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A Modified Trefftz Method for Two-Dimensional Laplace Equation Considering the Domain's Characteristic Length

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Abstract: A newly modified Trefftz method is developed to solve the exterior and interior Dirichlet problems for two-dimensional Laplace equation, which takes the characteristic length of problem domain into account. After introducing a circular artificial boundary which is uniquely determined by the physical problem domain, we can derive a Dirichlet to Dirichlet mapping equation, which is an exact boundary condition. By truncating the Fourier series expansion one can match the physical boundary condition as accurate as one desired. Then, we use the collocation method and the Galerkin method to derive linear equations system to determine the Fourier coefficients. Here, the factor of characteristic length ensures that the modified Trefftz method is stable. We use a numerical example to explore why the conventional Trefftz method is failure and the modified one still survives. Numerical examples with smooth boundaries reveal that the present method can offer very accurate numerical results with absolute errors about in the orders from 10^{-10} to 10^{-16}. The new method is powerful even for problems with complex boundary shapes, with discontinuous boundary conditions or with corners on boundary.

Keyword: Laplace equation, Artificial boundary condition, Modified Trefftz method, Characteristic length, Collocation method, Galerkin method, DtD mapping

1 Introduction

The Dirichlet problem of Laplace equation in plane domain is a classical one. Although for some simple domains with contour like as circle, ellipse, rectangle, etc., the exact solutions could be found, in general, for a given arbitrary plane domain the finding of closed-form solution is not an easy task. Indeed, the explicit solutions are exceptional. If one was chosen an arbitrary shape as the problem domain, the geometric complexity commences and then typically the numerical solutions would be required.

In this paper we consider numerical solutions of Laplace equation on an unbounded domain and also in a bounded domain, of which the boundary shape may be complicated, which in turn rendering the boundary value specified there also a rather complicated function.

The problems on exterior domain arise in many application fields. For example, the incompressible irrotational flow around a body is described by an exterior problem of Laplace equation. The difficulties in these problems are the unboundedness of the domain and the geometrical complexity of boundary. Even, there are many numerical methods to solve the exterior problems, the most popular one is the artificial boundary method. In the procedure of this method, an artificial boundary is introduced to divide the exterior domain S into a bounded part S_1 and an unbounded part S_2 as schematically shown in Fig. 1(a). The basic idea of artificial boundary has existed since the works by Feng (1980), Feng and Yu (1982) and Yu (1983) about twenty years ago. After a suitable artificial boundary condition is set up on the artificial boundary Γ_R in Fig. 1(a), the original exterior problem in the domain $S = S_1 \cup S_2$ is reduced to a new one defined on the bounded domain S_1. Indeed, on the boundary Γ_R there exists an exact boundary condition as first derived by Feng (1980):

$$\frac{\partial u(R,\theta)}{\partial r} = \int_0^{2\pi} K(\theta,\theta')u(R,\theta')d\theta', \tag{1}$$

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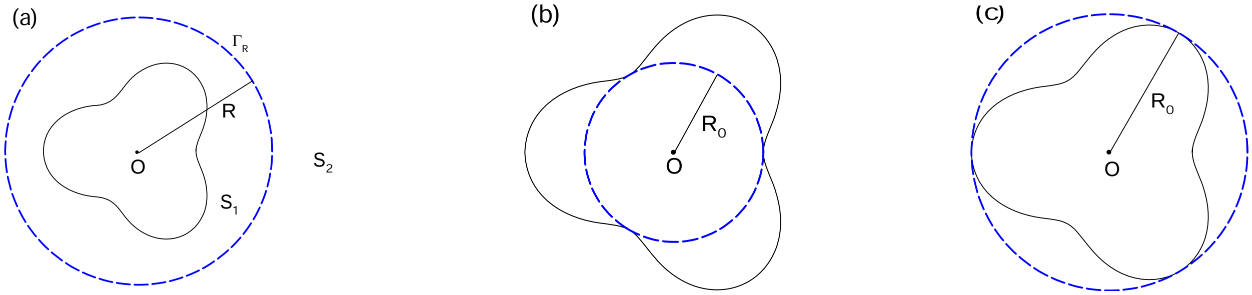

Figure 1: Schematically displaying the domains for (a) the usual artificial boundary method for exterior problem, (b) the present method for exterior problem, and (c) the present method for interior problem. R_0 specifies a characteristic length of the problem domain.

where

$$K(\theta,\theta') = \frac{1}{\pi R}\sum_{k=1}^{\infty} k\cos k(\theta-\theta') = -\frac{1}{4\pi R\sin^2\frac{(\theta-\theta')}{2}}. \tag{2}$$

The above mapping from the Dirichlet boundary function $u(R,\theta)$ onto the Neumann boundary function $\partial u(R,\theta)/\partial r$ on the artificial circle Γ_R is sometimes called the DtN mapping [Givoli, Patlashenko and Keller (1998)]. Usually, the elliptic problem is solved under a given boundary condition on a complicated boundary ∂S together with the above boundary condition on a simple fictitious boundary Γ_R by a proper numerical method, e.g., finite element method. Then, we may apply the boundary element method or other boundary type methods to treat the problem in S_2 by using the boundary data on Γ_R, which is already calculated from a previous method applying in S_1. About the combined methods for elliptic problems the readers may refer the monograph by Li (1998).

In order to reduce the computational cost and increase the computational accuracy, the bounded domain must not be too large and the artificial boundary condition must be a good approximation of the exact boundary condition on the artificial boundary. It is needless to say that the accuracy of the artificial boundary condition and the computational cost are closely related [Givoli, Patlashenko and Keller (1997); Han and Zheng (2002)]. Therefore, having an artificial boundary condition with high accuracy on a given artificial boundary becomes an effective and important method for solving partial differential equation in an unbounded domain which arises in various fields of engineering. In the last two decades, many researchers were worked on this issue for various problems by using different techniques. A detailed review concerning the numerical solutions of problems on unbounded domain through artificial boundary condition techniques can be found in [Tsynkov (1998); Han (2005)].

For the Laplace equation, Han and Wu (1985) have obtained the exact boundary conditions and a series of their approximations at an artificial boundary. For exterior problems of elliptic equation, Han and Zheng (2002) have employed a mixed finite element method and high-order local artificial boundary conditions to solve them. By considering a definite artificial circle determined by the problem domain, Liu (2007a) has developed a meshless regularized integral equation method for the Laplace equation in arbitrary plane domain, and Liu (2007b) extended these results to the Laplace problem defined in a doubly-connected region.

For a complicated domain the conventional methods usually require a large number of nodes and elements to match the geometrical shape. In order to overcome these difficulties, the meshless local boundary integral equation (LBIE) method [Atluri, Kim and Cho (1999)], and the meshless local Petrov-Galerkin (MLPG) method [Atluri and Shen (2002)] are proposed. Both methods use local weak forms and the integrals can be easily evaluated over regularly shaped domains, like as circles in 2D problems and spheres in 3D problems.

On the other hand, the method of fundamental solutions (MFS), also called the F-Trefftz method, utilizes the fundamental solutions as basis functions to expand the solution, which is another popularly used meshless method [Cho, Golberg, Muleshkov and Li (2004)]. In order to tackle of the ill-posedness of MFS, Jin (2004) has proposed a new numerical scheme for the solution of the Laplace and biharmonic equations subjected to noisy boundary data. A regularized solution was obtained by using the truncated singular value decomposition, with the regularization parameter determined by the L-curve method. Tsai, Lin, Young and Atluri (2006) have proposed a practical procedure to locate the sources in the use of MFS for various time independent operators, including Laplacian operator, Helmholtz operator, modified Helmholtz operator, and biharmonic operator. The procedure is developed through some systematic numerical experiments for relations among the accuracy, condition number, and source positions in different shapes of computational domains. By numerical experiments, they found that good accuracy can be achieved when the condition number approaches the limit of equation solver. The MFS has a broad application in engineering computations, for example, Cho, Golberg, Muleshkov and Li (2004), Hong and Wei (2005), Young, Chen and Lee (2005), Young and Ruan (2005), and Young, Tsai, Lin and Chen (2006).

Recently, Young, Chen, Chen and Kao (2007) have proposed a modified method of fundamental solutions (MMFS) for solving the Laplace problems, which implements the singular fundamental solutions to evaluate the solutions, and it can locate the source points on the real boundary as contrasted to the conventional MFS. Therefore, the major difficulty of the coincidence of the source and collocation points in the conventional MFS is thereby overcome, and the ill-posed nature of the conventional MFS disappears.

In general, the artificial boundary condition may be very rough, if one cannot design a special method to properly realize the exact boundary condition. In this paper, we will show how to design an artificial boundary condition with high accuracy on a given artificial boundary, which is uniquely determined by the domain of physical problem. The concept of characteristic length is proposed. Then, the original problem can be reduced to a boundary value problem on a fictitious circle, which allows us to obtain a series solution with high accuracy.

The other parts of present paper are arranged as follows. In Section 2 we derive the basic equations along a given artificial circle. By using the Dirichlet to Dirichlet (DtD) mapping technique, a newly modified Trefftz method is thus performed, which takes the characteristic length of the problem domain into account. In Section 3 we consider a direct collocation method to find the Fourier coefficients. Then, in Section 4 we apply the Galerkin method to find the Fourier coefficients. In Section 5 we use some examples to test the new methods, and also use a numerical example to explore the reason why the modified Trefftz method can survive, when the conventional Trefftz method is failure. Finally, we give some conclusions in Section 6.

2 Basic formulations and comments

In this paper we consider new methods to solve the Dirichlet problem under the boundary condition specified on a non-circular boundary:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r < \rho \text{ or } r > \rho, \; 0 \le \theta \le 2\pi, \quad (3)$$

$$u(\rho,\theta) = h(\theta), \; 0 \le \theta \le 2\pi, \quad (4)$$

where $h(\theta)$ is a given function, and $\rho = \rho(\theta)$ is a given contour describing the boundary shape of interior or exterior domain. The contour ∂S in the polar coordinates is given by $\partial S = \{(r,\theta)|r = \rho(\theta),\; 0 \le \theta \le 2\pi\}$, which is the boundary of the problem domain S. For exterior problem S is unbounded, and S is bounded for interior problem.

We can replace Eq. (4) by the following boundary condition:

$$u(R_0,\theta) = f(\theta), \; 0 \le \theta \le 2\pi, \quad (5)$$

where $f(\theta)$ is an unknown function to be determined, and R_0 is a given positive constant, such that the disk $D=\{(r,\theta)|r\le R_0,\ 0\le\theta\le 2\pi\}$ can cover S for interior problem, or for exterior problem it is inside in the complement of S, that is, $D\in\mathbb{R}^2/\overline{S}$. Specifically, we may let

$$R_0\le\rho_{\min}=\min_{\theta\in[0,2\pi]}\rho(\theta)\ \ \text{(exterior problem)},\tag{6}$$

$$R_0\ge\rho_{\max}=\max_{\theta\in[0,2\pi]}\rho(\theta)\ \ \text{(interior problem)}.\tag{7}$$

See Figs. 1(b) and 1(c). Because R_0 is uniquely determined by the contour of the considered problem by Eq. (6) or Eq. (7), we do not need to worry how to choose R_0. In the later, it will be clear that R_0 specifies a characteristic length of the problem domain, and also plays a major role to control the stability of our numerical method.

The above basic idea is to replace the original boundary condition (4) on a complicated contour by a simpler boundary condition (5) on a specified circle. However, we require to derive a new equation to solve $f(\theta)$. If this task can be performed and if the function $f(\theta)$ can be available, then the advantage of the present method is that we have a closed-form solution in terms of the Poisson integral:

$$u(r,\theta)=\pm\frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-R_0^2}{R_0^2-2R_0r\cos(\theta-\xi)+r^2}f(\xi)d\xi.\tag{8}$$

Here, R_0 can be viewed as the radius of an *artificial circle*, and $f(\theta)$ is an unknown function to be determined on that artificial circle. In the above and similarly in the below, the positive sign is used for the exterior problem, and conversely the minus sign is used for the interior problem.

By satisfying Eqs. (3) and (5), we can write a Fourier series expansion for $u(r,\theta)$:

$$u(r,\theta)=a_0+\sum_{k=1}^{\infty}\left[a_k\left(\frac{R_0}{r}\right)^{\pm k}\cos k\theta+b_k\left(\frac{R_0}{r}\right)^{\pm k}\sin k\theta\right],\tag{9}$$

where

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(\xi)d\xi,\tag{10}$$

$$a_k=\frac{1}{\pi}\int_0^{2\pi}f(\xi)\cos k\xi d\xi,\tag{11}$$

$$b_k=\frac{1}{\pi}\int_0^{2\pi}f(\xi)\sin k\xi d\xi.\tag{12}$$

By imposing the condition (4) on Eq. (9) we obtain

$$a_0+\sum_{k=1}^{\infty}B^k(\theta)[a_k\cos k\theta+b_k\sin k\theta]=h(\theta),\tag{13}$$

where

$$B(\theta):=\left(\frac{R_0}{\rho(\theta)}\right)^{\pm 1}.\tag{14}$$

Substituting Eqs. (10)-(12) into Eq. (13) leads to a first-kind Fredholm integral equation:

$$\int_0^{2\pi}K(\theta,\theta')f(\theta')d\theta'=h(\theta),\tag{15}$$

where

$$K(\theta,\theta')=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}B^k(\theta)\cos k(\theta-\theta')\tag{16}$$

is a kernel function. Liu (2007a, 2007c) has applied the regularization integral equation method to solve Eq. (15) for the Dirichlet boundary value problems. But in this paper, we are going to directly solve Eq. (9) to obtain the Fourier coefficients a_k and b_k as simple as possible.

Corresponding to the DtN mapping in Eq. (1), the one in Eq. (15) is a mapping from the Dirichlet boundary function $f(\theta)$ on the artificial circle onto the Dirichlet boundary function $h(\theta)$ on the real boundary ∂S. This mapping may be called the Dirichlet-to-Dirichlet mapping, and is abbreviated as the DtD mapping. Up to this point, it can be seen that our method is different from other artificial boundary condition methods. The reasons are given as follows. Comparing Figs. 1(a) and 1(b) for solving the exterior problems, we do not require to divide the unbounded domain S into two parts S_1 and S_2, and we just need to solve an exterior problem which is outside the boundary Γ_{R_0}. It has an analytical solution as shown by Eq. (8) or Eq. (9) when we can get $f(\theta)$ or the Fourier coefficients. This concept can be immediately extended to apply on the interior problems as that shown in Fig. 1(c). The artificial boundary set up here is uniquely determined by the domain of the problem through Eq. (6) or Eq. (7). This point is also different from other artificial boundary condition methods.

Eq. (15) is an exact boundary condition; however, it is difficult to directly inverse that equation to obtain the exact boundary data $f(\theta)$. In the paper by Liu (2007a) the regularization integral equation method was applied to solve Eq. (15), but in this paper, we are going to solve its equivalent part in Eq. (13) to obtain the Fourier coefficients directly. Without loss much accuracy we can consider a truncated version of Eq. (13).

It is known that for the Laplace equation in the two-dimensional bounded domain the set $\{1, r^k \cos k\theta, r^k \sin k\theta,\ k = 1, 2, \ldots\}$ forms the T-complete function, and the solution can be expanded by these bases [Kita and Kamiya (1995)]

$$u(r,\theta) = a_0 + \sum_{k=1}^{\infty} [a_k r^k \cos k\theta + b_k r^k \sin k\theta]. \quad (17)$$

It is simply a direct consequence of Eq. (9) by inserting $R_0 = 1$ for the interior problem by taking the minus sign there. As that done in Eq. (13), after imposing the boundary condition (4) on Eq. (17) one obtains

$$a_0 + \sum_{k=1}^{\infty} \rho^k(\theta)[a_k \cos k\theta + b_k \sin k\theta] = h(\theta). \quad (18)$$

This method is very simple and is called the Trefftz method [Kita and Kamiya (1995)] for the interior problem of Laplace equation.

The Trefftz method is designed to satisfy the governing equation and leaves the unknown coefficients determined by satisfying the boundary conditions through the collocation, the least square or the Galerkin method, etc. [Kita and Kamiya (1995); Kita, Kamiya and Iio (1999)]. Recently, Li, Lu, Huang and Cheng (2007) gave a very comprehensive comparison of the Trefftz, collocation and other boundary methods. They concluded that the collocation Trefftz method is the simplest algorithm and provides the most accurate solution with the best numerical stability.

As just mentioned, our new artificial boundary method has led to a simple solution procedure as that for the Trefftz method. The resultant governing Eq. (9) bears a certain similarity with that used in the Trefftz method; however, it needs to stress that our equation is based on the series solution with the boundary data on an artificial circle with a radius R_0, and this quantity R_0 is appeared explicitly in the series solution (9), but it does not appear in the Trefftz method. It may argue that the new method is just the Trefftz method. However, this is incorrect. By letting $R_0 = 1$, we indeed recover our method to the Trefftz method. But the converse is not true. R_0 is a characteristic length of the considered problem, and usually it is not equal to 1. The Trefftz method does not consider this characteristic length R_0 in the above two equations (17) and (18). In Section 5, a numerical example will be given for an interior problem, whose characteristic length is much larger than 1. When directly applying Eqs. (17) and (18) together with a collocation method as to be introduced below to solve that problem, the Trefftz method is failure.

Liu (2007d) has proposed a modified Trefftz method to calculate the Laplace problems under mixed-boundary conditions. Because the ill-posedness of the conventional Trefftz method is overcome by the new method, which can even be applied on the singular problem with a high accuracy never seen before. Liu (2007e) has employed the same idea to modify the direct Trefftz method for the two-dimensional potential problem, and Liu (2007f) used this idea to develop a highly accurate numerical method to calculate the Laplace equation in doubly-connected domains.

3 The collocation method

For Eq. (9) we have found that Eq. (13) can be used to determine the unknown coefficients. Therefore, the above mentioned techniques for the Trefftz method can be employed here to find the unknown Fourier coefficients. In this section we introduce the collocation method, and in the next section we will introduce the Galerkin method.

The series expansion in Eq. (13) is well suited in the range of $\theta \in [0, 2\pi]$. Hence, we may have an admissible function with finite terms

$$a_0 + \sum_{k=1}^{m} B^k(\theta)[a_k \cos k\theta + b_k \sin k\theta] = h(\theta), \quad 0 \le \theta \le 2\pi. \tag{19}$$

Our next task is to find $a_k,\ k = 0, 1, \ldots, m$ and $b_k,\ k = 1, \ldots, m$ from Eq. (19).

Eq. (19) is imposed at $n = 2m$ different collocated points θ_i on the interval with $0 \le \theta_i \le 2\pi$:

$$a_0 + \sum_{k=1}^{m} B^k(\theta_i)[a_k \cos k\theta_i + b_k \sin k\theta_i] = h(\theta_i). \tag{20}$$

It can be seen that the basic idea behind the collocation method is rather simple, and it has a great advantage of the flexibility to apply to different geometric shapes, and the simplicity for computer programming as to be shown below.

Let

$$\theta_i = i\Delta\theta, \ \ i = 1, \ldots, n, \tag{21}$$

where $\Delta\theta = 2\pi/(n-1)$. When the index i in Eq. (20) runs from 1 to n we obtain a linear equations system with dimensions $n = 2m+1$:

$$\begin{bmatrix} 1 & V(1) & \cdots & V(m) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_m \\ b_m \end{bmatrix} = \begin{bmatrix} h(\theta_0) \\ h(\theta_1) \\ h(\theta_2) \\ \vdots \\ h(\theta_{n-1}) \\ h(\theta_n) \end{bmatrix}. \tag{22}$$

where

$$V(1) = \begin{bmatrix} B(\theta_0)\cos\theta_0 & B(\theta_0)\sin\theta_0 \\ B(\theta_1)\cos\theta_1 & B(\theta_1)\sin\theta_1 \\ B(\theta_2)\cos\theta_2 & B(\theta_2)\sin\theta_2 \\ \vdots & \vdots \\ B(\theta_{n-1})\cos\theta_{n-1} & B(\theta_{n-1})\sin\theta_{n-1} \\ B(\theta_n)\cos\theta_n & B(\theta_n)\sin\theta_n \end{bmatrix},$$

$$V(m) = \begin{bmatrix} B^m(m\theta_0)\cos(m\theta_0) & B^m(m\theta_0)\sin(m\theta_0) \\ B^m(m\theta_1)\cos(m\theta_1) & B^m(m\theta_1)\sin(m\theta_1) \\ B^m(m\theta_2)\cos(m\theta_2) & B^m(m\theta_2)\sin(m\theta_2) \\ \vdots & \vdots \\ B^m(m\theta_{n-1})\cos(m\theta_{n-1}) & B^m(m\theta_{n-1})\sin(m\theta_{n-1}) \\ B^m(m\theta_n)\cos(m\theta_n) & B^m(m\theta_n)\sin(m\theta_n) \end{bmatrix}.$$

Corresponding to the uniformly distributed collocated points on the circle, θ_0 is a single independent collocated point which is supplemented to obtain $2m+1$ equations used to solve the $2m+1$ unknowns $(a_0, a_1, b_1, \cdots, a_m, b_m)$.

We denote the above equation by

$$\mathbf{Rc} = \mathbf{h},$$

where $\mathbf{c} = (a_0, a_1, b_1, \cdots, a_m, b_m)^{\mathrm{T}}$ is the vector of unknown coefficients. The superscript T signifies the transpose.

The conjugate gradient method can be used to solve the following normal equation:

$$\mathbf{Ac} = \mathbf{b}, \tag{23}$$

where

$$\mathbf{A} := \mathbf{R}^{\mathrm{T}}\mathbf{R}, \ \ \mathbf{b} := \mathbf{R}^{\mathrm{T}}\mathbf{h}. \tag{24}$$

Inserting the calculated $\mathbf{c}$ into Eq. (9) we can calculate $u(r, \theta)$ at any point in the problem domain by

$$u(r,\theta) = c_1 + \sum_{k=1}^{m} \left[c_{2k} \left(\frac{R_0}{r}\right)^{\pm k} \cos k\theta + c_{2k+1} \left(\frac{R_0}{r}\right)^{\pm k} \sin k\theta \right]. \tag{25}$$

Even we do not discuss the well-posedness of Eq. (23) here, the interesting readers may refer the paper by Liu (2007d), where one can see that the condition number can be greatly reduced by the modified Trefftz method than the original Trefftz method about thirty orders with $m = 20$.

4 The Galerkin method

In order to apply the Galerkin method on Eq. (19), we require $\mathbf{P}$ and $\mathbf{Q}$, which are $2m+1$-vectors given by

$$\mathbf{P} := \begin{bmatrix} 1 \\ B\cos\theta \\ B\sin\theta \\ B^2\cos 2\theta \\ B^2\sin 2\theta \\ \vdots \\ B^m\cos m\theta \\ B^m\sin m\theta \end{bmatrix}, \quad \mathbf{Q} := \begin{bmatrix} 1 \\ \cos\xi \\ \sin\xi \\ \cos 2\xi \\ \sin 2\xi \\ \vdots \\ \cos m\xi \\ \sin m\xi \end{bmatrix}. \tag{26}$$

In Eq. (13), 1, $\cos(k\theta)$ and $\sin(k\theta)$ can be employed as the weighting functions to minimize the residual function

$$R(\theta) = a_0 + \sum_{k=1}^{m} B^k(\theta)[a_k\cos k\theta + b_k\sin k\theta] - h(\theta). \tag{27}$$

Applying 1 and each $\cos(k\theta)$ and $\sin(k\theta)$ for $k = 1,\ldots,m$ on the above equation, integrating it from 0 to 2π and forcing each resultant to be zero, i.e.,

$$\langle R(\theta), 1\rangle = \int_0^{2\pi} R(\xi)d\xi = 0, \tag{28}$$

$$\langle R(\theta), \cos(k\theta)\rangle = \int_0^{2\pi} R(\xi)\cos(k\xi)d\xi = 0, \tag{29}$$

$$\langle R(\theta), \sin(k\theta)\rangle = \int_0^{2\pi} R(\xi)\sin(k\xi)d\xi = 0 \tag{30}$$

for each $k = 1,\ldots,m$, we can obtain a linear equations system of dimensions $n = 2m+1$:

$$\mathbf{Rc} = \mathbf{d}, \tag{31}$$

where

$$\mathbf{R} := \int_0^{2\pi} \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi, \tag{32}$$

$$\mathbf{d} := \int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi. \tag{33}$$

Now, we can apply the conjugate gradient method to solve the normal form of Eq. (31). Then, inserting the calculated **c** into Eq. (25) we can calculate $u(r,\theta)$ at any point in the problem domain.

5 Numerical examples

In this section we will apply the new methods on both exterior and interior problems.

5.1 Example 1 (exterior problem)

In this example we investigate a discontinuous boundary condition on the unit circle:

$$h(\theta) = \begin{cases} 1 & 0 \le \theta < \pi, \\ -1 & \pi \le \theta < 2\pi. \end{cases} \tag{34}$$

For this example an analytical solution is given by

$$u(x,y) = \frac{2}{\pi}\arctan\left(\frac{2y}{x^2+y^2-1}\right). \tag{35}$$

We have applied the collocation method with $m = 150$ and $\theta_0 = \pi$, and the Galerkin method with $m = 100$ on this example. In Fig. 2(a) we compare the exact solution with numerical solutions along a circle with radius 2.5. It can be seen that the numerical solutions are close to the exact solution. Furthermore, the numerical errors were plotted in Fig. 2(b), of which it can be seen that the collocation method is slightly accurate than the Galerkin method.

5.2 Example 2 (exterior problem)

In this example we consider a complex epitrochoid boundary shape

$$\rho(\theta) = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos(a\theta/b)}, \tag{36}$$

$$x(\theta) = \rho\cos\theta, \quad y(\theta) = \rho\sin\theta \tag{37}$$

with $a = 3$ and $b = 1$. The contour shape is already plotted in Fig. 1(b). The analytical solution is supposed to be

$$u(x,y) = \exp\left(\frac{x}{x^2+y^2}\right)\cos\left(\frac{y}{x^2+y^2}\right). \tag{38}$$

The exact boundary data can be easily derived by inserting Eqs. (36) and (37) into the above equation.

We have applied the new methods on this example by using $R_0 = 2$, $m = 50$ and $\theta_0 = 0$ for the collocation method, and $R_0 = \rho_{\min} = 3$ and $m = 20$ for the Galerkin method. In Fig. 3 we compare the exact solution with numerical solutions along a circle with radius 10. It can be seen that the numerical solutions are almost coincident with the

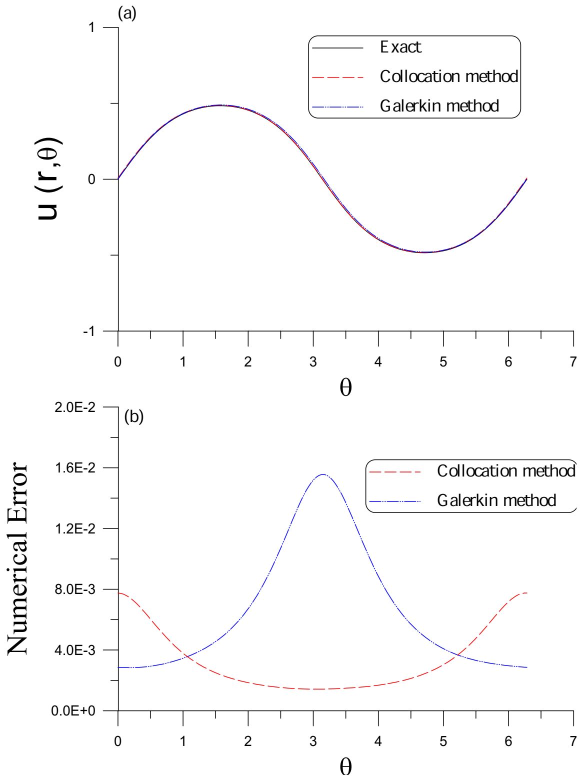

Figure 2: Comparing the exact solution and numerical solutions for Example 1 in (a), and the numerical errors are plotted in (b).

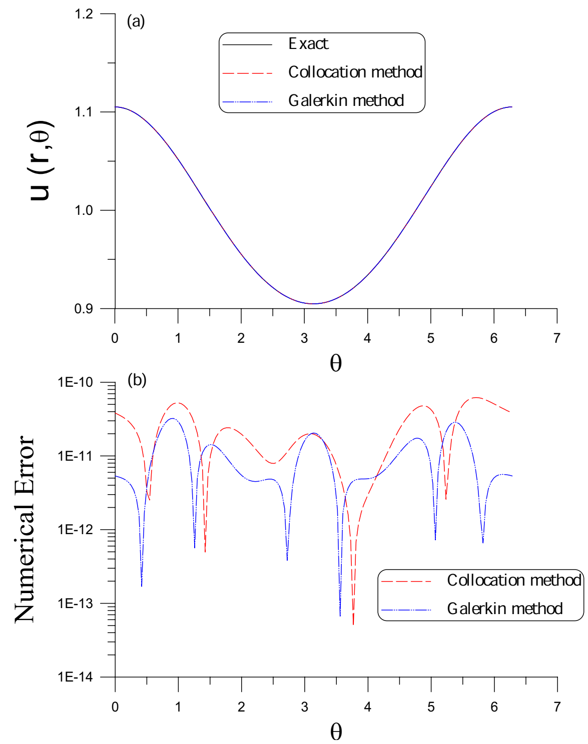

Figure 3: Comparing the exact solution and numerical solutions for Example 2 in (a), and the numerical errors are plotted in (b).

exact solution. The numerical errors are compared in Fig. 3(b), of which we can see that the Galerkin method is slightly accurate than the collocation method, and both methods have absolute errors smaller than 10^{-10}.

5.3 *Example 3 (interior problem)*

In this example we consider another epitrochoid boundary shape with $a = 4$ and $b = 1$. The inset in Fig. 4 displays the contour shape. We consider two analytical solutions:

$$u(x,y) = x^2 - y^2, \quad u(x,y) = e^x \cos y. \tag{39}$$

The exact boundary data can be obtained by inserting Eqs. (36) and (37) into the above equations.

In the numerical computations we have fixed $R_0 = \rho_{\max} = 6$, $m = 25$ and $\theta_0 = \pi/2$ for the collocation method, and $m = 4$ for the Galerkin method. In Fig. 4(a) we compare the numerical solutions with exact solution $u(x,y) = x^2 - y^2$ along a circle with radius $r = 3$, while the numerical errors are plotted in Fig. 4(b). In Fig. 5(a) we compare the numerical solutions with exact solution $u(x,y) = e^x \cos y$ along a circle with radius $r = 3$, while the numerical errors are plotted in Fig. 5(b). For all these cases very accurate numerical solutions are obtained with absolute errors smaller than 10^{-10}.

In Section 3 we have mentioned that when the characteristic length R_0 is larger than 1, applying the Trefftz method may be ineffective. We applying the collocation method to solve Eq. (18) with $n = 2m+1$ collocated points. For the second case, we use $m = 9$ and $m = 10$ to calculate the numerical solutions by the Trefftz method. The numerical results are compared with the exact one $u = e^x \cos y$ along a circle with radius $r = 3$. From Fig. 6, we find that large errors appear, especially that with $m = 10$. The result with $m = 9$ is the best one, and others m make more large errors.

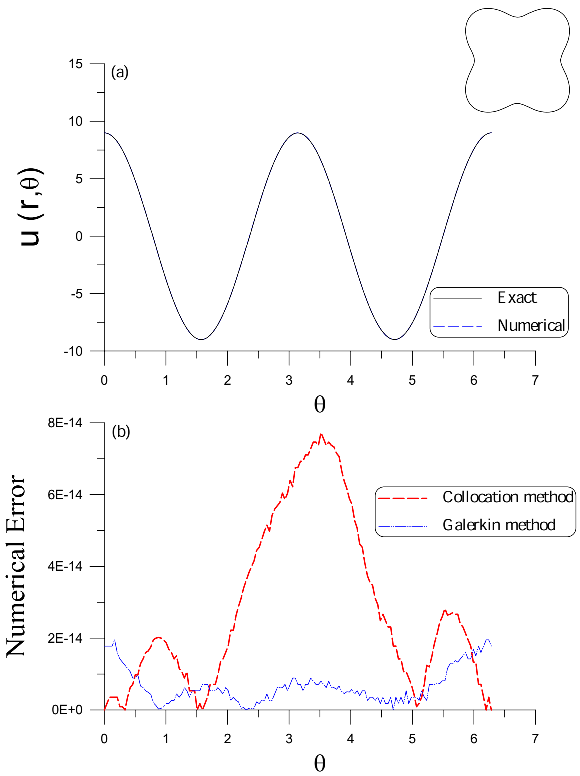

Figure 4: Comparing the exact solution and numerical solutions for Example 3 case 1 in (a), and the numerical errors are plotted in (b).

The main reason for the failure of the Trefftz method is that one uses a divergent series

$$a_0+\sum_{k=1}^{m}\rho^k(\theta)[a_k\cos k\theta+b_k\sin k\theta]=h(\theta), \quad (40)$$

where $\rho(\theta)>1$, to solve the unknown coefficients. Conversely, in our method the series

$$a_0+\sum_{k=1}^{m}B^k(\theta)[a_k\cos k\theta+b_k\sin k\theta]=h(\theta) \quad (41)$$

used to find the unknown coefficients is convergent, because by Eqs. (14) and (7) we have

$$B(\theta)=\frac{\rho(\theta)}{R_0}<1. \quad (42)$$

From this point we can understand that the characteristic lengh R_0 indeed plays a vital role to control the stability of numerical method. When the Trefftz method is sensitive to the collocation points, displaying unstable behavior, the present method is stable by adopting a characteristic factor R_0 in the algebraic equations.

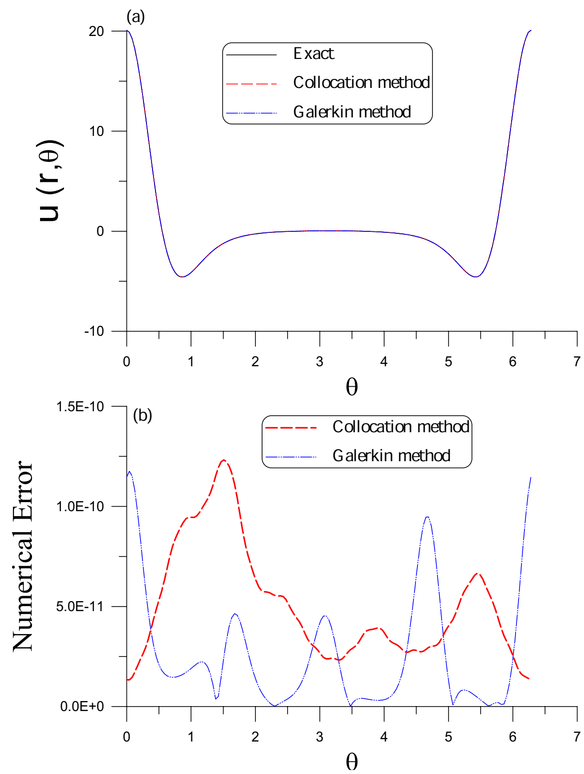

Figure 5: Comparing the exact solution and numerical solutions for Example 3 case 2 in (a), and the numerical errors are plotted in (b).

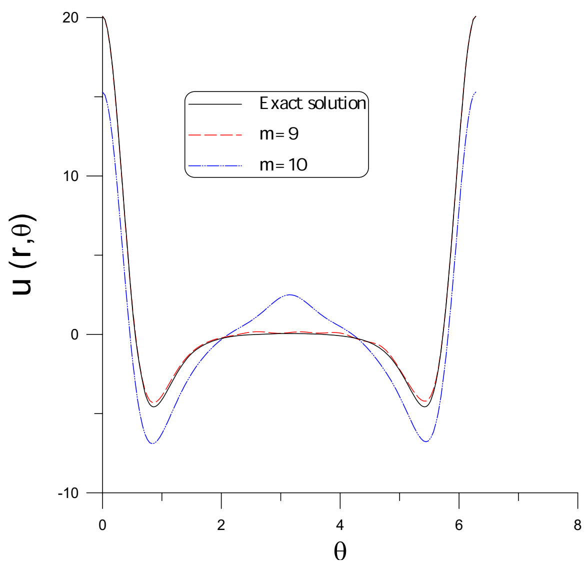

Figure 6: For Example 3 case 2, comparing the exact solution and the numerical solutions calculated by the conventional Trefftz method.

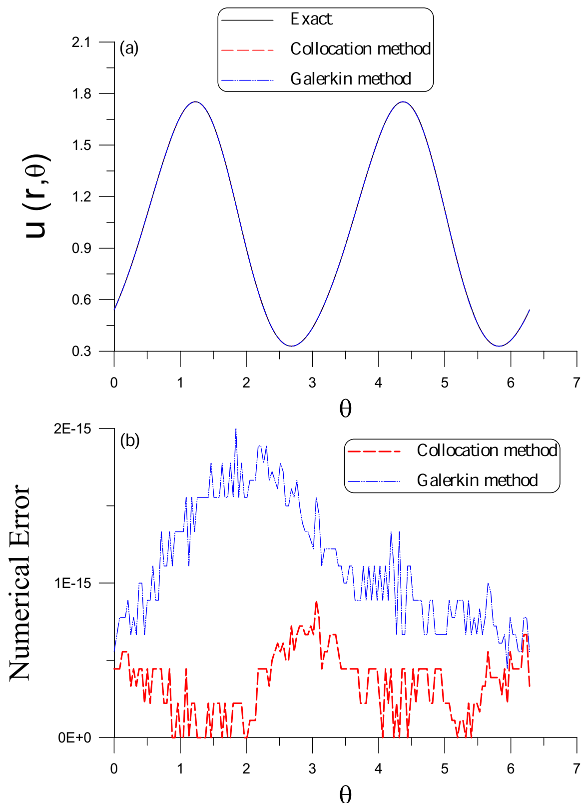

Figure 7: Comparing the exact solution and numerical solutions for Example 4 in (a), and the numerical errors are plotted in (b).

5.4 *Example 4 (interior problem)*

For this example the solution domain is a disk with a radius equal to 2. To illustrate the accuracy and stability of the new method we consider the following analytical solution [Jin (2004)]:

$$u(x,y) = \cos x \cosh y + \sin x \sinh y. \tag{43}$$

The exact boundary data can be easily derived by inserting $x = 2\cos\theta$ and $y = 2\sin\theta$ into the above equation.

In the numerical computations we have fixed $m = 50$ and $\theta_0 = 0$. In Fig. 7(a) we compare the exact solution with numerical solutions along a circle with radius 1. It can be seen that the numerical solution is very close to the exact solution, of which the L^2 error is about 10^{-16} and the absolute errors smaller than 2×10^{-15} are plotted in Fig. 7(b).

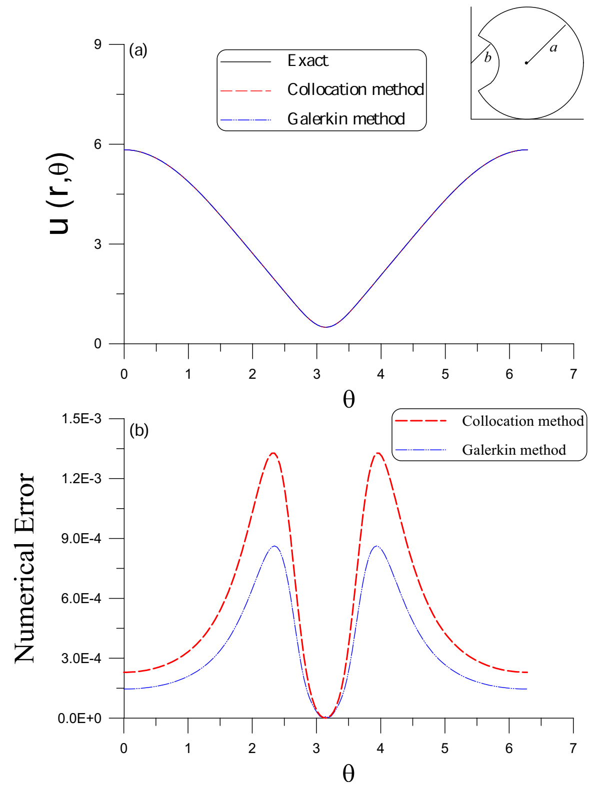

Figure 8: Comparing the exact solution and numerical solutions for Example 5 in (a), and the numerical errors are plotted in (b).

5.5 *Example 5 (interior problem)*

For this example an elastic torsion problem is solved for a circular shaft with a grove. In the inset of Fig. 8 a schematic plot of the cross section with grove is shown. We consider the following analytical solution [Timoshenko and Goodier (1961)]:

$$u(r,\theta) = \frac{b^2}{2} + (a^2 + ar\cos\theta)\left[1 - \frac{ab^2}{r^2 + 2ar\cos\theta + a^2}\right]. \tag{44}$$

Here, we have expressed the conjugate warping function $u(r,\theta)$ in a polar coordinate with the circular center as the original point. For the most solutions appeared in the literature, the original point is placed at the left-end point of the circle.

The contour shape of this problem is given by

$$\rho(\theta) = \begin{cases} -a\cos\theta - \sqrt{b^2 - a^2\sin\theta^2} & \pi - \phi \le \theta \le \pi + \phi, \\ a & \text{otherwise,} \end{cases} \tag{45}$$

where

$$\phi = \arccos\left(\frac{2a^2 - b^2}{2a^2}\right). \tag{46}$$

The boundary condition is obtained by inserting Eq. (45) for r into Eq. (44).

In the numerical computations we have fixed $a = 2$, $b = 1$, $m = 30$ and $\theta_0 = 0$. In Fig. 8(a) we compare the exact solution with numerical solutions along a circle with radius 1. It can be seen that the numerical solutions are close to the exact solution, of which the L^2 error is about 8.22×10^{-3} for the collocation method, and about 5.27×10^{-3} for the Galerkin method. The absolute errors are plotted in Fig. 8(b). Due to the corners appeared in the boundary, two peaks are exhibited in the absolute error curves.

6 Conclusions

In this paper we have proposed a new artificial boundary method to calculate the solutions of exterior and interior Laplace problems in arbitrary plane domains. In practice, it is a modified Trefftz method by taking the domain's characteristic length R_0 into account. The new factor R_0 is very important to stabilize the Trefftz method. The equations derived are easy to numerical implementation, which is similar to that for the conventional Trefftz method. The collocation method and the Galerkin method are used to find the unknown Fourier coefficients. The computational costs of the new methods are saving, especially that for the collocation method. The numerical examples show that the effectiveness of the new methods and the accuracy is very good. About this, we should note that both methods are highly accurate with the absolute errors smaller than 10^{-10} for the problems with smooth boundaries. When there appears discontinuity or singularity of boundary, the accuracy is decreased a certain amount, with the absolute error about in the order of 10^{-3}. Generally speaking, under the same m, the Galerkin method is accurate than the collocation method; however, the Galerkin method requires more computational cost to calculate the coefficients matrix $\mathbf{R}$ of the algebraic system. The new methods presented here possess several advantages than the conventional boundary-type solution methods, which including meshfree, singularity-free, semi-analyticity, efficiency, accuracy and stability.

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