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Two Dimensional Dynamic Green's Functions for Piezoelectric Materials

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Abstract: A formulation for two-dimensional self-similar anisotropic elastodyamics problems is generalized to piezoelectric materials. In the formulation the general solution of the displacements is expressed in terms of the eigenvalues and eigenvectors of a related eight-dimensional eigenvalue problem. The present formulation can be used to derive analytic solutions directly without the need of performing integral transforms as required in Cagniard-de Hoop method. The method is applied to derive explicit dynamic Green's functions. Some analytic results for hexagonal 6mm materials are also derived. Numerical examples for the quartz are illustrated.

Keyword: transient motion, piezoelectric material, dynamic Green's function

1 Introduction

The Green's functions for piezoelectric materials relate the mechanical displacements and electric potential at a point to the concentrated forces or charges applied at another point. The Green's functions are important analytically in understanding mechanical or electric behavior of loaded piezoelectric materials. They are also crucial numerically in constructing boundary integral equations for either static or dynamics analysis of finite piezoelectric solids. For example, Dziatkiewicz and Fedelinski (2007) applied the dual reciprocity boundary element method to free vibrations of two-dimensional piezoelectric structures using the static Green's function. The developed method was used to compute frequencies and mode shapes of natural vibrations of two-dimensional piezoelectric structures. Sanz, Solis and Dominguez (2007) have presented a general mixed boundary element formulation for three-dimensional piezoelectric fracture mechanics problems. It is thus highly desirable to obtain explicit expressions for the Green's functions so that they can be easily and accurately evaluated.

Piezoelectric solids are inherently anisotropic elastic. It is not surprising that many methods of analysis for piezoelectric solids are derived from those for anisotropic elastic solids. The Stroh formalism is widely recognized as an elegant and powerful method for two-dimensional anisotropic elastostatics. A distinctive feature of the Stroh formalism is that the general solution is provided in terms of the eigenvalues and eigenvectors of a constant six-dimensional matrix. The general solution contains three arbitrary complex functions. The functions can often be found by taking advantage of the orthogonality relations among the eigenvectors in conjunction with theories of analytic functions. The Stroh's formalism has been applied to yield the static Green's functions for various configurations (Ting, 1996). Generalization of the Stroh's formalism to piezoelectric materials has been given by Ting (Ting, 1996), leading to an eigenvalue problem of a constant eight-dimensional matrix. Wu (2000) extended the Stroh's formalism to treat the self-similar elastodynamic problems for anisotropic elastic material. The formulation is also based on a six-dimensional matrix, which, however, is a function of position and time. A major advantage of the formulation of Wu (2000) is that solutions can be derived directly without the need of performing integral transforms. In this paper the formulation of Wu (2000) is generalized to piezoelectric materials in the context of the quasistatic approximation. The generalized formulation is then utilized to derive the dynamic Green's functions of piezo-

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electric materials.

The two-dimensional dynamic Green's functions for transversely isotropic piezoelectric materials of class 6 mm have been derived by Daros and Antes (2000) using an integral transform inversion technique of Burridge (1967). Recently Wang and Zhang (2005) have obtained the two-dimensional dynamic Green's functions for general piezoelectric solids using Radon transform. The Green's functions, however, is in the form of a one-dimensional integral along the unit circle. The Green's functions reported in the present paper are also valid for general piezoelectric solids. Moreover, the Green's functions are in explicit form such that only the eigenvalue problem of an eight-dimensional matrix needs to be solved.

The plan of the paper is as follows. In section 2 the basic governing equations for linear piezoelectric materials are illustrated. An extension of the formulation (Wu, 2000) is developed for piezoelectric material in section 3. In section 4 the dynamic Green's functions in an infinite piezoelectric medium are obtained with the proposed formulation. Some analytic results for hexagonal 6mm materials are derived in section 5. Numerical examples are given in section 6. Some concluding remarks are finally given.

2 Basic Equations

For a linear piezoelectric solid, the elastic stress σ_{ij}, the elastic displacement u_i, the electric displacement D_i and the electric potential ϕ are related by

$$\sigma_{ij} = C_{ijks}u_{k,s} + e_{sij}\phi_{,s}, \tag{1}$$

$$D_i = e_{iks}u_{k,s} - \varepsilon_{is}\phi_{,s}, \tag{2}$$

where a subscript comma denotes partial differentiation with respect to coordinates, repeated indices imply summation from 1 to 3, C_{ijks} are the elastic stiffness, and e_{iks}, and ε_{is} are, respectively, the piezoelectric stress constants and permittivity constants. In the absence of body forces and free charges the balance laws under quasi-static approximation require

$$\sigma_{ij,j} = \rho\ddot{u}_i, \tag{3}$$

$$D_{i,i} = 0, \tag{4}$$

where ρ is the density and an overhead dot designates derivative with respect to time t.

By letting $\phi = u_4$ and $D_i = \sigma_{4i}$, Eqs. (1) and (2) can be expressed in terms of the generalized stress and generalized displacement as

$$\sigma_{Ij} = E_{IjKs}u_{K,s} \tag{5}$$

where the upper case subscripts range from 1 to 4, lower case subscripts from 1 to 3 and generalized electric-mechanical constants E_{IjKs} are defined as

$$E_{IjKs} = \begin{cases} C_{ijks}, & I,K = 1,2,3, \\ e_{sij}, & I = 1,2,3, \quad K = 4, \\ e_{iks}, & I = 4, \quad K = 1,2,3, \\ -\varepsilon_{is}, & I = 4, \quad K = 4. \end{cases}$$

Equations (3) and (4) can also be combined as

$$\sigma_{Ij,j} = \rho\delta^*_{IK}\ddot{u}_K \tag{6}$$

where $\delta^*_{IK} = \delta_{IK}$, $I,K = 1,2,3$, δ_{IK} being the Kronecker's delta and $\delta^*_{IK} = 0$, $I,K = 4$. Substitution of Eq. (5) into (6) yields the governing equations for the elastic displacement and the electric potential as

$$E_{IjKs}u_{K,sj} = \rho\delta^*_{IK}\ddot{u}_K. \tag{7}$$

3 Formulation

For two-dimensional problems in which the generalized displacement $\mathbf{u} = [u_1, u_2, u_3, \phi]^T$ are independent of x_3, Eq. (7) can be expressed as

$$\mathbf{Q}\mathbf{u}_{,11} + (\mathbf{R}+\mathbf{R}^T)\mathbf{u}_{,12} + \mathbf{T}\mathbf{u}_{,22} = \rho\hat{\mathbf{I}}\ddot{\mathbf{u}}, \tag{8}$$

where $\hat{\mathbf{I}}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are 4×4 matrices given by

$$\begin{aligned} \hat{\mathbf{I}} &= \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}, & \mathbf{Q} &= \begin{bmatrix} \mathbf{Q}^E & \mathbf{e}_{11} \\ \mathbf{e}_{11}^T & -\varepsilon_{11} \end{bmatrix}, \\ \mathbf{R} &= \begin{bmatrix} \mathbf{R}^E & \mathbf{e}_{21} \\ \mathbf{e}_{12}^T & -\varepsilon_{12} \end{bmatrix}, & \mathbf{T} &= \begin{bmatrix} \mathbf{T}^E & \mathbf{e}_{22} \\ \mathbf{e}_{22}^T & -\varepsilon_{22} \end{bmatrix}, \end{aligned} \tag{9}$$

in which $\mathbf{I}$ is the 3×3 identity matrix and the elements of 3×3 matrices $\mathbf{Q}^E$, $\mathbf{R}^E$, $\mathbf{T}^E$, and 3×1 matrices $\mathbf{e}_{ij}$ are

$$Q^E_{ik} = C_{i1k1}, \qquad R^E_{ik} = C_{i1k2},$$
$$T^E_{ik} = C_{i2k2}, \quad (\mathbf{e}_{ij})_s = e_{ijs}.$$

Consider the generalized displacement $\mathbf{u}$ in the following form

$$\mathbf{u}(x_1,x_2,t) = \tilde{\mathbf{u}}(w) + \psi(t)\mathbf{e}_4, \tag{10}$$

where $w(x_1,x_2,t)$ is defined implicitly by

$$\Delta(w,x_1,x_2,t) = wt - x_1 - p(w)x_2 = 0, \tag{11}$$

with $p(w)$ as an analytic function of w, $\psi(t)$ a function of t, and $\mathbf{e}_4 = [0,\ 0,\ 0,\ 1]^T$. It may be shown that the first derivatives of $\mathbf{u}(x_1,x_2,t)$ with respect to x_1, x_2, and t can be expressed as

$$\mathbf{u}_{,1} = \frac{\tilde{\mathbf{u}}'(w)}{\Delta'}, \quad \mathbf{u}_{,2} = \frac{p(w)}{\Delta'}\tilde{\mathbf{u}}'(w),$$
$$\hat{\mathbf{I}}\dot{\mathbf{u}} = -\frac{w}{\Delta'}\hat{\mathbf{I}}\tilde{\mathbf{u}}'(w), \tag{12}$$

and the second derivatives as

$$\begin{aligned} \mathbf{u}_{,11} &= \frac{1}{\Delta'}\frac{\partial}{\partial w}\left(\frac{\tilde{\mathbf{u}}'(w)}{\Delta'}\right), \\ \mathbf{u}_{,22} &= \frac{1}{\Delta'}\frac{\partial}{\partial w}\left(\frac{p(w)^2}{\Delta'}\tilde{\mathbf{u}}'(w)\right), \\ \mathbf{u}_{,12} &= \frac{1}{\Delta'}\frac{\partial}{\partial w}\left(\frac{p(w)}{\Delta'}\tilde{\mathbf{u}}'(w)\right), \end{aligned} \tag{13}$$

$$\hat{\mathbf{I}}\ddot{\mathbf{u}} = \hat{\mathbf{I}}\frac{1}{\Delta'}\frac{\partial}{\partial w}\left(\frac{w^2}{\Delta'}\tilde{\mathbf{u}}'(w)\right), \tag{14}$$

where $\tilde{\mathbf{u}}'(w)$ denotes the derivative of $\tilde{\mathbf{u}}(w)$ with respect to w, Δ' is given by

$$\Delta' = \frac{\partial \Delta(w,x_1,x_2,t)}{\partial w} = t - p'(w)x_2, \tag{15}$$

and $p'(w)$ is the derivative of $p(w)$ with respect to w. With Eqs. (13) and (14), Eq. (8) becomes

$$\frac{1}{\Delta'}\frac{\partial}{\partial w}\left\{ \left[\mathbf{Q} - \rho w^2\hat{\mathbf{I}} + p(w)\left(\mathbf{R}+\mathbf{R}^{\mathbf{T}}\right) + p(w)^2\mathbf{T}\right] \cdot \frac{1}{\Delta'}\tilde{\mathbf{u}}'(w)\right\} = 0 \tag{16}$$

Equation (16) shows that for the generalized displacement $\mathbf{u}$ given by Eq. (10) to be a solution to Eq. (8), $\tilde{\mathbf{u}}(w)$ must satisfy Eq. (16) and $\psi(t)$ is arbitrary.

Let $\tilde{\mathbf{u}}'(w)$ be expressed as

$$\tilde{\mathbf{u}}'(w) = f(w)\mathbf{a}(w), \tag{17}$$

where $f(w)$ is an arbitrary scalar function of w. It follows that $\mathbf{u}$ is a solution of Eq. (8) if

$$\mathbf{D}(p,w)\mathbf{a}(w) = 0, \tag{18}$$

where $D(p,w)$ is given by

$$\mathbf{D}(p,w) = \mathbf{Q} + p(\mathbf{R}+\mathbf{R}^T) + p^2\mathbf{T} - \rho w^2\hat{\mathbf{I}}. \tag{19}$$

For non-trivial solutions of $\mathbf{a}(w)$ we must have

$$|\mathbf{D}(p,w)| = 0, \tag{20}$$

where $|\mathbf{D}|$ is the determinant of $\mathbf{D}$. Equation (20) provides eight eigenvalues of p as a function of w, denoted by $p_\alpha(w)$, $\alpha = 1,2,\ldots,8$. The corresponding function $w_\alpha = w_\alpha(y_1,y_2)$ can be determined from Eq. (11) with $p(w)$ replaced by $p_\alpha(w)$. More conveniently we can substitute Eq. (11) into Eq. (19) and rewrite $\mathbf{D}$ as

$$\mathbf{D}(p,y_1,y_2) = \hat{\mathbf{Q}} + p(\hat{\mathbf{R}}+\hat{\mathbf{R}}^T) + p^2\hat{\mathbf{T}}, \tag{21}$$

where

$$\begin{aligned} \hat{\mathbf{Q}} &= \begin{bmatrix} \mathbf{Q}^E - \rho y_1^2\mathbf{I} & \mathbf{e}_{11} \\ \mathbf{e}_{11}^T & -\varepsilon_{11} \end{bmatrix}, \\ \hat{\mathbf{R}} &= \begin{bmatrix} \mathbf{R}^E - \rho y_1 y_2\mathbf{I} & \mathbf{e}_{21} \\ \mathbf{e}_{12}^T & -\varepsilon_{12} \end{bmatrix}, \\ \hat{\mathbf{T}} &= \begin{bmatrix} \mathbf{T}^E - \rho y_2^2\mathbf{I} & \mathbf{e}_{22} \\ \mathbf{e}_{22}^T & -\varepsilon_{22} \end{bmatrix} \end{aligned} \tag{22}$$

with $y_1 = x_1/t$ and $y_2 = x_2/t$. The function $p_\alpha(y_1,y_2)$ can be directly obtained by $|\mathbf{D}(p,y_1,y_2)| = 0$. The corresponding $w_\alpha(y_1,y_2)$ is simply given by Eq. (11) and the associated eigenvector $\mathbf{a}_\alpha(y_1,y_2)$ is determined by Eq. (18).

It is clear from Eq. (21) that complex roots of p in Eq. (18) and the corresponding $\mathbf{a}$ must appear in complex conjugate pairs. To further discuss the properties of p let

$$p = \tan\theta, \quad w = \frac{c}{\cos\theta}. \tag{23}$$

Substituting Eq. (23) into Eq. (18) gives

$$\mathbf{\Gamma}(\cos\theta,\sin\theta)\mathbf{a} = \rho c^2\hat{\mathbf{I}}\mathbf{a}, \tag{24}$$

where

$$\mathbf{\Gamma}(\cos\theta,\sin\theta) = \mathbf{Q}\cos^2\theta + (\mathbf{R}+\mathbf{R}^T)\cos\theta\sin\theta + \mathbf{T}\sin^2\theta. \tag{25}$$

Equation (24) can be further rewritten as

$$\mathbf{\Gamma}^E(\cos\theta,\sin\theta)\mathbf{a}^E=\rho c^2\mathbf{a}^E, \tag{26}$$

where

$$\Gamma^E_{ij}=\Gamma_{ij}-\frac{\Gamma_{i4}\Gamma_{4j}}{\Gamma_{44}},\quad i,j=1,2,3,$$
$$\mathbf{a}^E=\begin{bmatrix}a_1 & a_2 & a_3\end{bmatrix}^T.$$

For real p Eq. (26) shows that $\mathbf{a}^E$ is the polarization vector and c is the speed of the acoustic plane wave propagating in the direction of $(\cos\theta,\sin\theta)$. Equation (26) may also be expressed as

$$\left[\mathbf{\Gamma}^E(s_1,s_2)-\rho\mathbf{I}\right]\mathbf{a}^E=0$$

where

$$s_1=\cos\theta/c=1/w,\quad s_2=\sin\theta/c=p/w, \tag{27}$$

are the slowness components. For nontrivial $\mathbf{a}^E$,

$$\left|\mathbf{\Gamma}^E(s_1,s_2)-\rho\mathbf{I}\right|=0. \tag{28}$$

Equation (28) is an algebraic equation of degree six in s_1 and s_2, which describes a three-sheeted slowness surface in the (s_1,s_2) space. Equations (28), (23) and (27) suggest a graphical way for finding real $p's$. For a given w let the intersection points of the straight line $s_1=1/w$ and the slowness surface be $s_2^{(\alpha)}(s_1)$. The real p_α is simply given by $p_\alpha=s_2^{(\alpha)}(s_1)/s_1$. If $t\to\infty$ or $w\to 0$, the straight will not intersect the slowness surface and no real p_α exists. In this case p_α appear in four complex conjugated pairs. On the other hand as $t\to 0$ or $w\to\infty$, the straight will intersect the slowness surface at six points, providing six real p_α. From Eq. (19) the other two complex roots and the corresponding $\mathbf{a}^*$ may be shown to be

$$p^*=\frac{-\varepsilon_{12}+i\varepsilon}{\varepsilon_{22}},\quad \overline{p}^*,\ a^*=e_4 \tag{29}$$

where $\varepsilon=\sqrt{\varepsilon_{11}\varepsilon_{22}-\varepsilon_{12}^2}$ and $i=\sqrt{-1}$.

From Eq. (12), the general solution of the generalized displacement satisfying Eq. (8) may be represented as

$$\mathbf{u}(x_1,x_2,t)_{,1}=\sum_{\alpha=1}^{8}\frac{f_\alpha(w_\alpha)}{\Delta'_\alpha}\mathbf{a}_\alpha(w_\alpha), \tag{30}$$

$$\mathbf{u}(x_1,x_2,t)_{,2}=\sum_{\alpha=1}^{8}\frac{p_\alpha(w_\alpha)}{\Delta'_\alpha}f_\alpha(w_\alpha)\mathbf{a}_\alpha(w_\alpha), \tag{31}$$

$$\dot{\mathbf{u}}(x_1,x_2,t)=-\sum_{\alpha=1}^{8}\frac{w_\alpha}{\Delta'_\alpha}f_\alpha(w_\alpha)\mathbf{a}_\alpha(w_\alpha)+\dot{\psi}(t)\mathbf{e}_4, \tag{32}$$

Note that the wave surface can be parametrized as (Wu 2000)

$$x_1=\left(w-\frac{p_\alpha(w)}{p'_\alpha(w)}\right)t,\quad x_2=\frac{t}{p'_\alpha(w)}. \tag{33}$$

From Eq. (15)

$$\Delta'_\alpha=0, \tag{34}$$

upon arrival of the bulk waves.

The constitutive law of Eq. (5) can be expressed as

$$\mathbf{t}_1=\mathbf{Qu}_{,1}+\mathbf{Ru}_{,2}, \tag{35}$$
$$\mathbf{t}_2=\mathbf{R}^T\mathbf{u}_{,1}+\mathbf{Tu}_{,2}, \tag{36}$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are the generalized stress vectors given by $\mathbf{t}_1=(\sigma_{11},\sigma_{21},\sigma_{31},D_1)^T$ and $\mathbf{t}_2=(\sigma_{12},\sigma_{22},\sigma_{32},D_2)^T$. By substituting Eqs. (30) and (31) into Eqs. (35) and (36), the general solutions of the generalized stress vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ can be expressed as

$$\mathbf{t}_1(x_1,x_2,t)=\sum_{\alpha=1}^{8}\frac{1}{\Delta'_\alpha}[\rho w_\alpha^2\hat{\mathbf{I}}\mathbf{a}_\alpha(w_\alpha) - p_\alpha(w_\alpha)\mathbf{b}_\alpha(w_\alpha)]f_\alpha(w_\alpha), \tag{37}$$

$$\mathbf{t}_2(x_1,x_2,t)=\sum_{\alpha=1}^{8}\frac{f_\alpha(w_\alpha)}{\Delta'_\alpha}\mathbf{b}_\alpha(w_\alpha), \tag{38}$$

where

$$\begin{aligned}\mathbf{b}_\alpha(w)&=(\mathbf{R}^T+p_\alpha(w)\mathbf{T})\mathbf{a}_\alpha(w)\\ &=-\frac{1}{p}\left(\mathbf{Q}-\rho w^2\hat{\mathbf{I}}+p_\alpha(w)\mathbf{R}\right)\mathbf{a}_\alpha(w)\end{aligned}. \tag{39}$$

The second identity in Eq. (39) follows from Eq. (18). Introduce given by

$$\begin{aligned}\hat{\mathbf{b}}_\alpha(y_1,y_2) &= (\hat{\mathbf{R}}^T + p_\alpha \hat{\mathbf{T}})\mathbf{a}_\alpha \\ &= -\frac{1}{p_\alpha}(\hat{\mathbf{Q}} + p_\alpha \hat{\mathbf{R}})\mathbf{a}_\alpha\end{aligned}. \qquad (40)$$

The second line of Eq. (40) follows from Eq. (18). The vector $\hat{\mathbf{b}}_\alpha(y_1,y_2)$ is related to $\mathbf{b}_\alpha(w)$ by

$$\hat{\mathbf{b}}_\alpha(y_1,y_2) = \mathbf{b}_\alpha(w) - \rho w y_2 \hat{\mathbf{I}}\mathbf{a}_\alpha(w). \qquad (41)$$

It is noted that while $\hat{b}_\alpha$ is a function of y_1 and y_2, it is not a function of w alone for $y_2 \neq 0$. Equation (40) can be cast into the following eight-dimensional eigenvalue problem

$$\hat{\mathbf{N}}\hat{\boldsymbol{\xi}} = p\hat{\boldsymbol{\xi}}, \qquad (42)$$

where

$$\hat{\mathbf{N}} = \begin{pmatrix} \hat{\mathbf{N}}_1 & \hat{\mathbf{N}}_2 \\ \hat{\mathbf{N}}_3 & \hat{\mathbf{N}}_1^T \end{pmatrix}, \qquad \hat{\boldsymbol{\xi}} = \begin{pmatrix} \mathbf{a} \\ \hat{\mathbf{b}} \end{pmatrix},$$

$$\hat{\mathbf{N}}_1 = -\hat{\mathbf{T}}^{-1}\hat{\mathbf{R}}^T, \qquad \hat{\mathbf{N}}_2 = \hat{\mathbf{T}}^{-1},$$

$$\hat{\mathbf{N}}_3 = \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1}\hat{\mathbf{R}}^T - \hat{\mathbf{Q}}.$$

The p and $\hat{\boldsymbol{\xi}}$ are the eigenvalue and right eigenvector, respectively, of $\hat{\mathbf{N}}$. Since $\hat{\mathbf{N}}_2$ and $\hat{\mathbf{N}}_3$ are symmetric, the left eigenvector, $\hat{\boldsymbol{\eta}}$, of $\hat{\mathbf{N}}$ defined by

$$\hat{\mathbf{N}}^T\hat{\boldsymbol{\eta}} = p\hat{\boldsymbol{\eta}} \qquad (43)$$

is given by

$$\hat{\boldsymbol{\eta}} = \begin{pmatrix} \hat{\mathbf{b}} \\ \mathbf{a} \end{pmatrix}.$$

If the eigenvalues p_α and p_β are distinct, the corresponding left and right eigenvectors satisfy orthogonality relations

$$\hat{\boldsymbol{\eta}}_\alpha^T\hat{\boldsymbol{\xi}}_\beta = \mathbf{a}_\alpha^T\hat{\mathbf{b}}_\beta + \hat{\mathbf{b}}_\alpha^T\mathbf{a}_\beta = 0, \quad \alpha \neq \beta. \qquad (44)$$

For $y_2 = 0$, $\mathbf{b}_\alpha(y_1) = \hat{\mathbf{b}}_\alpha(y_1)$ and from Eq. (44)

$$\mathbf{a}_\alpha^T(y_1)\mathbf{b}_\beta(y_1) + \mathbf{b}_\alpha^T(y_1)\mathbf{a}_\beta(y_1) = 0, \quad \alpha \neq \beta. \qquad (45)$$

With Eqs. (30), (36) and (45), the function $f_\alpha(y_1)$ can be represented as

$$f_\alpha(y_1) = \frac{t}{\gamma_\alpha(y_1)}\Big[\mathbf{a}_\alpha^T(y_1)\mathbf{t}_2(x_1,t) + \mathbf{b}_\alpha^T(y_1)\mathbf{u}_{,1}(x_1,t)\Big], \qquad (46)$$

where

$$\gamma_\alpha(y_1) = 2\mathbf{a}_\alpha^T(y_1)\mathbf{b}_\alpha(y_1). \qquad (47)$$

A useful expression for calculating Δ'_α is as follows. Differentiating Eq. (18) with respect to w and pre-multiplying the result by $\mathbf{a}_\alpha^T$ gives

$$p'_\alpha(w)\mathbf{a}_\alpha^T(w)\left(\mathbf{R}^T + p_\alpha(w)\mathbf{T}\right)\mathbf{a}_\alpha(w) = \rho w \mathbf{a}_\alpha^T(w)\hat{\mathbf{I}}\mathbf{a}_\alpha(w). \qquad (48)$$

By using Eq. (39), Eq. (48) can be rewritten as

$$p'_\alpha(w) = \rho w \frac{\mathbf{a}_\alpha^T(w)\hat{\mathbf{I}}\mathbf{a}_\alpha(w)}{\mathbf{a}_\alpha^T(w)\mathbf{b}_\alpha(w)}. \qquad (49)$$

Substitution of Eq. (49) into Eq. (15) yields

$$\Delta'_\alpha = t\frac{\mathbf{a}_\alpha^T(w)\hat{\mathbf{b}}_\alpha(y_1,y_2)}{\mathbf{a}_\alpha^T(w)\mathbf{b}_\alpha(w)}, \qquad (50)$$

where Eq. (41) has been used.

The formulation developed so far is for the case where the displacement $\mathbf{u}$ is homogeneous of degree 0. If the displacement $\mathbf{u}$ is homogeneous of degree n, we can define a fictitious displacement $\mathbf{u}^*$ by (Eringen and Suhubi, 1975).

$$\mathbf{u}^*(x_1,x_2,t) = \begin{cases} \dfrac{\partial^n \mathbf{u}(x_1,x_2,t)}{\partial t^n}, & n > 0 \\ \displaystyle\int_0^t\int_0^{\tau_m}\cdots\int_0^{\tau_2}\mathbf{u}(x_1,x_2,t)\,d\tau_1\cdots d\tau_{m-1}d\tau_m, & n = -m,\ m > 0 \end{cases} \qquad (51)$$

The fictitious displacement $\mathbf{u}^*$ is homogeneous of degree 0 and the formulation applies.

4 Two-Dimensional Dynamic Green's Function

Consider a line impulse force $\mathbf{h}$ and a line impulse charge q which appear at the origin at time $t = 0$ in an infinite medium. The jump conditions for the generalized stress vector $\mathbf{t}_2$ and the continuity conditions for the displacement $\mathbf{u}$ at $x_2 = 0$ are given by

$$\mathbf{t}_2^+(x_1,t) - \mathbf{t}_2^-(x_1,t) = -\delta(x_1)\delta(t)\mathbf{F}, \qquad (52)$$

$$\frac{\partial}{\partial x_1}\mathbf{u}^+(x_1,t) - \frac{\partial}{\partial x_1}\mathbf{u}^-(x_1,t) = 0, \qquad (53)$$

where $\mathbf{F} = (h_1\ h_2\ h_3\ -q)^T$, $\delta(t)$ is the Dirac delta function and superscripts + and - denotes the limiting values as $x_2 \to 0^+$ and $x_2 \to 0^-$, respectively. The stresses are clearly homogeneous of degree -2 and the displacement $\mathbf{u}$ homogeneous of degree -1. From Eq. (50) the fictitious displacement $\mathbf{u}^*$ is given by

$$\mathbf{u}^*(x_1,x_2,t) = \int_0^t \mathbf{u}(x_1,x_2,t)d\tau_1, \tag{54}$$

From Eqs. (52) and (53) the jump conditions for the fictitious traction $\mathbf{t}_2^*$ and the continuity conditions for the fictitious displacement $\mathbf{u}^*$ at $x_2 = 0$ are

$$\mathbf{t}_2^*(x_1,t)^+ - \mathbf{t}_2^*(x_1,t)^* = -\delta(x_1)H(t)\mathbf{F}, \tag{55}$$

$$\frac{\partial}{\partial x_1}\mathbf{u}^*(x_1,t)^+ - \frac{\partial}{\partial x_1}\mathbf{u}^*(x_1,t)^- = 0. \tag{56}$$

where $H(t)$ is the Heaviside step function.

From Eqs. (55) and (56) for $t > 0$, Eq. (46) yields

$$f_\alpha(y_1)^+ - f_\alpha(y_1)^- = -\frac{\delta(y_1)}{\gamma_\alpha(y_1)}\mathbf{a}_\alpha^T(w_\alpha)\mathbf{F}. \tag{57}$$

If p_α is complex, from Plemelj formula the solution of $f_\alpha(w_\alpha)$ is given by

$$f_\alpha(w_\alpha) = \pm\frac{1}{2\pi i w_\alpha \gamma_\alpha(w_\alpha)}\mathbf{a}_\alpha^T(w_\alpha)\mathbf{F}, \tag{58}$$

where "+" should be taken if the imaginary part of p_α is positive and "-" is taken if the imaginary part of p_α is negative. If p_α is real,

$$f_\alpha(w_\alpha) = 0.$$

The fictitious velocity $\dot{u}^*$ is given by

$$\dot{\mathbf{u}}^*(x_1,x_2,t) = -\frac{1}{\pi}\text{Im}\left[\sum_{\alpha=1}^{n_+}\frac{\mathbf{a}_\alpha(w_\alpha)\mathbf{a}_\alpha^T(w_\alpha)}{\gamma_\alpha(w_\alpha)\Delta'_\alpha}\right]\mathbf{F} + \dot{\psi}^*(t)\mathbf{e}_4, \tag{59}$$

where n_+ is the number of p_α with positive imaginary parts and Im stands for the imaginary part.

From Eq. (28), as $t \to 0^+$, $\dot{u}^*$is given by

$$\dot{\mathbf{u}}^* = \left[-\frac{F_4}{2\pi\varepsilon t} + \dot{\psi}^*(t)\right]\mathbf{e}_4, \tag{60}$$

or

$$\dot{\phi}^*(t) = -\frac{F_4}{2\pi\varepsilon t} + \dot{\psi}^*(t). \tag{61}$$

If $\dot{\phi}^*$ is required to be bounded at $t = 0$, the function $\dot{\psi}^*(t)$ must be in the following form

$$\dot{\psi}^*(t) = \frac{F_4}{2\pi\varepsilon t} + h(t), \tag{62}$$

where $h(t)$ is a regular function of t. Since only the spatial variation of the electric potential $\dot{\phi}^*$ is of interest, we can let $h(t) = 0$. Substitution of Eqs. (60) and (54) into Eq. (59) leads to

$$\mathbf{u}(x_1,x_2,t) = \dot{\mathbf{u}}^*(x_1,x_2,t) = \mathbf{G}^+(x_1,x_2,t)\mathbf{F},$$

where $\mathbf{G}^+$ is the Green's function for $t > 0$ given by

$$\mathbf{G}^+(x_1,x_2,t) = -\frac{1}{\pi}\text{Im}\left[\sum_{\alpha=1}^{n_+}\frac{\mathbf{a}_\alpha(w_\alpha)\mathbf{a}_\alpha^T(w_\alpha)}{\gamma_\alpha(w_\alpha)\Delta'_\alpha}\right] + \frac{1}{2\pi\varepsilon t}\mathbf{e}_4\mathbf{e}_4^T. \tag{63}$$

Since as $t \to 0^+$, it may be shown that the fictitious displacement is

$$\mathbf{u}^*(x_1,x_2,0^+) = \frac{1}{2\pi\varepsilon}\text{Re}\left[\log(x_1 + p^*x_2)\right]\mathbf{e}_4 F_4, \tag{64}$$

where Re stands for the real part; while $\mathbf{u}^*(x_1,x_2,t) = 0$ as $t \to 0^-$. Equation (64) is a result of the quasi-static approximation which assumes that the electromagnetic wave speed is infinite. Consequently the electrostatic potential without piezoelectricity is induced instantaneously as the line charge is applied. The Green's function $\mathbf{G}(x_1,x_2,t)$ for $t > 0^-$ is thus given by

$$\mathbf{G}(x_1,x_2,t) = \mathbf{G}^+(x_1,x_2,t) + \frac{\delta(t)}{2\pi\varepsilon}\text{Re}\left[\log(x_1 + p^*x_2)\right]\mathbf{e}_4\mathbf{e}_4^T. \tag{65}$$

From Eq. (50) , Eq. (65) can be also expressed as

$$\mathbf{G}(x_1,x_2,t) = -\frac{1}{\pi t}\text{Im}\left[\sum_{\alpha=1}^{n_+}\frac{\mathbf{a}_\alpha(y_1,y_2)\mathbf{a}_\alpha^T(y_1,y_2)}{\hat{\gamma}_\alpha(y_1,y_2)}\right] + \frac{1}{2\pi\varepsilon}\left[\frac{1}{t} + \delta(t)\text{Re}\left[\log(x_1 + p^*x_2)\right]\right]\mathbf{e}_4\mathbf{e}_4^T \tag{66}$$

and $\hat{\gamma}_\alpha(y_1,y_2) = 2\mathbf{a}_\alpha^T(y_1,y_2)\hat{\mathbf{b}}(y_1,y_2)$. Thus the Green's function can be evaluated simply from the complex eigenvectors of Eq. (42). It should be noted that the Green's function matrix given by Eq. (42) is symmetric.

5 Analytic Results For Hexagonal 6mm materials

For hexagonal 6mm materials with the (x_1,x_2) plane as the isotropy plane, the matrix $\mathbf{D}$ given by Eq. (21) becomes

$$\mathbf{D} = \begin{pmatrix} D_{11} & D_{12} & 0 & 0 \\ D_{12} & D_{22} & 0 & 0 \\ 0 & 0 & D_{33} & D_{34} \\ 0 & 0 & D_{34} & D_{44} \end{pmatrix}, \tag{67}$$

where

$$\begin{aligned} D_{11} &= \hat{C}_{11} + 2p\hat{C}_{16} + p^2\tilde{C}_{66}, \\ D_{12} &= p(C_{11} - C_{66}), \\ D_{22} &= \hat{C}_{66} + 2p\hat{C}_{26} + p^2\hat{C}_{22}, \\ D_{33} &= \hat{C}_{55} + 2p\hat{C}_{45} + p^2\hat{C}_{44}, \\ D_{34} &= e_{15}(1+p^2), \\ D_{44} &= -\varepsilon_{11}(1+p^2). \end{aligned}$$

Here the contracted notation for the elastic constants is employed and

$$\begin{aligned} \left(\hat{C}_{11}, \hat{C}_{66}, \hat{C}_{55}\right) &= (C_{11}, C_{66}, C_{44}) - \rho y_1^2 (1,1,1), \\ \left(\hat{C}_{16}, \hat{C}_{26}, \hat{C}_{45}\right) &= -\rho y_1 y_2 (1,1,1), \\ \left(\tilde{C}_{66}, \hat{C}_{22}, \hat{C}_{44}\right) &= (C_{66}, C_{11}, C_{44}) - \rho y_2^2 (1,1,1). \end{aligned}$$

Equation (67) shows that the electric potential u_4 is coupled with the anti-plane displacement u_3 only. Thus for an anti-plane force and charge, the p-eigenvalues are determined by

$$\begin{aligned} D_{33}D_{44} - D_{34}^2 &= (1+p^2) \\ &\times \left[\left(\overline{C}_{44} - \rho y_1^2\right) - 2\rho y_1 y_2 p + \left(\overline{C}_{44} - \rho y_2^2\right) p^2\right] = 0, \end{aligned} \tag{68}$$

where $\overline{C}_{44} = C_{44} + e_{15}^2/\varepsilon_{11}$ is the stiffened elastic constants. The roots of p with positive imaginary parts satisfying Eq. (68) are given by

$$p_3 = \frac{y_1 y_2/c_s^2 + i\sqrt{(1-(y/c_s)^2)}}{1-(y_2/c_s)^2}, \quad y < c_s \quad \text{and} \quad p_4 = i \tag{69}$$

where $c_s = \sqrt{\overline{C}_{44}/\rho}$ is the shear wave speed and $y = \sqrt{y_1^2 + y_2^2}$. The corresponding eigenvectors are

$$\mathbf{a}_3 = \begin{pmatrix} \varepsilon_{11} \\ e_{15} \end{pmatrix} \quad \text{and} \quad \mathbf{a}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{70}$$

From Eqs. (40), (69) and (70), $\hat{\gamma}_3$ and $\hat{\gamma}_4$ are obtained as

$$\hat{\gamma}_3 = 2i\overline{C}_{44}\sqrt{(1-(y/c_s)^2)}\varepsilon_{11}^2 \quad \text{and} \quad \hat{\gamma}_{44} = -2i\varepsilon_{11}. \tag{71}$$

Substitution of (70) and (71) into Eq. (66) yields

$$\begin{aligned} G_{33} &= \frac{1}{2\pi t\overline{C}_{44}} \frac{H(c_s - y)}{\sqrt{(1-(y/c_s)^2)}}, \\ G_{34} &= G_{43} = \frac{e_{15}}{\varepsilon_{11}} G_{33}, \\ G_{44} &= \left(\frac{e_{15}}{\varepsilon_{11}}\right)^2 G_{33} + \frac{\delta(t)\log r}{2\pi\varepsilon_{11}}. \end{aligned}$$

6 Numerical Examples

To verify the present formulation, numerical calculations were first made for the hexagonal 6mm piezoelectric solids treated by Daros and Antes (2000). In that paper, three specific piezoelectric solids, $BaTiO_3$, PZT-6B and ZnO, were considered and the (x_1,x_3) plane was assumed to be the isotropy plane. For comparison purposes, the variations of the scaled components $\pi c_0 t G_{11}$ and $\pi c_0 t G_{22}$, where $c_0 = \sqrt{C_{44}/\rho}$, with the dimensionless variable $\overline{y} = x_2/(c_0 t)$ along $x_1 = 0$ for the three materials were computed as shown in figures 1~3. The present results are in close agreement with those in Daros and Antes (2000).

The Green's functions given by Eq. (65) were computed next for quartz, which is a crystal of trigonal 32 symmetry class. The Green's functions may be expressed in the following dimensionless form:

$$\overline{G}_{ij}(\psi,\tau) = \begin{cases} (\pi C_0 r/c_0) G_{ij}(x_1,x_2,t), & i,j = 1,2,3, \\ (\pi e_0 r/c_0) G_{ij}(x_1,x_2,t), & i = 4,\ j = 1,2,3 \\ & \text{or } i = 1,2,3,\ j = 4 \\ (\pi\varepsilon_0 r/c_0) G_{ij}(x_1,x_2,t), & i = 4,\ j = 4 \end{cases} \tag{72}$$

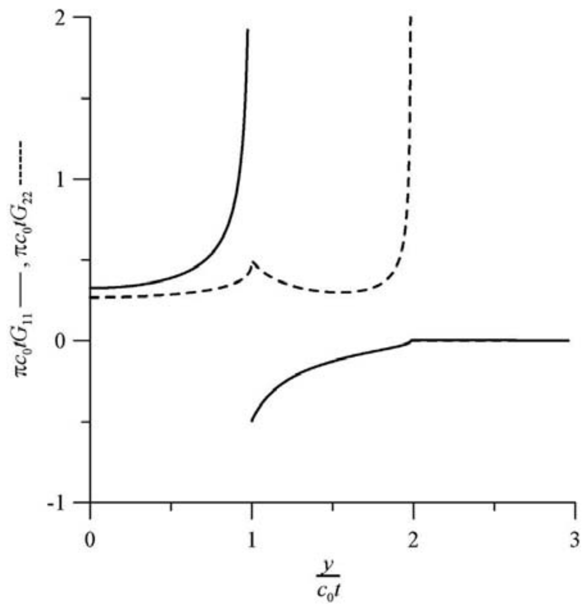

Figure 1: Displacement components $\pi c_0 t G_{11}$ and $\pi c_0 t G_{22}$ for $BaTiO_3$ along $x_1 = 0$.

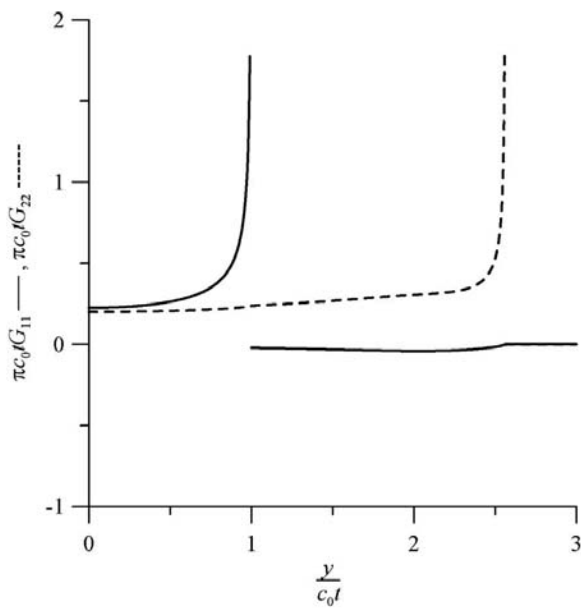

Figure 2: Displacement components $\pi c_0 t G_{11}$ and $\pi c_0 t G_{22}$ for PZT-6B along $x_1 = 0$.

where $c_0 = \sqrt{C_0/\rho}$, $\tau = tc_0/r$, $r = \sqrt{x_1^2 + x_2^2}$, and $\psi = \tan^{-1}(x_2/x_1)$. Here C_0, e_0 and $\varepsilon_0 = e_0^2/C_0$, respectively, are certain reference elastic constant, piezoelectric stress constant and permittivity. The elastic stiffness constants $\mathbf{C}$, the piezoelectric stress constants $\mathbf{e}$, and dielectric constants

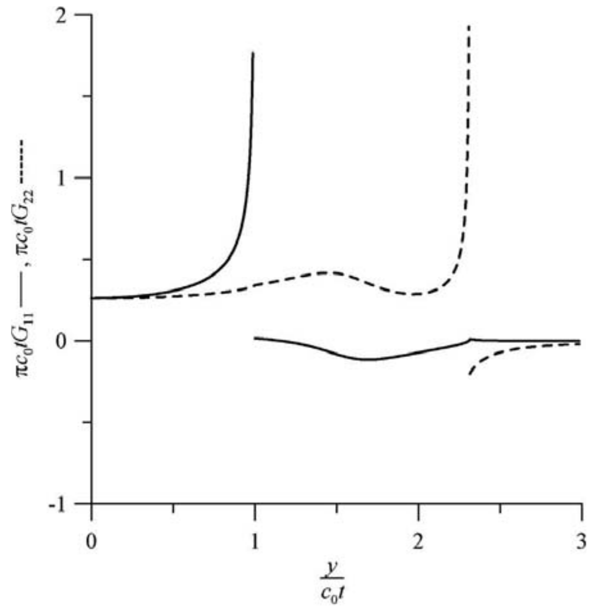

Figure 3: Displacement components $\pi c_0 t G_{11}$ and $\pi c_0 t G_{22}$ for ZnO along $x_1 = 0$.

$\boldsymbol{\varepsilon}$ of quartz used for calculations were:

$$\mathbf{C} = \begin{bmatrix} 86.74 & 6.97 & 11.9 & -17.91 & 0 & 0 \\ 6.97 & 86.74 & 11.9 & 17.91 & 0 & 0 \\ 11.9 & 11.9 & 107.2 & 0 & 0 & 0 \\ -17.91 & 17.91 & 0 & 57.93 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.93 & -17.91 \\ 0 & 0 & 0 & 0 & -17.91 & 39.885 \end{bmatrix} \text{GPa}, \quad (73)$$

$$\mathbf{e} = \begin{bmatrix} -0.171 & 0.171 & 0 & 0.0406 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0406 & 0.171 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{C/m}^2, \quad (74)$$

and

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 39.21 & 0 & 0 \\ 0 & 39.21 & 0 \\ 0 & 0 & 41.03 \end{bmatrix} \times 10^{-12} \text{ Farads/m.} \quad (75)$$

The reference material constants of quartz were selected as $C_0 = C_{44}$, $e_0 = e_{12}$.

Figure 4 displays the wave surface of quartz in the infinite region for the observational angle, ψ, between 0° and 90°. The three wavefronts are denoted by L, FT, and ST. Figures 5∼8 show the components of Green's functions for the observational angle $\psi = 36°$. The components $\overline{G}_{ij},\ i,j = 1,2,3$, are given in figures 5 and 6. These components correspond to the displacements due to a unit line force. The responses resemble those for an elastic material. However, as shown in the inserts in figures 5 and 6, small disturbances appear even before the arrival of the fastest bulk L-wave. Moreover, the disturbances are appreciable only when the L-wave is approached. The phenomenon is a result of the electro-mechanical coupling effect of the piezoelectric material and the quasistatic approximation, in which the electro-magnetic wave speed is assumed infinite. The components $\overline{G}_{ij}$, $i = 1,2,3$, $j = 4$, corresponding to the displacements due to a unit line charge, are shown in figures 7. Again disturbances appear before the arrival of the L-wave. However, in contrast to those due to the mechanical force, the disturbances occur soon after the electric potential is applied and the variations are more pronounced. The component $\overline{G}_{44}$, which is the electric potential due to a line charge, is shown in figure 8. The component exhibits similar features as those of the displacements displayed in figure 7.

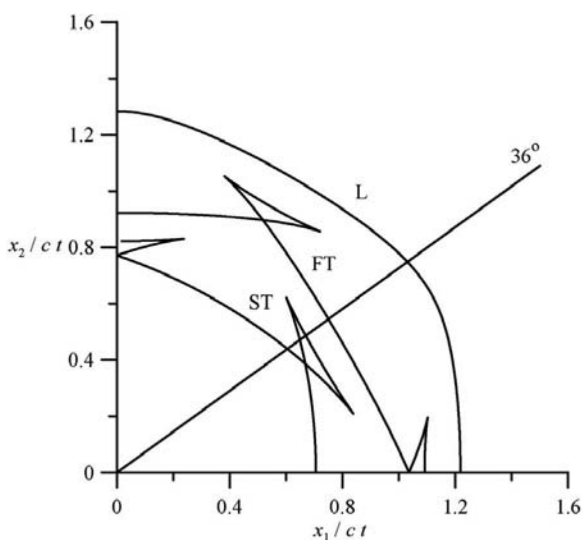

Figure 4: Wavefronts and the angle of observation for quartz.

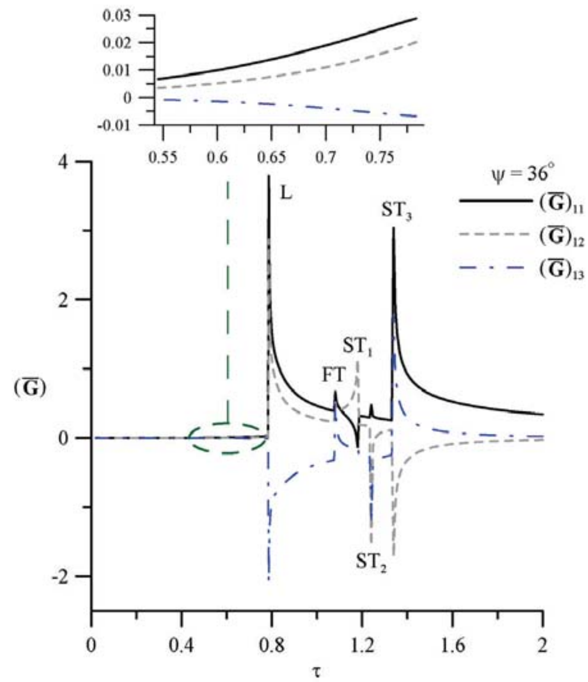

Figure 5: The components of Green's functions $(\overline{\mathbf{G}})_{11}$, $(\overline{\mathbf{G}})_{12}$ and $(\overline{\mathbf{G}})_{13}$ for $\psi = 36°$.

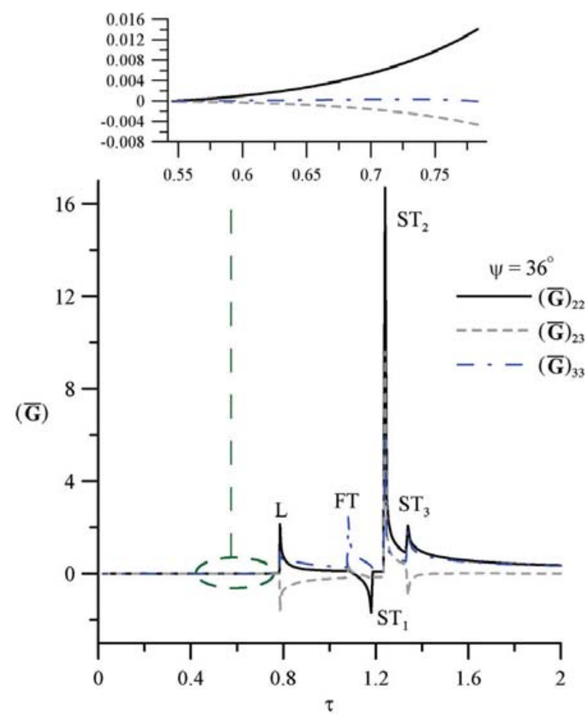

Figure 6: The components of Green's functions $(\overline{\mathbf{G}})_{22}$, $(\overline{\mathbf{G}})_{23}$ and $(\overline{\mathbf{G}})_{33}$ for $\psi = 36°$.

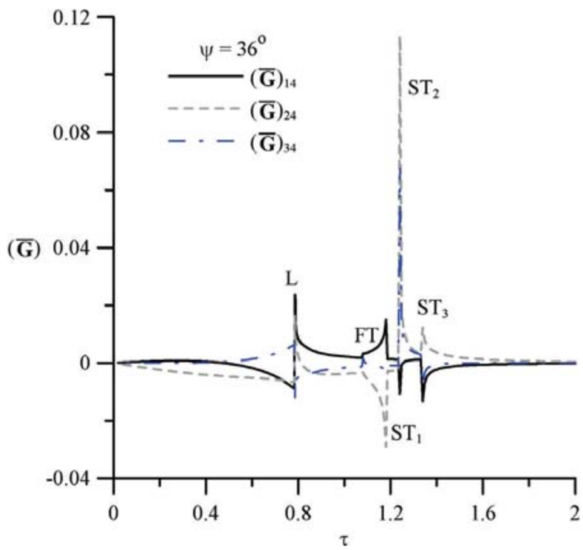

Figure 7: The components of Green's functions $(\overline{\mathbf{G}})_{14}$, $(\overline{\mathbf{G}})_{24}$ and $(\overline{\mathbf{G}})_{34}$ for $\psi = 36^{o}$.

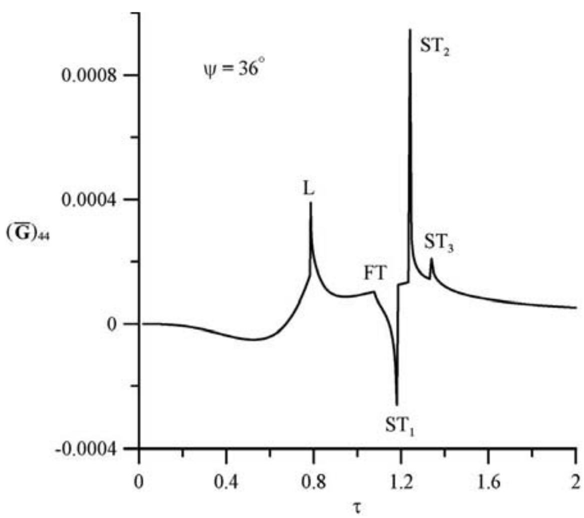

Figure 8: The components of Green's functions $(\overline{\mathbf{G}})_{44}$, for $\psi = 36^{o}$.

7 Concluding Remarks

A formulation developed by Wu (2000) for two-dimensional anisotropic elastodynamics is extended to treat general piezoelectric materials. The formulation does not require integral transforms and can be used to yield the displacement or stress fields in the time domain directly. The formulation is applied to derive analytic expressions for Green's functions of infinite piezoelectric media. The Green's functions can be simply calculated using the eigenvalues and eigenvectors of an eight by eight matrix. Numerical examples provided for several piezoelectric materials show that the dynamic responses can be accurately computed by the present formulation.

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