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New Integrating Methods for Time-Varying Linear Systems and Lie-Group Computations

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Abstract: In many engineering applications the Lie group calculation is very important. With this in mind, the subject of this paper is for an in-depth investigation of time-varying linear systems, and its accompanied Lie group calculations. In terms of system matrix $\mathbf{A}$ in Eq. (11) and a one-order lower fundamental solution matrix associated with the sub-state matrix function $\mathbf{A}_s^s$, we propose two methods to nilpotentlize the time-varying linear systems. As a consequence, we obtain two different calculations of the general linear group. Then, the nilpotent systems are further transformed to a unique new system $\dot{\mathbf{Z}}(t) = \mathbf{B}(t)\mathbf{Z}(t)$, which having a special simple $\mathbf{B}(t) \in sl(n+1,\mathbb{R})$ with $\mathbf{B}_s^s$ and B_0^0 vanishing. Correspondingly, we get a third calculation of the general linear group. By using the nilpotent property we can develop quite simple numerical scheme of nilpotent type to calculate the state transition matrix. We also develop a Lie-group solver in terms of the exponential mapping of $\mathbf{B}$. Several numerical examples were employed to assess the performance of proposed schemes. Especially, the new Lie-group solver is very stable and highly accurate.

Keyword: Lie-group solver, Nilpotent matrix, Time-varying linear system, Quadratic invariant

1 Introduction

Lie group is a differentiable manifold, endowed a group structure that is compatible with the underlying topology of the manifold. The main purpose of the Lie-group solver is for providing a better algorithm that retains the orbit generated from numerical solution on the manifold which associated with the Lie-group.

The general linear group is a Lie group, whose manifold is an open subset $GL(n+1,\mathbb{R}) := \{\mathbf{G} \in \mathbb{R}^{(n+1)\times(n+1)} | \det \mathbf{G} \neq 0\}$ of the linear space of all $(n+1)\times(n+1)$ nonsingular matrices. Thus, $GL(n+1,\mathbb{R})$ is also an $(n+1)\times(n+1)$-dimensional manifold. The group composition is given by the matrix multiplication [Weyl (1966)].

The general linear group $GL(n+1,\mathbb{R})$ gives uniquely a real Lie algebra $gl(n+1,\mathbb{R})$. Consider a one-parameter subgroup $\mathbf{G}(t)$, $t \in \mathbb{R}$, of the general linear group $GL(n+1,\mathbb{R})$, which is a curve passing through the group identity at $t = 0$,

$$\mathbf{G}(0) = \mathbf{I}_{n+1}, \tag{1}$$

and which left acts on the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, resulting in a congruence of curves in $\mathbb{R}^{n+1}$,

$$\mathbf{X}(t) = \mathbf{G}(t)\mathbf{X}(0) = [\mathbf{G}(t)\mathbf{G}^{-1}(t_0)]\mathbf{X}(t_0), \ \ t_0, t \in \mathbb{R}. \tag{2}$$

Owing to the closure property of the Lie group, $\mathbf{G}(t)\mathbf{G}^{-1}(t_0)$ also belongs to $GL(n+1,\mathbb{R})$. When t_0 is put very close to t, $\mathbf{G}(t)\mathbf{G}^{-1}(t_0)$ is very close to the identity $\mathbf{I}_{n+1}$. Moreover,

$$\mathbf{A}(t) := \frac{\partial}{\partial t}[\mathbf{G}(t)\mathbf{G}^{-1}(t_0)]\bigg|_{t_0=t} = \dot{\mathbf{G}}(t)\mathbf{G}^{-1}(t) \tag{3}$$

defines a string of tangent vectors on the tangent space at the group identity of the group manifold, more precisely, a continuously singly parametrized series of one-dimensional subalgebra of the real Lie algebra $gl(n+1,\mathbb{R})$.

Differentiating Eq. (2), setting $t_0 = t$ and then using Eq. (3) yields

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t). \tag{4}$$

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The flow generated by such a $gl(n+1,\mathbb{R})$ vector field is the congruence of curves resulting from solving the time-varying linear system (4). Essentially, it is very important that when one wants to establish a local coordinate on a smooth manifold on which a finite dimensional Lie group of transformations is acting, the solution of linear time-varying differential equations is necessary.

Due to Eqs. (3) and (1), $\mathbf{G}(t)$ is a fundamental solution matrix of the system of ordinary differential equations in Eq. (4), and

$$\mathbf{\Phi}(t,t_0) := \mathbf{G}(t)\mathbf{G}^{-1}(t_0) \tag{5}$$

is the state transition matrix [Rugh (1993)]. In general, it may not be possible to derive an analytic, closed-form expression of $\mathbf{\Phi}(t,t_0)$ associated with arbitrary matrix $\mathbf{A}(t)$. In the time-varying case, we usually use a power series expansion, called the Peano-Baker formula [Rugh (1993)], to express $\mathbf{\Phi}(t,t_0)$ by

$$\begin{aligned}\mathbf{\Phi}(t,t_0) &= \mathbf{I}_{n+1} + \int_{t_0}^{t} \mathbf{A}(\tau_1)d\tau_1 \\ &+ \int_{t_0}^{t}\int_{t_0}^{\tau_1} \mathbf{A}(\tau_1)\mathbf{A}(\tau_2)d\tau_2 d\tau_1 + \cdots \\ &+ \int_{t_0}^{t}\int_{t_0}^{\tau_1}\cdots\int_{t_0}^{\tau_{n-1}} \mathbf{A}(\tau_1)\mathbf{A}(\tau_2)\cdots\mathbf{A}(\tau_n) \\ &\cdot d\tau_n\cdots d\tau_2 d\tau_1 + \cdots \end{aligned} \tag{6}$$

Follows from Eq. (3) a linear matrix differential equation:

$$\dot{\mathbf{G}}(t) = \mathbf{A}(t)\mathbf{G}(t), \;\; \mathbf{G}(0) = \mathbf{I}_{n+1}, \tag{7}$$

whose differential structure, as observed by Hausdorff (1906), can be transferred to the differential structure for the underlying Lie algebra $\boldsymbol{\sigma}(t) \in gl(n+1,\mathbb{R})$ with

$$\dot{\boldsymbol{\sigma}}(t) = \sum_{k=0}^{\infty} \frac{B_k}{k!}\mathrm{ad}_{\boldsymbol{\sigma}}^k \mathbf{A}(t), \;\; \boldsymbol{\sigma}(0) = \mathbf{0}_{n+1}, \tag{8}$$

where $\mathbf{0}_{n+1}$ denotes the $(n+1)$-order zero matrix, B_k are the Bernoulli numbers, and the adjoint operator in $\mathrm{ad}_{\mathbf{x}}\mathbf{y}$ with $\mathbf{x},\mathbf{y} \in gl(n+1,\mathbb{R})$ is defined by the following iterated commutation [Isidori (1989)],

$$\mathrm{ad}_{\mathbf{x}}^0\mathbf{y} = \mathbf{y}, \;\; \mathrm{ad}_{\mathbf{x}}^k\mathbf{y} = [\mathbf{x}, \mathrm{ad}_{\mathbf{x}}^{k-1}\mathbf{y}], \;\; k \in \mathbb{N}. \tag{9}$$

Here, $[\mathbf{x},\mathbf{y}] = \mathbf{xy} - \mathbf{yx}$ denotes the Lie commutator. Magnus (1954) has shown that

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \int_0^t \mathbf{A}(\tau_1)d\tau_1 \\ &+ \frac{1}{2}\int_0^t\int_0^{\tau_1}[\mathbf{A}(\tau_1),\mathbf{A}(\tau_2)]d\tau_2 d\tau_1 \\ &+ \frac{1}{4}\int_0^t\int_0^{\tau_1}\int_0^{\tau_2}[[\mathbf{A}(\tau_3),\mathbf{A}(\tau_2)],\mathbf{A}(\tau_1)]d\tau_3 d\tau_2 d\tau_1 \\ &+ \frac{1}{12}\int_0^t\int_0^{\tau_1}\int_0^{\tau_1}[\mathbf{A}(\tau_3),[\mathbf{A}(\tau_2),\mathbf{A}(\tau_1)]]d\tau_3 d\tau_2 d\tau_1 \\ &+ \cdots, \end{aligned} \tag{10}$$

and proved that $\mathbf{G}(t) = \exp\boldsymbol{\sigma}(t)$ is a solution of Eq. (7).

No matter which formula, Eq. (6) or Eq. (10), is used to calculate the state transition matrix, we need to calculate many multivariate integrals of matrix functions and infinite sum. Especially, the latter further requires to calculate a large number of commutators and a single exponential of a complicated matrix function argument. On the other hand, when one uses Eq. (8) to calculate the state transition matrix, one may require to solve a highly nonlinear differential equations system.

In the past few years a number of Lie-group algorithms have been proposed and studied, for example, the rigid-frame technique of Crouch and Grossman (1993), the Runge-Kutta-Munthe-Kaas scheme by Munthe-Kaas (1998), the Fer expansion by Iserles (1984), and the Magnus expansions by Iserles and Norsett (1999) and Iserles, Norsett and Rasmussen (2001). In addition these methods, the use of Wei-Norman formula [Wei and Norman (1964)] to establish the Lie group is fairly popular in the control and system theory [Sastry (1999); Altafini (2005)]. The Wei-Norman formula provides an explicit relation between the Magnus expansion and a complete product of exponentials expansion. Unfortunately, such a relation is given by a set of nonlinear differential equations, more difficult to solve than the Magnus expansion, which is obtained by a local solution of a linear time-varying system expressed by means of a single exponential.

In Section 2 of this paper we will propose two new methods to transform the time-varying linear system (4) to the index two nilpotent time-varying linear systems:

$$\dot{\mathbf{Y}}(t) = \mathbf{N}_i(t)\mathbf{Y}(t), \;\; \mathbf{N}_i^2(t) = \mathbf{0}, \;\; i = 1,2.$$

Due to this nilpotent property of $\mathbf{N}_i(t)$, the new systems are more easy to treat than the original system (4). Furthermore, we will derive the sufficient conditions about the state matrix function $\mathbf{A}$, such that there exists a quadratic invariant of the time-varying linear system.

In Section 3 we will give some examples of the quadratic invariants for the second-order system, the third-order linear differential equation and the fourth-order linear differential equation.

In Section 4, we further reduce the nilpotent systems introduced in Section 2 to a simple linear system with vanishing $\mathbf{B}_s^s$ and B_0^0. We will prove that the quadratic invariants can be classified into two types: Minkowskian and Euclidean.

Corresponding to the results in Sections 2 and 3, two nilpotent type algorithms and a new Lie-group solver are developed in Section 5. The results to be developed may facilitate us to develop some new techniques to solve the Lie-group equation (7). The nilpotent form is first developed by Liu (2006a) for the $SO(3)$ system. This paper extends these results to the general linear Lie-group.

In Section 6 we will use some examples to show that the newly developed numerical algorithms are better to preserve the system's invariants and also the group structures.

Finally, the conclusions are made in Section 7.

2 Nilpotentlizations of time-varying linear system

In this section we propose two new methods to transform the time-varying linear system (4) to the index two nilpotent time-varying linear systems. The state matrix $\mathbf{A}$ is a function of $t \in \mathbb{R}$ with dimensions $(n+1)\times(n+1)$, and is decomposed into the following form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_s^s & \mathbf{A}_0^s \\ (\mathbf{A}_s^0)^T & A_0^0 \end{bmatrix}. \tag{11}$$

Here, $\mathbf{A}(t)$ is assumed to be that Eq. (4) has a unique solution, and there are no any other constraints on the $n \times n$ sub-matrix function $\mathbf{A}_s^s(t)$, on the $n \times 1$ matrix functions $\mathbf{A}_0^s(t)$ and $\mathbf{A}_s^0(t)$, and on the scalar function $A_0^0(t)$.

2.1 An indefinite-metric method

We first prove the following results.

Theorem 1. Corresponding to the linear system (4), if we consider the following variable transformation

$$\mathbf{Y} = \begin{bmatrix} \mathbf{G}_n^{-1} & \mathbf{0}_{n\times 1} \\ \mathbf{V}^T\mathbf{G}_n^{-1} & -\eta \end{bmatrix} \mathbf{X}, \tag{12}$$

where

$$\dot{\mathbf{G}}_n = \mathbf{A}_s^s\mathbf{G}_n, \;\; \mathbf{G}_n(0) = \mathbf{I}_n, \tag{13}$$

$$\eta := \exp\left[-\int_0^t A_0^0(\xi)d\xi\right], \tag{14}$$

$$\mathbf{U} := \frac{1}{\eta}\mathbf{G}_n^{-1}\mathbf{A}_0^s, \tag{15}$$

$$\mathbf{V} := \int_0^t \eta(\xi)\mathbf{G}_n^T(\xi)\mathbf{A}_s^0(\xi)d\xi, \tag{16}$$

then there exists a linear system

$$\dot{\mathbf{Y}}(t) = \mathbf{N}_1(t)\mathbf{Y}(t), \tag{17}$$

where

$$\mathbf{N}_1 := \begin{bmatrix} \mathbf{U}\mathbf{V}^T & -\mathbf{U} \\ \mathbf{V}^T\mathbf{U}\mathbf{V}^T & -\mathbf{V}^T\mathbf{U} \end{bmatrix}, \tag{18}$$

satisfying

$$\mathrm{tr}\mathbf{N}_1 = 0, \;\; \mathbf{N}_1^2 = \mathbf{0}, \tag{19}$$

is a zero trace nilpotent matrix function with index two.

Proof. Corresponding to the $\mathbf{A}$ given in Eq. (11), let us decompose $\mathbf{X}$ into

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix} = \begin{bmatrix} X^1 \\ \vdots \\ X^n \\ X^0 \end{bmatrix}. \tag{20}$$

From Eqs. (4) and (11) by inserting the above $\mathbf{X}$ we obtain

$$\dot{\mathbf{X}}^s = \mathbf{A}_s^s\mathbf{X}^s + X^0\mathbf{A}_0^s, \tag{21}$$

$$\dot{X}^0 = (\mathbf{A}_s^0)^T\mathbf{X}^s + A_0^0X^0. \tag{22}$$

The superscript T denotes the transpose, and hence, the term $(\mathbf{A}_s^0)^T\mathbf{X}^s$ is the inner product of $\mathbf{A}_s^0$ and $\mathbf{X}^s$, which is more frequently written as $\mathbf{A}_s^0\cdot\mathbf{X}^s$.

The integral of the first equation leads to

$$\mathbf{X}^s(t)=\mathbf{G}_n(t)\mathbf{X}^s(0)+\int_0^t\mathbf{G}_n(t)\mathbf{G}_n^{-1}(\xi)\mathbf{A}_0^s(\xi)X^0(\xi)d\xi, \quad (23)$$

where $\mathbf{G}_n$ is an $n\times n$ transformation matrix satisfying Eq. (13).

Substituting Eq. (23) into Eq. (22) we obtain

$$\dot{X}^0(t)=A_0^0(t)X^0(t)+\mathbf{U}_1^T(t)\mathbf{X}^s(0)+\int_0^t\mathbf{U}_1^T(t)\mathbf{U}_2(\xi)X^0(\xi)d\xi, \quad (24)$$

where

$$\mathbf{U}_1:=\mathbf{G}_n^T\mathbf{A}_s^0=\frac{1}{\eta}\dot{\mathbf{V}}, \quad (25)$$

$$\mathbf{U}_2:=\mathbf{G}_n^{-1}\mathbf{A}_0^s=\eta\mathbf{U}. \quad (26)$$

Here, $\mathbf{U}$ and $\mathbf{V}$ were defined in Eqs. (15) and (16).

Introducing the integrating factor η as defined by Eq. (14) and a new scalar variable given as follows:

$$W^0:=\eta X^0, \quad (27)$$

Eq. (24), after multiplying both the sides by $\eta(t)$ and replacing $X^0(\xi)$ by $W^0(\xi)/\eta(\xi)$, changes to

$$\dot{W}^0(t)=\eta(t)\mathbf{U}_1^T(t)\mathbf{X}^s(0)+\int_0^t\eta(t)\mathbf{U}_1^T(t)\frac{1}{\eta(\xi)}\mathbf{U}_2(\xi)W^0(\xi)d\xi. \quad (28)$$

Integrating Eq. (28) and using Eqs. (25) and (26) we obtain

$$W^0(t)=W^0(0)+\mathbf{V}^T(t)\mathbf{X}^s(0)+\int_0^t[\mathbf{V}^T(t)-\mathbf{V}^T(\xi)]\mathbf{U}(\xi)W^0(\xi)d\xi. \quad (29)$$

Left multiplying Eq. (29) by $[\mathbf{U}^T\ \ \mathbf{V}^T\mathbf{U}]^T$ we obtain an $(n+1)$-vectorial integral equation:

$$\begin{bmatrix}\mathbf{U}W^0\\ \mathbf{V}^T\mathbf{U}W^0\end{bmatrix}=\begin{bmatrix}\mathbf{U}\mathbf{V}^T & -\mathbf{U}\\ \mathbf{V}^T\mathbf{U}\mathbf{V}^T & -\mathbf{V}^T\mathbf{U}\end{bmatrix}\left\{\int_0^t\begin{bmatrix}\mathbf{U}(\xi)W^0(\xi)\\ \mathbf{V}^T(\xi)\mathbf{U}(\xi)W^0(\xi)\end{bmatrix}d\xi+\begin{bmatrix}\mathbf{X}^s(0)\\ -W^0(0)\end{bmatrix}\right\}. \quad (30)$$

Let $\mathbf{N}_1$ be defined by Eq. (18), and let

$$\mathbf{Y}(t)=\begin{bmatrix}\mathbf{Y}^s(t)\\ Y^0(t)\end{bmatrix}:=\int_0^t\begin{bmatrix}\mathbf{U}(\xi)W^0(\xi)\\ \mathbf{V}^T(\xi)\mathbf{U}(\xi)W^0(\xi)\end{bmatrix}d\xi+\begin{bmatrix}\mathbf{X}^s(0)\\ -W^0(0)\end{bmatrix}, \quad (31)$$

and then using Eq. (30) we obtain a linear equations system as that given by Eq. (17), where $\mathbf{N}_1$ can be proved to satisfy Eq. (19).

In terms of $\mathbf{U}_2$ defined by Eq. (26), $\mathbf{X}^s$ in Eq. (23) can be written as

$$\mathbf{X}^s(t)=\mathbf{G}_n(t)\left[\mathbf{X}^s(0)+\int_0^t\mathbf{U}_2(\xi)X^0(\xi)d\xi\right]. \quad (32)$$

By means of Eqs. (27) and (26) it further changes to

$$\mathbf{X}^s(t)=\mathbf{G}_n(t)\left[\mathbf{X}^s(0)+\int_0^t\mathbf{U}(\xi)W^0(\xi)d\xi\right], \quad (33)$$

which being compared with $\mathbf{Y}^s$ in Eq. (31) readily leads to

$$\mathbf{X}^s=\mathbf{G}_n\mathbf{Y}^s. \quad (34)$$

On the other hand, by differentiating Eq. (31) and taking the second row we have

$$\dot{Y}^0=\mathbf{V}^T\mathbf{U}W^0. \quad (35)$$

The term $\dot{Y}^0$ as shown in Eq. (17) with $\mathbf{N}_1$ defined by Eq. (18) is equal to

$$\dot{Y}^0=\mathbf{V}^T\mathbf{U}\mathbf{V}^T\mathbf{Y}^s-\mathbf{V}^T\mathbf{U}Y^0. \quad (36)$$

By equating Eqs. (35) and (36) and using Eq. (27) it follows that

$$X^0=\frac{W^0}{\eta}=\frac{1}{\eta}[\mathbf{V}^T\mathbf{Y}^s-Y^0]. \quad (37)$$

Thus, from Eqs. (34) and (37) we obtain the relation between $\mathbf{X}$ and $\mathbf{Y}$, given as follows:

$$\begin{bmatrix}\mathbf{X}^s\\ X^0\end{bmatrix}=\begin{bmatrix}\mathbf{G}_n & \mathbf{0}_{n\times 1}\\ \frac{1}{\eta}\mathbf{V}^T & \frac{-1}{\eta}\end{bmatrix}\begin{bmatrix}\mathbf{Y}^s\\ Y^0\end{bmatrix}, \quad (38)$$

or the inverse relation given as follows:

$$\begin{bmatrix} \mathbf{Y}^s \\ Y^0 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_n^{-1} & \mathbf{0}_{n\times 1} \\ \mathbf{V}^T\mathbf{G}_n^{-1} & -\eta \end{bmatrix} \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix}. \tag{39}$$

This completes the proof of this theorem. □

Theorem 2. $\mathbf{N}_1$ defined by Eq. (17) satisfies

$$\mathbf{N}_1^T\mathbf{h} + \mathbf{h}\mathbf{N}_1 = \begin{bmatrix} \mathbf{V}\mathbf{U}^T + \mathbf{U}\mathbf{V}^T & -\mathbf{U} \\ -\mathbf{U}^T & 0 \end{bmatrix}, \tag{40}$$

where

$$\mathbf{h} := \begin{bmatrix} \mathbf{I}_n - \mathbf{V}\mathbf{V}^T & \mathbf{V} \\ \mathbf{V}^T & -1 \end{bmatrix} \tag{41}$$

is an indefinite matrix.

Proof. Substituting Eq. (18) for $\mathbf{N}_1$ and Eq. (41) for $\mathbf{h}$ into Eq. (40) and through some calculations we can prove Eq. (40). For any nonzero $\mathbf{Y} = (\mathbf{Y}^s, Y^0) \in \mathbb{R}^{n+1}$ we have

$$\begin{bmatrix} (\mathbf{Y}^s)^T & Y^0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_n - \mathbf{V}\mathbf{V}^T & \mathbf{V} \\ \mathbf{V}^T & -1 \end{bmatrix} \begin{bmatrix} \mathbf{Y}^s \\ Y^0 \end{bmatrix} = \|\mathbf{Y}^s\|^2 - (\mathbf{V}^T\mathbf{Y}^s - Y^0)^2, \tag{42}$$

where $\|\mathbf{Y}^s\|^2 := (\mathbf{Y}^s)^T\mathbf{Y}^s$ denotes the squared norm of $\mathbf{Y}^s$. Since the right-hand side may be positive, zero or negative, $\mathbf{h}$ is indefinite. □

Theorem 3. The fundamental matrix $\mathbf{G}_{n+1}$ for Eq. (7) has the following representation:

$$\mathbf{G}_{n+1} = \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \frac{1}{\eta}\mathbf{V}^T & \frac{-1}{\eta} \end{bmatrix} \mathbf{H}\mathbf{g}, \tag{43}$$

where $\mathbf{H} \in SL(n+1,\mathbb{R})$ is the fundamental matrix for Eq. (17), satisfying

$$\dot{\mathbf{H}}(t) = \mathbf{N}_1(t)\mathbf{H}(t), \;\; \mathbf{H}(0) = \mathbf{I}_{n+1}, \tag{44}$$

and

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{45}$$

is the metric tensor of the $(n+1)$-dimensional Minkowski space.

Proof. With $\mathbf{H}$ satisfying Eq. (44), the solution of Eq. (17) can be expressed by

$$\mathbf{Y}(t) = \mathbf{H}(t)\mathbf{Y}(0). \tag{46}$$

Substituting it into Eq. (38) and using $\mathbf{Y}(0) = \mathbf{g}\mathbf{X}(0)$ resulting from Eq. (12) by inserting $t = 0$, we obtain

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \frac{1}{\eta}\mathbf{V}^T & \frac{-1}{\eta} \end{bmatrix} \mathbf{H}\mathbf{g}\mathbf{X}(0), \tag{47}$$

which upon comparing with the solution of Eq. (4), i.e., $\mathbf{X}(t) = \mathbf{G}_{n+1}(t)\mathbf{X}(0)$ with $\mathbf{G}_{n+1}(t)$ satisfying Eq. (7), we obtain Eq. (43). □

Theorem 4. For system (17) if $\dot{\mathbf{V}} = \mathbf{U}$ holds, then

$$\mathbf{N}_1^T\mathbf{h} + \mathbf{h}\mathbf{N}_1 + \dot{\mathbf{h}} = \mathbf{0}, \tag{48}$$

$$\mathbf{H}^T\mathbf{h}\mathbf{H} = \mathbf{g}, \tag{49}$$

and

$$\mathbf{Y}^T(t)\mathbf{h}(t)\mathbf{Y}(t) = \mathbf{Y}^T(0)\mathbf{g}\mathbf{Y}(0) \tag{50}$$

is a quadratic invariant. Similarly, under the same condition,

$$\mathbf{X}^T(t)\boldsymbol{\eta}_a(t)\mathbf{X}(t) = \mathbf{X}^T(0)\mathbf{g}\mathbf{X}(0) \tag{51}$$

is a quadratic invariant of system (4), where

$$\boldsymbol{\eta}_a = \begin{bmatrix} \mathbf{G}_n^{-T}\mathbf{G}_n^{-1} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -\eta^2 \end{bmatrix}. \tag{52}$$

At the same time, we have

$$\mathbf{A}^T\boldsymbol{\eta}_a + \boldsymbol{\eta}_a\mathbf{A} + \dot{\boldsymbol{\eta}}_a = \mathbf{0}, \tag{53}$$

$$\mathbf{G}_{n+1}^T\boldsymbol{\eta}_a\mathbf{G}_{n+1} = \mathbf{g}. \tag{54}$$

Proof. Taking the time derivative of Eq. (41) and using the condition of $\dot{\mathbf{V}} = \mathbf{U}$ we obtain

$$\dot{\mathbf{h}} = \begin{bmatrix} -\mathbf{U}\mathbf{V}^T - \mathbf{V}\mathbf{U}^T & \mathbf{U} \\ \mathbf{U}^T & 0 \end{bmatrix}, \tag{55}$$

which combined with Eq. (40) leads to Eq. (48). Taking the time derivative of $\mathbf{H}^T\mathbf{h}\mathbf{H}$, then substituting Eq. (44) for $\dot{\mathbf{H}}$ and using Eq. (48) we obtain

$$\frac{d}{dt}[\mathbf{H}^T\mathbf{h}\mathbf{H}] = \mathbf{0}_{n+1},$$

which indicates that $\mathbf{H}^T\mathbf{h}\mathbf{H}$ is a constant matrix. At $t = 0$ we have $\mathbf{H}^T(0)\mathbf{h}(0)\mathbf{H}(0) = \mathbf{g}$, and thus Eq. (49) is proved.

Substituting Eq. (46) for $\mathbf{Y}$ into the quadratic form $\mathbf{Y}^T\mathbf{hY}$ and using Eq. (49) and $\mathbf{g}=\mathbf{h}(0)$ we can prove that $\mathbf{Y}^T\mathbf{hY}=\mathbf{Y}^T(0)\mathbf{gY}(0)$ is a quadratic invariant of system (17).

Substituting Eq. (12) for $\mathbf{Y}$ into Eq. (50) we can obtain Eq. (51). Taking the time derivative of Eq. (51) and inserting Eq. (4) for $\dot{\mathbf{X}}$, we can obtain Eq. (53). Substituting Eq. (43) for $\mathbf{G}_{n+1}$ into the left-hand side of Eq. (54) and using Eq. (49) and $\mathbf{g}^2=\mathbf{I}_{n+1}$ we can obtain the right-hand side of Eq. (54). □

In Table 1 we compare the above two $\mathbf{X}$ and $\mathbf{Y}$ systems under the condition of $\dot{\mathbf{V}}=\mathbf{U}$, where the Lie algebras, Lie groups and the invariants are compared. It is clear that these systems are different representations of the same linear systems in the Minkowskian type space. Now, we apply the above theorem to the linear Lorentzian system [Liu (2002)] as a direct result.

Corollary 1. Eq. (4) is a linear Lorentzian system if

$$\mathbf{A}^T\mathbf{g}+\mathbf{gA}=\mathbf{0}. \tag{56}$$

For this system we have a quadratic invariant

$$\mathbf{X}^T(t)\mathbf{gX}(t)=\mathbf{X}^T(0)\mathbf{gX}(0), \tag{57}$$

and $\mathbf{G}_{n+1}$ satisfies

$$\mathbf{G}_{n+1}^T\mathbf{g}\mathbf{G}_{n+1}=\mathbf{g}. \tag{58}$$

Proof. It is easy to check that the general form for such an $\mathbf{A}$ satisfying Eq. (56) is

$$\mathbf{A}=\begin{bmatrix} \mathbf{A}_s^s & \mathbf{A}_0^s \\ (\mathbf{A}_0^s)^T & 0 \end{bmatrix}, \tag{59}$$

where $\mathbf{A}_s^s$ is a skew-symmetric matrix function with dimensions $n\times n$. Therefore, by Eq. (13) we have $\mathbf{G}_n=\mathbf{R}$, where $\mathbf{R}$ is an orthogonal matrix satisfying $\mathbf{R}^T\mathbf{R}=\mathbf{I}_n$. On the other hand, by Eq. (14) we have $\eta=1$ because of $A_0^0=0$. According to these two results, from Eq. (52) we obtain $\boldsymbol{\eta}_a=\mathbf{g}$. Inserting it into Eqs. (51) and (54) we can prove Eqs. (57) and (58) immediately. □

From this corollary we can see that the results presented in Theorem 4 are the extensions of the famous linear Lorentzian system. This extended system has the general form in Eq. (4) but with the following constraint on its state matrix:

$$\mathbf{A}_0^s=\eta^2\mathbf{G}_n\mathbf{G}_n^T\mathbf{A}_s^0, \tag{60}$$

which is obtained by inserting Eqs. (15) and (16) into $\dot{\mathbf{V}}=\mathbf{U}$.

2.2 A positive-definite metric method

In this section we will nilpotentlize the time-varying linear system (4) by a different method. We next prove the following results.

Theorem 5. Corresponding to the linear system (4), if we consider the following variable transformation

$$\mathbf{Y}=\begin{bmatrix} \mathbf{G}_n^{-1} & \mathbf{0}_{n\times 1} \\ -\mathbf{V}^T\mathbf{G}_n^{-1} & \eta \end{bmatrix}\mathbf{X}, \tag{61}$$

then there exists a linear system

$$\dot{\mathbf{Y}}(t)=\mathbf{N}_2(t)\mathbf{Y}(t), \tag{62}$$

where

$$\mathbf{N}_2:=\begin{bmatrix} \mathbf{UV}^T & \mathbf{U} \\ -\mathbf{V}^T\mathbf{UV}^T & -\mathbf{V}^T\mathbf{U} \end{bmatrix}, \tag{63}$$

satisfying

$$\operatorname{tr}\mathbf{N}_2=0,\ \ \mathbf{N}_2^2=\mathbf{0}, \tag{64}$$

is a zero trace nilpotent matrix function with index two.

Proof. In addition to Eq. (30), we can left multiply Eq. (29) by $[\mathbf{U}^T\ \ -\mathbf{V}^T\mathbf{U}]^T$ to obtain another $(n+1)$-vectorial integral equation:

$$\begin{bmatrix} \mathbf{U}W^0 \\ -\mathbf{V}^T\mathbf{U}W^0 \end{bmatrix}=\begin{bmatrix} \mathbf{UV}^T & \mathbf{U} \\ -\mathbf{V}^T\mathbf{UV}^T & -\mathbf{V}^T\mathbf{U} \end{bmatrix} \left\{\int_0^t \begin{bmatrix} \mathbf{U}(\xi)W^0(\xi) \\ -\mathbf{V}^T(\xi)\mathbf{U}(\xi)W^0(\xi) \end{bmatrix} d\xi+\begin{bmatrix} \mathbf{X}^s(0) \\ W^0(0) \end{bmatrix}\right\}. \tag{65}$$

Let $\mathbf{N}_2$ be defined by Eq. (63), and let

$$\mathbf{Y}(t)=\begin{bmatrix} \mathbf{Y}^s(t) \\ Y^0(t) \end{bmatrix} :=\int_0^t \begin{bmatrix} \mathbf{U}(\xi)W^0(\xi) \\ -\mathbf{V}^T(\xi)\mathbf{U}(\xi)W^0(\xi) \end{bmatrix} d\xi+\begin{bmatrix} \mathbf{X}^s(0) \\ W^0(0) \end{bmatrix}, \tag{66}$$

Table 1: Under the condition of $\dot{\mathbf{V}} = \mathbf{U}$ the comparisons of three linear systems in the Minkowskian type space

Variables	Equations	Metrics	Lie Algebras	Lie Groups	Invariant Forms
$\mathbf{X}$	$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$	$\boldsymbol{\eta}_a$	$\mathbf{A}^T\boldsymbol{\eta}_a + \boldsymbol{\eta}_a\mathbf{A} + \dot{\boldsymbol{\eta}}_a = \mathbf{0}$	$\mathbf{G}_{n+1}^T\boldsymbol{\eta}_a\mathbf{G}_{n+1} = \mathbf{g}$	$\mathbf{X}^T\boldsymbol{\eta}_a\mathbf{X}$
$\mathbf{Y}$	$\dot{\mathbf{Y}} = \mathbf{N}_1\mathbf{Y}$	$\mathbf{h}$	$\mathbf{N}_1^T\mathbf{h} + \mathbf{h}\mathbf{N}_1 + \dot{\mathbf{h}} = \mathbf{0}$ $\operatorname{tr}\mathbf{N}_1 = 0,\ \mathbf{N}_1^2 = \mathbf{0}$	$\mathbf{H}^T\mathbf{h}\mathbf{H} = \mathbf{g}$	$\mathbf{Y}^T\mathbf{h}\mathbf{Y}$
$\mathbf{Z}$	$\dot{\mathbf{Z}} = \mathbf{B}\mathbf{Z}$	$\mathbf{g}$	$\mathbf{B}^T\mathbf{g} + \mathbf{g}\mathbf{B} = \mathbf{0}$ $\operatorname{tr}\mathbf{B} = 0$	$\mathbf{G}_{n+1}^T\mathbf{g}\mathbf{G}_{n+1} = \mathbf{g}$	$\mathbf{Z}^T\mathbf{g}\mathbf{Z}$

and then from Eq. (65) we obtain a linear equations system as that given by Eq. (62), where $\mathbf{N}_2$ can be proved to satisfy Eq. (64).

The proof of

$$\mathbf{X}^s = \mathbf{G}_n\mathbf{Y}^s$$

is similar to that given in Theorem 1. From Eq. (66) by differentiating and taking the second row we have

$$\dot{Y}^0 = -\mathbf{V}^T\mathbf{U}W^0.$$

The term $\dot{Y}^0$ as shown in Eq. (62) with $\mathbf{N}_2$ defined by Eq. (63) is equal to

$$\dot{Y}^0 = -\mathbf{V}^T\mathbf{U}\mathbf{V}^T\mathbf{Y}^s - \mathbf{V}^T\mathbf{U}Y^0.$$

From the above two equations and Eq. (27) one has

$$X^0 = \frac{W^0}{\eta} = \frac{1}{\eta}[\mathbf{V}^T\mathbf{Y}^s + Y^0].$$

Then, we obtain the relation between $\mathbf{X}$ and $\mathbf{Y}$:

$$\begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \frac{1}{\eta}\mathbf{V}^T & \frac{1}{\eta} \end{bmatrix} \begin{bmatrix} \mathbf{Y}^s \\ Y^0 \end{bmatrix}, \tag{67}$$

or the inverse relation:

$$\begin{bmatrix} \mathbf{Y}^s \\ Y^0 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_n^{-1} & \mathbf{0}_{n\times 1} \\ -\mathbf{V}^T\mathbf{G}_n^{-1} & \eta \end{bmatrix} \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix}. \tag{68}$$

This completes the proof of this theorem. □

Theorem 6. $\mathbf{N}_2$ defined by Eq. (62) satisfies

$$\mathbf{N}_2^T\mathbf{k} + \mathbf{k}\mathbf{N}_2 = \begin{bmatrix} \mathbf{V}\mathbf{U}^T + \mathbf{U}\mathbf{V}^T & \mathbf{U} \\ \mathbf{U}^T & 0 \end{bmatrix}, \tag{69}$$

where

$$\mathbf{k} := \begin{bmatrix} \mathbf{I}_n + \mathbf{V}\mathbf{V}^T & \mathbf{V} \\ \mathbf{V}^T & 1 \end{bmatrix} \tag{70}$$

is a positive definite matrix.

Proof. Substituting Eq. (63) for $\mathbf{N}_2$ and Eq. (70) for $\mathbf{k}$ into the left-hand side of Eq. (69) and through some calculations we obtain the right-hand side of Eq. (69). The positive definiteness of $\mathbf{k}$ can be proved as follows. For any nonzero $\mathbf{Y} = (\mathbf{Y}^s, Y^0) \in \mathbb{R}^{n+1}$ we have

$$\begin{bmatrix} (\mathbf{Y}^s)^T & Y^0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_n + \mathbf{V}\mathbf{V}^T & \mathbf{V} \\ \mathbf{V}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{Y}^s \\ Y^0 \end{bmatrix} = \|\mathbf{Y}^s\|^2 + (\mathbf{V}^T\mathbf{Y}^s + Y^0)^2.$$

Since the right-hand side is positive, by definition $\mathbf{k}$ is positive definite. □

Theorem 7. The fundamental matrix $\mathbf{G}_{n+1}$ for Eq. (7) has the following representation:

$$\mathbf{G}_{n+1} = \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \frac{1}{\eta}\mathbf{V}^T & \frac{1}{\eta} \end{bmatrix}\mathbf{K}, \tag{71}$$

where $\mathbf{K} \in SL(n+1,\mathbb{R})$ is the fundamental matrix for Eq. (62), satisfying

$$\dot{\mathbf{K}}(t) = \mathbf{N}_2(t)\mathbf{K}(t),\ \ \mathbf{K}(0) = \mathbf{I}_{n+1}. \tag{72}$$

Proof. With $\mathbf{K}$ satisfying Eq. (72), the solution of Eq. (62) can be expressed by

$$\mathbf{Y}(t) = \mathbf{K}(t)\mathbf{Y}(0). \tag{73}$$

Substituting it into Eq. (67) and using $\mathbf{Y}(0) = \mathbf{X}(0)$ we obtain

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \frac{1}{\eta}\mathbf{V}^T & \frac{1}{\eta} \end{bmatrix}\mathbf{K}\mathbf{X}(0), \tag{74}$$

which when comparing with the solution of Eq. (4), i.e., $\mathbf{X}(t) = \mathbf{G}_{n+1}(t)\mathbf{X}(0)$ with $\mathbf{G}_{n+1}(t)$ satisfying Eq. (7), we obtain Eq. (71). Since $\mathrm{tr}\,\mathbf{N}_2 = 0$ and $\det\mathbf{K}(0) = 1$, by the Abel formula we have $\det\mathbf{K}(t) = 1$ for all t, that is, $\mathbf{K} \in SL(n+1,\mathbb{R})$. □

Theorem 8. For system (62) if $\dot{\mathbf{V}} = -\mathbf{U}$ holds, then

$$\mathbf{N}_2^T\mathbf{k} + \mathbf{k}\mathbf{N}_2 + \dot{\mathbf{k}} = \mathbf{0}, \tag{75}$$

$$\mathbf{K}^T\mathbf{k}\mathbf{K} = \mathbf{I}_{n+1}, \tag{76}$$

and

$$\mathbf{Y}^T(t)\mathbf{k}(t)\mathbf{Y}(t) = \mathbf{Y}^T(0)\mathbf{I}_{n+1}\mathbf{Y}(0) \tag{77}$$

is a quadratic invariant. Similarly, under the same condition,

$$\mathbf{X}^T(t)\boldsymbol{\eta}_b(t)\mathbf{X}(t) = \mathbf{X}^T(0)\mathbf{I}_{n+1}\mathbf{X}(0) \tag{78}$$

is a quadratic invariant of system (4), where

$$\boldsymbol{\eta}_b = \begin{bmatrix} \mathbf{G}_n^{-T}\mathbf{G}_n^{-1} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \eta^2 \end{bmatrix}. \tag{79}$$

At the same time, we have

$$\mathbf{A}^T\boldsymbol{\eta}_b + \boldsymbol{\eta}_b\mathbf{A} + \dot{\boldsymbol{\eta}}_b = \mathbf{0}, \tag{80}$$

$$\mathbf{G}_{n+1}^T\boldsymbol{\eta}_b\mathbf{G}_{n+1} = \mathbf{I}_{n+1}. \tag{81}$$

Proof. Taking the time derivative of Eq. (70) and using the condition of $\dot{\mathbf{V}} = -\mathbf{U}$ we obtain

$$\dot{\mathbf{k}} = \begin{bmatrix} -\mathbf{U}\mathbf{V}^T - \mathbf{V}\mathbf{U}^T & -\mathbf{U} \\ -\mathbf{U}^T & 0 \end{bmatrix}. \tag{82}$$

Upon combining it with Eq. (69) we obtain Eq. (75). Taking the time derivative of $\mathbf{K}^T\mathbf{k}\mathbf{K}$, then substituting Eq. (72) for $\dot{\mathbf{K}}$ and using Eq. (75) we obtain

$$\frac{d}{dt}[\mathbf{K}^T\mathbf{k}\mathbf{K}] = \mathbf{0}_{n+1}.$$

Due to $\mathbf{K}^T\mathbf{k}\mathbf{K} = \mathbf{I}_{n+1}$ at $t = 0$, Eq. (76) is proved.

Substituting Eq. (73) for $\mathbf{Y}$ into the quadratic form $\mathbf{Y}^T\mathbf{k}\mathbf{Y}$ and using Eq. (76) and $\mathbf{k}(0) = \mathbf{I}_{n+1}$ we can prove that $\mathbf{Y}^T\mathbf{k}\mathbf{Y} = \mathbf{Y}^T(0)\mathbf{I}_{n+1}\mathbf{Y}(0)$ is a quadratic invariant of system (62).

Substituting Eq. (61) for $\mathbf{Y}$ into Eq. (77) we can obtain Eq. (78). Taking the time derivative of Eq. (78) and inserting Eq. (4) for $\dot{\mathbf{X}}$, we can obtain Eq. (80). Substituting Eq. (71) for $\mathbf{G}_{n+1}$ into the left-hand side of Eq. (54) and using Eq. (76) we can obtain the right-hand side of Eq. (81). □

In Table 2 we compare the above two $\mathbf{X}$ and $\mathbf{Y}$ systems under the condition of $\dot{\mathbf{V}} = -\mathbf{U}$, where the Lie algebras, Lie groups and the invariants are compared. It is clear that these systems are different representations of the same linear systems in the Euclidean type space. We apply the above theorem to the linear skew system as a direct result.

Corollary 2. Eq. (4) is a linear skew system if

$$\mathbf{A}^T\mathbf{I}_{n+1} + \mathbf{I}_{n+1}\mathbf{A} = \mathbf{0}. \tag{83}$$

For this system we have a quadratic invariant:

$$\mathbf{X}^T(t)\mathbf{I}_{n+1}\mathbf{X}(t) = \mathbf{X}^T(0)\mathbf{I}_{n+1}\mathbf{X}(0), \tag{84}$$

and $\mathbf{G}_{n+1}$ satisfies the orthogonal condition:

$$\mathbf{G}_{n+1}^T\mathbf{I}_{n+1}\mathbf{G}_{n+1} = \mathbf{I}_{n+1}. \tag{85}$$

Proof. It is easy to check that the general form for such an $\mathbf{A}$ satisfying Eq. (83) is

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_s^s & \mathbf{A}_0^s \\ -(\mathbf{A}_0^s)^T & 0 \end{bmatrix}, \tag{86}$$

where $\mathbf{A}_s^s$ is a skew-symmetric matrix function with dimensions $n \times n$. Therefore, by Eq. (13) we have $\mathbf{G}_n = \mathbf{R}$, where $\mathbf{R}$ is an orthogonal matrix satisfying $\mathbf{R}^T\mathbf{R} = \mathbf{I}_n$. On the other hand, by Eq. (14) we have $\eta = 1$ because of $A_0^0 = 0$. According to these two results, from Eq. (79) we obtain $\boldsymbol{\eta}_b = \mathbf{I}_{n+1}$. Inserting it into Eqs. (78) and (81) one is easily proved Eqs. (84) and (85). □

From this corollary we can see that the results presented in Theorem 8 are the extensions of the famous linear skew system. This extended system endows the general form in Eq. (4) with the following constraint:

$$\mathbf{A}_0^s = -\eta^2\mathbf{G}_n\mathbf{G}_n^T\mathbf{A}_s^0, \tag{87}$$

which is obtained by inserting Eqs. (15) and (16) into $\dot{\mathbf{V}} = -\mathbf{U}$.

Table 2: Under the condition of $\dot{\mathbf{V}} = -\mathbf{U}$ the comparisons of three linear systems in the Euclidean type space

Variables	Equations	Metrics	Lie Algebras	Lie Groups	Invariant Forms
$\mathbf{X}$	$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$	$\boldsymbol{\eta}_b$	$\mathbf{A}^T\boldsymbol{\eta}_b + \boldsymbol{\eta}_a\mathbf{A} + \dot{\boldsymbol{\eta}}_b = \mathbf{0}$	$\mathbf{G}_{n+1}^T\boldsymbol{\eta}_b\mathbf{G}_{n+1} = \mathbf{I}_{n+1}$	$\mathbf{X}^T\boldsymbol{\eta}_b\mathbf{X}$
$\mathbf{Y}$	$\dot{\mathbf{Y}} = \mathbf{N}_2\mathbf{Y}$	$\mathbf{k}$	$\mathbf{N}_2^T\mathbf{k} + \mathbf{k}\mathbf{N}_2 + \dot{\mathbf{k}} = \mathbf{0}$ $\operatorname{tr}\mathbf{N}_2 = 0,\ \mathbf{N}_2^2 = \mathbf{0}$	$\mathbf{K}^T\mathbf{k}\mathbf{K} = \mathbf{I}_{n+1}$	$\mathbf{Y}^T\mathbf{k}\mathbf{Y}$
$\mathbf{Z}$	$\dot{\mathbf{Z}} = \mathbf{B}\mathbf{Z}$	$\mathbf{I}_{n+1}$	$\mathbf{B}^T\mathbf{I}_{n+1} + \mathbf{I}_{n+1}\mathbf{B} = \mathbf{0}$ $\operatorname{tr}\mathbf{B} = 0$	$\mathbf{G}_{n+1}^T\mathbf{I}_{n+1}\mathbf{G}_{n+1} = \mathbf{I}_{n+1}$	$\mathbf{Z}^T\mathbf{I}_{n+1}\mathbf{Z}$

3 Examples of quadratic invariants

In this section we are going to apply Theorems 4 and 8 to derive the sufficient conditions for the existence of quadratic invariants for some lower order linear systems, and the quadratic invariants are written out explicitly. Also, we demonstrate the use of Theorems 3 and 7 by an example. Of course, Theorems 4 and 8 are applicable to any linear system when the corresponding conditions are satisfied.

3.1 Second-order system

Let us first consider the second-order linear differential equations system with

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{88}$$

For this case we have

$$G(t) = \exp\left[\int_0^t a_{11}(\xi)d\xi\right], \tag{89}$$

$$\eta(t) = \exp\left[-\int_0^t a_{22}(\xi)d\xi\right], \tag{90}$$

such that from Eqs. (60) and (87) the conditions for the existence of quadratic invariants are

$$\begin{aligned} a_{12} &= \eta^2 G^2 a_{21} \qquad (91)\\ &= \exp\left(2\int_0^t [a_{11}(\xi) - a_{22}(\xi)]d\xi\right) a_{21}, \end{aligned}$$

$$\begin{aligned} a_{12} &= -\eta^2 G^2 a_{21} \qquad (92)\\ &= -\exp\left(2\int_0^t [a_{11}(\xi) - a_{22}(\xi)]d\xi\right) a_{21}. \end{aligned}$$

Under condition (91) the quadratic invariant is

$$G^{-2}(t)x_1^2(t) - \eta^2(t)x_2^2(t) = x_1^2(0) - x_2^2(0). \tag{93}$$

To prove it, we write

$$\begin{aligned} &\frac{d}{dt}[G^{-2}x_1^2 - \eta^2 x_2^2] \\ &= -2G^{-3}\dot{G}x_1^2 + 2G^{-2}x_1\dot{x}_1 - 2\eta\dot{\eta}x_2^2 - 2\eta^2 x_2\dot{x}_2 \\ &= -2a_{11}G^{-2}x_1^2 + 2G^{-2}x_1(a_{11}x_1 + a_{12}x_2) \\ &\qquad + 2a_{22}\eta^2 x_2^2 - 2\eta^2 x_2(a_{21}x_1 + a_{22}x_2) \\ &= 2a_{12}G^{-2}x_1x_2 - 2a_{21}\eta^2 x_2x_1 \\ &= 2a_{21}\eta^2 G^2 G^{-2}x_1x_2 - 2a_{21}\eta^2 x_2x_1 \\ &= 0, \end{aligned}$$

where $\dot{x}_1 = a_{11}x_1 + a_{12}x_2$, $\dot{x}_2 = a_{21}x_1 + a_{22}x_2$, and Eq. (91) were used.

Similarly, under condition (92) we can prove the following quadratic form is an invariant:

$$G^{-2}(t)x_1^2(t) + \eta^2(t)x_2^2(t) = x_1^2(0) + x_2^2(0). \tag{94}$$

Under a special condition of $a_{11} = a_{22}$, we have $\eta G = 1$, and thus the condition for the existence of quadratic invariant is $a_{12} = a_{21}$ or $a_{12} = -a_{21}$. For example, for

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2}(\cos t - e^{\sin t}) & \sin^2 t \\ \sin^2 t & \frac{1}{2}(\cos t - e^{\sin t}) \end{bmatrix}, \tag{95}$$

the invariant is

$$\eta^2(t)[x_1^2(t) - x_2^2(t)] = x_1^2(0) - x_2^2(0), \tag{96}$$

where

$$\eta(t) = \exp\left[-\frac{1}{2}\int_0^t (\cos\xi - e^{\sin\xi})d\xi\right]. \tag{97}$$

Similarly, for

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2}(\cos t - e^{\sin t}) & \sin^2 t \\ -\sin^2 t & \frac{1}{2}(\cos t - e^{\sin t}) \end{bmatrix}, \tag{98}$$

the invariant is

$$\eta^2(t)[x_1^2(t)+x_2^2(t)] = x_1^2(0)+x_2^2(0). \quad (99)$$

From these examples it can be seen that Theorems 4 and 8 can help us to judge which system possesses a quadratic invariant and to derive that invariant.

3.2 *Third-order system*

Then, let us consider the third-order linear differential equation:

$$\frac{d^3x}{dt^3} - A_0^0(t)\frac{d^2x}{dt^2} - A_2^0(t)\frac{dx}{dt} - A_1^0(t)x = 0. \quad (100)$$

Let $x_1 := x$, $x_2 := \dot{x}$ and $x_0 := \ddot{x}$, and then we obtain a three-dimensional time-varying linear system (4) with the state matrix being

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ A_1^0 & A_2^0 & A_0^0 \end{bmatrix}. \quad (101)$$

For this case

$$\mathbf{G}_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \ \ \eta(t) = \exp\left[-\int_0^t A_0^0(\xi)d\xi\right], \quad (102)$$

such that we have

$$\dot{\mathbf{V}} = \eta\mathbf{G}_2^T \begin{bmatrix} A_1^0 \\ A_2^0 \end{bmatrix} = \eta \begin{bmatrix} A_1^0 \\ tA_1^0 + A_2^0 \end{bmatrix}, \quad \mathbf{U} = \frac{1}{\eta}\mathbf{G}_2^{-1}(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\eta}\begin{bmatrix} -t \\ 1 \end{bmatrix}. \quad (103)$$

Under the condition of $\dot{\mathbf{V}} = \mathbf{U}$, i.e.,

$$A_1^0 = -\frac{t}{\eta^2}, \ \ A_2^0 = \frac{1}{\eta^2} - tA_1^0 = \frac{1+t^2}{\eta^2}, \quad (104)$$

the system has a quadratic invariant:

$$x_1^2(t) - 2tx_1(t)x_2(t) + (1+t^2)x_2^2(t) - \eta^2(t)x_0^2(t) = x_1^2(0)+x_2^2(0)-x_0^2(0). \quad (105)$$

Similarly, under the condition of $\dot{\mathbf{V}} = -\mathbf{U}$, i.e.,

$$A_1^0 = \frac{t}{\eta^2}, \ \ A_2^0 = -\frac{1}{\eta^2} - tA_1^0 = -\frac{1+t^2}{\eta^2}, \quad (106)$$

the system has a quadratic invariant:

$$x_1^2(t) - 2tx_1(t)x_2(t) + (1+t^2)x_2^2(t) + \eta^2(t)x_0^2(t) = x_1^2(0)+x_2^2(0)+x_0^2(0). \quad (107)$$

3.3 *Fourth-order system*

We use the following example to demonstrate the use of Theorems 3 and 7, which is the fourth-order linear differential equation:

$$\frac{d^4x}{dt^4} - A_0^0(t)\frac{d^3x}{dt^3} - A_3^0(t)\frac{d^2x}{dt^2} - A_2^0(t)\frac{dx}{dt} - A_1^0(t)x = 0. \quad (108)$$

Let $x_1 := x$, $x_2 := \dot{x}$, $x_3 := \ddot{x}$ and $x_0 := d^3x/dt^3$, and then we obtain a four-dimensional time-varying linear system (4) with the state matrix being

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_1^0 & A_2^0 & A_3^0 & A_0^0 \end{bmatrix}. \quad (109)$$

For this case it is easy to write

$$\mathbf{G}_3 = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad (110)$$

which is a fundamental matrix corresponding to the submatrix $\mathbf{A}_s^s$ in Eq. (109),

$$\mathbf{A}_s^s = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \quad (111)$$

By Theorem 3, we have

$$\mathbf{G}_3 = \begin{bmatrix} \mathbf{G}_2 & \mathbf{0}_{2\times 1} \\ \frac{1}{\eta}\mathbf{V}^T & \frac{-1}{\eta} \end{bmatrix}\mathbf{Hg}, \quad (112)$$

where $\mathbf{G}_2$ is defined by Eq. (102) and $\mathbf{H}$ is the fundamental matrix for Eq. (17), satisfying

$$\dot{\mathbf{H}} = \mathbf{N}_1\mathbf{H}, \ \ \mathbf{H}(0) = \mathbf{I}_3, \quad (113)$$

and

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & -1 \end{bmatrix}. \quad (114)$$

In view of Eq. (111) we have $\eta = 1$, $\mathbf{A}_0^s = (0,1)^T$, $\mathbf{A}_s^0 = (0,0)^T$, and thus by Eqs. (15) and (16) we obtain

$$\mathbf{U} = \begin{bmatrix} -t \\ 1 \end{bmatrix}, \ \ \mathbf{V} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (115)$$

These make $\mathbf{N}_1$ having a simple form:

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 0 & t \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{116}$$

Correspondingly, solving Eq. (113) we van get

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}. \tag{117}$$

Inserting Eqs. (117) and (114) into Eq. (112) we have

$$\mathbf{G}_3 = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{118}$$

which recovers to Eq. (110) again.

Similarly, by Theorem 7 we have

$$\mathbf{G}_3 = \begin{bmatrix} \mathbf{G}_2 & \mathbf{0}_{2\times 1} \\ \frac{1}{\eta}\mathbf{V}^T & \frac{1}{\eta} \end{bmatrix} \mathbf{K}, \tag{119}$$

where

$$\dot{\mathbf{K}} = \mathbf{N}_2\mathbf{K}, \quad \mathbf{K}(0) = \mathbf{I}_3, \tag{120}$$

$$\mathbf{N}_2 = \begin{bmatrix} 0 & 0 & -t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{121}$$

Correspondingly, solving Eq. (120) we can get

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & \frac{-t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}. \tag{122}$$

Inserting Eq. (122) into Eq. (119) we have

$$\mathbf{G}_3 = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \tag{123}$$

which recovers to Eq. (110) again.

The above derivations indicate that Theorems 3 and 7 can help us to construct the higher-order fundamental matrix from the lower-order fundamental matrix and other fundamental matrices solved from the nilpotent systems.

Now, we turn our attention to the quadratic invariants of the present example. From Eq. (110) it follows that

$$\mathbf{G}_3^{-1} = \begin{bmatrix} 1 & -t & \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}, \tag{124}$$

such that we have

$$\dot{\mathbf{V}} = \eta \mathbf{G}_3^T \begin{bmatrix} A_1^0 \\ A_2^0 \\ A_3^0 \end{bmatrix} = \eta \begin{bmatrix} A_1^0 \\ tA_1^0 + A_2^0 \\ \frac{t^2}{2}A_1^0 + tA_2^0 + A_3^0 \end{bmatrix},$$

$$\mathbf{U} = \frac{1}{\eta}\mathbf{G}_3^{-1}(t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} \frac{t^2}{2} \\ -t \\ 1 \end{bmatrix}. \tag{125}$$

Under the condition of $\dot{\mathbf{V}} = \mathbf{U}$, i.e.,

$$\begin{aligned} A_1^0 &= \frac{t^2}{2\eta^2}, \\ A_2^0 &= -\frac{t}{\eta^2} - tA_1^0 = -\frac{2t+t^3}{2\eta^2}, \\ A_3^0 &= \frac{1}{\eta^2} - tA_2^0 - \frac{t^2}{2}A_1^0 = \frac{(2+t^2)^2}{4\eta^2}, \end{aligned} \tag{126}$$

the system has a quadratic invariant:

$$\begin{aligned} &x_1^2(t) - 2tx_1(t)x_2(t) + (1+t^2)x_2^2(t) + t^2x_1(t)x_3(t) \\ &- (2t+t^3)x_2(t)x_3(t) + \left(1+\frac{t^2}{2}\right)^2 x_3^2(t) \\ &- \eta^2(t)x_0^2(t) = x_1^2(0) + x_2^2(0) + x_3^2(0) - x_0^2(0). \end{aligned} \tag{127}$$

Similarly, under the condition of $\dot{\mathbf{V}} = -\mathbf{U}$, i.e.,

$$\begin{aligned} A_1^0 &= -\frac{t^2}{2\eta^2}, \\ A_2^0 &= \frac{t}{\eta^2} - tA_1^0 = \frac{2t+t^3}{2\eta^2}, \\ A_3^0 &= -\frac{1}{\eta^2} - tA_2^0 - \frac{t^2}{2}A_1^0 = -\frac{(2+t^2)^2}{4\eta^2}, \end{aligned} \tag{128}$$

the system has a quadratic invariant:

$$x_1^2(t) - 2tx_1(t)x_2(t) + (1+t^2)x_2^2(t) + t^2x_1(t)x_3(t) - (2t+t^3)x_2(t)x_3(t) + \left(1+\frac{t^2}{2}\right)^2 x_3^2(t) + \eta^2(t)x_0^2(t) = x_1^2(0) + x_2^2(0) + x_3^2(0) + x_0^2(0). \tag{129}$$

4 A new Lie-group solver

Before utilizing the property of nilpotent matrix to develop numerical methods for time-varying linear system, we give a more effective and unique representation of systems (17) and (62) and then system (4).

Theorem 9. The nilpotent system (17) can be transformed to a new system:

$$\dot{\mathbf{Z}}(t) = \mathbf{B}(t)\mathbf{Z}(t), \tag{130}$$

where

$$\mathbf{B} := \begin{bmatrix} \mathbf{0}_n & \mathbf{U} \\ \dot{\mathbf{V}}^T & 0 \end{bmatrix}, \tag{131}$$

$$\mathbf{Z} := \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{V}^T & -1 \end{bmatrix} \mathbf{Y}. \tag{132}$$

Proof. $\mathbf{N}_1$ as defined by Eq. (18) is a nilpotent matrix with index two, i.e., $\mathbf{N}_1^2 = \mathbf{0}$. Accordingly, we can transform it to a upper-triangular matrix by considering a similar transformation $\mathbf{Q}^{-1}\mathbf{N}_1\mathbf{Q}$, where

$$\mathbf{Q} = \mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{V}^T & -1 \end{bmatrix}. \tag{133}$$

Through some calculations we obtain

$$\mathbf{Q}^{-1}\mathbf{N}_1\mathbf{Q} = \begin{bmatrix} \mathbf{0}_n & \mathbf{U} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}. \tag{134}$$

Now considering the variable transformation (132) and using Eq. (17), we can derive

$$\dot{\mathbf{Z}} = [\mathbf{Q}^{-1}\mathbf{N}_1\mathbf{Q} - \mathbf{Q}^{-1}\dot{\mathbf{Q}}]\mathbf{Z}.$$

By means of Eqs. (134) and (133) we can obtain Eq. (130) with its $\mathbf{B}$ given by Eq. (131). □

A similar result holds for system (62).

Theorem 10. The nilpotent system (62) can also be transformed to Eq. (130) with $\mathbf{B}$ still given by Eq. (131) but with

$$\mathbf{Z} := \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{V}^T & 1 \end{bmatrix} \mathbf{Y}. \tag{135}$$

Proof. $\mathbf{N}_2$ as defined by Eq. (63) is a nilpotent matrix with index two, i.e., $\mathbf{N}_2^2 = \mathbf{0}$. Similarly, we can transform it to a upper-triangular matrix by

$$\mathbf{Q}^{-1}\mathbf{N}_2\mathbf{Q} = \begin{bmatrix} \mathbf{0}_n & \mathbf{U} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}, \tag{136}$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ -\mathbf{V}^T & 1 \end{bmatrix}, \quad \mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{V}^T & 1 \end{bmatrix}. \tag{137}$$

Now considering the variable transformation (135) and using Eq. (62), we can get

$$\dot{\mathbf{Z}} = [\mathbf{Q}^{-1}\mathbf{N}_2\mathbf{Q} - \mathbf{Q}^{-1}\dot{\mathbf{Q}}]\mathbf{Z}.$$

By means of Eqs. (136) and (137) we obtain Eq. (130) again with its $\mathbf{B}$ still given by Eq. (131). □

System (130) is the simplest representation of the time-varying linear system (4), of which both the spatial part $\mathbf{B}_s^s$ and the scalar part B_0^0 vanish. So in addition to these two representations (43) and (71) we may obtain a better representation of the fundamental matrix for Eq. (7) in terms of the fundamental matrix for Eq. (130) as follows.

Theorem 11. The fundamental matrix $\mathbf{G}_{n+1}$ for Eq. (7) has the following representation:

$$\mathbf{G}_{n+1} = \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \frac{1}{\eta} \end{bmatrix} \mathbf{M}_{n+1}, \tag{138}$$

where $\mathbf{M}_{n+1} \in SL(n+1,\mathbb{R})$ is the fundamental matrix for Eq. (130), satisfying

$$\dot{\mathbf{M}}_{n+1}(t) = \mathbf{B}(t)\mathbf{M}_{n+1}(t), \ \ \mathbf{M}_{n+1}(0) = \mathbf{I}_{n+1}. \tag{139}$$

Proof. We first note that either by Eqs. (16) and (39) or by Eqs. (135) and (68) we obtain the same

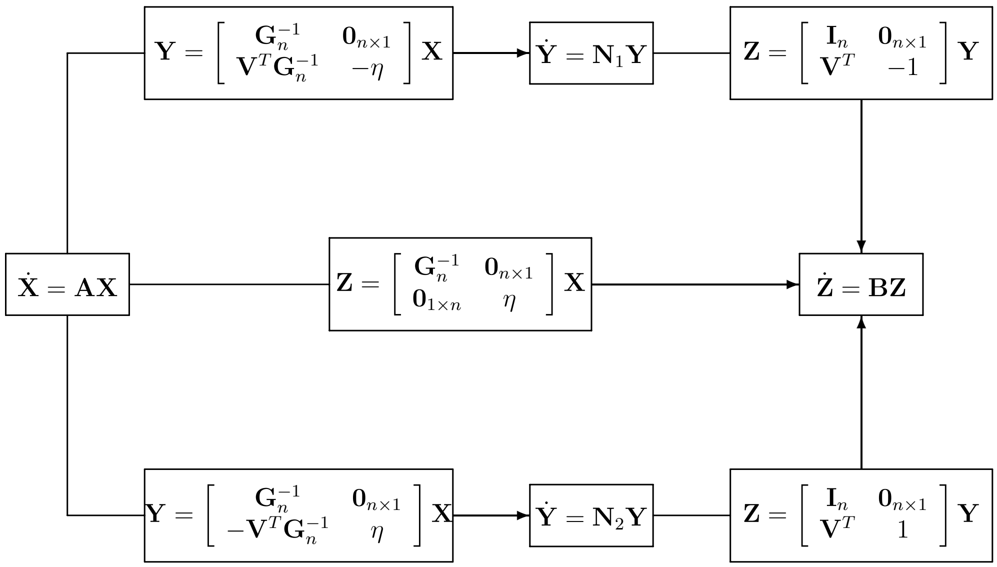

Figure 1: A summary of the transformations for the time-varying linear systems.

transformation from $\mathbf{X}$ to $\mathbf{Z}$ as shown in Fig. 1, or from $\mathbf{Z}$ to $\mathbf{X}$ as given by

$$\mathbf{X} = \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \frac{1}{\eta} \end{bmatrix}\mathbf{Z}. \tag{140}$$

With $\mathbf{M}_{n+1}$ satisfying Eq. (139), the solution of Eq. (130) can be expressed by

$$\mathbf{Z}(t) = \mathbf{M}_{n+1}(t)\mathbf{Z}(0). \tag{141}$$

Substituting it into Eq. (140) and using $\mathbf{Z}(0) = \mathbf{X}(0)$ we obtain

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \frac{1}{\eta} \end{bmatrix}\mathbf{M}_{n+1}\mathbf{X}(0), \tag{142}$$

which upon comparing with the solution of Eq. (4), i.e., $\mathbf{X}(t) = \mathbf{G}_{n+1}(t)\mathbf{X}(0)$ with $\mathbf{G}_{n+1}(t)$ satisfying Eq. (7), we obtain Eq. (138). Since $\operatorname{tr}\mathbf{B} = 0$ and $\det \mathbf{M}_{n+1}(0) = 1$, by the Abel formula we have $\det \mathbf{M}_{n+1}(t) = 1$ for all $t \in \mathbb{R}$, that is, $\mathbf{M}_{n+1} \in SL(n+1,\mathbb{R})$. □

In Fig. 1 we summarize the three transformations from system (4) to system (130). Substituting Eq. (140) into Eq. (4) we can get

$$\dot{\mathbf{Z}} = [\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} - \mathbf{Q}^{-1}\dot{\mathbf{Q}}]\mathbf{Z}.$$

where

$$\mathbf{Q} := \begin{bmatrix} \mathbf{G}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \frac{1}{\eta} \end{bmatrix}, \ \mathbf{Q}^{-1} := \begin{bmatrix} \mathbf{G}_n^{-1} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \eta \end{bmatrix}.$$

Then through some derivations and using Eqs. (11), (13) and (14) we can also derive system (130). From Theorems 9 and 10 we can derive the following results.

Corollary 3. For system (130) if the condition $\dot{\mathbf{V}} = \mathbf{U}$ holds, then there exists a quadratic invariant

$$\mathbf{Z}^T(t)\mathbf{g}\mathbf{Z}(t) = \mathbf{Z}^T(0)\mathbf{g}\mathbf{Z}(0) \tag{143}$$

Proof. Under the condition of $\dot{\mathbf{V}} = \mathbf{U}$, the system matrix $\mathbf{B}$ in Eq. (131) satisfies Eq. (56). Therefore, the result in Eq. (143) follows directly from Corollary 1. □

In Table 1 we compare the above system under the same condition of $\dot{\mathbf{V}} = \mathbf{U}$ with that in Theorem 4, where the Lie algebras, Lie groups and the invariants are compared. It is clear that those systems are different representations of the linear systems in the Minkowskian type space [Liu (2001)].

Corollary 4. For system (130) if the condition $\dot{\mathbf{V}} = -\mathbf{U}$ holds, then there exists a quadratic invariant

$$\mathbf{Z}^T(t)\mathbf{I}_{n+1}\mathbf{Z}(t) = \mathbf{Z}^T(0)\mathbf{I}_{n+1}\mathbf{Z}(0) \tag{144}$$

Proof. Under the condition of $\dot{\mathbf{V}} = -\mathbf{U}$, the system matrix $\mathbf{B}$ in Eq. (131) satisfies Eq. (83). Therefore, the result in Eq. (144) follows directly from Corollary 2. □

In Table 2 we compare the above system under the same condition of $\dot{\mathbf{V}} = -\mathbf{U}$ with that in Theorem 8, where the Lie algebras, Lie groups and the invariants are compared. It is clear that those systems are different representations of the linear systems in the Euclidean type space.

The above two Corollaries 3 and 4 indicate that when the linear system (4) possesses the quadratic invariants, the simplest forms are represented by the Z-quadratics in Eqs. (143) and (144). The first quadratic invariant is know as the Minkowskian separation, which is further classified into three types: space-like if $\mathbf{Z}^T\mathbf{g}\mathbf{Z} > 0$, light-like if $\mathbf{Z}^T\mathbf{g}\mathbf{Z} = 0$, and time-like if $\mathbf{Z}^T\mathbf{g}\mathbf{Z} < 0$. Refer Liu (2001). Conversely, for the second invariant in Eq. (144), there is only one type of $\mathbf{Z}^T\mathbf{I}_{n+1}\mathbf{Z} > 0$, which describes an invariant set on the n-dimensional sphere $\mathbb{S}^n$ with a radius $\sqrt{\mathbf{Z}^T(0)\mathbf{I}_{n+1}\mathbf{Z}(0)}$ determinated by the initial condition.

5 Numerical methods

In this section, we will derive numerical methods for systems (17) and (62) by utilizing the nilpotent matrix property. Here we will use $\mathbf{N}$ to denote $\mathbf{N}_1$ or $\mathbf{N}_2$. By means of the Peano-Baker formula the state transition matrix $\mathbf{\Phi}(t,t_0)$ for system (17) or (62), which maps the state vector $\mathbf{Y}(t_0)$ at time t_0 to the state vector $\mathbf{Y}(t)$ at time t, can be expressed as:

$$\begin{aligned}\mathbf{\Phi}(t,t_0) = \mathbf{I}_{n+1} &+ \int_{t_0}^{t}\mathbf{N}(\tau_1)d\tau_1 \\ &+ \int_{t_0}^{t}\int_{t_0}^{\tau_1}\mathbf{N}(\tau_1)\mathbf{N}(\tau_2)d\tau_2 d\tau_1 + \cdots \\ &+ \int_{t_0}^{t}\int_{t_0}^{\tau_1}\cdots\int_{t_0}^{\tau_{n-1}}\mathbf{N}(\tau_1)\mathbf{N}(\tau_2)\cdots\mathbf{N}(\tau_n) \\ &\cdot d\tau_n\cdots d\tau_2 d\tau_1 + \cdots \quad (145)\end{aligned}$$

For developing a numerical scheme we search a state transition matrix from state $\mathbf{Y}_\ell$ at time t_ℓ to state $\mathbf{Y}_{\ell+1}$ at time $t_{\ell+1}$ with $\Delta t = t_{\ell+1} - t_\ell$ small enough. Upon letting t_0 to be t_ℓ and t to be $t_{\ell+1}$ in the above integrals, then approximating of which by the trapezoidal rule and taking advantage of $\mathbf{N}^2(t) = \mathbf{0}$ for all $t \in \mathbb{R}$, we obtain

$$\begin{aligned}\mathbf{\Phi}(t_{\ell+1},t_\ell) = \mathbf{I}_{n+1} &+ \frac{\Delta t}{2}[\mathbf{N}(t_\ell) + \mathbf{N}(t_{\ell+1})] \\ &+ \frac{(\Delta t)^2}{4}\mathbf{N}(t_{\ell+1})\mathbf{N}(t_\ell). \quad (146)\end{aligned}$$

Substituting it into

$$\mathbf{Y}_{\ell+1} = \mathbf{\Phi}(t_{\ell+1},t_\ell)\mathbf{Y}_\ell, \quad (147)$$

results in a numerical scheme for systems (17) and (62):

$$\begin{aligned}\mathbf{Y}_{\ell+1} = \Bigg(\mathbf{I}_{n+1} &+ \frac{\Delta t}{2}[\mathbf{N}(t_\ell) + \mathbf{N}(t_{\ell+1})] \\ &+ \frac{(\Delta t)^2}{4}\mathbf{N}(t_{\ell+1})\mathbf{N}(t_\ell)\Bigg)\mathbf{Y}_\ell. \quad (148)\end{aligned}$$

Then, by means of Eq. (38) or (67) we can calculate $\mathbf{X}$ forward step-by-step.

It should be emphasized that the matrix resulting from the Peano-Baker formula is not equal to $\exp\int_{t_0}^{t}\mathbf{N}(\tau)d\tau$, and is not guaranteed to be an element of $SL(n+1,\mathbb{R})$ even $\mathbf{N}$ is an element of the Lie algebra $sl(n+1,\mathbb{R})$, i.e., $\mathrm{tr}\,\mathbf{N} = 0$. In order to obtain this type numerical scheme which preserving $SL(n+1,\mathbb{R})$, let us return to systems (17) and (62). The resulting $\mathbf{N}$ makes us easily to derive the so-called Lie-group scheme as follows:

$$\mathbf{Y}_{\ell+1} = \exp[\Delta t\mathbf{N}(\bar{\ell})]\mathbf{Y}_\ell = [\mathbf{I}_{n+1} + \Delta t\mathbf{N}(\bar{\ell})]\mathbf{Y}_\ell, \quad (149)$$

where $\mathbf{N}(\bar{\ell}) = \mathbf{N}(t_\ell + \Delta t/2)$. The higher order terms disappear due to $\mathbf{N}^k = \mathbf{0}$, $k \geq 2$. Obviously, $\mathbf{I}_{n+1} + \Delta t\mathbf{N}(\bar{\ell}) \in SL(n+1,\mathbb{R})$.

The above method can be also applied to the calculations of $\mathbf{H}$ in Eq. (44) or $\mathbf{K}$ in Eq. (72), which is denoted by $\mathbf{M}$ for unifying the notations. In Eq. (43) or Eq. (71) we have expressed $\mathbf{G}_{n+1}$ in terms of $\mathbf{G}_n$ and $\mathbf{M}$, the latter of which can be calculated forward step-by-step with

$$\mathbf{M}_{n+1}(\ell+1) = [\mathbf{I}_{n+1} + \Delta t\mathbf{N}(\bar{\ell})]\mathbf{M}_{n+1}(\ell). \quad (150)$$

However, we need to know $\mathbf{G}_n$ in advance. This can be achieved by using Eq. (43) or Eq. (71)

again but with n replaced by $n-1$. We can repeat this process by decreasing n until $\mathbf{G}_{n-1}$ was obtained in a closed-form for an n.

Corresponding to the nilpotent type algorithms (148) and (149), we can also employ Theorems 9, 10 and 11 to develop the so-called Lie-group solver. In Eq. (138) we have expressed $\mathbf{G}_{n+1}$ in terms of $\mathbf{G}_n$ and $\mathbf{M}_{n+1}$, the latter of which can be calculated by

$$\mathbf{M}_{n+1}(\ell+1) = \exp[\Delta t \mathbf{B}(\bar{\ell})]\mathbf{M}_{n+1}(\ell) = \begin{bmatrix} M1 & M2 \\ M3 & M4 \end{bmatrix} \mathbf{M}_{n+1}(\ell), \tag{151}$$

where

$$M1 = \mathbf{I}_n + \frac{\cosh\left(\Delta t\sqrt{\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}\right)-1}{\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}\mathbf{U}(\bar{\ell})\dot{\mathbf{V}}^T(\bar{\ell})$$

$$M2 = \frac{\sinh\left(\Delta t\sqrt{\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}\right)}{\sqrt{\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}}\mathbf{U}(\bar{\ell})$$

$$M3 = \frac{\sinh\left(\Delta t\sqrt{\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}\right)}{\sqrt{\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}}\dot{\mathbf{V}}^T(\bar{\ell})$$

$$M4 = \cosh\left(\Delta t\sqrt{\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}\right).$$

In the above, if $\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell}) < 0$, then cosh and sinh are replaced by cos and sin and the term $\sqrt{\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}$ is replaced by $\sqrt{-\mathbf{U}(\bar{\ell})\cdot\dot{\mathbf{V}}(\bar{\ell})}$. If we replace the above $\mathbf{M}_{n+1}$ by $\mathbf{Z}$ we have a numerical method to calculate the solution of Eq. (130). Then, with the aid of Eq. (140) we have a numerical method to calculate the solution of Eq. (4). On the other hand, the combination of Eqs. (151) and (138) led to a new Lie-group solver to calcualte the linear group $\mathbf{G}_{n+1}$.

6 Numerical examples

6.1 Example 1

We first consider a definite example of

$$\frac{d^3x}{dt^3} + \frac{dx}{dt} + \frac{2}{e^t+2}\left(x + \frac{d^2x}{dt^2}\right) = 0, \tag{152}$$

whose solution is

$$x(t) = \left(\dot{x}(0) + \frac{1}{3}[x(0)+\ddot{x}(0)]\right)\sin t + \left(\frac{1}{3}x(0) - \frac{2}{3}\ddot{x}(0)\right)\cos t + \frac{1}{3}[x(0)+\ddot{x}(0)](1+e^{-t}). \tag{153}$$

We apply schemes (148) and (149) to calculate $\mathbf{Y}$ and Eq. (38) to calculate $\mathbf{X}$. The numerical errors in terms of the differences of numerical solutions to closed-form solution for x are shown in Fig. 2. Here we fix the time stepsize to be $\Delta t = 0.01$ sec in all calculations. It can be seen that scheme (149) gives better numerical result than scheme (148). Under the same condition we also apply scheme (151) to this problem. After $\mathbf{Z}$ is calculated, we use Eq. (140) to calculate $\mathbf{X}$. It can be seen that scheme (151) gives better numerical result than both schemes (149) and (148).

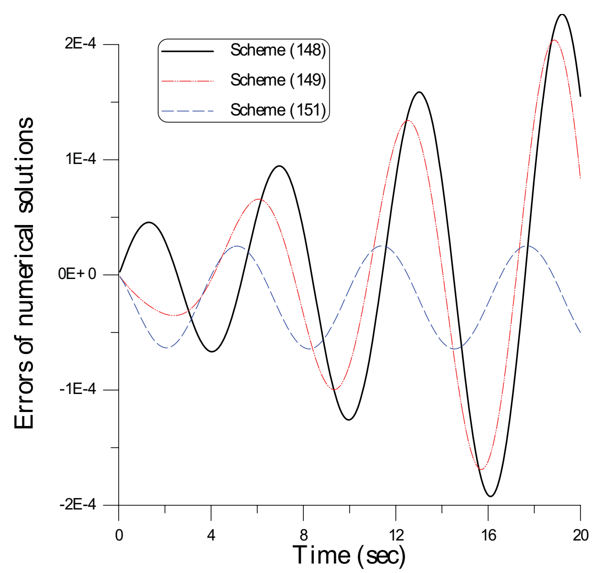

Figure 2: For Example 1 we comparing the numerical errors by schemes (148), (149) and (151).

6.2 Example 2

Then, let us consider a third-order linear differential equation:

$$\frac{d^3x}{dt^3} + \frac{1}{1+t}\frac{d^2x}{dt^2} - \frac{1+t^2}{(1+t)^2}\frac{dx}{dt} - \frac{t}{(1+t)^2}x = 0.$$

(154)

This equation has a quadratic invariant:

$$x_1^2(t)-2tx_1(t)x_2(t)+(1+t^2)x_2^2(t)+(1+t)^2x_0^2(t) = x_1^2(0)+x_2^2(0)+x_0^2(0). \quad (155)$$

We apply scheme (149) to calculate **Y** and then Eq. (38) to calculate **X**. Starting from an initial point $(x_1(0),x_2(0),x_0(0))=(1,1,1)$ we plot the time histories of $(x_1(t),x_2(t),x_0(t))$ in Fig. 3(a). The numerical error of invariant in terms of

$$\text{Error} := |x_1^2(t)-2tx_1(t)x_2(t)+(1+t^2)x_2^2(t) + (1+t)^2x_0^2(t)-[x_1^2(0)+x_2^2(0)+x_0^2(0)]| \quad (156)$$

is shown in Fig. 3(b). It can be seen that scheme (149) implies a good persistence of the invariant.

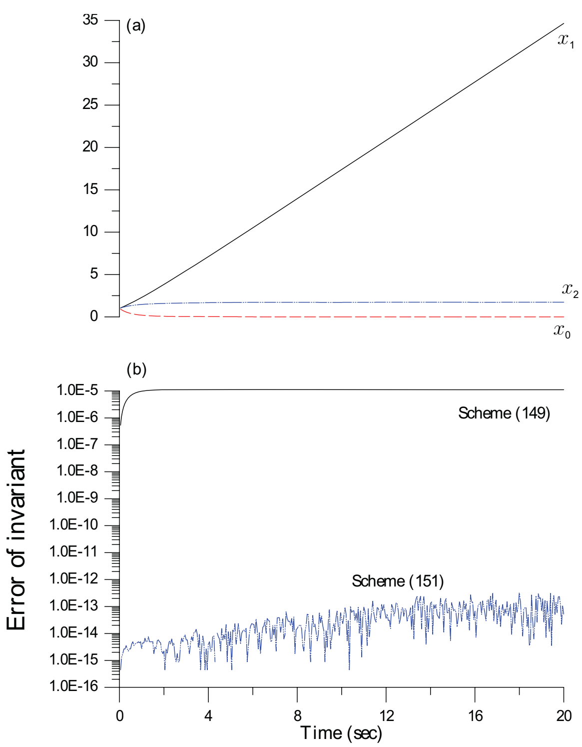

Figure 3: For Example 2: (a) displaying the time histories of solutions, and (b) comparing the numerical errors of invariant by schemes (149) and (151).

When we apply scheme (151) on this problem, it is surprisingly that this scheme is very good to retain the invariant with the error smaller than 10^{-13}.

6.3 *Example 3*

The following example

$$\dot{\mathbf{G}}_3(t)=\begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix}\mathbf{G}_3(t),$$
$$\mathbf{G}_3(0)=\mathbf{I}_3 \quad (157)$$

is of $SO(3)$ flow equation [Liu (2006a)].

For this example, it is immediately to obtain $\eta=1$ and

$$\mathbf{G}_2(t)=\begin{bmatrix} \cos\overline{\omega}_3(t) & -\sin\overline{\omega}_3(t) \\ \sin\overline{\omega}_3(t) & \cos\overline{\omega}_3(t) \end{bmatrix}, \quad (158)$$

where

$$\overline{\omega}_3(t)=\int_0^t \omega_3(\xi)d\xi. \quad (159)$$

Therefore, we have

$$\mathbf{U}=\begin{bmatrix} \omega_2\cos\overline{\omega}_3-\omega_1\sin\overline{\omega}_3 \\ -\omega_2\sin\overline{\omega}_3-\omega_1\cos\overline{\omega}_3 \end{bmatrix}, \quad \dot{\mathbf{V}}=-\mathbf{U}, \quad (160)$$

and **B** is read as

$$\mathbf{B}=\begin{bmatrix} 0 & 0 & M5 \\ 0 & 0 & M6 \\ M7 & M8 & 0 \end{bmatrix}, \quad (161)$$

where

$$M5=\omega_2\cos\overline{\omega}_3-\omega_1\sin\overline{\omega}_3$$
$$M6=-\omega_2\sin\overline{\omega}_3-\omega_1\cos\overline{\omega}_3$$
$$M7=-\omega_2\cos\overline{\omega}_3+\omega_1\sin\overline{\omega}_3$$
$$M8=\omega_2\sin\overline{\omega}_3+\omega_1\cos\overline{\omega}_3.$$

Now, we can apply scheme (151) to calculate $\mathbf{G}_3$. For definite we consider $\omega_3=\Omega-\omega$, $\omega_2=-\sin\Omega t$ and $\omega_1=\cos\Omega t$. For this case the closed-form solution of $\mathbf{G}_3$ has been derived by Liu (2006a).

For this $SO(3)$ flow, it is utmost important that the numerical method can retain the orthogonality of $\mathbf{G}_3$, that is, $\mathbf{G}_3^{\mathrm{T}}\mathbf{G}_3=\mathbf{I}_3$. The error of orthogonality is defined as $\|\mathbf{G}_3^{\mathrm{T}}\mathbf{G}_3-\mathbf{I}_3\|$ with $\mathbf{G}_3$ calculated by numerical method. Under the parameters of $\omega = 2$ and $\Omega = 3$, the numerical error by using the numerical scheme (151) is shown in Fig. 4(a), where $\Delta t = 0.01$ sec was used. It can be seen that even up to a very large rotation the error is still smaller than 2×10^{-13}. A highly accurate result is obtained by this new Lie-group solver.

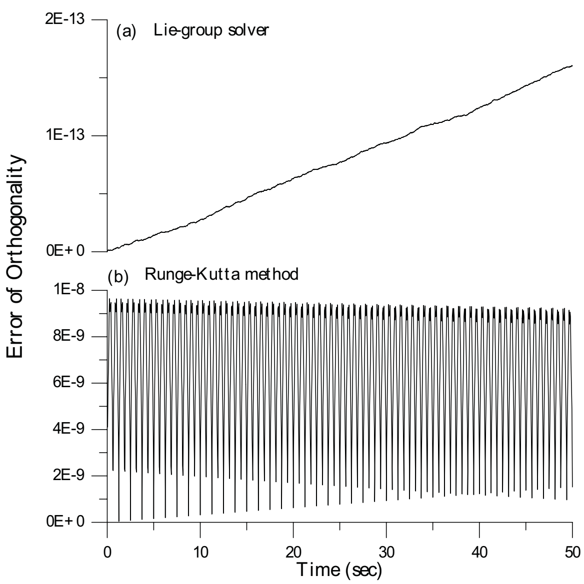

Figure 4: For Example 3 of the $SO(3)$ flow we comparing the numerical errors of orthogonality by (a) the Lie-group solver (151) and (b) the Runge-Kutta method.

Under the same parameters as that used in the above, we plot the numerical result calculated by the fourth-order Runge-Kutta method with a post projection as shown by Liu (2006b) in Fig. 4(b). It can be seen that this method preserves the orthogonality rather well; however, the Runge-Kutta method is still less accurate about five orders than our Lie-group solver. Through these investigations, the scheme based on Eq. (151) is a very stable one in both the aspects of accuracy and preservation of orthogonality.

6.4 Example 4

The following example

$$\dot{\mathbf{G}}_4(t) = \begin{bmatrix} 0 & t\sin\frac{\pi t}{4} & 0 & 0 \\ -t\sin\frac{\pi t}{4} & 0 & t\sin\frac{\pi t}{2} & 0 \\ 0 & -t\sin\frac{\pi t}{2} & 0 & t\sin\frac{3\pi t}{4} \\ 0 & 0 & -t\sin\frac{3\pi t}{4} & 0 \end{bmatrix} \mathbf{G}_4(t),$$
$$\mathbf{G}_4(0) = \mathbf{I}_4 \tag{162}$$

is of $SO(4)$ flow equation. Iserles, , Norsett and Rasmussen (2001) have used it to show that the Runge-Kutta method fails to preserve the orthogonality condition.

For this example we first note that $\mathbf{G}_2$ has a closed-form solution given as follows:

$$\mathbf{G}_2 = \begin{bmatrix} M9 & M10 \\ M11 & M12 \end{bmatrix}. \tag{163}$$

where

$$M9 = \cos\left(\frac{16}{\pi^2}\sin\frac{\pi t}{4} - \frac{4t}{\pi}\cos\frac{\pi t}{4}\right)$$
$$M10 = \sin\left(\frac{16}{\pi^2}\sin\frac{\pi t}{4} - \frac{4t}{\pi}\cos\frac{\pi t}{4}\right)$$
$$M11 = -\sin\left(\frac{16}{\pi^2}\sin\frac{\pi t}{4} - \frac{4t}{\pi}\cos\frac{\pi t}{4}\right)$$
$$M12 = \cos\left(\frac{16}{\pi^2}\sin\frac{\pi t}{4} - \frac{4t}{\pi}\cos\frac{\pi t}{4}\right).$$

Then we apply scheme (151) with $n = 2$ to calculate $\mathbf{M}_3$ and Eq. (138) with $n = 2$ to calculate $\mathbf{G}_3$. After that we apply scheme (151) with $n = 3$ to calculate $\mathbf{M}_4$ and Eq. (138) with $n = 3$ to calculate $\mathbf{G}_4$. The numerical results are shown in Fig. 5. Figure 5(a) displays the $(1,1)$ component of $\mathbf{G}_4$ calculated by our numerical scheme (151) together with Eq. (138). On the other hand we also apply the scheme by the Peano-Baker formula with directly inserting the above $\mathbf{A}$ into Eq. (145) up to third order. Figure 5(b) shows the difference of the numerical results obtained from the Peano-Baker scheme to our scheme. The orthogonality error in terms of $\|\mathbf{G}_4^T\mathbf{G}_4 - \mathbf{I}_4\|$ for our scheme and for the Peano-Baker scheme are shown in Figs. 5(c) and 5(d), respectively. The error of our Lie-group solver is almost nil by design. This, however, is not the case with the Peano-Baker scheme, which displays unstable behavior after about 13 seconds as shown in Fig. 5(b), and reveals exponentially-growing orthogonality error as shown in Fig. 5(d). It demonstrates that this new Lie-group solver possesses significantly better stability properties than other conventional schemes.

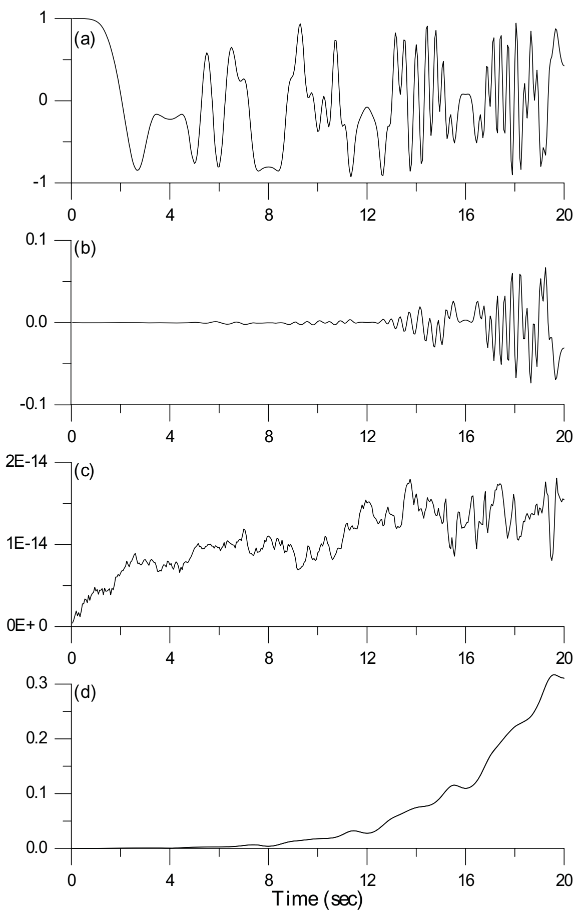

Figure 5: For Example 4 of the $SO(4)$ flow: (a) The $(1,1)$ component of the numerical solution calculated by the Lie-group solver (151), (b) the difference of the numerical solution by the Peano-Baker formula up to the third order to that in (a), and the numerical errors of orthogonality for (c) scheme (151), and (d) the Peano-Baker formula.

7 Conclusions

We have developed new methods to transform the time-varying linear systems into the nilpotent linear systems with index two. In the derivations an indefinite metric and a positive definite metric play the key roles to underpin the Lie algebras and Lie groups behavior as well as the invariant forms of the new systems. The sufficient conditions for the existence of quadratic invariants are derived, which can be classified into two types: the Minkowskian and the Euclidean invariants.

According to these results we have developed two different representations and calculations of the general linear group $\mathbf{G}_{n+1} \in GL(n+1,\mathbb{R})$. Then, the nilpotent systems were further transformed to a new system (130) having a special simple $\mathbf{B}(t) \in sl(n+1,\mathbb{R})$ with $\mathbf{B}_s^s$ and B_0^0 vanishing. Correspondingly, we derived a third representation and calculation of the general linear group $\mathbf{G}_{n+1}$. The last one is called the Lie-group solver, which is simpler than the nilpotent type algorithms, beacuse its state matrix $\mathbf{B}$ includes $\dot{\mathbf{V}}$ without needing a further integral, rather than that the $\mathbf{V}$ appeared in the nilpotent matrix $\mathbf{N}$, which needs an integral as shown in Eq. (16).

The accuracy of the Lie-group solver is much better than the scheme by directly applying the discretized Peano-Baker formula to the original system or the fourth-order Runge-Kutta method. Numerical examples confirmed that our scheme can preserve the group properties very well. The present Lie-group computation method is effective allowing us to sequentially construct the higher order Lie-group from its one-order lower Lie-group as demonstrated by Example 4 of the computation of $\mathbf{G}_4$ by using an exact $\mathbf{G}_2$. Of course, if $\mathbf{G}_2$ is also not known exactly, we can start from $\mathbf{G}_1$ for the computation of $\mathbf{G}_4$. In the Lie-group computations, the Lie-group solver possesses significantly better stability properties than other non Lie-group integrators. As a byproduct, the accuracy of the Lie-group solver is also better than other conventional schemes.

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