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Wind Set-down Relaxation

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Abstract: We developed analytical solutions to the wind set-down and the wind set-down relaxation problems. The response of the ocean to the wind blowing over a long-narrow and linearly sloping shallow basin is referred to as wind set-down. The shoreline exhibits oscillatory behavior when the wind calms down and the resulting problem is referred to as wind set-down relaxation. We use an existing hodograph-type transformation that was introduced to solve the nonlinear shallow-water wave equations analytically for long wave propagation and obtain an explicit-transform analytical solution for wind set-down. For the wind set-down relaxation, the nonlinear shallow-water wave equations are solved analytically as an initial-boundary value problem, with forced initial data derived from our wind set-down solution.

Keyword: Wind set-down, wind set-down relaxation, shallow-water wave equations, hodograph transformation.

1 Introduction

We consider a long-narrow and shallow basin with a linearly sloping bottom (Fig. 1) such as the Gulf of Suez, the Gulf of Elat or the bay of Baja California. The mouth of the basin is connected to the sea which is practically infinitely deep compared to the basin depth at the transition. If a moderate wind blows seaward, there is a steady-state solution for the sea surface height [Nof and Paldor (1992)]. The solution results from the balance of the wind stress at the top of the water column with the vertically integrated pressure gradient and the problem is called the wind set-down. As long as the wind remains blowing in the same direction, the steady-state solution will continue to hold. If the wind suddenly calms down, water accelerates in the shoreward-direction under the pressure gradient that exists because of the difference in elevation between the shoreline and the sea boundary, since now there is no longer any wind stress. Then the water surface exhibits oscillatory behavior and the phenomenon is referred to as wind set-down relaxation.

Nof and Paldor (1992) solved the steady-state wind set-down problem and established an implicit-analytical solution. Later, Gelb, Gottlieb, and Paldor (1997) used the implicit-analytical solution of Nof and Paldor (1992) as an initial condition and solved the wind set-down relaxation problem numerically, employing the nonlinear shallow-water wave (NSW) equations. The main difficulty in solving the NSW equations is the moving singularity of the equations at the shoreline [Carrier and Greenspan (1958); Gelb, Gottlieb, and Paldor (1997)].

Moreover, the analytical solution of the NSW equations over a sloping beach is a classical problem in shallow-water wave dynamics. The major advance for the analytical solution of the NSW equations was presented by Carrier and Greenspan (1958). Carrier and Greenspan (1958) introduced a hodograph-type transformation which reduced the NSW equations into a single second-order linear partial differential equation (PDE) and solved the initial value problem (IVP) of a periodic long wave propagating over a sloping beach. The transformation introduced by Carrier and Greenspan (1958) had limited application since it was not possible to use any

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given wave profile as an initial condition. Recently, methodologies to handle IVP solutions of the NSW equations were presented in Carrier, Wu, and Yeh (2003), Kânoğlu (2004), and Kânoğlu and Synolakis (2006) considering more general initial conditions. In addition, boundary value problem (BVP) solution of the NSW equations was presented by Synolakis (1986, 1987) for a sloping beach connected to a constant-depth region.

In this study, we employ the hodograph-type transformation for the spatial and the spatial+temporal variables presented by Carrier, Wu, and Yeh (2003) to solve the wind set-down and the wind set-down relaxation problems respectively. Even though it is a classical approach to solve the NSW equations either as an IVP or a BVP for long wave propagation problem, here we will proceed with the initial-boundary value problem (IBVP) solution. First, we obtain an explicit-analytical solution of the former problem in the transform space. This solution will be called explicit-transform solution hereon. Then we use this solution as an initial condition to the latter problem and obtain analytical solution, again.

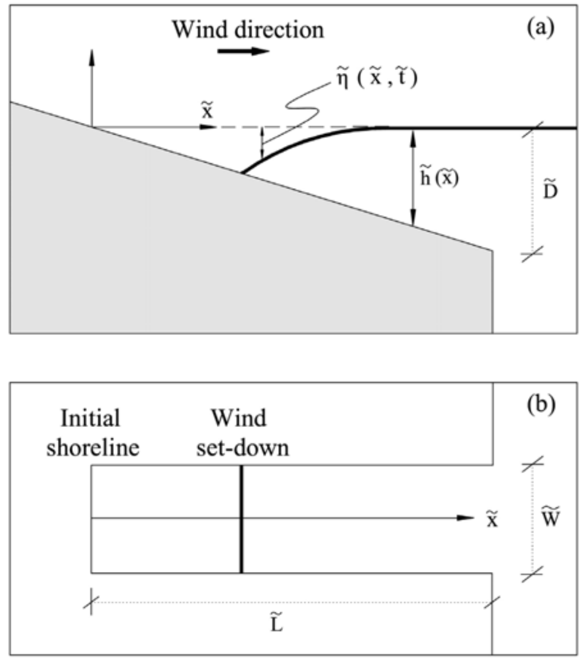

Figure 1: Definition sketch for a long-narrow ($\widetilde{W}/\widetilde{L} \ll 1$) and shallow ($\widetilde{D}/\widetilde{L} \ll 1$) basin: (a) cross section, (b) top view.

2 Mathematical Formulation

The steady-state nonlinear response of the ocean to the wind blowing over a long-narrow and shallow basin (Fig. 1) is referred to as wind set-down problem and is governed by a nonlinear equation. In dimensionless form, the governing equation is

$$-(h+\eta)\eta_x+\gamma=0, \tag{1}$$

subject to the boundary condition $\eta(x=1)=0$ [Nof and Paldor (1992)]. The boundary condition implies that the sea level is fixed at the transition of the basin to the much deeper and larger sea. Nof and Paldor (1992) derived the governing equation neglecting the dynamics in the cross-basin direction as well as the Coriolis force in a similar fashion to the wind set-up problem presented by Csanady (1982). Here $h(x)=x$ and $\eta=\eta(x)$ represent the undisturbed water of variable-depth and the free-surface elevation respectively. The origin of the coordinate system is chosen to be at the initial shoreline and x increases seaward. Dimensionless variables are introduced through

$$x=\frac{\widetilde{x}}{\widetilde{L}},\quad h=\frac{\widetilde{h}}{\widetilde{D}},\quad \eta=\frac{\widetilde{\eta}}{\widetilde{D}},\quad \gamma=\frac{\widetilde{L}\,\widetilde{\tau}_x}{\widetilde{D}^2\widetilde{g}\,\widetilde{\rho}_w}.$$

Here the characteristic length and depth scales are the basin length $\widetilde{L}$ and the transition depth $\widetilde{D}$ respectively. γ is a parameter determined by the geometry of the basin and the wind stress. The dimensional quantities $\widetilde{\tau}_x$, $\widetilde{\rho}_w$, and $\widetilde{g}$ are the stress component induced by the wind in the x-direction, the density of the water, and the gravitational acceleration respectively.

Nof and Paldor (1992) inverted the governing equation (Eq. 1) for $dx/d\eta$, thus transforming it into a linear equation in x. The resultant linear equation under the boundary condition $\eta(x=1)=0$ was solved, and the initial sea surface height $\eta(x)$ was determined implicitly [Nof and Paldor (1992)]. Their solution can be rearranged into the following form

$$x = -\gamma - \eta(x) + (1+\gamma)\exp\left(\frac{\eta(x)}{\gamma}\right), \tag{2}$$

using dimensionless variables and considering our coordinate system. It is important to note that this implicit solution Eq. 2 requires nonlinear iterations to obtain $\eta(x)$. In addition, the shoreline wind set-down position can not be directly determined using Eq. 2.

Once the wind calms down, since there is no corresponding wind stress, the water accelerates under the pressure gradient. The resulting problem is called the wind set-down relaxation. The NSW equations

$$u_t + uu_x + \eta_x = 0, \tag{3a}$$

$$[u(x+\eta)]_x + \eta_t = 0, \tag{3b}$$

can be used to describe the dynamics of the subsequent water motion as suggested by Gelb, Gottlieb, and Paldor (1997). Additional nondimensionalizations

$$t = \frac{\widetilde{t}}{\widetilde{L}/\sqrt{\widetilde{g}\,\widetilde{D}}}, \quad u = \frac{\widetilde{u}}{\sqrt{\widetilde{g}\,\widetilde{D}}},$$

are introduced for time and velocity respectively. Gelb, Gottlieb, and Paldor (1997) presented numerical solution to the wind set-down relaxation problem using Chebyshev and MacCormack numerical schemes with the initial condition taken from the implicit-analytical solution Eq. 2 of the steady-state wind set-down problem given by Nof and Paldor (1992).

We want to proceed with the analytical solution of the wind set-down relaxation problem. However, nonexistence of an explicit-analytical solution for the wind set-down problem prevents proceeding with the analytical solution. Therefore we will attempt differently to obtain an explicit-transform analytical solution to allow further analysis of this problem.

We use the hodograph-type transformation for the spatial variable

$$x = \sigma^2 - \eta, \tag{4}$$

as suggested by Carrier, Wu, and Yeh (2003). Then, the governing equation (Eq. 1) takes the form:

$$(\sigma^2+\gamma)\eta_\sigma - 2\gamma\sigma = 0, \tag{5}$$

for $\eta(\sigma)$ in the transform σ-space and the boundary condition is translated into $\eta(\sigma=1)=0$. The transform governing equation (Eq. 5) has the following exact solution with the prescribed boundary condition:

$$\eta(\sigma) = \gamma \ln\left(\frac{\sigma^2+\gamma}{1+\gamma}\right). \tag{6}$$

This is an explicit-transform solution for the steady-state problem in terms of the transform variable σ. Once the solution is obtained in the transform σ-space, it is straightforward to obtain the corresponding solution in the physical x-space using the combination of Eq. 6 and Eq. 4. The explicit solution (Eq. 6) can be evaluated for a specific σ to find $\eta(\sigma)$ and resultant $\eta(\sigma)$ together with σ gives the corresponding x through Eq. 4. One example of such a solution is presented in Fig. 2. Even though equal increments are chosen for σ in the transform space, conversion to the physical space generates unequal increments for x (as shown in Fig. 2) because of the nonlinear transformation. Note also that not only the solution of the wind set-down problem does not require nonlinear iterations to obtain η, unlike Eq. 2, but also the shoreline position –wind set-down position– is now defined at $\sigma=0$, i.e., $\eta(\sigma=0)=\gamma\ln(\gamma/(1+\gamma))$. Note that Eq. 6 can be converted into Eq. 2 given by Nof and Paldor (1992) with short algebra, using Eq. 4.

Once we obtain an explicit-transform analytical solution for the wind set-down problem, we now propose to solve the NSW equations as an IBVP in order to obtain an analytical solution for the relaxation problem. The transformation for the spatial variable x given in Eq. 4 is complemented with the transformation for the temporal variable

$$t = \lambda + u, \tag{7}$$

again as in Carrier, Wu, and Yeh (2003) in order to proceed with the IBVP solution of the NSW

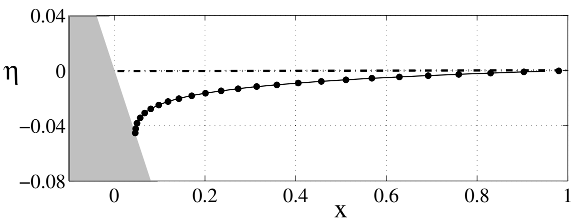

Figure 2: The steady-state wind set-down solution for $\gamma = 0.01$. Circles and solid line represent the explicit-transform analytical solution Eq. 6 together with Eq. 4 and the implicit-analytical solution Eq. 2 of Nof and Paldor (1992) respectively.

equations. Under the hodograph transformations Eq. 4 and Eq. 7, the NSW equations Eq. 3a and Eq. 3b are transformed into

$$\left(\sigma^2 u\right)_\sigma + 2\sigma\phi_\lambda = 0, \tag{8a}$$

$$u_\lambda + \frac{1}{2\sigma}\phi_\sigma = 0. \tag{8b}$$

Defining a potential function,

$$\phi(\sigma,\lambda) = \eta(\sigma,\lambda) + \frac{1}{2}\left[u(\sigma,\lambda)\right]^2, \tag{9}$$

Eq. 8a and Eq. 8b can further be reduced into a single second-order linear PDE for $\phi(\sigma,\lambda)$, eliminating u:

$$4\sigma\phi_{\lambda\lambda} - (\sigma\phi_\sigma)_\sigma = 0. \tag{10}$$

The nonlinear hodograph-type transformation Eq. 4 and Eq. 7 not only reduces the NSW equations into a single second-order linear PDE, but also the singularity of the moving shoreline is avoided. Moreover, the moving shoreline position where the sea surface intersects the sloping bottom is now fixed at the point $\sigma = 0$ in the transform (σ, λ)-space. Recently, Carrier and Yeh (2005) considered tsunami source with finite crest length and developed analytical solution for its evolution over a constant depth based on linear shallow-water wave theory. Carrier and Yeh (2005) were able to introduce a convenient change of variables which converts the linear shallow-water wave equation into a similar equation as Eq. 10 and they solved with the Fourier-Bessel transform as in Carrier, Wu, and Yeh (2003).

We will now attempt to solve Eq. 10 as IBVP. The initial conditions are defined from the steady-state wind set-down solution Eq. 6, i.e., initial wave profile $\eta(\sigma,\lambda=0)$ given in Eq. 6 with zero initial velocity $u(\sigma,\lambda=0)=0$. These conditions yield

$$\phi_\lambda(\sigma,\lambda=0) = 0, \tag{11a}$$

$$\phi(\sigma,\lambda=0) = \eta(\sigma,\lambda=0), \tag{11b}$$

for the potential function ϕ through Eq. 8a and the definition of the potential (Eq. 9) respectively. The importance of the requirement of an explicit-transform analytical solution for $\eta(\sigma,\lambda=0)$ for the steady-state wind set-down problem is now clear. Without the explicit analytical solution for wind set-down, it is not possible to proceed with the analytical solution of the NSW equations. Note also that Eq. 7 requires $\lambda = 0$ for $t = 0$ since the initial velocity is zero. In addition to these initial conditions, a bounded solution at the shoreline and undisturbed sea surface at the toe of the slope (at the basin mouth) $\eta(x=1,t)=0$ [Gelb, Gottlieb, and Paldor (1997)] require $\phi(\sigma,\lambda)$ to be finite at the shoreline and $\eta(\sigma=1,\lambda)=0$ in the transform space respectively.

After defining proper initial and boundary conditions in the transform (σ,λ)-space, the solution for Eq. 10 is now a classical separation of variables problem. $\phi(\sigma,\lambda) = F(\sigma)G(\lambda)$ gives

$$4\frac{G_{\lambda\lambda}}{G} = \frac{F_{\sigma\sigma}}{F} + \frac{1}{\sigma}\frac{F_\sigma}{F} = -4\alpha^2,$$

with a real constant α. The ordinary differential equation for $G(\lambda)$ is $G_{\lambda\lambda} + \alpha^2 G = 0$ with the general solution $G(\lambda) = c_1\cos(\alpha\lambda) + c_2\sin(\alpha\lambda)$. Here c_1 and c_2 are the arbitrary constants and application of the initial condition Eq. 11a leads $G_\lambda(0) = 0$. Therefore $c_2 = 0$ implying

$$G(\lambda) = c_1\cos(\alpha\lambda). \tag{12}$$

The differential equation $\sigma^2 F_{\sigma\sigma} + \sigma F_\sigma + 4\alpha^2\sigma^2 F = 0$ for $F(\sigma)$ is the Bessel's equation of order zero and it has the general solution $F(\sigma) = c_3 J_0(2\alpha\sigma) + c_4 Y_0(2\alpha\sigma)$. Boundness at the shoreline requires $c_4 = 0$. Further, the condition $\eta(\sigma = 1, \lambda) = 0$ applied at $\lambda = 0$ implies $\phi(\sigma = 1, \lambda = 0) = 0$ through the definition of the potential function (Eq. 9) since $u(\sigma = 1, \lambda = 0) = 0$ as explained previously. The eigenvalues of the problem are determined from the condition $\phi(\sigma = 1, \lambda = 0) = 0$ as $\alpha_n = z_n/2$ $(n = 1, 2, 3, \ldots)$ where the constants z_n are the zeros of the Bessel function of order zero, $J_0(z)$. So the solution for the differential equation for $F(\sigma)$ is given as:

$$F(\sigma) = c_3 J_0(z_n \sigma). \tag{13}$$

Now, we can construct the series solution of the problem using Eq. 12 and Eq. 13 as

$$\phi(\sigma, \lambda) = \sum_{n=1}^{\infty} K_n J_0(z_n \sigma) \cos\left(\frac{1}{2} z_n \lambda\right). \tag{14}$$

The nonhomogeneous initial condition Eq. 11b will be imposed last as usual for separation of variables:

$$\phi(\sigma, 0) = \eta(\sigma, 0) = \gamma \ln\left(\frac{\sigma^2 + \gamma}{1 + \gamma}\right).$$

Therefore, the Bessel coefficients K_n are determined through

$$\sum_{n=1}^{\infty} K_n J_0(z_n \sigma) = \gamma \ln\left(\frac{\sigma^2 + \gamma}{1 + \gamma}\right).$$

Multiplication of both sides with $\sigma J_0(z_m \sigma)$ and integrating [Watson (1944)] we get

$$K_n = \frac{2\gamma}{[J_1(z_n)]^2} \int_0^1 \omega \ln\left(\frac{\omega^2 + \gamma}{1 + \gamma}\right) J_0(z_n \omega) d\omega. \tag{15}$$

Finally, insertion of Eq. 15 into Eq. 14 gives the complete analytical solution for the wind set-down relaxation problem;

$$\phi(\sigma, \lambda) = \sum_{n=1}^{\infty} \frac{2\gamma}{[J_1(z_n)]^2} J_0(z_n \sigma) \cos\left(\frac{1}{2} z_n \lambda\right) \times \int_0^1 \omega \ln\left(\frac{\omega^2 + \gamma}{1 + \gamma}\right) J_0(z_n \omega) d\omega. \tag{16}$$

After obtaining the solution, we can now resolve the whole flow-field, especially the physical characteristics of the shoreline motion. Combining Eq. 8b and Eq. 16 we compute $u(\sigma, \lambda)$:

$$u(\sigma, \lambda) = \frac{1}{\sigma} \sum_{n=1}^{\infty} \frac{2\gamma}{[J_1(z_n)]^2} J_1(z_n \sigma) \sin\left(\frac{1}{2} z_n \lambda\right) \times \int_0^1 \omega \ln\left(\frac{\omega^2 + \gamma}{1 + \gamma}\right) J_0(z_n \omega) d\omega. \tag{17}$$

Once $u(\sigma, \lambda)$ is known, $\eta(\sigma, \lambda)$ can be evaluated through the definition of the potential function (Eq. 9) as $\eta(\sigma, \lambda) = \phi(\sigma, \lambda) - \frac{1}{2}[u(\sigma, \lambda)]^2$. Back transformation to the physical (x,t)-space is possible using Eq. 4 and Eq. 7. Since $\sigma = 0$ at the shoreline, shoreline velocity u_s can be evaluated through

$$u_s(\lambda) = \sum_{n=1}^{\infty} \frac{\gamma z_n}{[J_1(z_n)]^2} \sin(\frac{1}{2} z_n \lambda) \times \int_0^1 \omega \ln\left(\frac{\omega^2 + \gamma}{1 + \gamma}\right) J_0(z_n \omega) d\omega,$$

from Eq. 17 considering $\lim_{\sigma \to 0} [J_1(z_n \sigma)/\sigma] = \frac{1}{2} z_n$. The shoreline position x_s is now given as:

$$x_s(\lambda) = -\eta_s(\lambda) = \frac{1}{2} [u_s(\lambda)]^2 - \phi(\sigma = 0, \lambda),$$

at the respective time

$$t(\sigma = 0, \lambda) = \lambda + u_s(\lambda).$$

3 Results and Discussions

An interval of $0.01 \leq \gamma \leq 0.02$ was suggested in Gelb, Gottlieb, and Paldor (1997) for the nondimensional parameter γ. We use $\gamma = 0.01$, to evaluate some physical properties of the wind set-down relaxation problem, but also to compare the analytical solution with the existing numerical solution of Gelb, Gottlieb, and Paldor (1997). Fig. 3(a) compares the analytical solution for the shoreline position with the numerical results. The analytical solution agrees with the existing numerical solution. Further, we evaluated the power spectral density of the shoreline motion and presented in Fig. 3(b). The power spectral density is defined by $d_k = |f_k|^2$ where f_k is the discrete Fourier transform of the shoreline wave height η_s

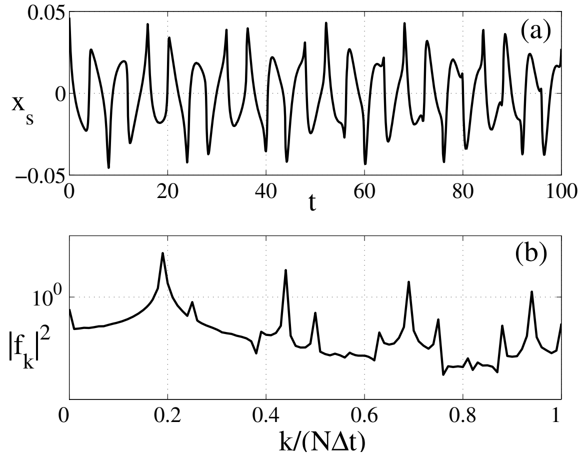

Figure 3: (a) Temporal variation and (b) power spectral density of the shoreline position x_s for $\gamma = 0.01$. Note that $x_s = -\eta_s$. Figures 3(a) and 3(b) correspond to Fig. 3 and Fig. 4 of Gelb, Gottlieb, and Paldor (1997) respectively.

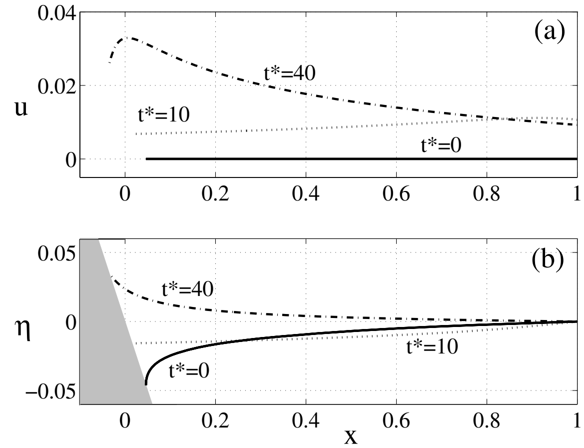

Figure 4: Spatial and temporal variation of (a) the velocity and (b) the free-surface elevation at $t^* = 0$ (solid line), 10 (dotted line), and 40 (dash-dotted line) for $\gamma = 0.01$. Figures 4(a) and 4(b) correspond to Fig. 8(a) and Fig. 8(b) for $t^* = 10$ and Fig. 9(a) and Fig. 9(b) for $t^* = 40$ of Gelb, Gottlieb, and Paldor (1997) respectively.

[Gelb, Gottlieb, and Paldor (1997)]:

$$f_k = \sum_{j=1}^{N-1} (\eta_{s,j} - \eta_{s,average}) e^{-i2\pi jk/N}; \quad k = 0,1,\ldots,N-1.$$

The corresponding nondimensional frequency is $k/N\Delta t$ with $\Delta t = T/N$ (T is the total time and $N = 2^m$ with positive integer m). We again obtain agreement with Gelb, Gottlieb, and Paldor (1997). We also evaluated the spatial variation of the depth-averaged velocity and the surface height at some specific times t^* using the Newton-Raphson iterations, as proposed by Synolakis (1986, 1987) and employed recently by Kânoğlu (2004). We determined the value λ^* for which $t(\sigma,\lambda^*) - t^* = 0$ from the algorithm

$$\lambda_{i+1} = \lambda_i - \frac{t(\sigma,\lambda_i) - t^*}{1 + u_\lambda(\sigma,\lambda_i)},$$

for a given σ. The spatial and temporal variations of the velocity and the free-surface elevation are presented in Fig. 4(a) and Fig. 4(b) respectively. The results compare well with the numerical solutions of Gelb, Gottlieb, and Paldor (1997) except when $t^* = 10$. We observed that velocities close to the shoreline exhibit difficulties for both Chebyshev and MacCormack methods for $t^* = 10$ in the numerical solution of Gelb, Gottlieb, and Paldor (1997).

4 Conclusions

We transformed the governing equation for the wind set-down problem using the hodograph-type transformation for the spatial variable and obtained explicit-transform analytical solution of the problem. This explicit-transform solution is crucial to proceed with the analytical solution of the relaxation problem since the existing solution was an implicit one. Then we used the complete hodograph-type transformation for the spatial and temporal variables to reduce the NSW equations into a single second-order linear PDE. We provided IBVP solution to this single second-order linear PDE, rather than the existing IVP and BVP solutions, to obtain the wind set-down relaxation solution. We imposed the explicit-transform wind set-down solution that we have developed as an initial condition to this reduced equation with the other proper initial and boundary conditions to solve the IBVP. We evaluated certain physical flow-field properties.

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