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A Systematic Approach for the Development of Weakly–Singular BIEs

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Abstract: Straight-forward systematic derivations of the weakly singular boundary integral equations (BIEs) are presented, following the simple and directly-derived approach by Okada, Rajiyah, and Atluri (1989b) and Han and Atluri (2002). A set of weak-forms and their algebraic combinations have been used to avoid the hyper-singularities, by directly applying the "intrinsic properties" of the fundamental solutions. The systematic decomposition of the kernel functions of BIEs is presented for regularizing the BIEs. The present approach is general, and is applied to developing weakly-singular BIEs for solids and acoustics successfully.

Keyword: boundary integration equation, regularization, decomposition.

1 Introduction

The boundary integral equations have distinct advantages over domain approaches, especially for problems where singularities or infinite domains are involved. In the past decades, the integral equations have been the subject of extensive investigations. The focus in these derivations is on the "fundamental solution" in a linear elastic isotropic solid, viz., the Kelvin solution for a unit point load applied at an arbitrary location, in an arbitrary direction, in an infinite linear elastic solid. The Kelvin solution is well-understood, and is "singular". In the Kelvin solution, for a 3-D solid, it is well-known that the displacement-vector is "weakly-singular", and the stress-tensor is "strongly-singular". In the classical formulation, the integral equation for the displacement vector at any point is "strongly singular". By differentiating the displacement BIE, the integral equation for the traction is obtained which is 'hyper-singular". Much has been written in the last 10∼15 years on the "regularization" of the tBIE [i.e., render the "hyper-singular" tBIE into a "weakly-singular" tBIE], through what appears to be laborious mathematical exercises and "manipulations". This literature is too large to discuss here, but excellent summaries may be found in [Cruse and Richardson (1996); Bonnet, Maier, Polizzotto (1998); Li and Mear (1998)].

In the present paper, we present a systematic approach for developing the BIEs by avoiding the hyper singularity. It is well-understood that BIEs are formulated by writing the global-weak-form of the momentum balance law with the use of the fundamental solution as the test functions. It is clear that there is no physical hyper singularity existing in both displacement and stress tensors. The hyper-singularity is introduced when the stress tensor is differentiated by applying the divergence theorem. We also observe that the divergence of the stress tensor keeps the Dirac-delta property even when each component of its gradients is hyper-singular. The Dirac-delta property can be directly applied to avoid the hyper-singularity if the test functions and its combination are so chosen that all introduced terms of the gradients are kept in the form of the divergence of the stress tensor. The traditional displacement BIE has been well formulated in which such principal is kept to avoid the hyper-singularity. However, such principal has not been kept while formulating the traditional traction BIE by differentiating the displacement BIE directly. Thereafter all hyper-singular terms are kept together properly for applying the Dirac-delta property, which requires various laborious treatments.

On the other hand, as far back as 1989, Okada, Rajiyah, and Atluri (1989a,b, 1991) have proposed a way to *directly derive* integral equations

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for gradients of displacements, by taking the gradient of the displacement of the fundamental solution as the test function, rather than the displacement for the displacement BIE and then differentiating it with respect to **x** as is most common is literature. All hyper-singular terms are kept in form of the divergence of the stress tensor for the direct application of the Dirac-delta property. Thus, the directly derived BIE for the displacement gradients [Okada, Rajiyah and Atluri (1989a,b)] is only "strongly-singular", as opposed to being "hyper-singular", which has been successfully applied for solving nonlinear elastic problems with finite deformation, as known as the field BIE. In 2003, Han and Atluri (2003a) have extended to formulate the traction BIEs in the same manner, by applying the constitutive equations to the BIEs for the displacement gradients. It is essential that the stress field of the fundamental solution is taken as the test function. The traction BIE has also been fully regularized in a very straight-forward and simple manner [Han and Atluri (2003a)]. In the present paper, such a derivation procedure has been formulated to be a general approach for developing non-hyper BIEs systematically in: i) choosing proper test functions for the weak-forms to avoid the hyper singularity; ii) decomposing the kernel functions into two curl-free and divergence-free parts; iii) applying the decomposed functions for the regularization of the BIEs. This procedure is not limited to elasticity and has been used to develop the non-hyper BIEs for acoustics [Qian, Han and Atluri (2004), Qian, Han, Ufimtsev and Atluri (2004)], fracture mechanics [Han and Atluri (2002, 2003b)], and their meshless approaches[Atluri (2004), Atluri, Han and Shen (2003), Atluri, Han and Rajendran (2004), Atluri, Liu and Han (2006)].

The structure of the paper is as follows. In Section 2, the non-hyper-singular BIEs are directly derived, following Okada, Rajiyah and Atluri (1989a,b) and Han and Atluri (2003a). We discuss the general approach to avoid the hyper singularities in developing BIEs in Section 3. In Section 4, the fundamental solution in term of the Galerkin vector potential is revisited, and the decomposition of the kernel functions of the fundamental solution is also presented. The systematic method for regularization of BIEs is presented in Section 5, and some conclusions are summarized in Section 6.

2 Non-hyper-singular BIEs in Elasticity

The governing equations of momentum balance in a solid undergoing small displacements are

$$\sigma_{ij,i} + f_i = 0; \quad \sigma_{ij} = \sigma_{ji} \quad ()_{,i} \equiv \frac{\partial}{\partial \xi_i} \tag{2.1}$$

For the present, we ignore the body forces f_i (but include them later, when necessary). Thus, (2.1) are reduced to:

$$\sigma_{ij,i} = 0 \quad \text{in } \Omega \tag{2.2}$$

For a homogeneous linear elastic isotropic homogeneous solid, the constitutive equation is

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} = E_{ijkl}u_{k,l} \tag{2.3}$$

where

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k)} \tag{2.4}$$

and

$$E_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.5}$$

with λ and μ being the Lame's constants.

Let u_i be the trial functions for displacements, to satisfy Eq. (2.2), in terms of u_i, when Eqs. (2.3)-(2.5) are used. Let $\overline{u}_j$ be the test functions to satisfy the momentum balance laws in terms of u_i, in a weak form. The weak form of the equilibrium Eq. (2.2) can then be written as,

$$\int_\Omega \sigma_{ij,i}\overline{u}_j d\Omega = 0 \text{ or } \int_\Omega (E_{ijmn}u_{m,n})_i\overline{u}_j d\Omega = 0 \tag{2.6}$$

Applying the divergence theorem two times in Eq. (2.6), we obtain:

$$\int_{\partial\Omega} n_i E_{ijmn}u_{m,n}\overline{u}_j\, dS - \int_{\partial\Omega} n_n E_{ijmn}u_m\overline{u}_{j,i}\, dS$$

$$+\int_{\Omega} u_m (E_{ijmn}\overline{u}_{j,i})_{,n}\, d\Omega = 0 \quad (2.7)$$

Instead of the ***scalar*** weak form of Eq. (2.2), as in Eq. (2.6), we may also write a ***vector*** weak form of Eq. (2.2), by using the tensor test functions $\overline{u}_{i,k}$ as:

$$\int_{\Omega} \sigma_{ij,i}\overline{u}_{j,k} d\Omega = 0 \quad k = 1,2,3 \quad (2.8)$$

By applying divergence theorem *three times* in Eq. (2.8), we may write:

$$\int_{\partial\Omega} n_i E_{ijmn} u_{m,n}\overline{u}_{j,k}\, dS - \int_{\partial\Omega} n_k E_{ijmn} u_{m,n}\overline{u}_{j,i}\, dS + \int_{\partial\Omega} n_n E_{ijmn} u_{m,k}\overline{u}_{j,i}\, dS - \int_{\Omega} u_{m,k}(E_{ijmn}\overline{u}_{j,i})_{,n}\, d\Omega = 0 \quad (2.9)$$

Consider a body of an infinite/semi-infinite extent containing the solid of the interest, subject to a point force at a generic location $\mathbf{x}$ in the direction $\mathbf{e}^p$. The fundamental solution, in infinite/semi-infinite space, of the stress field, denoted by $\boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi})$, at any point $\boldsymbol{\xi}$ due to this point load at $\mathbf{x}$, is generated by the balance law:

$$\nabla\cdot\boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi}) + \delta(\mathbf{x},\boldsymbol{\xi})\mathbf{e}^p = \mathbf{0} \text{ or } \sigma^{*p}_{ij,i}(\mathbf{x},\boldsymbol{\xi}) + \delta_{pj}(\mathbf{x},\boldsymbol{\xi}) = 0 \quad (2.10)$$

It should be noted that the fundamental solution in Eq. (2.10) is general. It is not limited to the well-known Kelvin solution. It could be one of any stress distributions for a number of problems of practical importance, derived from the Kelvin solution by methods of synthesis and superposition, such as the fundamental solution in a semi-infinite solid. The governing equation in Eq. (2.10) is the most important property of the solutions for developing BIEs and canceling the singularities. It is demonstrated in the following sections.

Without losing the generality, the corresponding displacement field is denoted by $u^{*p}_i(\mathbf{x},\boldsymbol{\xi})$. By taking this displacement solution as the test function $\overline{u}_i(\boldsymbol{\xi})$, and with the consideration of its properties in Eq. (2.10), we re-write Eqs. (2.7), (2.9), respectively, as,

$$u_p(\mathbf{x}) = \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u^{*p}_j(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} u_m(\boldsymbol{\xi}) t^{*p}_m(\mathbf{x},\boldsymbol{\xi})\, dS \quad (2.11)$$

$$\begin{aligned} -u_{p,k}(\mathbf{x}) &= \int_{\partial\Omega} n_i(\boldsymbol{\xi}) E_{ijmn} u_{m,n}(\boldsymbol{\xi}) u^{*p}_{j,k}(\mathbf{x},\boldsymbol{\xi})\, dS \\ &\quad - \int_{\partial\Omega} n_k(\boldsymbol{\xi}) E_{ijmn} u_{m,n}(\boldsymbol{\xi}) u^{*p}_{j,i}(\mathbf{x},\boldsymbol{\xi})\, dS \\ &\quad + \int_{\partial\Omega} n_n(\boldsymbol{\xi}) E_{ijmn} u_{m,k}(\boldsymbol{\xi}) u^{*p}_{j,i}(\mathbf{x},\boldsymbol{\xi})\, dS \\ &= \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u^{*p}_{j,k}(\mathbf{x},\boldsymbol{\xi})\, dS \\ &\quad + \int_{\partial\Omega} D_t u_m(\boldsymbol{\xi}) e_{nkt}\sigma^{*p}_{nm}(\mathbf{x},\boldsymbol{\xi})\, dS \end{aligned} \quad (2.12)$$

where the surface tangential operator D_t is defined as,

$$D_t = n_r e_{rst}\frac{\partial}{\partial \xi_s} \quad (2.13)$$

It clearly shows that the hyper-singularities in Eqs. (2.7), (2.9) are cancelled through the use of the properties of the fundamental solution. Eq. (2.11) is the well-known traditional displacement BIE. However, Eq. (2.12) is the displacement gradient BIE containing no hyper-singularity, which is quite different from the traditional hyper-singular traction BIE. It should be noted that the integral equations for $u_p(\mathbf{x})$ and $u_{p,k}(\mathbf{x})$ as in Eqs. (2.11) and (2.12) are derived independently of each other. On the other hand, if we derive the integral equation for displacement-gradients, by directly differentiating $u_p(\mathbf{x})$ in Eq. (2.11), i.e. by differentiating,

$$u_p(\mathbf{x}) = \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u^{*p}_j(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} u_m(\boldsymbol{\xi}) t^{*p}_m(\mathbf{x},\boldsymbol{\xi})\, dS \quad (2.14)$$

with respect to x_k, we obtain:

$$u_{p,k}(\mathbf{x}) = \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u^{*p}_{j,k}(\mathbf{x},\boldsymbol{\xi})\, dS - \int_{\partial\Omega} u_m(\boldsymbol{\xi}) t^{*p}_{m,k}(\mathbf{x},\boldsymbol{\xi})\, dS \quad (2.15)$$

Thus, Eq. (2.14) is hypersingular, since $t^{*p}_{m,k}(\mathbf{x},\boldsymbol{\xi})$ is of $O(r^{-3})$ for 3D problems. On the other hand, the directly derived integral equations for $u_{p,k}(\mathbf{x},\boldsymbol{\xi})$ as in Eq. (2.12) contain only singularities of $O(r^{-2})$.

Eqs. (2.11) and (2.12) were originally given in [Okada, Rajiyah, and Atluri (1989a,b)], and the notion of using unsymmetric weak-forms of the differential equations, to obtain integral representations for displacements, was presented in [Atluri (1985)]. It has also been extended for tractions by applying the constitutive equation [Han and Atluri (2003a)], as

$$
\begin{aligned}
-E_{abpk}u_{p,k}(\mathbf{x}) = E_{abpk}\int_{\partial\Omega} t_j(\boldsymbol{\xi})u^{*p}_{j,k}(\mathbf{x},\boldsymbol{\xi})\,dS \\
+E_{abpk}\int_{\partial\Omega} D_t u_m(\boldsymbol{\xi})e_{nkt}\sigma^{*p}_{nm}(\mathbf{x},\boldsymbol{\xi})\,dS \quad (2.16)
\end{aligned}
$$

Then Eq. (2.15) can be re-written as,

$$
\begin{aligned}
-\sigma_{ab}(\mathbf{x}) = \int_{\partial\Omega} t_p(\boldsymbol{\xi})\Theta^{*p}_{ab}(\mathbf{x},\boldsymbol{\xi})\,dS \\
+\int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})\Sigma^{*}_{abpq}(\mathbf{x},\boldsymbol{\xi})\,dS \quad (2.17)
\end{aligned}
$$

where by definition,

$$
\begin{aligned}
\Theta^{*p}_{ab}(\mathbf{x},\boldsymbol{\xi}) = E_{abkl}u^{*k}_{p,l}(\mathbf{x},\boldsymbol{\xi}) \\
\Sigma^{*}_{abpq}(\mathbf{x},\boldsymbol{\xi}) = E_{abkl}e_{nlp}\sigma^{*k}_{nq}(\mathbf{x},\boldsymbol{\xi})
\end{aligned} \quad (2.18)
$$

The function $\Theta^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})$ is identical to the stress field if the Kelvin solution is used.

Contracting Eq. (2.16) with $n_a(\mathbf{x})$ on the boundary, we have

$$
\begin{aligned}
-t_b(\mathbf{x}) = \int_{\partial\Omega} t_p(\boldsymbol{\xi})n_a(\mathbf{x})\Theta^{*p}_{ab}(\mathbf{x},\boldsymbol{\xi})\,dS \\
+\int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})n_a(\mathbf{x})\Sigma^{*}_{abpq}(\mathbf{x},\boldsymbol{\xi})\,dS \quad (2.19)
\end{aligned}
$$

where the traction is defined as,

$$
t_b(\mathbf{x}) = n_a(\mathbf{x})\sigma^{(}_{ab}\mathbf{x}) \quad (2.20)
$$

3 Discussion of Non-hyper-singular BIEs

Before discussing the non-hyper-singular traction BIE in Eq. 3, we examine the displacement BIE in Eq. (2.11) which is non-hyper-singular derived in a straight-forward manner. The governing equations of momentum balance in Eq. (2.1) contain three independent equations, for 3D solid problems. We applied a vector test function $u^{*p}_i(\mathbf{x},\boldsymbol{\xi})$ to three equations and their summation yields a scalar weakform for the displacement in the direction $\mathbf{e}^p$, as the displacement BIE in Eq. (2.11). One advantage by using three weak-forms is that the properties of the fundamental solution in Eq. (2.10) can be applied directly to avoid the hyper-singularity appearing in the displacement BIE in Eq. (2.11). As shown in Eq. (2.7), *the key point to make it possible is to put all hyper-singular terms together in the form of the governing equations, after applying the divergence theorem to the summation of three weak-forms.*

To develop traction BIEs, we apply the gradients of the displacement field of the fundamental solution $u^{*p}_{i,j}(\mathbf{x},\boldsymbol{\xi})$ as a tenser test function to the governing equations. It means that nine weak-forms are used and their summation yields a vector weakform for the gradients of the displacement in the direction $\mathbf{e}^p$, shown in Eq. (2.12). Such summation results in a term in the form of the governing equations in the weakform in Eq. (2.9), which enables the direct use of the property of the fundamental solution to avoid the hype-singularity. In addition, we also apply the constitutive equations to the BIEs for the gradients in Eq. (2.12) and obtain the BIEs for the stress in Eq. (2.16). It means that the actual test function we apply to the governing equation is the stress field of the fundamental solution $\Theta^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})$ (as its variant form) shown in Eq. (2.17). It needs to be pointed out that the non-hyper-singular displacement and traction BIEs in the present paper are very general because no special properties of the fundamental solutions are used to cancel the singularities except the basic governing equations in Eq. (2.10). Thus, the present BIEs are applicable to any fundamental solutions and their various forms of Eq. (2.10). If the Kelvin solution of the infinite extent is used, these BIEs become the widely-used displacement and traction BIEs. They become the non-hyper-singular BIEs for the semi-infinite extent, if the Mindlin solution [Mindlin (1936)] is used.

In contrast, the traditional traction BIE was derived by differentiating the displacement BIE directly as shown in Eq. (2.14), and ignoring the properties of the fundamental solutions. Thus, the hyper-singularities have been introduced into the final BIEs which can be cancelled in a simple manner. Much research has been done to deal with the hyper-singularities, such as the use of kernel decomposition with high-order numerical quadrature scheme, special forms of the fundamental solutions, and particular auxiliary solutions. All these techniques are limited to one particular fundamental solution, which is hard to be extended to other fundamental solutions.

4 Fundamental Solutions in Terms of Galerkin Vector Potential for Displacements

Consider a linear elastic, homogeneous, isotropic body in a domain Ω, with boundary $\partial\Omega$. The Lame' constants of the linear elastic isotropic body are λ and μ; and the corresponding Young's modulus and Poisson's ratio are E and υ, respectively. We use Cartesian coordinates ξ_i, and the attendant base vectors $\mathbf{e}_i$, to describe the geometry in Ω. The solid is assumed to undergo infinitesimal deformations. The displacement vector, strain-tensor, and the stress-tensor in the elastic body are denoted as $\mathbf{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, respectively. It is well known that the displacement vector, which is a continuous function of $\boldsymbol{\xi}$, can be derived, in general, from the Galerkin vector potential φ such that:

$$\mathbf{u} = \nabla^2\varphi - \frac{1}{2(1-\upsilon)}\nabla(\nabla\cdot\varphi) \tag{4.1a}$$

$$= A\nabla^2\varphi + \nabla^2\varphi - \nabla(\nabla\cdot\varphi) \tag{4.1b}$$

$$= \mathbf{u}^\Phi + \mathbf{u}^\Psi \tag{4.1c}$$

where, by definition,

$$\mathbf{u}^\Phi = A\nabla\Phi = A\nabla^2\varphi \tag{4.2}$$

$$\mathbf{u}^\Psi = \nabla\times\Psi = -\nabla\times\nabla\times\varphi = \nabla^2\varphi - \nabla(\nabla\cdot\varphi) \tag{4.3}$$

$$A = \frac{\mu}{\lambda+2\mu} = \frac{1-2\upsilon}{2(1-\upsilon)} \tag{4.4}$$

where Φ is a scalar potential, and Ψ is a vector potential, such that:

$$\Phi = \nabla\cdot\varphi \tag{4.5}$$

$$\Psi = -(\nabla\times\varphi) \tag{4.6}$$

Thus the displacement vector has been decomposed into two parts, $\mathbf{u}^\Phi$ and $\mathbf{u}^\Psi$, which posses the properties,

$$\nabla\times\mathbf{u}^\Phi = A\nabla\times\nabla\Phi = 0 \quad \text{curl-free} \tag{4.7}$$

$$\nabla\cdot\mathbf{u}^\Psi = \nabla\cdot\nabla\times\Psi = 0 \quad \text{divergence-free} \tag{4.8}$$

It is well-known that a divergence free tensor must be a curl of another divergence free tensor, and a curl-free tensor must be a gradient of a scalar potential. As a particular case, the fundamental solution of Eq. (2.10) also possesses such properties, which, and its derivatives in various forms as the kernel functions, can be decomposed into: i) a divergence-free part, and ii) a curl-free part. Thereafter, the singularities in the BIEs can be reduced to be tractable, by i) applying the Stokes' theorem, or ii) utilizing the Cauchy Principal value (CPV) integral.

In the present paper, the Kelvin solution is taken to demonstrate the decomposition procedure. The similar derivation has been successfully applied for solving problems of acoustics governed by the Helmholtz differential equation by Qian, Han and Atluri (2004) and Qian, Han, Ufimtsev and Atluri (2004).

Consider the fundamental solution of Eq. (2.10), the corresponding Galerkin vector displacement potentials are given as,

$$\varphi^{*p} = (1-\upsilon)F^{*(p)}\mathbf{e}^p \text{ no summation on } p \tag{4.9}$$

where for the Kelvin solution in an infinite extent,

$$F^* = \frac{r}{8\pi\mu(1-\upsilon)} \text{ for 3D problems} \tag{4.10a}$$

and

$$F^* = \frac{-r^2\ln r}{8\pi\mu(1-\upsilon)} \text{ for 2D problems} \tag{4.10b}$$

where $r = \|\boldsymbol{\xi} - \mathbf{x}\|$. The derivatives of function F can be found in Appendix A.

One may re-write Eq. (2.10) in term of F^* as,

$$\mu(1-\upsilon)F^*_{,kkll}+\delta(\mathbf{x},\boldsymbol{\xi})=0 \tag{4.11}$$

The corresponding displacements are derived from the Galerkin vector displacement potential, using (4.1a), as:

$$u_i^{*p}(\mathbf{x},\boldsymbol{\xi})=(1-\upsilon)\delta_{pi}F^*_{,kk}-\frac{1}{2}F^*_{,pi} \tag{4.12}$$

which is weakly singular and requires no further treatment.

The gradients of the displacements in Eq. (4.12) are:

$$u_{i,j}^{*p}(\mathbf{x},\boldsymbol{\xi})=(1-\upsilon)\delta_{pi}F^*_{,kkj}-\frac{1}{2}F^*_{,pij} \tag{4.13}$$

which contains the singularity of $O(r^{-2})$ for 3D or $O(r^{-1})$ for 2D. All kernel functions containing these gradients need to be regularized, including $\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})$, $\Theta_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})$, $\Sigma^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})$. For the case of the Kelvin solution, the second one is identical to the stress field $\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})$.

One may decompose the stress field from its definition, given by:

$$\begin{aligned}\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) \equiv& E_{ijkl}u_{k,l}^{*p}\\ =&\mu[(1-\upsilon)\delta_{pi}F^*_{,kkj}+\upsilon\delta_{ij}F^*_{,pkk}-F^*_{,pij}]\\ &+\mu(1-\upsilon)\delta_{pj}F^*_{,kki}\end{aligned} \tag{4.14}$$

which is not divergence-free due to the last term, as

$$\sigma_{ij,i}^{*p}(\mathbf{x},\boldsymbol{\xi})=\mu(1-\upsilon)\delta_{pj}F^*_{,kkii}=-\mu\delta_{pj}\delta(\mathbf{x},\boldsymbol{\xi}) \tag{4.15}$$

We define two functions ϕ_{ij}^{*p} and ψ_{ij}^{*p} to decompose the stress field $\sigma_{ij,i}^{*p}(\mathbf{x},\boldsymbol{\xi})$, as

$$\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})=\phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})+\psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) \tag{4.16a}$$

$$\phi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\equiv-\mu(1-\upsilon)\delta_{pj}F^*_{,kki} \text{ curl-free} \tag{4.16b}$$

$$\psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\equiv\mu[(1-\upsilon)\delta_{pi}F^*_{,kkj}+\upsilon\delta_{ij}F^*_{,pkk}-F^*_{,pij}] \text{ divergence-free} \tag{4.16c}$$

where, as a divergence free tensor, $\psi^*(\mathbf{x},\xi)$ must be a curl of another divergence-free tensor. We choose to rewrite it in term of F^* from Eq. (4.16c), as:

$$\begin{aligned}\psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})&=\mu e_{tis}[(1-\upsilon)e_{tpj}F^*_{,kk}-e_{tkj}F^*_{,pk}]_{,s}\\ &\equiv e_{ist}G^{*p}_{tj,s}\end{aligned} \tag{4.17}$$

where, by definition,

$$G_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})=\mu[(1-\upsilon)e_{ipj}F^*_{,kk}-e_{ikj}F^*_{,pk}] \tag{4.18}$$

which is weakly singular.

The third kernel function, $\Sigma^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})$, can be decomposed from its definition in Eq. (4.14), written in terms of F^* as:

$$\begin{aligned}&\Sigma^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})=E_{ijkl}e_{nlp}\sigma^{*k}_{nq}(\mathbf{x},\boldsymbol{\xi})\\ &=\mu^2[(e_{inp}F_{,jqn}-e_{inp}\delta_{jq}F_{,bbn}+e_{int}e_{tqk}e_{jpm}F_{,kmn})\\ &\quad+\upsilon(e_{inq}\delta_{jp}F_{,bbn}+e_{jnq}\delta_{ip}F_{,bbn})]\end{aligned} \tag{4.19}$$

We also have the divergence of Σ^*_{ijpq} as:

$$\Sigma^*_{ijpq,i}(\mathbf{x},\boldsymbol{\xi})=\mu^2\upsilon e_{jiq}F_{,bbip}\equiv\Lambda^*_{ijpq,i}(\mathbf{x},\boldsymbol{\xi}) \tag{4.20}$$

where, by definition,

$$\Lambda^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})=\mu^2\upsilon e_{jiq}F_{,bbp} \text{ curl-free} \tag{4.21}$$

Thus, one may obtain the divergence-free part by subtracting the curl-free part from the kernel function $\Sigma^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})$, as

$$\begin{aligned}&\mathrm{K}^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})=\Sigma^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})-\Lambda^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})\\ &=\mu^2e_{int}[(\delta_{tp}F_{,jq}-\delta_{tp}\delta_{jq}F_{,bb}+e_{tqk}e_{jpm}F_{,km})\\ &\quad+\upsilon(\delta_{tq}\delta_{jp}F_{,bb}+e_{tpm}e_{jmq}F_{,bb})]_{,n}\\ &\equiv e_{int}H^*_{tjpq,n}(\mathbf{x},\boldsymbol{\xi}) \text{ divergence-free}\end{aligned} \tag{4.22}$$

and its companion divergence-free tensor is given, by definition, as

$$\begin{aligned}&H^*_{ijpq}(\mathbf{x},\boldsymbol{\xi})=\mu^2[(\delta_{ip}F_{,jq}-\delta_{ip}\delta_{jq}F_{,bb}+e_{iqk}e_{jpm}F_{,km})\\ &\quad+\upsilon(\delta_{iq}\delta_{jp}F_{,bb}+e_{ipm}e_{jmq}F_{,bb})]\\ &=\mu^2[-\delta_{ij}F_{,pq}+2\delta_{ip}F_{,jq}+2\delta_{jq}F_{,ip}-\delta_{pq}F_{,ij}\\ &\quad-2\delta_{ip}\delta_{jq}F_{,bb}+2\upsilon\delta_{iq}\delta_{jp}F_{,bb}\\ &\quad+(1-\upsilon)\delta_{ij}\delta_{pq}F_{,bb})]\end{aligned}$$

(4.23)

which is weakly singular.

From Eqs. (4.20) and (4.22), some properties of the kernel functions can be found as following:

$$\nabla \cdot \Sigma^*(\mathbf{x},\boldsymbol{\xi}) = \nabla \cdot \Lambda^*(\mathbf{x},\boldsymbol{\xi}) \tag{4.24a}$$

$$\nabla \cdot \mathrm{K}^*(\mathbf{x},\boldsymbol{\xi}) = \mathbf{0} \tag{4.24b}$$

$$\mathrm{K}^*(\mathbf{x},\boldsymbol{\xi}) = \nabla \times H^*(\mathbf{x},\boldsymbol{\xi}) \tag{4.24c}$$

For the strong singularity of the curl-free part, we write the weak form of Eq. (2.10) over the domain, using a constant c as a test function, as

$$\int_{\Omega} \sigma^{*p}_{ij,i}(\mathbf{x},\boldsymbol{\xi})d\Omega + \delta_{pj} = 0 \tag{4.25a}$$

and apply the divergence theorem once,

$$\int_{\partial\Omega} n_i(\boldsymbol{\xi})\sigma^{*p}_{ij}(\mathbf{x},\boldsymbol{\xi})dS + \delta_{pj} = 0 \tag{4.25b}$$

or

$$\int_{\partial\Omega} t^{*p}_{j}(\mathbf{x},\boldsymbol{\xi})dS + \delta_{pj} = 0 \quad \mathbf{x} \in \Omega \tag{4.25c}$$

Eq. (4.25c) is the well-known "basic identity" of the fundamental solution $\boldsymbol{\sigma}^{*p}(\mathbf{x},\boldsymbol{\xi})$. Eq. (4.25c) is simply an affirmation of the force balance law for Ω: if the point load is applied at a point $\mathbf{x} \in \Omega$ when Ω is entirely embedded in an infinite space, the tractions exerted by the surrounding infinite body on the finite-body Ω should be equilibrated with the applied point force at $\mathbf{x}$ inside Ω.

Once the point $\mathbf{x}$ approaches a smooth boundary, i.e. $\mathbf{x} \in \partial\Omega$, the first term in Eq. (4.25d) can be written in a Cauchy Principal value (CPV) integral, denoted by $\int^{\mathrm{CPV}}$, as,

$$\lim_{\mathbf{x}\to\partial\Omega}\int_{\partial\Omega} t^{*p}_{j}(\mathbf{x},\boldsymbol{\xi})dS = \int^{\mathrm{CPV}}_{\partial\Omega} t^{*p}_{j}(\mathbf{x},\boldsymbol{\xi})dS - \frac{1}{2}\delta_{pj} \tag{4.26a}$$

and thus, one obtains:

$$\int^{\mathrm{CPV}}_{\partial\Omega} t^{*p}_{j}(\mathbf{x},\boldsymbol{\xi})dS + \frac{1}{2}\delta_{pj} = 0 \quad \mathbf{x} \in \partial\Omega \tag{4.26b}$$

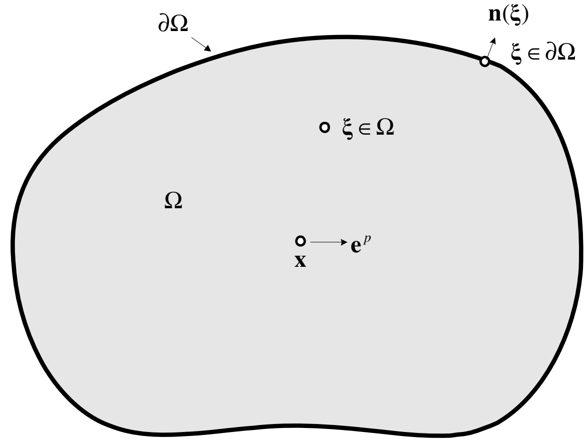

Figure 1: A solution domain with source point $\mathbf{x}$ and target point $\boldsymbol{\xi}$

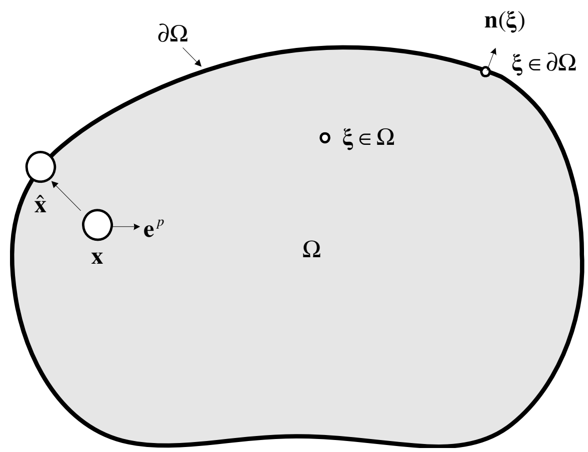

Figure 2: A loading point $\mathbf{x}$ approaching the boundary

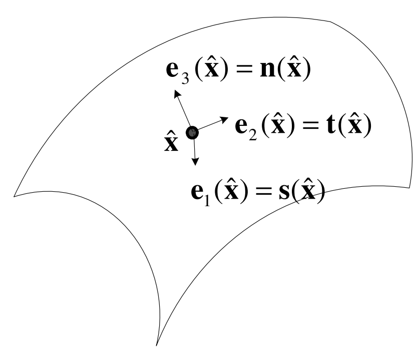

Figure 3: Local coordinates at a boundary point $\hat{\mathbf{x}}$

The second term on the right hand-side of Eq. (4.26a) results from the principal value of the singular integral involving t_j^{*p}, which has a $O(\frac{1}{r^2})$ singularity. Eq. (4.26b) may also be physically explained as below. σ_{ij}^{*p} (and thus t_j^{*p}) are solutions due to a point load applied in an infinite space. In reality, the point load can be assumed to be distributed over a small-sphere, of radius ε, in an infinite body. The tractions distributed over this sphere, that result in a point load, are of $O(\frac{1}{\varepsilon^2})$; while the surface area of the sphere is $O(\varepsilon^2)$. As long as this sphere is inside Ω, and while Ω is a part of the infinite space, the load applied on Ω is still unity. Suppose $\mathbf{x} \to \hat{\mathbf{x}}$ at $\partial\Omega$ shown in Fig. 2, then the sphere of radius ε is centered at the boundary. As long as the boundary is smooth, only one-half of the sphere of radius ε is actually inside Ω, when $\mathbf{x} \to \hat{\mathbf{x}}$ at $\partial\Omega$. Thus while the load applied, in infinite space, on a sphere of radius ε at $\hat{\mathbf{x}} \in \partial\Omega$, is still unity, the actual load applied on Ω is only $\frac{1}{2}$. Thus we obtain Eq. (4.26b).

We can write Eq. (4.25c) for $\mathbf{x} \in \partial\Omega$, with Eq. (4.26), as:

$$\int_{\partial\Omega}^{\text{CPV}} t_j^{*p}(\mathbf{x},\xi)dS + \frac{1}{2}\delta_{pj} = 0 \qquad \mathbf{x} \in \partial\Omega \tag{4.27}$$

From Eq. (4.16), we also see that

$$-\int_{\partial\Omega}^{\text{CPV}} n_i(\xi)\sigma_{ij}^{*p}(\mathbf{x},\xi)dS = \int_{\partial\Omega}^{\text{CPV}} n_i(\xi)\phi_{ij}^{*p}(\mathbf{x},\xi)dS \tag{4.28a}$$

or

$$-\int_{\partial\Omega}^{\text{CPV}} t_j^{*p}(\mathbf{x},\xi)dS = \int_{\partial\Omega}^{\text{CPV}} n_i(\xi)\phi_{ij}^{*p}(\mathbf{x},\xi)dS \tag{4.28b}$$

for both $\mathbf{x} \in \Omega$ and $\mathbf{x} \in \partial\Omega$.

The corresponding equations for Ψ_{ij}^{*p} can also be written as,

$$\int_{\partial\Omega} n_i(\boldsymbol{\xi})\Psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})dS = 0 \tag{4.29}$$

and

$$\int_{\partial\Omega} n_i(\boldsymbol{\xi})\Psi_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi})\cdot u_j^(\mathbf{x})dS = 0 \tag{4.30}$$

5 Regularization of BIEs

The traction BIE in Eq. 3 may be satisfied in a *weak-form* at $\partial\Omega$, using a Petrov-Galerkin scheme, as:

$$\begin{aligned}
&-\int_{\partial\Omega} w_b(\mathbf{x})t_b(\mathbf{x})dS_x \\
&= \int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} t_q(\boldsymbol{\xi})n_a(\mathbf{x})\Theta_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})dS_\xi \\
&+ \int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})n_a(\mathbf{x})\Sigma_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi
\end{aligned} \tag{5.1}$$

where $w_b(\mathbf{x})$ is a test function. With the use of Eqs. (4.16) and (4.22), the weakform can be rewritten as,

$$\begin{aligned}
&-\int_{\partial\Omega} w_b(\mathbf{x})t_b(\mathbf{x})dS_x = \int_{\partial\Omega} w_b(\mathbf{x})dS_x \\
&\cdot \int_{\partial\Omega} t_q(\boldsymbol{\xi})n_a(\mathbf{x})[\Psi_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi}) \; - \phi_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})]dS_\xi \\
&+ \int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})n_a(\mathbf{x})\mathrm{K}_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi
\end{aligned} \tag{5.2}$$

With the fact that

$$\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial \xi_i}, \tag{5.3}$$

one may rewrite Eq. (5.2) with the decomposition of the kernel functions as:

$$\begin{aligned}
&-\frac{1}{2}\int_{\partial\Omega} t_b(\mathbf{x})w_b(\mathbf{x})dS_x \\
&= -\int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} t_q(\boldsymbol{\xi})D_a G_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \\
&- \int_{\partial\Omega} t_q(\boldsymbol{\xi})\, dS_\xi \int_{\partial\Omega}^{CPV} n_a(\mathbf{x})w_b(\mathbf{x})\phi_{ab}^{*q}(\mathbf{x},\boldsymbol{\xi})dS_x \\
&- \int_{\partial\Omega} w_b(\mathbf{x})dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi})D_a H_{abpq}^{*}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi
\end{aligned} \tag{5.4}$$

where G_{ab}^{*q} is defined in Eq.(4.18); ϕ_{ab}^{*q} is defined in Eq. (4.16), and H_{abpq}^{*} is defined in Eq. (4.23).

If $w_b(\mathbf{x})$ is continuous, one may use Stokes' theorem, and re-write Eq. (5.4) as:

$$-\frac{1}{2}\int_{\partial\Omega} t_b(\mathbf{x})w_b(\mathbf{x})dS_x$$

$$= \int_{\partial\Omega} D_a w_b(\mathbf{x}) dS_x \int_{\partial\Omega} t_q(\boldsymbol{\xi}) G^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi - \int_{\partial\Omega} t_q(\boldsymbol{\xi})\, dS_\xi \int^{CPV}_{\partial\Omega} n_a(\mathbf{x}) w_b(\mathbf{x}) \phi^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi}) dS_x + \int_{\partial\Omega} D_a w_b(\mathbf{x}) dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi}) H^*_{abpq}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \tag{5.5}$$

If the test function $w_b(\mathbf{x})$ is chosen to be identical to a function that is energy-conjugate to t_b, namely, the trial function $\hat{u}_b(\mathbf{x})$, we generate the symmetric Galerkin BEM as:

$$-\frac{1}{2}\int_{\partial\Omega} t_b(\mathbf{x}) \hat{u}_b(\mathbf{x}) dS_x = \int_{\partial\Omega} D_a \hat{u}_b(\mathbf{x}) dS_x \int_{\partial\Omega} t_q(\boldsymbol{\xi}) G^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi - \int_{\partial\Omega} t_q(\boldsymbol{\xi})\, dS_\xi \int^{CPV}_{\partial\Omega} n_a(\mathbf{x}) \hat{u}_b(\mathbf{x}) \phi^{*q}_{ab}(\mathbf{x},\boldsymbol{\xi}) dS_x + \int_{\partial\Omega} D_a \hat{u}_b(\mathbf{x}) dS_x \int_{\partial\Omega} D_p u_q(\boldsymbol{\xi}) H^*_{abpq}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \tag{5.6}$$

The results in Eq. (5.5) are similar to those reported in [Li and Mear (1998)] but are different from those in Li and Mear (1998) in the kernel functions appearing in Eq. (5.6). However, here, we obtain these results in a very straightforward and simple manner.

Regarding the displacement BIE in Eq. (2.11), only the term containing the stress field of the fundamental solutions is strongly singular and needs the treatment for the numerical implementation. We also consider the possibility of satisfying the dBIE, at $\partial\Omega$, in *a weak form*, through a general Petrov-Galerkin scheme, and write a weak-form for Eq. (2.11) as:

$$\frac{1}{2}\int_{\partial\Omega} w_p(\mathbf{x}) u_j(\mathbf{x}) dS_x = \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u^{*p}_j(\mathbf{x},\boldsymbol{\xi})\, dS_\xi - \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} n_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) \psi^{*p}_{ij}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi + \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int^{CPV}_{\partial\Omega} n_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) \phi^{*p}_{ij}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \tag{5.7}$$

Applying Stokes' theorem to Eq. (5.7), we have

$$\frac{1}{2}\int_{\partial\Omega} w_p(\mathbf{x}) u_p(\mathbf{x}) dS_x = \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u^{*p}_j(\mathbf{x},\boldsymbol{\xi})\, dS_\xi + \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int_{\partial\Omega} D_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) G^{*p}_{ij}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi + \int_{\partial\Omega} w_p(\mathbf{x}) dS_x \int^{CPV}_{\partial\Omega} n_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) \phi^{*p}_{ij}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \tag{5.8}$$

where G^{*p}_{ij} is defined in Eq.(4.18).

If $w_p(\mathbf{x})$ is chosen to be identical to a function which is energy-conjugate to u_p, viz., the trial function $\hat{t}_p(\mathbf{x})$, we obtain the symmetric Galerkin dBEM, as

$$\frac{1}{2}\int_{\partial\Omega} \hat{t}_p(\mathbf{x}) u_p(\mathbf{x}) dS_x = \int_{\partial\Omega} \hat{t}_p(\mathbf{x}) dS_x \int_{\partial\Omega} t_j(\boldsymbol{\xi}) u^{*p}_j(\mathbf{x},\boldsymbol{\xi})\, dS_\xi + \int_{\partial\Omega} \hat{t}_p(\mathbf{x}) dS_x \int_{\partial\Omega} D_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) G^{*p}_{ij}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi + \int_{\partial\Omega} \hat{t}_p(\mathbf{x}) dS_x \int^{CPV}_{\partial\Omega} n_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) \phi^{*p}_{ij}(\mathbf{x},\boldsymbol{\xi})\, dS_\xi \tag{5.9}$$

After the regularization, both displacement and traction BIEs are weakly singular. All integrals can be performed by using the ordinary Gaussian quadrature scheme, which requires no special numerical techniques to deal with the hyper or strong singularities. The BIEs developed here have been successfully applied to solving fracture problems [Han and Atluri (2003a)] and acoustic problems [Qian, Han and Atluri (2004)].

6 Closure

A systematic approach for developing boundary integral equations has been presented to avoid the hyper singularities completely. We have also presented a systematic way to decompose the corresponding kernel functions into divergence-free and companion curl-free parts, with the use of the properties of the fundamental solutions, for regularizing the BIEs. The fully regularized BIEs contain weakly singular integrals which are numerical tractable. The present approach has been demonstrated by developing the weakly-singular displacement and traction BIEs in a linear elastic solid undergoing small displacements, in a simple and straight-forward manner.

The present approach follows the methodologies presented in Okada, Rajiyah, and Atluri (1989b) and extended in Han and Atluri (2003a). It is general and could be extended for developing BIEs for other PDEs for acoustics, electro- magnetics, finite elasticity, and the MLPG approach.

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Appendix A: Derivatives of the Kelvin solution

In a 3-dimensional linear elastic homogeneous body we can easily derive the derivatives of F^*, as:

$$F^* = \frac{r}{8\pi\mu(1-\upsilon)} \text{ for 3D problems} \tag{A-1}$$

$$F^*_{,p} = \frac{r_{,p}}{8\pi\mu(1-\upsilon)} \tag{A-2a}$$

$$F^*_{,pi} = \frac{1}{8\pi\mu(1-\upsilon)r}(\delta_{pi} - r_{,p}r_{,i}) \tag{A-2b}$$

$$F^*_{,kk} = \frac{1}{4\pi\mu(1-\upsilon)r} \tag{A-2c}$$

$$F^*_{,pij} = -\frac{1}{8\pi\mu(1-\upsilon)r^2} \cdot (\delta_{pi}r_{,j} + \delta_{pj}r_{,i} + \delta_{ij}r_{,p} - 3r_{,p}r_{,i}r_{,j}) \tag{A-2d}$$

$$F^*_{,kki} = -\frac{1}{4\pi\mu(1-\upsilon)r^2}r_{,i} \tag{A-2e}$$

$$F^*_{,kkij} = -\frac{1}{4\pi\mu(1-\upsilon)r^3}(\delta_{ij} - 3r_{,i}r_{,j}) \tag{A-2f}$$

and for a 2-dimensional body,

$$F^* = \frac{-r^2\ln r}{8\pi\mu(1-\upsilon)} \tag{A-3}$$

$$F^*_{,p} = -\frac{1}{8\pi\mu(1-\upsilon)}(r + 2r\ln r)r_{,p} \tag{A-4a}$$

$$F^*_{,pi} = -\frac{1}{8\pi\mu(1-\upsilon)}[\delta_{pi}(1 + 2\ln r) + 2r_{,p}r_{,i}] \tag{A-4b}$$

$$F^*_{,kk} = -\frac{1}{2\pi\mu(1-\upsilon)}(1 + \ln r) \tag{A-4c}$$

$$F^*_{,pij} = -\frac{1}{4\pi\mu(1-\upsilon)r} \cdot (\delta_{pi}r_{,j} + \delta_{pj}r_{,i} + \delta_{ij}r_{,p} - 2r_{,p}r_{,i}r_{,j}) \tag{A-4d}$$

$$F^*_{,kki} = -\frac{1}{2\pi\mu(1-\upsilon)r}r_{,i} \tag{A-4e}$$

$$F^*_{,kkij} = -\frac{1}{2\pi\mu(1-\upsilon)r^2}(\delta_{ij} - 2r_{,i}r_{,j}) \tag{A-4f}$$

The singularity in each of the derivatives can be seen, for 3D problems, as:

$$F^*_{,kkij} \propto O(\frac{1}{r^3}) \tag{A-5a}$$

$$F^*_{,kki} \propto O(\frac{1}{r^2}) \tag{A-5b}$$

$$F^*_{,kk} \propto O(\frac{1}{r}) \tag{A-5c}$$

$$F^*_{,pk} \propto O(\frac{1}{r}) \tag{A-5d}$$

and for 2D problems as:

$$F^*_{,kkij} \propto O(\frac{1}{r^2}) \tag{A-6a}$$

$$F^*_{,kki} \propto O(\frac{1}{r}) \tag{A-6b}$$

$$F^*_{,kk} \propto O(\ln r) \tag{A-6c}$$

$$F^*_{,pk} \propto O(\ln r) \tag{A-6d}$$

The kernel functions for 3D problems are given as:

$$u_i^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{16\pi(1-\upsilon)r}[(3-4\upsilon)\delta_{ip} + r_{,i}r_{,p}] \tag{A-7}$$

$$G_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{8\pi(1-\upsilon)r}[(1-2\upsilon)e_{ipj} + e_{ikj}r_{,k}r_{,p}] \tag{A-8}$$

$$\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{8\pi(1-\upsilon)r^2} \cdot [(1-2\upsilon)(\delta_{ij}r_{,p} - \delta_{ip}r_{,j} - \delta_{jp}r_{,i}) - 3r_{,i}r_{,j}r_{,p}] \tag{A-9}$$

$$H_{ijpq}^{*}(\mathbf{x},\boldsymbol{\xi}) = \frac{\mu}{8\pi(1-\upsilon)r}[4\upsilon\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} - 2\upsilon\delta_{ij}\delta_{pq} + \delta_{ij}r_{,p}r_{,q} + \delta_{pq}r_{,i}r_{,j} - 2\delta_{ip}r_{,j}r_{,q} - \delta_{jq}r_{,i}r_{,p}] \tag{A-10}$$

and for 2D plane strain problems as:

$$u_i^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{8\pi(1-\upsilon)}[-(3-4\upsilon)\ln r\delta_{ip} + r_{,i}r_{,p}] \tag{A-11}$$

$$G_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{4\pi(1-\upsilon)} \cdot [-(1-2\upsilon)\ln r\, e_{ipj} + e_{ikj}r_{,k}r_{,p}] \tag{A-12}$$

$$\sigma_{ij}^{*p}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{4\pi(1-\upsilon)r} \cdot [(1-2\upsilon)(\delta_{ij}r_{,p} - \delta_{ip}r_{,j} - \delta_{jp}r_{,i}) - 2r_{,i}r_{,j}r_{,p}] \tag{A-13}$$

$$H_{ijpq}^{*}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{4\pi(1-\upsilon)} \cdot [-4\upsilon\ln r\delta_{iq}\delta_{jp} + \ln r\delta_{ip}\delta_{jq} + 2\upsilon\ln r\delta_{ij}\delta_{pq} + \delta_{ij}r_{,p}r_{,q} + \delta_{pq}r_{,i}r_{,j} - 2\delta_{ip}r_{,j}r_{,q} - \delta_{jq}r_{,i}r_{,p}] \tag{A-14}$$