it is assumed that the mesh for displacement field has been arrived at independently and this mesh itself is not necessarily open for refinement. The mesh refinement process begins with the definition of a coarse mesh for all the random fields of interest. Upon completing the reliability analysis of the structure with this mesh, indicators of relative importance of different random field elements are also obtained. Based on this, one could enhance the number of random field elements in the important zones and

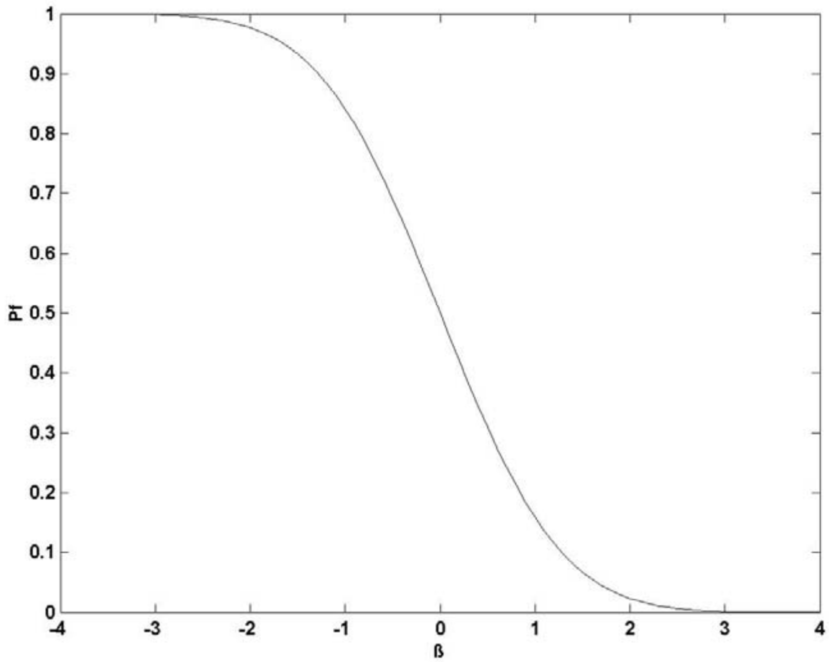


Figure 6: Relationship between probability of failure and reliability index

deplete the number of elements in the least important regions. Throughout the present study however, we adopt structured meshes that are refined using successive bisection as and when needed.

7 Finite Element based Reliability Analysis

Through appropriate transformations, the basic random field/process variables obtained through random field discretisation are transformed to spatial element random variables/ temporal random load time history variables. Finite element analysis (static/free vibration/forced vibration) is carried out using suitable element formulations to compute the performance function and the gradients of the performance function with respect to the required random variables. Structural reliability analysis is carried out using first-order reliability method (FORM) with improvements as has been mentioned in Section 3. In reliability analysis using first-order reliability method, optimization algorithms are commonly used to obtain the design point and the corresponding reliability index or safety index β. In the present work the Rackwitz and Fiessler method (Rackwitz and Fiessler, 1978) is used to solve the optimization problem.

In FORM, an approximation to the probability of failure is obtained by linearising the limit-state surface at the design point. The existence of multiple design points may cause the following problems in FORM: first, the optimization algorithm may converge to a local design point; in this case, the FORM solutions will miss the region of dominant contribution to the failure probability integral and, hence, the corresponding approximations will be in gross error; second, even if the global design point is found, there could be significant contributions to the failure probability integral from the neighborhoods of local points approximating the limit-state surface only at the global design point will not account for these contributions. In the present research work, search for multiple design points of a reliability problem is carried out using the method of artificial barriers proposed by Der Kiureghian and Dakesian (1998). The computation of multinormal integrals is a necessary step for estimating the probability of failure of structural systems. In the present work, an approach using the product of one-dimensional normal integrals is used is used to get an approximate value of the multinormal integral. The method is called product of conditional marginals (PCM) method and was proposed by Pandey (1998).

Adaptive stochastic finite element reliability analysis of linear structural problems (static/free vibration/forced vibration) is carried out using formulations to compute the performance function and the gradients of the performance function with respect to the required element random variables or random load time history variables.

For reliability analysis of structures under stochastic static loads the performance function defined in terms of displacement at a given node $\hat{\boldsymbol{x}}$ is given by,

$$g = u^* - u(\hat{\boldsymbol{x}}, \boldsymbol{\Psi}) \tag{10}$$

Here, u^* is the permissible displacement and $\boldsymbol{\Psi}$ is the vector of non-Gaussian random variables obtained using EOLE method of discretisation of the relevant random field meshes.

In the context of vibration problems, one of the commonly used design criteria is related to the avoidance of resonance in periodically driven systems. Thus, if we consider a system driven harmonically, at a frequency ω^*, the designer would wish to keep the nearest structural natural frequency away from ω^*. Or, alternatively, one may limit the resonant amplitude within prescribed thresholds. In the former case, one can define the performance function as the region $\omega^* - \Delta\omega \leq \omega_i(\boldsymbol{\Psi}) \leq \omega^* + \Delta\omega$, where, $\omega_i(\boldsymbol{\Psi})$ is the structural natural frequency centered around ω^* which needs to be avoided. Since the natural frequencies of randomly parametered systems are themselves random, one can define the failure problem as,

$$P_f = P\left[(\omega_1^* - \omega_i(\boldsymbol{\Psi}) \leq 0) \bigcap (\omega_i(\boldsymbol{\Psi}) - \omega_2^* \leq 0)\right] \tag{11}$$

where, $\omega_1^* = \omega^* - \Delta\omega$ and $\omega_2^* = \omega^* + \Delta\omega$. This problem itself can be perceived as a problem in parallel system reliability with,

$$g_1 = \omega_1^* - \omega_i(\boldsymbol{\Psi}) \tag{12}$$

$$g_2 = \omega_i(\boldsymbol{\Psi}) - \omega_2^* \tag{13}$$

Thus one gets,

$$P_F = P[\omega_i \in (\omega^* - \Delta\omega, \omega^* + \Delta\omega)] \tag{14}$$

$$= P[\omega^* - \Delta\omega \leq \omega_i \leq \omega^* + \Delta\omega] \tag{15}$$

$$= P[\omega_1^* \leq \omega_i \leq \omega_2^*] \tag{16}$$

Here, $\Delta\omega = 0.5*$ half power bandwidth=$\eta\omega^*$, ω^*=resonance frequency, η=damping ratio corresponding to the mode considered.

$$P_F = P\left[\bigcap_{k=1}^{2} g_k(\boldsymbol{\Psi}) \leq 0\right] \tag{17}$$

Next we consider the reliability analysis of structures in forced vibration under stochastic excitations. The performance function here is defined in terms of peak displacement at a given node $\hat{\boldsymbol{x}}$ in given time duration, which can be expressed as,

$$g = u^* - \max_{0 \leq t \leq T} |u(\hat{\boldsymbol{x}}, t, \boldsymbol{\Psi})| \tag{18}$$

The gradients of the performance function are computed using direct differentiation method. Details of formulations developed and used for reliability analysis of static, free vibration and forced vibration problems including gradients of structural matrices with respect to parameters of interest are given in the work by Manjuprasad (2005).

In reliability analysis using FORM, $|\beta_{k+1} - \beta_k| \langle 0.001$ and $\left|g\left(u^{k+1}\right) - g\left(u^k\right)\right| \langle 0.001$ are used as stopping criteria. In addition, maximum number of iterations is set equal to 25. In the method of bulges, $|\beta_k/\beta_1| \langle 1.5$ is used as a stopping criterion in addition to a maximum number of iterations equal to 5.

8 Summary of the Steps

We summarize here the various steps involved in SFEM based reliability analysis using adaptively refined random field meshes:

1. Input data: This includes the following details:
   a. Structural geometry and boundary conditions.
   b. Probabilistic characteristics of structural properties, applied loads, and performance limits. This description could be in terms of types of first order pdf, its parameters, and the definition of auto-covariance and cross-covariance functions.
   c. Number of performance functions (N_P) and their details.
   d. Details of displacement field discretisation including element types, number of elements and mesh configuration.
   e. Details of the initial coarse mesh for the different random fields.
   f. Tolerances on computation of reliability index using Rackwitz and Fiessler method, number of iterations and maximum number of design points (N_D), number of terms in of expansion using EOLE method.

2. Reliability estimation: A loop over performance function runs at the outset. Within this loop a sub-loop over multiple design points is implemented. Within each loop for a design point, reliability index is computed using FORM in conjunction with Rackwitz-Fiessler algorithm. The required performance function and its gradients are computed using SFEM. For the i^{th} performance function we obtain β_{ij} and DP_{ij}, with $j = 1\ldots k_j \leq N_D$, where β_{ij} is the j^{th} reliability index associated with the i^{th} design point DP_{ij} for the i^{th} performance function. The i^{th} performance function is now linearized around each of the design point DP_{ij}, $j = 1\ldots k_j \leq N_D$ and a set of hyperplanes HP_{ij}, $j = 1\ldots k_j \leq N_D$ are obtained. Subsequent to the completion of the loop over all the performance functions we would be left with a collection of hyperplanes each of which represents a plane that is obtained by linearising one of the performance functions around one of its design points. We now perform a system reliability analysis using the PCM method paying due consideration to the configuration of the hyperplanes to be in series or parallel format. It must be noted that the set of performance functions defined through the hyperplanes originating from a given performance function are all in series while the performance functions themselves could be in series or parallel.

3. Computation of GIM: The GIMs as given by Eq. 5 are now computed with R=total number of design points (DP_{ij}) evaluated in the previous step. Similarly, in computing the weights as in Eq. 7, the resulting reliability indices (β_{ij}) computed earlier are used. This leads to assignment of importance measures to each of the elements in the non-Gaussian vector $\boldsymbol{\Psi}$.

4. Mesh refinement: The GIMs calculated in the previous step point spatial regions in which each off the elements of the random fields $\boldsymbol{w}(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x},t)$ need further refinement. Based on this we employ a combination of mesh refinement and derefinement to arrive at the revised definition of random field mesh configuration.

5. Stopping criteria: Repeat steps (1) to (4) till satisfactory convergence on reliability estimate is obtained.

9 Numerical Examples and Discussion

The procedures discussed in the previous sections are illustrated with the help of the following three problems:

1. Failure due to excessive displacements of a statically loaded one span beam. The elastic modulus of the beam is modeled as a lognormal random field.

2. Failure of a cantilever beam with random elastic modulus due to occurrence of resonance.

3. Failure due to excessive displacements of a cantilever beam under random dynamic excitation. Here, the elastic modulus and mass density of the beam are treated as lognormal random fields and load is considered as a stationary Gaussian random process.

The numerical examples illustrate the following range of issues relevant to the problem:

1. Effect of choice of Nin EOLE method of discretisation.
2. Mesh refinement with respect to alternative performance functions as well as multiple performance functions
3. Mesh refinement issues that arise in static analysis, free vibration analysis, and forced vibration analysis using direct integration.
4. Treatment of more than one random field within the same structure.
5. Use of one-dimensional beam and two-dimensional plane stress finite elements
6. Treatment of one-dimensional and two-dimensional random fields

Example 1

In this example, we consider the beam structure shown in Fig. 7. The data for this beam are as given in the paper by Liu and Liu (1993). Thus, we take the span of the beam as 9.76 m (32 ft) and the load to be static with uniformly distributed intensity of 116.7 N/m (8k/ft). The elastic modulus is assumed to be a homogeneous non-Gaussian random field with a lognormal first order pdf and auto-covariance function of the form in Eq. 3. Consequently, the flexural rigidity of the beam also becomes a random field and we take its mean value to be 465706.41 N-m^2 (1,125,000 k-ft^2) and a coefficient of variation (COV) of 0.20. The structure is modeled using a set of Euler-Bernoulli beam elements. The following parametric studies have been conducted to bring out different aspects of mesh refinement procedure.

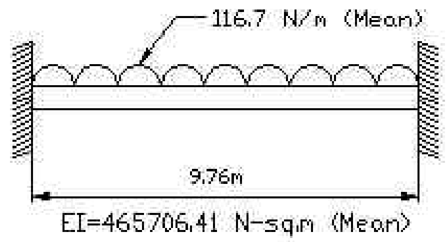


Figure 7: Geometry and loading of fixed end beam for Example 1

Study 1: Adaptive Refinement of Random Field Mesh

Here, we take $a = 0.25L, COV = 0.2$ and $r = 5$. We focus attention on the influence of varying number of elements in the random field mesh. We define the performance function in terms of midspan deflection namely $g(\Psi) = \delta^* - \delta_{mid}$. We assume that δ^*=7.62 mm (0.3 inch). We begin by taking 32 elements (with total degrees of freedom =96) for the displacement field mesh and consider the effect of varying the number of elements in the random field mesh with their lengths being held uniform. These results are shown in Table 1a in which the number of elements in the random field mesh is varied from 8 to 32 and the notional reliability index β is computed for all the mesh configurations. To describe the displacement field mesh and random field mesh we characterize each mesh configuration by a pair of numbers (N_1, N_2), where, N_1=number of elements in the displacement field mesh and N_2=number of elements in the random field mesh. For the mesh configuration (32, 32) we obtain β=2.0087 and this value of β is taken to be the reference β^* in computing η_β. From Table 1a it can be seen that η_β reduces monotonically as the number of elements in the random field mesh are increased from 8 to 32. The effect of adaptively refining the random field mesh is studied in Tables 1b and 1c.

In obtaining the results in Table 1b we begin by analyzing the GIM for the uniform (32,8) mesh (Fig. 8). The resulting refined random field mesh is shown in Fig. 9. For this configuration we obtain a value of β=2.0091 and η_β=0.0199 (Table 1b). A comparison of η_β for non-uniform (32, 12) mesh (Table 1b, η_β=0.0199) and uniform (32,12)

Table Table 1a: Uniform refinement of RF mesh

N_1	N_2	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert} \times 100$
32	8	2.0126	0.1942
	12	2.0101	0.0697
	14	2.0096	0.0448
	16	2.0093	0.0299
	18	2.0091	0.0191
	20	2.0089	0.0100
	22	2.0088	0.0050
	32	2.0087(β^*)	0.0000

Table Table 1b: Adaptive refinement of RF mesh using bisection of elements

N_1	N_2	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert} \times 100$
32	12	2.0091	0.0199
	14	2.0089	0.0010
	16	2.0089	0.0010

Table Table 1c: Adaptive refinement of RF mesh by relocation of nodes

N_1	N_2	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert} \times 100$
32	8	2.0138	0.2539
	16	2.0091	0.0199

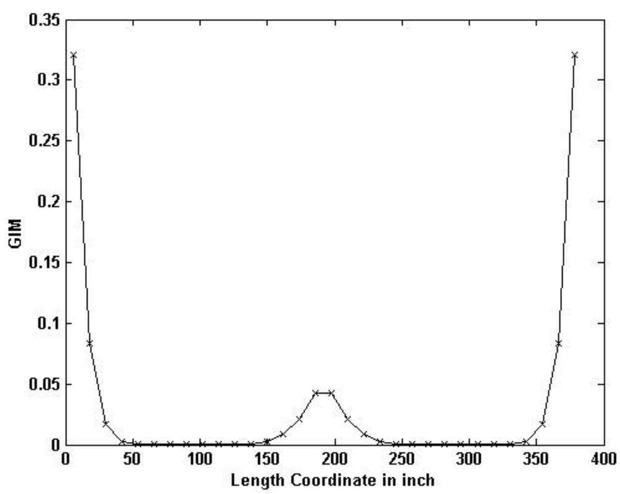


Figure 8: GIM plot for (32, 8) uniform mesh

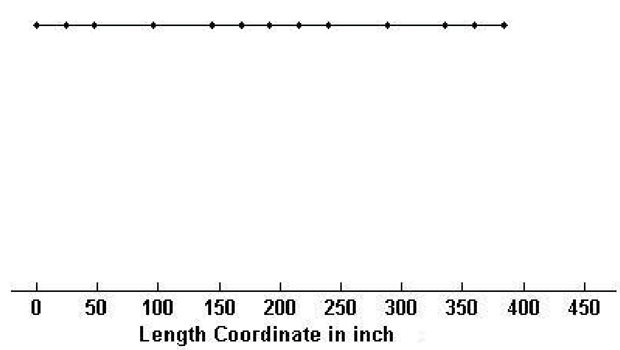


Figure 9: Adaptive 12-element RF mesh for elastic modulus based on GIM plot for (32, 8) uniform mesh

mesh (Table 1a, η_β=0.0697) clearly shows a reduction inη_β by a factor of 3.5 due to adaptive refinement of the random field mesh for the problem considered. Based on the refined (32, 12) mesh the GIM obtained is shown in Fig. 10, which, in turn, leads to the refined (32, 14) mesh shown in Fig. 11. Further refinements based on the GIM for the (32, 14) mesh (Fig. 12) leads to the refined mesh (32, 16) as shown in Fig. 13. At this stage we observe that η_β and β have converged to a value of 0.001 and 2.0089, respectively, and one could stop further refinement at this stage.

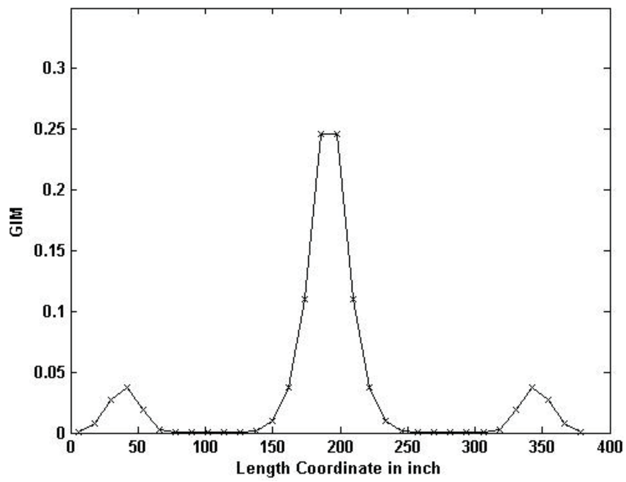


Figure 10: GIM plot for (32, 12) adaptive mesh

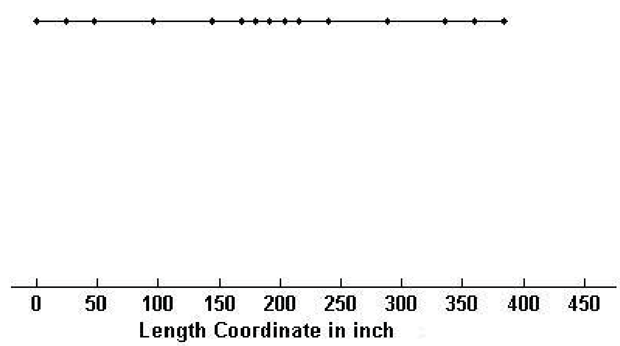


Figure 11: Adaptive 14-element RF mesh based on GIM plot for (32, 12) adaptive mesh

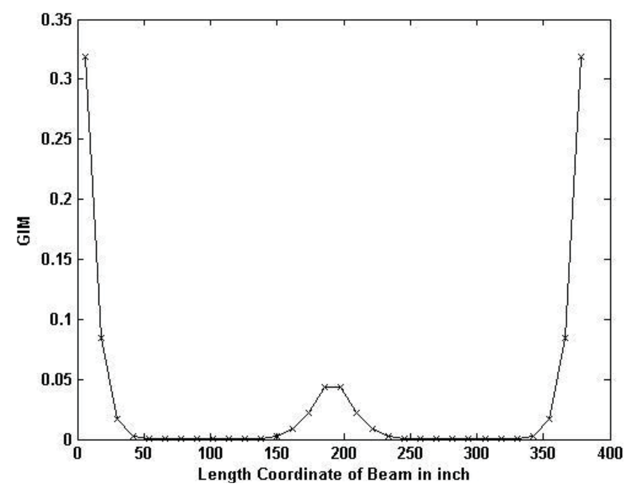


Figure 12: GIM plot for (32, 14) adaptive mesh

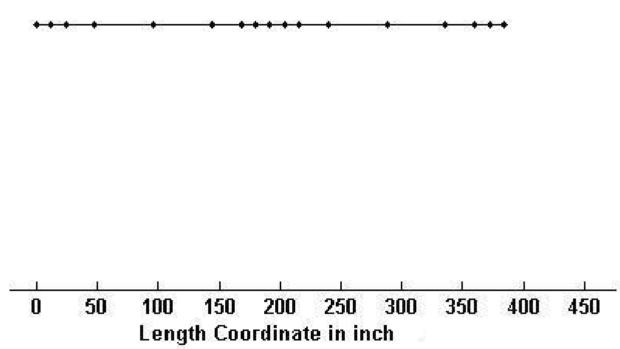


Figure 13: Adaptive 16-element RF mesh based on GIM plot for (32, 14) adaptive mesh

In studies reported in Table 1b, the mesh refinement was carried out by successive bisection, without making efforts to remove nodes that are located in regions of low GIM. The question of relocating nodes is considered in Table 1c. Here, based on GIM for the uniform (32, 8) mesh (Fig. 8), we obtain the random field mesh as shown in Fig. 14a. Here, nodes that lie in regions of low GIM ($50 < x <150$ and $250 < x < 350$) have been removed and shifted to regions of high GIM. Figure 14b shows the node locations for a uniform mesh for comparison with the adaptive mesh. Similar results beginning with the uniform (32, 16) mesh and its GIM plot (Fig. 15) leading to non-uniform (32, 16) mesh with relocated nodes are shown in Fig 16. A comparison of Tables 1a and 1c shows that by relocating the nodes based on GIM for the uniform mesh, one gets mixed results. Thus, for (32, 8) mesh the refinement leads to a deterioration of η_β from 0.1942

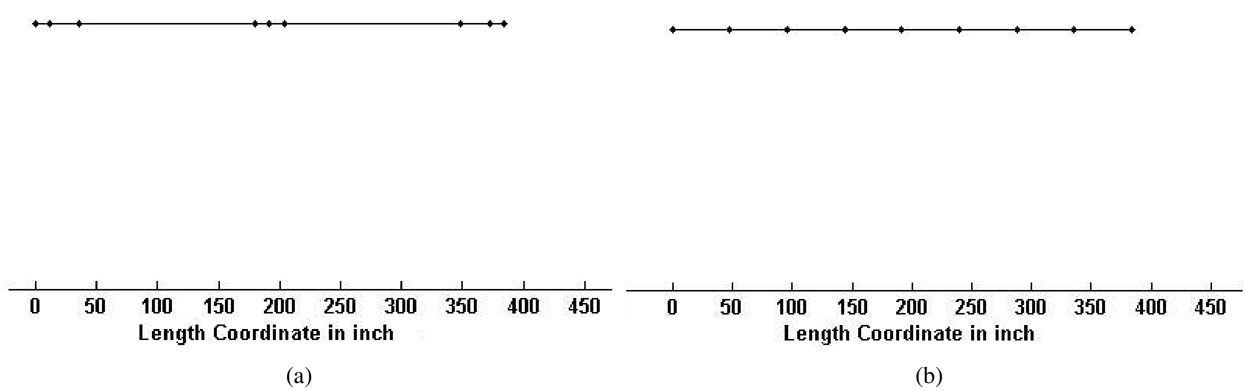


Figure 14: (a) Adaptive 8-element RF mesh using relocation of nodes based on GIM plot for (32,8) uniform mesh (b) Uniform 8-element RF mesh

to 0.2539 while for (32, 16) mesh one obtains an improvement from 0.1205 to 0.0199. This disparity could be explained if one considers the variance of error of discretisation with N=8+1=9 and N=16+1=17 random field nodes. This variation is shown in Fig. 17 wherein the variance for the case N=9 (non-uniform relocated nodes) is seen to be substantially higher than the result for N=9 with uniform placement of nodes. On the other hand, similar results for N=17 (uniform and relocated) do not show as much difference as is observed for N=9. This points towards the usefulness of information on variance of error in EOLE indicating if a refinement strategy involving relocation of nodes could be acceptable or not.

Study 2: Adaptive Refinement of Displacement Field Mesh

The formulation of GIM in this study is mainly motivated by the need to rank the element random variables in terms of their relative importance to the reliability calculations. Thus, it would appear that this measure would mainly be applicable to the refinement of random field mesh. However, the reliability calculations itself would crucially depend on the details of displacement field mesh. Thus, question would arise on the possible usefulness of GIM in relocating the nodes associated with the displacement field mesh. Limited studies have been carried out to explore this issue and the results are summarized in Tables 2a and 2b.

In this study, again we take $a = 0.25L, COV = 0.2$ and $r = 5$. We define the performance function in terms of midspan deflection namely $g(\Psi) = \delta^* - \delta_{mid}$. We assume that δ^*=7.62 mm (0.3 inch). We begin by fixing 32 elements for the random field mesh and consider the effect of varying the number of elements in the displacement field mesh to 8, 16 and 32, with their lengths being held uniform. The results of reliability analysis are shown in Table 2a. Now beginning from GIM plots for (8, 32) and (16, 32) meshes (Figs. 18 and 19) we relocate the nodes of the displacement field meshes as shown in Figs. 20 and 21 respectively. The corresponding results of reliability analysis are shown in Table 2b. A comparison of Tables 2a and 2b shows that the adaptive refinement of displacement field mesh based on GIM indeed leads to significant improvement in η_β.

Study 3: Adaptive Refinement with Multiple Random Field Meshes

In this study we consider the same beam problem, but now carrying a spatially varying random load. We model the load as a stationary Gaussian random field and the elastic modulus of the beam as a lognormal random field. The load is

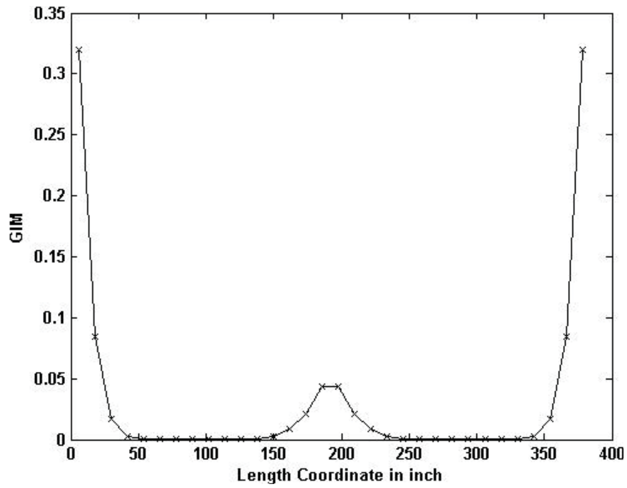


Figure 15: GIM plot for (32, 16) uniform mesh

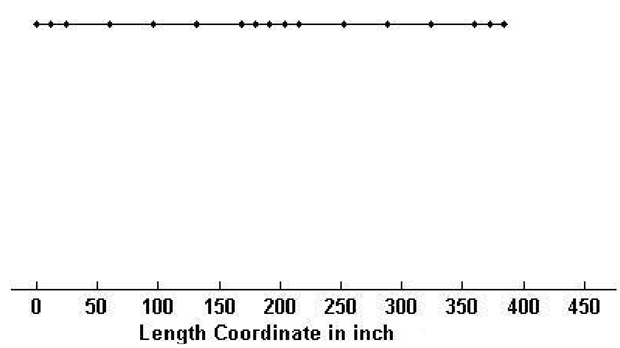


Figure 16: Adaptive 16-element RF mesh using relocation of nodes based on GIM plot for (32, 16) uniform mesh

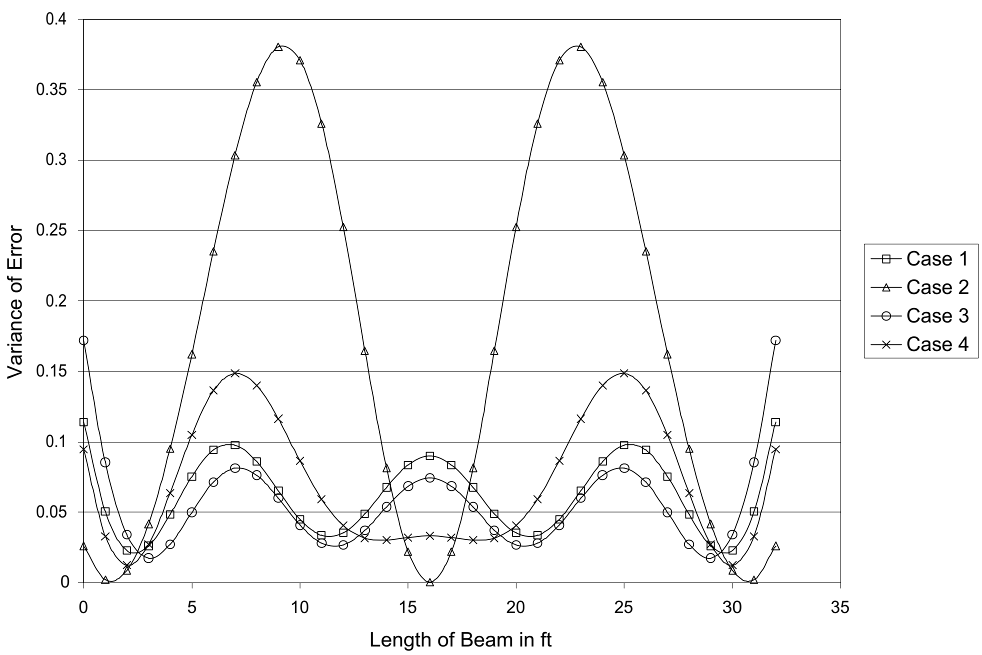


Figure 17: Variation of the variance of error of discretisation using EOLE along the length of the beam, Case (1) uniform 8-element RF mesh, Case (2) adaptive 8-element RF mesh with relocation of nodes, Case (3) uniform 16-element RF mesh, Case (4) adaptive 16-element RF mesh with relocation of nodes; number of terms of expansion r=5; correlation length factor $a = 0.25L$; number of finite elements $N^* = 32$.

Table Table 2a: Uniform refinement of DF mesh

N_1	N_2	DOF	$\boldsymbol{\beta}$	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert} \times 100$
8	32	24	1.9867	1.0952
16		48	2.0044	0.2141
32		96	2.0087($\boldsymbol{\beta}^*$)	0.0000

Table Table 2b: Adaptive refinement of DF mesh by relocation of nodes

N_1	N_2	DOF	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert} \times 100$
8	32	24	2.0238	0.7517
16		48	2.0095	0.0398

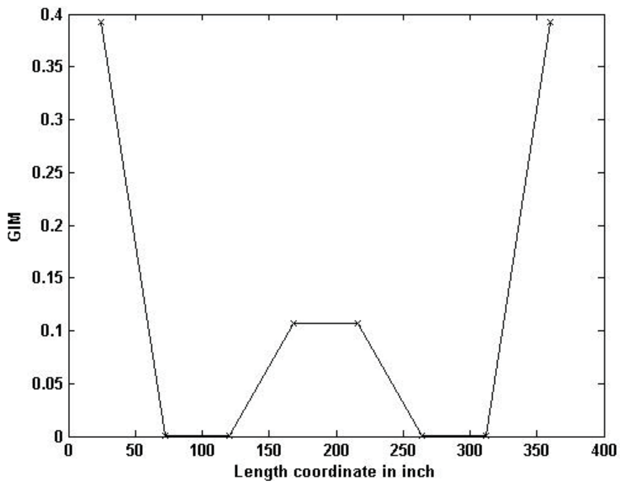


Figure 18: GIM plot for (8, 32) uniform mesh

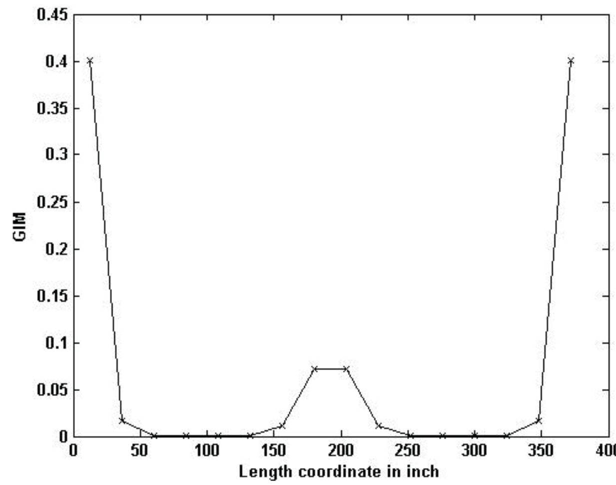
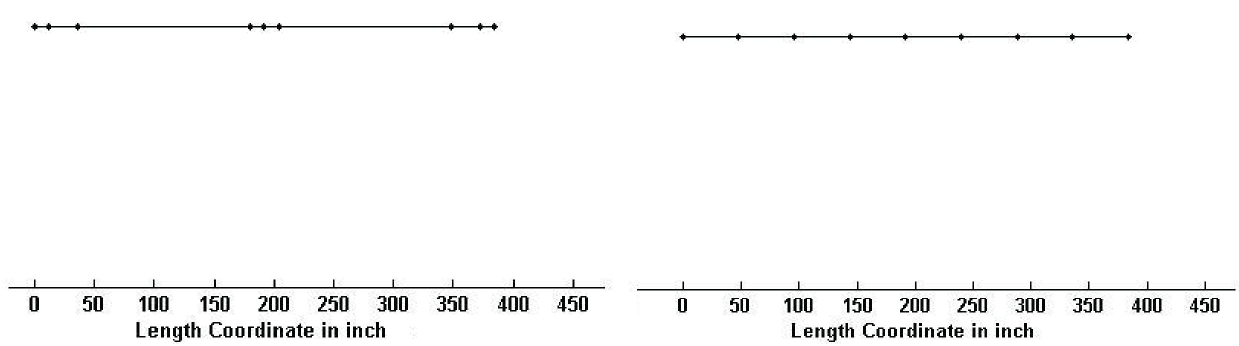


Figure 19: GIM plot for (16, 32) uniform mesh



(a) (b)

Figure 20: (a) Adaptive 8-element DF mesh using relocation of nodes based on GIM plot for (8, 32) uniform mesh (b) Uniform 8-element DF mesh

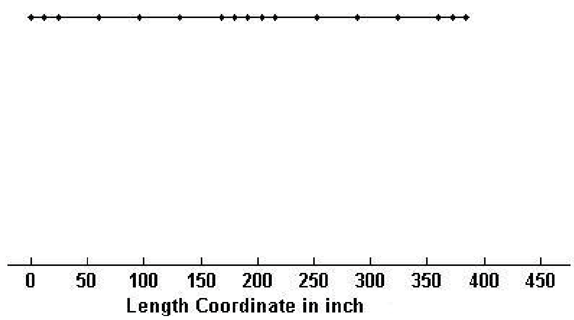


Figure 21: Adaptive 16-element DF mesh using relocation of nodes based on GIM plot for (16, 32) uniform mesh

modeled as $q(x) = q_0(1+\varepsilon f(x))$ with q_0=116.7 N/m (8k/ft) and ε=0.2. Here, $f(x)$ is a zero mean stationary Gaussian random field with covariance function given by Eq. 3, with a=0.25L. Since elastic modulus of the beam is a random field, the flexural rigidity of the beam is a random field and we take its mean value to be 465706.41 N-m^2 (1,125,000 k-ft^2) with a coefficient of variation of 0.20 and $a = 0.25L$. The spatially varying elastic modulus $E(x)$ and the distributed load $q(x)$ are discretised using EOLE with r=5. We define the performance function in terms of midspan deflection namely $g(\Psi) = \delta^* - \delta_{mid}$. We assume that δ^*= 6.35 mm (0.25 inch). The problem on hand consists of arriving at refined meshes for the random fields $E(x)$ and $q(x)$. We introduce a notation (N_1, N_2, N_3) to characterize a mesh with N_1=number elements in the displacement field mesh, N_2=number of elements in the random field mesh for $E(x)$ and N_3=number of elements in the random field mesh for $q(x)$.

Table 3 shows results for (32, 8, 8), (32, 14, 12) and (32, 32, 32) uniform meshes. Using the (32, 32, 32) uniform mesh, the reference notional reliability index β^* is determined as 1.3595. The last row in Table 3 provides results for (32, 14, 12) mesh in which the random fields $E(x)$ and $q(x)$ have been adaptively refined based on GIM for (32, 8, 8) uniform mesh. The plots of GIM for $E(x)$ and $q(x)$ are shown respectively in Figs. 22 and 23. It is clear that the locations needing refinement for $E(x)$ and $q(x)$ fields are dissimilar. Figs. 24 and 25 respectively show the location of nodes for adaptively refined random field meshes for $E(x)$ and $q(x)$. The benefit of refining the random field mesh is clearly seen in Table 3 in which η_β is seen to drop from a value of 2.2803 for a (32,14,12) uniform mesh to a value of 0.4634 for the (32,14,12) adaptively refined mesh.

Study 4: Adaptive Refinement with Independent Multiple Performance Functions

Studies 1–5 have been conducted with respect to the beam provided with fixed supports at two ends. In this study we consider the same beam problem, but now provided with hinged supports at two ends. We consider a performance function with respect to admissible rotations at the left support in addition to the performance criteria with respect to midspan deflection. It must be noted however that here the two performance functions are considered independent of each other. The elastic modulus is assumed to be homogeneous non-Gaussian random field with a lognormal first order pdf and auto-covariance function of the form in Eq. 3. In the numerical work, we take $COV = 0.2$, $r = 5$ and N^*=32. We begin by evaluating β with uniform (32, 32) mesh for the two performance functions namely $g_1(\Psi) = \delta^* - \delta_{mid}$ and $g_2(\Psi) = \theta^* - \theta_{left}$; see Table 4 for the de-

Table 3: Refinement of multiple RF meshes for elastic modulus and distributed load

N_1	N_2 (E)	N_3 (q)	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert}\times 100$	Remarks
32	8	8	1.2816	5.7301	Uniform RF mesh
32	14	12	1.3285	2.2803	
32	32	32	1.3595(β^*)	0.0	
32	14	12	1.3532	0.4634	Adaptive RF mesh

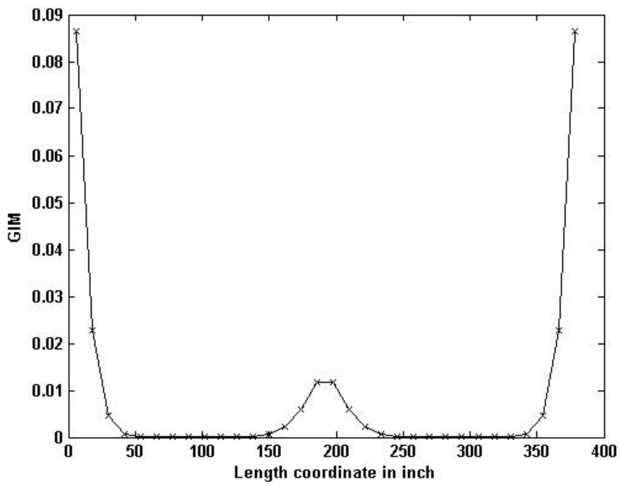


Figure 22: GIM plot for elastic modulus with (32, 8) uniform mesh

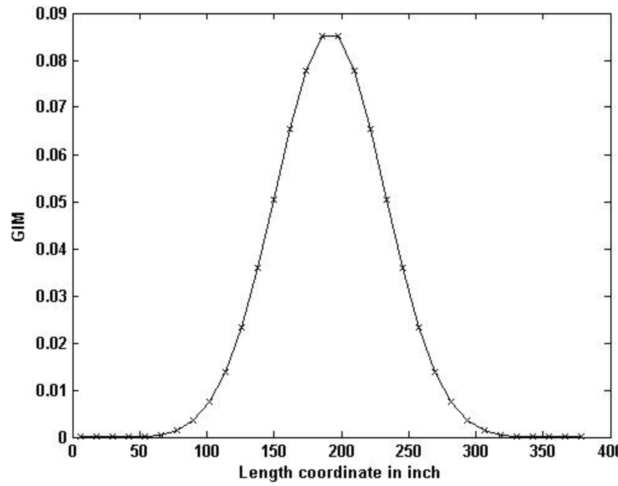


Figure 23: GIM plot for distributed load with (32, 8) uniform mesh

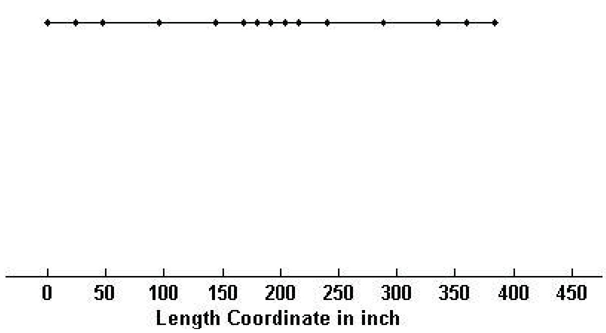


Figure 24: Adaptive 14-element RF mesh for elastic modulus based on GIM plot for (32, 8) uniform mesh

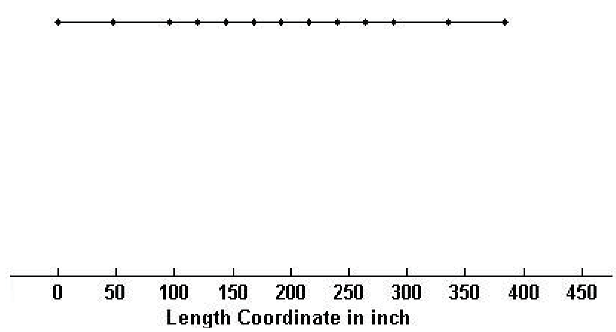


Figure 25: Adaptive 12-element RF mesh for distributed load based on GIM plot for (32, 8) uniform mesh

Table 4: RF mesh refinement with multiple independent performance functions in terms of midspan deflection and left support rotation

N_1	N_2	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert}\times 100$	Remarks
PF in terms of midspan deflection, $g_1(\Psi) = \delta^* - \delta_{mid}$				
32	08	2.72750	0.00073	Uniform RF mesh
32	12	2.72749	0.00037	
32	32	2.72748 (β^*)	0.00000	
32	12	2.72748	0.00000	Adaptive RF mesh
PF in terms of support rotation, $g_2(\Psi) = \theta^* - \theta_{left}$				
32	08	2.87714	0.00070	Uniform RF mesh
32	12	2.87714	0.00070	
32	32	2.87712 (β^*)	0.00000	
32	12	2.87712	0.00000	Adaptive RF mesh

tails.

We assume that δ^*=45.72 mm (1.8 inch) and θ^*=0.015 radian. Next we evaluate the GIM for the two performance functions starting with (32, 8) uniform mesh (see Figs. 26 and 27). Upon successive bisection of important regions we arrive at the refined mesh as shown in Figs. 28 and 29. The corresponding estimates of β are provided in Table 4. It is of interest to note that the GIM variation depends upon the performance function criteria used: for criteria on midspan deflection, the midspan region of the beam requires adaptive refinement whereas, for the performance criteria with respect to the left end rotation the important regions are skewed to the left. Furthermore, a comparison of Fig. 26 with Fig. 8 reveals the influence of changing boundary conditions from fixed ends to hinged ends on the process of random field refinement. Thus it emerges that although a beam that is fixed at the two ends and a beam that is hinged at two ends are symmetric with respect to the midspan, the regions for refining the random field mesh for the purpose of reliability analysis turn out to be different for the two structures even when there is no asymmetry either in loading condition or in the definition of the performance function. This observation points towards the capacity of the refinement procedure defined herein to unravel the important features of discretisation that are difficult at the outset to predict.

Study5: Adaptive Refinement with Mutually Dependent Multiple Performance Functions

In this study, we consider the two performance functions $g_1(\Psi) = \delta^* - \delta_{mid}$ and $g_2(\Psi) = \theta^* - \theta_{left}$ to be in series. This would mean that the beam is considered to have failed if either the midspan deflection exceeds a permissible limit or the rotation at the left end support exceeds a permissible limit. It may be noted that the two performance functions have been earlier studied separately in Study 4. It is important to note that the two performance functions here are mutually dependent. As has already been noted, when considered separately the two performance functions lead to notably different adaptively refined mesh configurations for the elastic modulus random field. Thus it would be of interest to understand the nature of adaptive refinement of the random field mesh when the two performance functions are considered as being in series. In this numerical illustration, we assume that $COV = 0.2$, $r = 5$ and N^*=32. We assume that δ^*= 45.72 mm (1.8 inch) and θ^*=0.015 radian. As before, we use the notation (N_1, N_2) to denote N_1 elements in the displacement field mesh and N_2 elements in the random field mesh. We define β^* with respect to (32,32) uniform mesh. Results for (32, 8), (32, 12) and (32, 32) uniform mesh are summarized in Table 5. It was found during the numerical work that for each of the two performance functions only one design point was considered relevant.

Starting with GIM for (32,8) uniform mesh (see

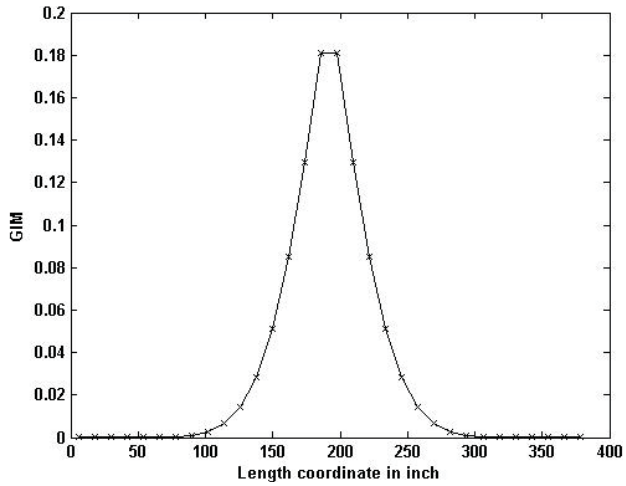


Figure 26: GIM plot for (32, 8) uniform mesh with performance function in terms of midspan deflection

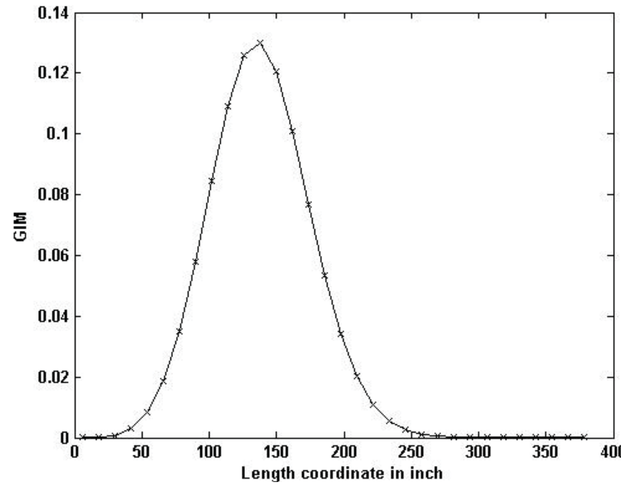


Figure 27: GIM plot for (32, 8) uniform mesh with performance function in terms of left-support rotation

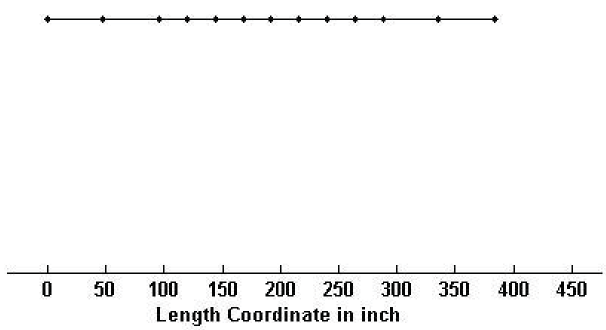


Figure 28: Adaptive 12-element RF mesh for elastic modulus based on GIM plot for (32, 8) uniform mesh with performance function in terms of midspan deflection

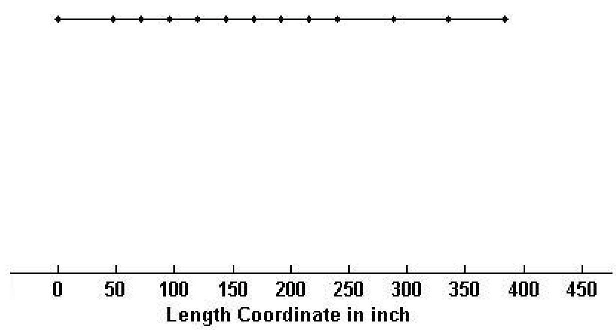


Figure 29: Adaptive 12-element RF mesh for elastic modulus based on GIM plot for (32, 8) uniform mesh with performance function in terms of left-support rotation

Table 5: RF mesh refinement with multiple mutually dependent performance functions in terms of midspan deflection and left support rotation

N_1	N_2	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert} \times 100$	Remarks
32	8	2.58011	0.001163	Uniform RF mesh
32	12	2.58010	0.000775	
32	32	2.58008(β^*)	0.000000	
32	12	2.58008	0.000000	Adaptive RF mesh

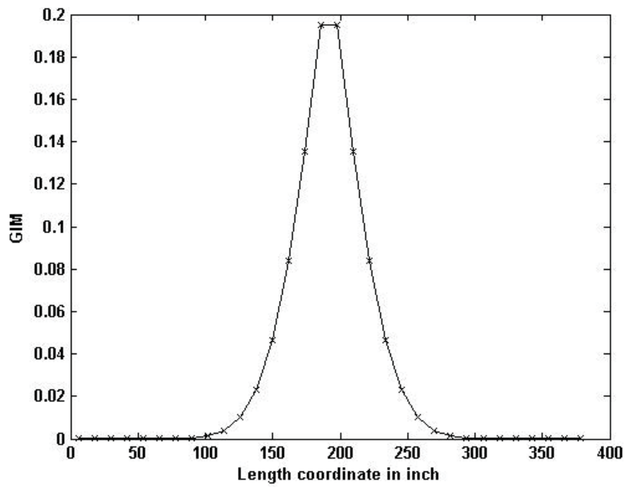


Figure 30: GIM plot for (32, 8) uniform mesh with multiple mutually dependent performance functions in terms of midspan deflection and left support rotation

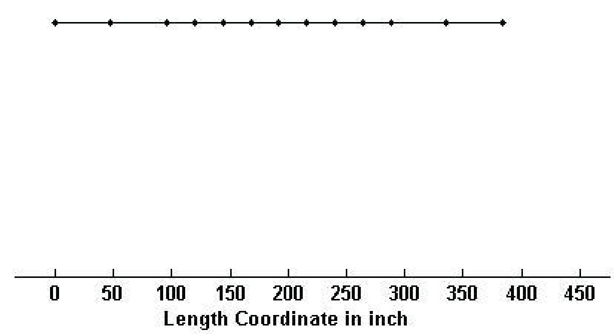


Figure 31: Adaptive 12-element RF mesh for elastic modulus based on GIM plot for (32, 8) uniform mesh with multiple mutually dependent performance functions in terms of midspan deflection and left support rotation

Fig. 30) we obtain an adaptively refined (32,12) mesh (see Fig 31). The result of reliability analysis using this mesh is also shown in Table 5. It may be observed that the GIM shown in Fig. 30 closely resembles the plot when only $g_1(x)$ is considered (see Fig. 26). This observation points towards stronger influence of the performance function with respect to translation than with respect to rotation. As might be expected, the estimate on system reliability (β=2.58008) is lesser than the two reliability indices when the performances with respect to displacement and rotation are considered separately (β=2.72748 and β=2.87712 respectively; see Table 4).

Example 2

In this example we study the reliability of a cantilever beam with performance function defined in terms of the fundamental natural frequency of the beam. The beam structure under study is shown in Fig. 32 and, Nagesh Iyer and Appa Rao (2002) have studied this beam problem earlier in the context of adaptive mesh refinements for deterministic vibration analysis.

We model the beam structure using 4-noded rectangular plane stress elements. The elastic modulus of the beam is modeled as a 2D homogeneous isotropic random field with a lognormal first order pdf and covariance function as given in Eq. 4. The mean value of the elastic modulus is taken as1.0E5 MPa and a coefficient of variation of 0.20 is assumed. The Poisson's ratio, mass density and damping ratio are taken to be 0.3, 1.0E-6 kg/mm^3 and 0.02 respectively. The random field discretisation is carried out using 4-noded quadrilateral elements and adapting EOLE method in conjunction with Nataf's transformation technique. Since the elastic properties of the beam are random in nature, the natural frequencies of the beam turn out to be random in nature. We examine the probability of occurrence of resonance in the beam structure when it is subjected to a harmonic excitation of frequency f^*=1760 Hz; see Section 7 for the basic motivation for defining this performance criteria. We define the failure as the event that the first natural frequency lies in the interval (f_1^*, f_2^*) with $f_1^* = f^* - \Delta f$ and $f_2^* = f^* + \Delta f$. It may be noted that in the absence of randomness in the elastic modulus, the first natural frequency of the beam has a deterministic value of 1653 Hz. The probability of failure is now given by, $P_f = P[f_1^* \leq f_1 \leq f_2^*]$. Thus, the problem can

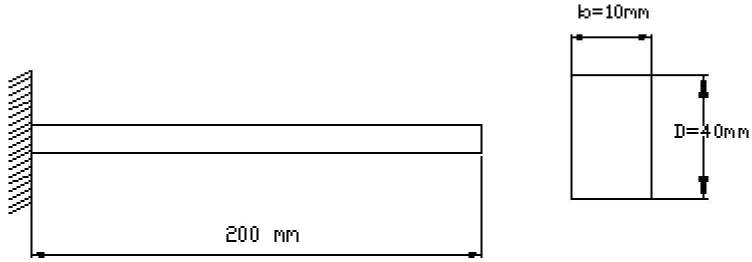


Figure 32: Geometry of cantilever beam for Example 2

be presumed as a problem in parallel system reliability involving two elements. The ingredients for solving this problem have already been summarized in Section 3.

Here, we take $a = 0.25L$ and $r = 5$. We introduce a notation (N_1^*, N_2^*, N_1, N_2) to denote a mesh configuration with $N_1^* \times N_2^*$ elements for the displacement field mesh and $N_1 \times N_2$ for the random field mesh. Here, N_1^* and N_1 denote the number of elements along the length direction of the beam and N_2^* and N_2 denote the number of elements along the depth direction of the beam. We begin by evaluating the reference notional reliability index β^* by using a (10, 2, 20, 2) uniform mesh. Next, by holding the displacement field mesh at 10×2 uniform mesh, we evaluate β for different uniform random field meshes starting with 5×2 and going up to 10×2. These results are shown in Table 6a. The GIM for the (10, 2, 5, 2) mesh is shown in Fig. 33. Based on the GIM plot we now successively refine the random field mesh adaptively to 6×2 and 7×2 as shown in Figs. 34 and 35 respectively. The corresponding results of reliability analysis are shown in Table 6b.

It was found during the numerical work that for each of the two performance functions only one design point was considered relevant. A comparison of Tables 6a and 6b reveals the advantages of adaptive refinement of random field mesh in terms of being able to achieve lesser values η_β with coarser mesh configuration itself.

Example 3

In this example we consider reliability analysis problem involving random dynamic excitation and multiple random field models for spatial variations in structural properties. The example structure considered is a cantilever beam shown in Fig. 36.

The load $f(t)$ due a tip concentrated load is modeled as a zero mean stationary Gaussian random process with a PSD given by band limited white noise with values of variance=10^4 kg^2, lower cut-off frequency=0 rad/s and upper cut-off frequency = 1000 rad/s. The beam structure is modeled using Euler-Bernoulli beam elements. Viscous damping with a proportional damping model is considered for the beam. The elastic modulus and mass density of the beam are modeled as mutually dependent homogeneous non-Gaussian random fields with a lognormal first order pdf and auto-covariance function of the form in Eq. 3. Mean value of elastic modulus is taken as 2.1E5 MPa and a coefficient of variation 0.20 is assumed. Mean value mass density is taken as 7.8E-6 kg/mm^3 and a coefficient of variation 0.20 considered. The Poisson's ratio is taken as 0.3 and damping ratio of first and second modes are assumed to be 0.02 and 0.01 respectively. Furthermore, the applied load is taken to be independent of the random fields for elastic modulus $E(x)$ and mass density $\rho(x)$.

The performance function is defined in terms of the maximum value of the tip displacement measured over specified time duration after the beam has reached a steady state. In implementing the reliability analysis procedure, the load $f(t)$ is replaced by a Fourier series with random coefficients, with number of terms equal to 10. The set of basic random variables includes the coefficients present in the Fourier expansion for the load in addition to the set of basic random variables emerging during the discretisation of $E(x)$

Table Table 6a: Uniform RF mesh refinement with performance function in terms of fundamental natural frequency

$N_1^* \times N_2^*$	$N_1 \times N_2$	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert} \times 100$
10×2	5×2	2.6941	0.8611
	6×2	2.6812	0.3781
	7×2	2.6783	0.2696
	10×2	2.6746	0.1310
	20×2	2.6711(β^*)	0.0000

Table Table 6b: Adaptive RF mesh refinement for cantilever beam with performance function in terms of fundamental natural frequency

$N_1^* \times N_2^*$	$N_1 \times N_2$	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert} \times 100$
10×2	6×2	2.6715	0.0150
	7×2	2.6709	0.0075

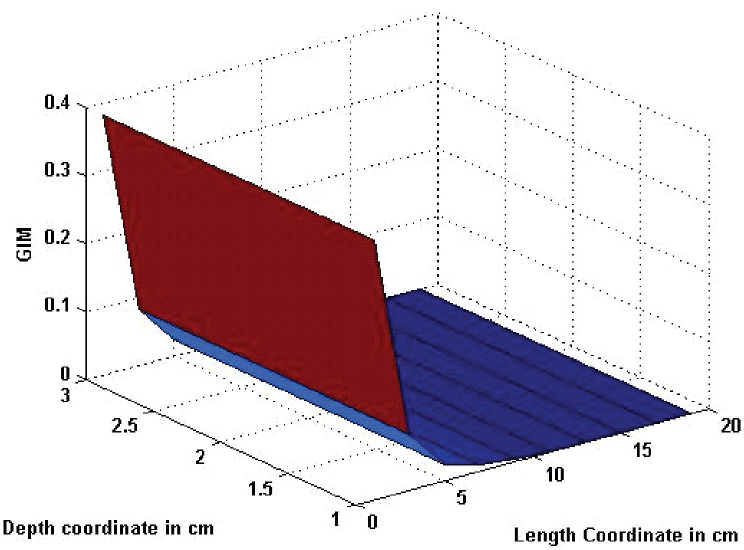


Figure 33: GIM plot for (10, 2, 5, 2) uniform mesh

and $\rho\,(x)$. It is important to note here that, in addition to choosing the details of displacement field and random field meshes, one needs to select the time step of integration as well. We adopt a direct integration approach for solving this problem using Newmark's method with parameters with δ=0.5, γ=0.25, (corresponding to average acceleration method).

Here again, we introduce the notation (N_1, N_2, N_3) to denote number of displacement field elements by N_1, number of random field elements in $E\,(x)$ by N_2 and number of random field elements in $\rho\,(x)$ by N_3. We define β^* using a (10, 10, 10) uniform mesh with time step Δt=0.05s. This result along with results for (10, 5, 5), (10, 5, 8) and (10, 10, 10) uniform meshes with a coarser time step of integration Δt=0.1s are shown in Table 7a.

By holding Δt at 0.1s we now adaptively refine the random field meshes using GIM derived from (10, 5, 5) uniform mesh with Δt=0.1s. Figs. 37, 38 and 39 show the GIM plots for the random fields representing elastic modulus, mass density, and also the random variables generated from the process $f\,(t)$ by fixing time t. A comparison of Figs. 37 and 38 show that the GIM values associated with $E\,(x)$ are negligibly small as compared

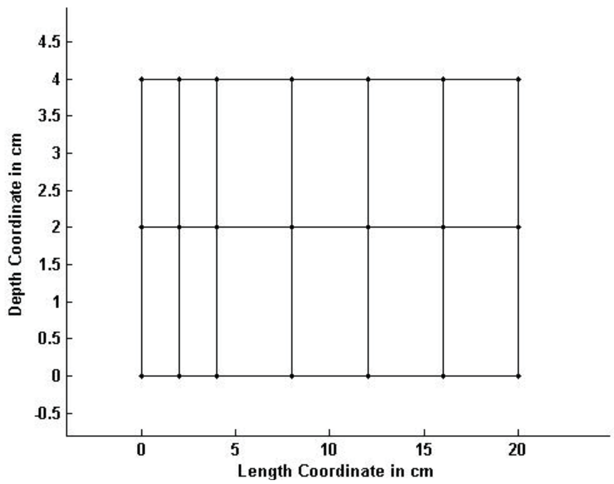


Figure 34: Adaptive 6×2 RF mesh for elastic modulus based on GIM plot for (10, 2, 5, 2) uniform mesh

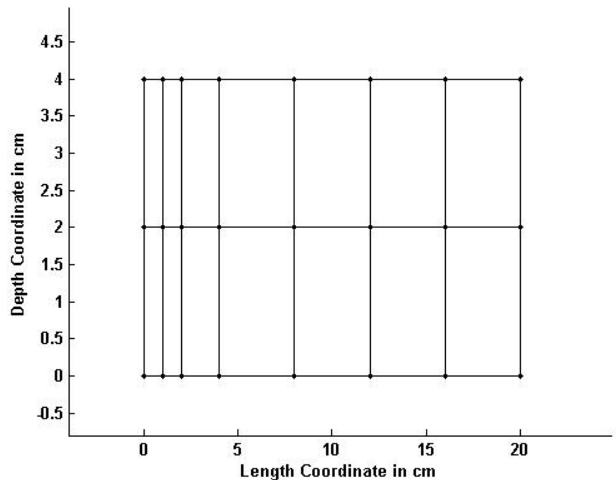


Figure 35: Adaptive 7×2 RF mesh for elastic modulus based on GIM plot for (10, 2, 6, 2) adaptive mesh

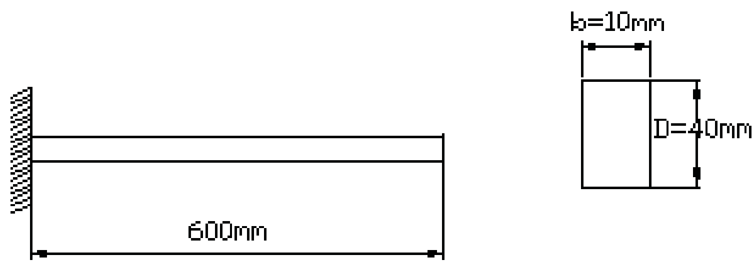


Figure 36: Geometry of cantilever beam with tip point load for Example 3

Table Table 7a: Uniform refinement of RF mesh for mass density of cantilever beam subjected to stationary random excitation due to tip point load

N_1	Δt *s*	N_2	N_3	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert}\times 100$
10	0.10	5	5	2.0392	1.3211
		5	8	2.0396	1.3017
		10	10	2.0402	1.2727
	0.05	10	10	2.0665(β^*)	0.0000

Table Table 7b: Adaptive refinement of RF mesh for mass density of cantilever beam subjected to stationary random excitation due to tip point load

N_1	Δt *s*	N_2	N_3	β	$\eta_\beta\,(\%) = \frac{\lvert\beta-\beta^*\rvert}{\lvert\beta^*\rvert}\times 100$
10	0.10	5	8	2.0610	0.2661

with GIM for $\rho\,(x)$. The spatial variation of GIM also shows notably different trends but this difference itself is of marginal interest given the very low values of GIM associated with $E\,(x)$. The results on reliability index obtained using adaptively refined mesh with Δt=0.1s and (10, 5, 8) mesh is shown in Table 7b. It should be noted that the adaptive refinement here is restricted only to the mass density field (see Fig. 40 for details of refined mesh for $\rho\,(x)$). The mesh for elastic modulus field is uniform in nature. Also, it needs to be emphasized that $\Delta t$is held uniform at

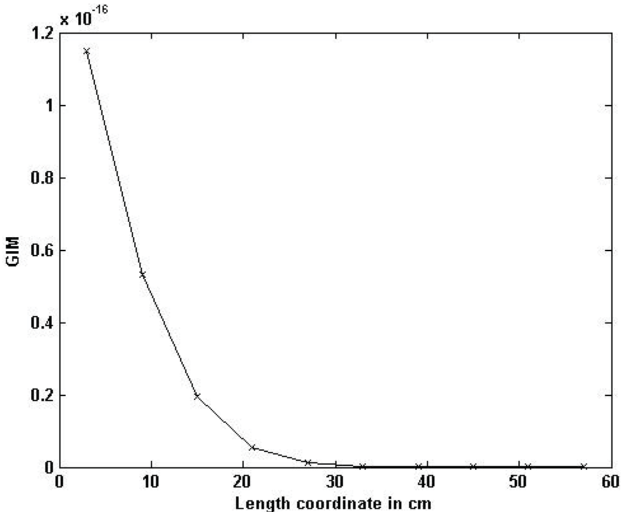


Figure 37: GIM plot for elastic modulus with (10, 5, 5) uniform mesh and time step =0.1s

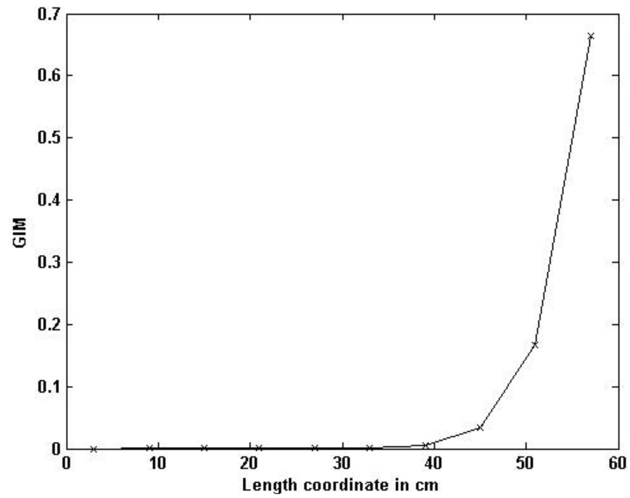


Figure 38: GIM plot for mass density with (10, 5, 5) uniform mesh and time step =0.1s

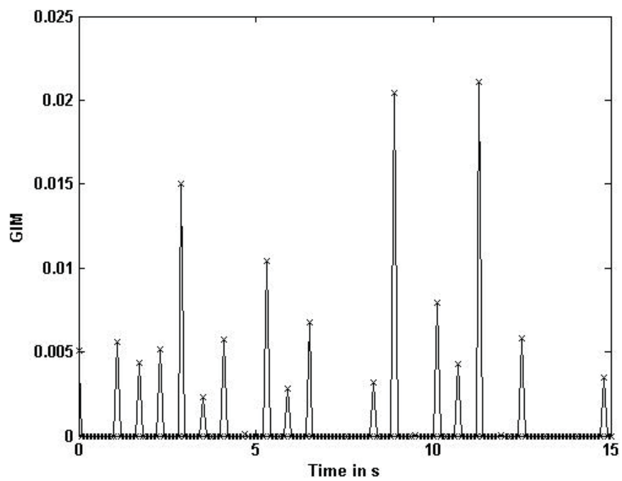


Figure 39: GIM plot for applied force at different time instants with (10, 5, 5) uniform mesh and time step =0.1s

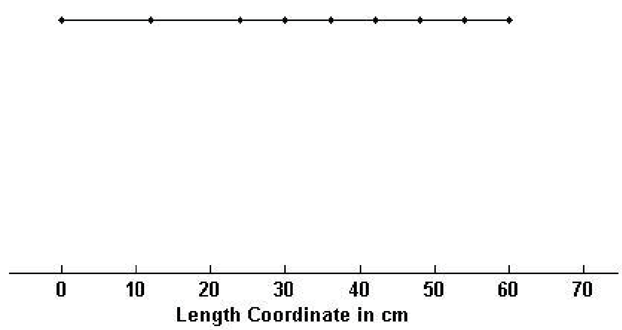


Figure 40: Adaptive 8-element RF mesh for mass density based on GIM plot for (10, 5, 5) uniform mesh; time step =0.1s

a coarser value of 0.1s. It is of interest to note that, based on GIM shown in Fig. 39 it is possible, at least in principle, to identify the regions along the time axis where the step sizes of integration could be adaptively refined. This aspect of refinement however has not been investigated further in this study. A comparison of Tables 7a and 7b reveals that the reliability index obtained using adaptive random field mesh refinement assuming a coarser Δt, matches well with the results on reliability index with a finer time step size (Δt=0.05s) and a finer displacement field and random field meshes (10,10,10 uniform mesh). This again points towards usefulness of mesh refinement procedures discussed here.

10 Conclusions

The problem of reliability analysis of linear static/dynamic systems with spatially varying random structural properties and loads that could vary randomly in space and time is considered.

The reliability itself is defined in terms of a set of one or more performance functions. The tools used for reliability analysis are essentially fashioned after first order reliability methods with a few refined features and stochastic finite element analysis procedures. An important aspect of the problem is associated with treatment of structural properties as a vector of mutually dependent non-Gaussian random fields. For the purpose of reliability analysis these random fields need to be discretised into a set of equivalent random variables. The accuracy of reliability analysis crucially depends upon details of discretisation procedures used. The present study chiefly addresses this issue. Specifically, we have developed a procedure to adaptively refine the random field mesh so as to obtain improved estimates of structural reliability. This adaptive procedure is based upon the definition of global importance measures. The procedures developed can take into account presence of multiple design points and structural reliability defined in terms of multiple performance functions. In order to evaluate the merit of an adaptively refined random field mesh we have utilized the result from a fairly refined mesh as a benchmark. It should be emphasized that such a benchmark is useful only to demonstrate the utility of refinement procedures proposed herein and these procedures themselves do not inherently depend upon the availability of such a benchmark.

The numerical examples presented cover both static and dynamic behaviour, one- and two-dimensional random fields and displacement field elements, simultaneous presence of more than one random field, space-time randomness associated with loads and multiple performance functions. These examples have clearly illustrated that it is possible to obtain better estimates of structural reliability by using adaptively refined random field meshes than by using uniform meshes with higher number of nodes. The study also points towards the possible usefulness of GIMs considered herein for the purpose of adaptive displacement field mesh refinement and adaptive time stepping in integrating the equations of motion. This aspect however, has not been explored further in this paper.

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