 CMES, vol.21, no.3, pp.239-254, 2007

An Inverse Problem in Estimating Simultaneously the Time-Dependent Applied Force and Moment of an Euler-Bernoulli Beam

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Abstract: An inverse forced vibration problem, based on the Conjugate Gradient Method (CGM), (or the iterative regularization method), is examined in this study to estimate simultaneously the unknown time-dependent applied force and moment for an Euler-Bernoulli beam by utilizing the simulated beam displacement measurements. The accuracy of this inverse problem is examined by using the simulated exact and inexact displacement measurements. The numerical experiments are performed to test the validity of the present algorithm by using different types of applied force and moment, sensor locations and measurement errors. Results show that excellent estimations on the applied force and moment can be obtained with any arbitrary initial guesses.

Nomenclature

E	Young's modulus
$F(t)$	applied force
I	moment of inertia
J	functional defined by equation (3)
J'_i	gradient of functional defined by equations (16) and (17)
$M(t)$	applied moment
P_i	directions of descent defined by equation (5)
t	time
x	axial coordinate
$y(x,t)$	estimated displacement
$Y(x_i,t)$	measured displacement

Greeks

β_i	search step size defined by equation (11)
γ_i	conjugate coefficient defined by equation (6)
$\lambda_i(x,t)$	adjoint functions defined by equation (14)
$\Delta y_i(x,t)$	sensitivity functions defined by equations (7) and (8)
ε	convergence criterion
ρ	mass density
ω	random number
σ	standard deviation of measurement errors

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1 Introduction

Beams are very important elements in civil, mechanical, and aeronautical engineering. For instance, bridges, cutting tools, robot arms, helicopter rotor blades and spacecraft antennae are all examples of structures that may be modeled as an Euler-Bernoulli beam. The vibration problem of the Euler-Bernoulli beam is very well developed and explored in details in many engineering applications.

The direct solutions for an Euler-Bernoulli beam problem are concerned in determining the beam displacements at any positions and times when the initial and boundary conditions, beam parameters, applied force and moment are all specified. For instance, Andreaus, Batra and Porfiri (2005) applied the Meshless Local Petrov-Galerkin (MLPG) method to examine the vibrations of cracked Euler-Bernoulli beams. Vinod, Gopalakrishnan and Ganguli (2006) applied the spectral finite element formulation for a rotating uniform Euler-Bernoulli beam subjected to small duration impact.

In contrast, the inverse forced vibration problem for an Euler-Bernoulli beam that is going to be discussed here involves the simultaneous determination of the time-dependent applied force and moment from the knowledge of the displacement measurements at the some specified positions x and times t. The inverse problem tends to be ill-posed, in the sense that small variations in the measured data can excite large excursion in the estimated values. For this reason a suitable algorithm should be chosen to avoid the ill-posed phenomena.

For the inverse vibration problems, the textbook by Gladwell (1986) contains a general presentation of the inverse problem for undamped vibrating system. Desanghere and Snoeys (1985) applied a condition number in forced identification problems and observed it is a reliable indicator for ill-conditioned matrix. Stevens (1987) provided an overview in identifying the forces for the case of linear vibratory system. Starek and Inman (1991,1992,1995) have analyzed an inverse eigenvalue problem in estimating the coefficient matrices. Bateman, Carne and Gregory (1991) presented two force reconstruction techniques, i.e., the sum of weighted acceleration and the deconvolution techniques to evaluate the impact test. Michaels and Pao (1985) have shown an iterative method of deconvolution, which determined the inverse source problem for an oblique force on an elastic plate. Ma, Tuan, Lin and Liu (1998) utilized the Kalman filter with a recursive estimator to determine the impulsive loads in a single-degree-of-freedom (SDOF) as well as for a multiple-degree-of-freedom (MDOF) lumped-mass systems. Huang applied the CGM in estimating the unknown time-dependent external force with time-dependent system parameters for a single-degree-of-freedom (SDOF) problem [Huang (2001)] and for a multiple-degree-of-freedom vibration system [Huang (2005)] and obtained good estimations. The above inverse problems belong to the general vibration system.

Regarding to the study of inverse problems for the Euler-Bernoulli beam, Chan and Ashebo (2006) used the Singular Value Decomposition (SVD) method in an inverse problem in identifying the moving forces on continuous bridges. Lee (2006) applied the CGM to solve the inverse problem in estimating the shear force between the tapered probe and sample during the scanning process of scanning near-field optical microscope (SNOM). Chang and Guo (2007) considered an inverse problem for the Euler–Bernoulli beam equation, with one end clamped and with torque input at the other end, the objective is to estimate the beam spatial varying coefficients by using the measurements for the displacement and the angle velocity at the non-clamped end of the beam.

Based on the above reviews, the discussions of the inverse forced vibration problems in estimating simultaneously both the time-dependent applied force and moment at the free end, using the Conjugate Gradient Method (CGM), have never been examined in the literature. For this reason, the purpose of this study is to establish an algorithm based on the CGM to estimate simultaneously the unknown time-dependent applied force and moment in an inverse forced vibration problem for an Euler-Bernoulli beam.

The technique of the CGM has been applied to many different inverse and optimal control problems [Chao, Chen and Lin (2001), Huang and Li (2006), Huang and Lo (2006), Huang and Wu (2006), Huang, Jan, Li and Shih (2007), Huang and Chen (2007)] and has been proven itself to be a very powerful algorithm. The CGM is also called an iterative regularization method, which means the regularization procedure is performed during the iterative processes. The CGM derives basis from the perturbational principles [Alifanov (1994)] and transforms the inverse problem into the solution of three problems, namely, the direct problem, the sensitivity problem and the adjoint problem, which will be discussed in detail in the next few sections.

Finally the inverse solutions with three different types of applied force and moment, sensor arrangements and measurement errors will be considered to show the validity of using the CGM in the present inverse forced vibration problem.

2 The Direct Problem

To illustrate the methodology for developing expressions for use in determining simultaneously the unknown time-dependent external force and moment for an Euler-Bernoulli beam in a forced vibration problem, the following problem is considered.

The initial displacement and velocity conditions of the system are $\tilde{y}(\tilde{x},0) = \frac{\partial \tilde{y}(\tilde{x},0)}{\partial \tilde{t}} = 0$, at $\tilde{t} = 0$. When $\tilde{t} > 0$, the boundary condition at $\tilde{x} = 0$ is clamped and the boundary condition at $\tilde{x} = \tilde{L}$ is subjected to the unknown time-dependent force $\tilde{F}(\tilde{t})$ and moment $\tilde{M}(\tilde{t})$, respectively.

The mathematical formulation of this forced vibration problem of an Euler-Bernoulli beam is given by:

$$\tilde{E}\tilde{I}\frac{\partial^4 \tilde{y}(\tilde{x},\tilde{t})}{\partial \tilde{x}^4} + \tilde{\rho}\tilde{A}\frac{\partial^2 \tilde{y}(\tilde{x},\tilde{t})}{\partial \tilde{t}^2} = 0 \qquad (1a)$$

Equation (1a) is subjected to the following boundary and initial conditions

$$\tilde{y}(0,\tilde{t}) = \frac{\partial \tilde{y}(0,\tilde{t})}{\partial \tilde{x}} = 0; \qquad \text{at } \tilde{x} = 0,\ \tilde{t} > 0 \qquad (1b,c)$$

$$\frac{\partial^2 \tilde{y}(\tilde{L},\tilde{t})}{\partial \tilde{x}^2} = \tilde{M}(t); \qquad \text{at } \tilde{x} = \tilde{L},\ \tilde{t} > 0 \qquad (1d)$$

$$\frac{\partial^3 \tilde{y}(\tilde{L},\tilde{t})}{\partial \tilde{x}^3} = \tilde{F}(t); \qquad \text{at } \tilde{x} = \tilde{L},\ \tilde{t} > 0 \qquad (1e)$$

$$\tilde{y}(\tilde{x},0) = \frac{\partial \tilde{y}(\tilde{x},0)}{\partial \tilde{t}} = 0; \qquad \text{at } \tilde{t} = 0 \qquad (1f,g)$$

here $\tilde{x}$ and $\tilde{t}$ are the axial and time coordinates, respectively, and $\tilde{L}$ is the length of the beam, $\tilde{y}$ is the deflection, $\tilde{E}$ is Young's modulus, $\tilde{A}$ is the cross-sectional area of beam, $\tilde{I}$ is the moment of inertia of $\tilde{A}$, $\tilde{\rho}$ is the mass density of material, $\tilde{F}(t)$ and $\tilde{M}(t)$ indicate the unknown applied force and moment, respectively.

If the following dimensionless quantities are defined

$$x = \frac{\tilde{x}}{\tilde{L}}; \qquad y = \frac{\tilde{y}}{\tilde{L}}; \qquad t = \sqrt{\frac{\tilde{E}\tilde{I}}{\tilde{\rho}\tilde{A}\tilde{L}^4}}\tilde{t};$$

$$F(t) = \frac{\tilde{L}^2}{\tilde{E}\tilde{I}}\tilde{F}(\tilde{t}); \quad M(t) = \frac{\tilde{L}}{\tilde{E}\tilde{I}}\tilde{M}(\tilde{t})$$

The dimensionless formulation of the Euler-Bernoulli beam can be expressed as:

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \qquad (2a)$$

subject to the following boundary and initial conditions

$$y(0,t) = \frac{\partial y(0,t)}{\partial x} = 0; \qquad \text{at } x = 0,\ t > 0 \qquad (2b,c)$$

$$\frac{\partial^2 y(1,t)}{\partial x^2} = M(t); \qquad \text{at } x = 1,\ t > 0 \qquad (2d)$$

$$\frac{\partial^3 y(1,t)}{\partial x^3} = F(t); \qquad \text{at } x = 1,\ t > 0 \qquad (2e)$$

$$y(x,0) = \frac{\partial y(x,0)}{\partial t} = 0; \qquad \text{at } t = 0 \qquad (2f,g)$$

The system under consideration here is shown in Figure 1. The direct problem considered here is concerned with the determination of the displacements of the beam, $y(x,t)$, when the initial and boundary conditions, and the applied time-dependent force $F(t)$ and moment $M(t)$ are all given. Here the finite difference method is used to solve equations (2).

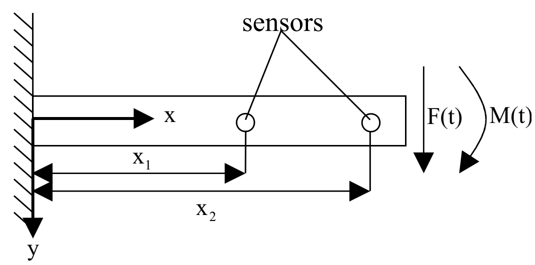


Figure 1: A clamped Euler-Bernoulli beam with unknown $F(t)$ and $M(t)$ applied at $x = 1$

3 The Inverse Problem

For the inverse vibration problem consider here, the time-dependent applied force $F(t)$ and moment $M(t)$ are regarded as being unknown, but everything else in equation (2) are known. In addition, displacement measurement with time at two appropriate locations x_1 and x_2 are considered available.

Let the measured beam displacements with time at x_1 and x_2 be denoted by $Y(x_1,t)$ and $Y(x_2,t)$, here $t = 0$ to t_f and t_f represents the final time of measurements. The present inverse vibration problem can be stated as follows: by utilizing the above mentioned measured beam displacements $Y(x_1,t)$ and $Y(x_2,t)$, the unknown time-dependent applied force $F(t)$ and moment $M(t)$ are to be estimated simultaneously.

In this work no real measured displacements were utilized, instead, the simulated values of $Y(x_1,t)$ and $Y(x_2,t)$ are generated by using the exact applied force $F(t)$ and moment $M(t)$, then try to retrieve the time-dependent applied force $F(t)$ and moment $M(t)$ by using $Y(x_1,t)$ and $Y(x_2,t)$ and the technique of the Conjugate Gradient Method (CGM).

The solution of this inverse vibration problem is to be obtained in such a way that the following functional is minimized:

$$J[F(t),M(t)] = \int_{t=0}^{t_f} \left\{ [y(x_1,t)-Y(x_1,t)]^2 + [y(x_2,t)-Y(x_2,t)]^2 \right\} \mathrm{d}t \quad (3)$$

here, $y(x_1,t)$ and $y(x_2,t)$ are the estimated or computed beam displacements at x_1 and x_2 with time t. These quantities are determined from the solution of the direct problem given previously by using an estimated force and moment for the exact values.

4 Conjugate Gradient Method for Minimization

The following iterative process based on the conjugate gradient method [Alifanov (1994)] is now used for the estimation of time-dependent applied force $F(t)$ and moment $M(t)$ by minimizing the functional $J[F(t),M(t)]$

$$F^{n+1}(t) = F^n(t) - \beta_1^n P_1^n(t); \quad \text{for } n = 0,1,2,\ldots \quad (4a)$$

$$M^{n+1}(t) = M^n(t) - \beta_2^n P_2^n(t); \quad \text{for } n = 0,1,2,\ldots \quad (4b)$$

where β_1^n and β_2^n are the search step sizes in going from iteration n to iteration $n+1$, and $P_1^n(t)$ and $P_2^n(t)$ are the directions of descent (i.e. search directions) given by

$$P_1^n(t) = J'^n_1(t) + \gamma_1^n P_1^{n-1}(t) \quad (5a)$$

$$P_2^n(t) = J'^n_2(t) + \gamma_2^n P_2^{n-1}(t) \quad (5b)$$

which are a conjugation of the gradient directions $J'^n_1(t)$ and $J'^n_2(t)$ at iteration n and the directions of descent $P_1^{n-1}(t)$ and $P_2^{n-1}(t)$ at iteration $n-1$.

The conjugate coefficients are determined from

$$\gamma_1^n = \frac{\int_{t=0}^{t_f} (J'^n_1)^2 \mathrm{d}t}{\int_{t=0}^{t_f} (J'^{n-1}_1)^2 \mathrm{d}t}; \quad \text{with } \gamma_1^0 = 0 \quad (6a)$$

$$\gamma_2^n = \frac{\int_{t=0}^{t_f} (J'^n_2)^2 \mathrm{d}t}{\int_{t=0}^{t_f} (J'^{n-1}_2)^2 \mathrm{d}t}; \quad \text{with } \gamma_2^0 = 0 \quad (6b)$$

We note that when $\gamma_1^n = \gamma_2^n = 0$ for any n, in equations (6), the directions of descent $P_1^n(t)$ and $P_2^n(t)$ become the gradient directions, i.e. the "Steepest descent method, SDM" is obtained. The convergence of CGM in minimizing the functional J is guaranteed in Lasdon, Mitter and Warren (1967).

To perform the iterations according to equation (4), the step sizes β_1^n and β_2^n and the gradient functions $J'^n_1(t)$ and $J'^n_2(t)$ need to be calculated. In order to develop expressions for the determination of these quantities, two "sensitivity problems" and an "adjoint problem" need to be constructed as described below.

4.1 Sensitivity Problems and Search Step Sizes

The present inverse problem involves two unknown functions, $F(t)$ and $M(t)$, in order to derive the sensitivity problem for each unknown function, we should perturb the unknown function one at a time.

Firstly, it is assumed that when F(t) undergoes a variation $\Delta F(t)$, y(x,t) are perturbed by $\Delta y_1(x,t)$. Then replacing in the direct problem $F(t)$ by $F(t)+\Delta F(t)$, $y(x,t)$ by $y(x,t)+\Delta y_1(x,t)$, subtracting the resulting expressions from the direct problem and neglecting the second-order terms, the following sensitivity problem for the sensitivity function Δy_1 is obtained.

$$\frac{\partial^4 \Delta y_1(x,t)}{\partial x^4} + \frac{\partial^2 \Delta y_1(x,t)}{\partial t^2} = 0 \quad (7a)$$

subject to the following boundary and initial conditions

$$\Delta y_1(0,t) = \frac{\partial \Delta y_1(0,t)}{\partial x} = 0; \qquad \text{at } x=0,\ t>0 \qquad \text{(7b,c)}$$

$$\frac{\partial^2 \Delta y_1(1,t)}{\partial x^2} = 0; \qquad \text{at } x=1,\ t>0 \qquad \text{(7d)}$$

$$\frac{\partial^3 \Delta y_1(1,t)}{\partial x^3} = \Delta F(t); \qquad \text{at } x=1,\ t>0 \qquad \text{(7e)}$$

$$\Delta y_1(x,0) = \frac{\partial \Delta y_1(x,0)}{\partial t} = 0; \qquad \text{at } t=0 \qquad \text{(7f,g)}$$

Similarly, by perturbing $M(t)$ with $\Delta M(t)$ and $y(x,t)$ with $\Delta y_2(x,t)$, the second sensitivity problem can be obtained as

$$\frac{\partial^4 \Delta y_2(x,t)}{\partial x^4} + \frac{\partial^2 \Delta y_2(x,t)}{\partial t^2} = 0 \qquad \text{(8a)}$$

subject to the following boundary and initial conditions

$$\Delta y_2(0,t) = \frac{\partial \Delta y_2(0,t)}{\partial x} = 0; \qquad \text{at } x=0,\ t>0 \qquad \text{(8b,c)}$$

$$\frac{\partial^2 \Delta y_2(1,t)}{\partial x^2} = \Delta M(t); \qquad \text{at } x=1,\ t>0 \qquad \text{(8d)}$$

$$\frac{\partial^3 \Delta y_2(1,t)}{\partial x^3} = 0; \qquad \text{at } x=1,\ t>0 \qquad \text{(8e)}$$

$$\Delta y_2(x,0) = \frac{\partial \Delta y_2(x,0)}{\partial t} = 0; \qquad \text{at } t=0 \qquad \text{(8f,g)}$$

The technique of finite difference method is used to solve the above two sensitivity problems.

The functional $J(F^{n+1}, M^{n+1})$ at iteration $n+1$ is obtained by rewriting equation (3) as

$$J[F(t),M(t)] = \int_{t=0}^{t_f} [y(x_1,t;F^n-\beta_1^n P_1^n, M^n-\beta_2^n P_2^n) - Y(x_1,t)]^2 \,\mathrm{d}t + \int_{t=0}^{t_f} [y(x_2,t;F^n-\beta_1^n P_1^n, M^n-\beta_2^n P_2^n) - Y(x_2,t)]^2 \,\mathrm{d}t \qquad (9)$$

where we have replaced $F^{n+1}(t)$ and $M^{n+1}(t)$ by the expression given by equation (4). If the estimated displacements $y(x_1,t;F^n-\beta_1^n P_1^n, M^n-\beta_2^n P_2^n)$ and $y(x_2,t;F^n-\beta_1^n P_1^n, M^n-\beta_2^n P_2^n)$ are linearized by a Taylor expansion, equation (9) takes the form

$$J[F(t),M(t)] = \int_{t=0}^{t_f} \Big[y(x_1,t;F^n,M^n) - \beta_1^n \Delta y_1(x_1,t;P_1^n) - \beta_2^n \Delta y_2(x_1,t;P_2^n) - Y(x_1,t)\Big]^2 \mathrm{d}t + \int_{t=0}^{t_f} \Big[y(x_2,t;F^n,M^n) - \beta_1^n \Delta y_1(x_2,t;P_1^n) - \beta_2^n \Delta y_2(x_2,t;P_2^n) - Y(x_2,t)\Big]^2 \mathrm{d}t \qquad (10)$$

where $y(x_1,t;F^n,M^n)$ and $y(x_2,t;F^n,M^n)$ are the solutions of the direct problem by using estimate force and moment for exact values at x_1 and x_2 with time t. The sensitivity functions $\Delta y_1(x_1,t;P_1^n)$, $\Delta y_1(x_2,t;P_1^n)$, $\Delta y_2(x_1,t;P_2^n)$ and $\Delta y_2(x_2,t;P_2^n)$ are taken as the solutions of problems (7) and (8) at x_1 and x_2 with time t by letting $\Delta F(t) = P_1^n(t)$ and $\Delta M(t) = P_2^n(t)$, respectively [Alifanov (1994)].

Equation (10) is differentiated with respect to β_1^n and β_2^n, respectively, and equating them equal to zero to obtain two independent equations. After solving these two equations, the search step sizes β_1^n and β_2^n can be determined as:

$$\beta_1^n = (C_3C_5 - C_2C_4)/(C_3C_3 - C_1C_2) \qquad \text{(11a)}$$

$$\beta_2^n = (C_3C_4 - C_1C_5)/(C_3C_3 - C_1C_2) \qquad \text{(11b)}$$

where

$$C_1 = \int_{t=0}^{t_f} \left[\Delta y_1(x_1,t)^2 + \Delta y_1(x_2,t)^2\right] \mathrm{d}t \qquad \text{(11c)}$$

$$C_2 = \int_{t=0}^{t_f} \left[\Delta y_2(x_1,t)^2 + \Delta y_2(x_2,t)^2\right] \mathrm{d}t \qquad \text{(11d)}$$

$$C_3 = \int_{t=0}^{t_f} \Big[\Delta y_1(x_1,t)\times\Delta y_2(x_1,t) + \Delta y_1(x_2,t)\times\Delta y_2(x_2,t)\Big]\mathrm{d}t \quad (11e)$$

$$C_4 = \int_{t=0}^{t_f} \Big\{[y(x_1,t)-Y(x_1,t)]\Delta y_1(x_1,t) + [y(x_2,t)-Y(x_2,t)]\Delta y_1(x_2,t)\Big\}\mathrm{d}t \quad (11f)$$

$$C_5 = \int_{t=0}^{t_f} \Big\{[y(x_1,t)-Y(x_1,t)]\Delta y_2(x_1,t) + [y(x_2,t)-Y(x_2,t)]\Delta y_2(x_2,t)\Big\}\mathrm{d}t \quad (11g)$$

4.2 Adjoint Problem and Gradient Equation

To obtain the adjoint problem, equation (2) is multiplied by a Lagrange multiplier (or adjoint function) $\lambda_1(x,t)$. The resulting expression is integrated over the correspondent space and time domains, then the result is added to the right hand side of equation (3) to yield the following expression for the functional J[F(t),M(t)]:

$$J[F(t),M(t)] = \int_{t=0}^{t_f} \left\{[y(x_1,t)-Y(x_1,t)]^2+[y(x_2,t)-Y(x_2,t)]^2\right\}\mathrm{d}t + \int_{t=0}^{t_f}\int_{x=0}^{1} \lambda_1(x,t)\left[\frac{\partial^4 y(x,t)}{\partial x^4}+\frac{\partial^2 y(x,t)}{\partial t^2}\right]\mathrm{d}x\mathrm{d}t \quad (12)$$

The variation ΔJ_1 is obtained by perturbing F by $F+\Delta F$ and y by $y+\Delta y_1$ in equation (12), subtracting the original equation (12) from the resulting expression and neglecting the second-order terms. It thus finds

$$\Delta J_1[F(t),M(t)] = \int_{t=0}^{t_f} \Big\{2[y(x_1,t)-Y(x_1,t)]\Delta y_1 + 2[y(x_2,t)-Y(x_2,t)]\Delta y_1\Big\}\mathrm{d}t + \int_{t=0}^{t_f}\int_{x=0}^{1} \lambda_1(x,t)\left[\frac{\partial^4 \Delta y_1(x,t)}{\partial x^4}+\frac{\partial^2 \Delta y_1(x,t)}{\partial t^2}\right]\mathrm{d}x\mathrm{d}t \quad (13a)$$

which can be rearranged as

$$\Delta J_1[F(t),M(t)] = \int_{t=0}^{t_f}\int_{x=0}^{1} \Big\{2[y(x,t)-Y(x,t)]\Delta y_1 \cdot [\delta(x-x_1)+\delta(x-x_2)]\Big\}\mathrm{d}x\mathrm{d}t + \int_{t=0}^{t_f}\int_{x=0}^{1} \lambda_1(x,t)\left[\frac{\partial^4 \Delta y_1(x,t)}{\partial x^4}+\frac{\partial^2 \Delta y_1(x,t)}{\partial t^2}\right]\mathrm{d}x\mathrm{d}t \quad (13b)$$

where $\delta(\cdot)$ is the Dirac delta function.

In equation (13b), the integrands containing adjoint function $\lambda_1(x,t)$ on the right hand side are integrated by parts; the initial and boundary conditions of the sensitivity problem are utilized. The vanishing of the integrands leads to the following adjoint problem for the determination of $\lambda_1(x,t)$:

$$\frac{\partial^4 \lambda_1(x,t)}{\partial x^4}+\frac{\partial^2 \lambda_1(x,t)}{\partial t^2} + 2[y(x,t)-Y(x,t)][\delta(x-x_1)+\delta(x-x_2)] = 0 \quad (14a)$$

subject to the following boundary and final time conditions

$$\lambda_1(0,t) = \frac{\partial \lambda_1(0,t)}{\partial x} = 0; \qquad \text{at } x=0,\ t>0 \quad (14b,c)$$

$$\frac{\partial^2 \lambda_1(1,t)}{\partial x^2} = \frac{\partial^3 \lambda_1(1,t)}{\partial x^3} = 0; \qquad \text{at } x=1,\ t>0 \quad (14d,e)$$

$$\lambda_1(x,t_f) = \frac{\partial \lambda_1(x,t_f)}{\partial t} = 0; \qquad \text{at } t=t_f \quad (14f,g)$$

The adjoint problem differs from the standard initial value problem in that the final time conditions at time $t = t_f$ are specified instead of the traditional initial conditions. However, the above adjoint problem can be transformed into an initial value problem by the transformation of the time variable as $\tau = t_f - t$. The standard technique of finite difference method can then be used to solve the above adjoint problem.

Finally, the following integral term is left

$$\Delta J = \int_{t=0}^{t_f} -\lambda_1(1,t)\Delta F(t)\mathrm{d}t \tag{15a}$$

From definition [Alifanov (1994)], the functional increment can be presented as

$$\Delta J = \int_{t=0}^{t_f} J'\Delta F(t)\mathrm{d}t \tag{15b}$$

A comparison of equations (15a) and (15b) leads to the following expression for the gradient of functional J':

$$J'_1[F(t)] = -\lambda_1(1,t) \tag{16}$$

Similarly, to derive the adjoint problem for the case when perturbing M(t), equation (2) is multiplied by a second Lagrange multiplier (or adjoint function) $\lambda_2(x,t)$. By following the same procedure as described previously, we finally find that the solutions for adjoint equation of $\lambda_2(x,t)$ are identical to that for $\lambda_1(x,t)$. This implies that the adjoint equations need to be solved only once since $\lambda_1(x,t) = \lambda_2(x,t)$. Finally the gradient equation for $M(t)$ can be obtained as

$$J'_2[M(t)] = -\frac{\partial \lambda_1(1,t)}{\partial x} \tag{17}$$

4.3 Stopping Criterion

If the problem contains no measurement errors, the traditional check condition is specified as

$$J\left[F^{n+1}(t), M^{n+1}(t)\right] < \varepsilon \tag{18}$$

where ε is a small-specified number. However, the measured beam displacements may contain measurement errors. Therefore, it is not expected that the functional equation (3) will be equal to zero at the final iteration step. Following the experiences of the author [Alifanov (1994)], the discrepancy principle is adopted as the stopping criterion, i.e. it is assumed that the residuals for the displacements may be approximated by

$$y(x_1,t) - Y(x_1,t) = y(x_2,t) - Y(x_2,t) \approx \sigma \tag{19}$$

where σ is the standard deviation of the displacement measurements, which is assumed to be a constant. Substituting equation (19) into equation (3), the following expression is obtained for stopping criterion ε:

$$\varepsilon = 2\sigma^2 t_f \tag{20}$$

Finally the stopping criterion is given by equation (18) with ε determined from equation (20).

5 Computational Procedure

The computational procedure for the solution of this inverse forced vibration problem in determining simultaneously $F(t)$ and $M(t)$ using CGM can be summarized as follows:

Suppose $F^n(t)$ and $M^n(t)$ are available at iteration n.

Step 1. Solve the direct problems given by equations (2) for $y(x,t)$.

Step 2. Examine the stopping criterion given by equation (20). Continue the iteration if it is not satisfied.

Step 3. Solve the adjoint problem given by equations (14) for $\lambda_1(x,t)$.

Step 4. Compute the gradient equations of the functional $J'_1(t)$ and $J'_2(t)$ from equations (16) and (17), respectively.

Step 5. Compute the conjugate coefficients γ_1^n and γ_2^n and directions of descent P_1^n and P_2^n from equations (6) and (5), respectively.

Step 6. Set $\Delta F = P_1^n$ and $\Delta M = P_2^n$. Solve the sensitivity problems given by equations (7) and (8) for Δy_1 and Δy_2, respectively.

Step 7. Compute the search step sizes β_1^n and β_2^n from equations (11).

Step 8. Compute the new estimations for $F^{n+1}(t)$ and $M^{n+1}(t)$ from equations (4) and return to step 1.

6 Results and Discussion

The CGM is utilized in this study to show its validity in estimating simultaneously the applied external force $F(t)$ and moment $M(t)$ for an Euler-Bernoulli beam with no prior information on the functional form of the unknown quantities.

To illustrate the accuracy of the CGM in predicting $F(t)$ and $M(t)$ in this work from the knowledge of the displacement recordings, two specific examples, involving different form of external $F(t)$ and $M(t)$, are considered here.

In order to compare the results for situations considering random measurement errors, the normally distributed uncorrelated errors with zero mean and constant standard deviation are considered. The simulated inexact measured displacements data $Y(x_i,t)$ can be expressed as

$$Y(x_i,t) = Y(x_i,t)_{exact} + \omega\sigma \tag{21}$$

where $Y(x_i,t)_{exact}$ are the solution of the direct vibration problem with an exact applied force $F(t)$ and moment $M(t)$; σ is the standard deviation of the measured displacements and ω is a random variable that is generated by subroutine DRNNOR of the IMSL (1987) and will be within -2.576 to 2.576 for a 99% confidence bound.

The initial conditions for displacement and velocity are both assumed zero, i.e. $y(x,0) = \frac{\partial y(x,0)}{\partial t} = 0$, at $t = 0$. After the dimensionless analysis, the length of Euler-Bernoulli beam becomes unity and total time becomes $t_f = 5000$. Here the space and time increments for finite difference equation are taken as $\Delta x = 0.01$ and $\Delta t = 50$, respectively. Therefore a total of 200 unknown discretized applied force and moment are to be determined in the present study.

One of the advantages of using the CGM to solve the inverse problems is that the initial guesses of the unknown quantities can be chosen arbitrarily. In all the test cases considered here the initial guesses of $F(t)$ and $M(t)$ are taken as $F(t)^0 = M(t)^0 = 0.0$. The numerical experiments in determining simultaneously $F(t)$ and $M(t)$ by the inverse analysis using the CGM are now illustrated below.

(A) Numerical test case 1

In the first numerical test case, the exact dimensionless time-dependent applied force $F(t)$ and moment $M(t)$ for an Euler-Bernoulli beam are given as:

$$F(t) = 50000 \times \sin\left(\frac{\pi t}{t_f}\right); \quad 0 \le t \le t_f \tag{22a}$$

$$M(t) = -\left[\sin\left(\frac{\pi t}{t_f}\right) + 0.5 \times \sin\left(\frac{2\pi t}{t_f}\right)\right]; \quad 0 \le t \le t_f \tag{22b}$$

The inverse analysis is first performed by using exact displacement measurements, i.e. $\sigma = 0.0$, measured at $x_1 = 0.5$ and $x_2 = 0.95$. When the stopping criterion is set as $\varepsilon = 2.2 \times 10^{-6}$, after 27 iterations the inverse solutions are converged, J is calculated as 2.14×10^{-6} and CPU time at Pentium IV-3 GHz PC is about 3 seconds. Figures 2(a) and 2(b) indicate the measured and estimated displacements at $x_1 = 0.5$ and $x_2 = 0.95$, respectively. The exact and estimated applied force $F(t)$ and moment $M(t)$ are shown in Figures 3(a) and 3(b), respectively. ERR1 and ERR2 represent the relative errors between the exact and estimated force and moment, respectively, and are calculated as ERR1 = 4.66×10^{-2} % and ERR2 = 7.16×10^{-1} %, respectively. ERR3 and ERR4 indicate the relative errors between the measured and estimated displacements at x_1 and x_2, respectively, and are calculated as ERR3 = 1.6×10^{-2} % and ERR4 = 3.82×10^{-2} %, respectively. Here ERR1, ERR2, ERR3 and ERR4 are defined as

$$\text{ERR1}\% = \sum_{j=1}^{100} \left| \frac{\hat{F}(t_j) - F(t_j)}{F(t_j)} \right| \div (100) \times 100\% \tag{23a}$$

$$\text{ERR2\%} = \sum_{j=1}^{100} \left| \frac{\hat{M}(t_j) - M(t_j)}{M(t_j)} \right| \div (100) \times 100\% \tag{23b}$$

$$\text{ERR3\%} = \sum_{j=1}^{100} \left| \frac{y(x_1,t_j) - Y(x_1,t_j)}{Y(x_1,t_j)} \right| \div (100) \times 100\% \tag{23c}$$

$$\text{ERR4\%} = \sum_{j=1}^{100} \left| \frac{y(x_2,t_j) - Y(x_2,t_j)}{Y(x_2,t_j)} \right| \div (100) \times 100\% \tag{23d}$$

here j represents the index of discreted time and $\hat{F}(t_j)$ and $\hat{M}(t_j)$ indicate the estimated applied force and moment, respectively.

From Figures 2 and 3 and errors for ERR1 to ERR4 it is concluded that the present algorithm has been applied successfully in the inverse vibration problem in estimating simultaneously the time-dependent applied force and moment since the estimated results are very accurate.

Next, it is of interest to discuss what will be happened when the measured positions are changed, i.e. measurement positions are now at x_1 = 0.4 and x_2 = 0.9. The same calculation conditions are used and the number of iterations under this situation is 10 with $\varepsilon = 4 \times 10^{-5}$ and CPU time at Pentium IV-3 GHz PC is about 1 seconds.

The estimated $F(t)$ and $M(t)$ are plotted in Figures 4(a) and 4(b), respectively. It is obvious that the estimated $F(t)$ and $M(t)$ are both accurate enough since ERR1 to ERR4 are obtained as ERR1 = 2.48×10^{-1} % , ERR2 = 1.97 %, ERR3 = 2.76×10^{-1} % and ERR4 = 1.02×10^{-1} %, respectively. The above numerical experiments suggested that the measured positions can be varied while accurate estimations can still be obtained. Moreover, based on a series of numerical experiments, it is suggested that the sensors should be placed no less than $x = 0.25$ for a good estimation due to the clamped condition at $x = 0$.

Finally, let us discuss the influence of the measurement errors on the inverse solutions. First the measurement error for the displacements measured by sensors is taken as $\sigma = 2.6 \times 10^{-3}$ (about 5 % of the average measured displacements at x_1 and x_2) with x_1 = 0.5 and x_2 = 0.95. After 3 iterations the estimations can be obtained and are plotted in Figures 5(a) and 5(b) for the estimated applied force and moment, respectively. The relative errors ERR1 to ERR4 are calculated as ERR1 = 5.79 % , ERR2 = 6.25 %, ERR3 = 5.06 % and ERR4 = 5.44 %, respectively. Second, the measurement error for the displacements measured by sensors is increased to $\sigma = 5.2 \times 10^{-3}$ (about 10 % of the average measured displacements at x_1 and x_2). After only 3 iterations the applied force and moment can be obtained and plotted in Figures 6(a) and 6(b), respectively. The relative errors ERR1 to ERR4 are calculated as ERR1 = 7.74 % , ERR2 = 7.74 %, ERR3 = 7.18 % and ERR4 = 7.74%, respectively. From those results we learned that the reliable inverse solutions can still be obtained when measurement errors are considered.

(B). Numerical test case 2

The calculation conditions for Euler-Bernoulli beam used in test case 2 are the same as were used in test case 1 and the unknown applied force $F(t)$ and moment $M(t)$ are given as shown in the following functions:

$$F(t) = 60000 \times \sin\left(\frac{2\pi t}{t_f}\right); \quad 0 \le t \le t_f \tag{24a}$$

$$M(t) = \begin{cases} 0; & \text{at } 0 \le t < 10 \\ -3; & \text{at } 10 \le t < 40 \\ 2; & \text{at } 40 \le t < 70 \\ -1 & \text{at } 70 \le t \le t_f \end{cases} \tag{24b}$$

The inverse analysis is first performed by using $\sigma = 0.0$ and measured at x_1 = 0.5 and x_2 = 0.95. When the stopping criterion is set as $\varepsilon = 2.7 \times 10^{-2}$, after 33 iterations the inverse solutions can be obtained and J is calculated as 2.68×10^{-2}. The CPU time at Pentium IV-3 GHz PC is about 3 seconds. Figures 7(a) and 7(b) show the measured and estimated displacements at x_1 = 0.5 and x_2 = 0.95, respectively, while the exact and estimated applied force $F(t)$ and moment $M(t)$ are given in Figures 8(a) and 8(b), respectively. The relative errors ERR1 to ERR4 are calculated as ERR1 =

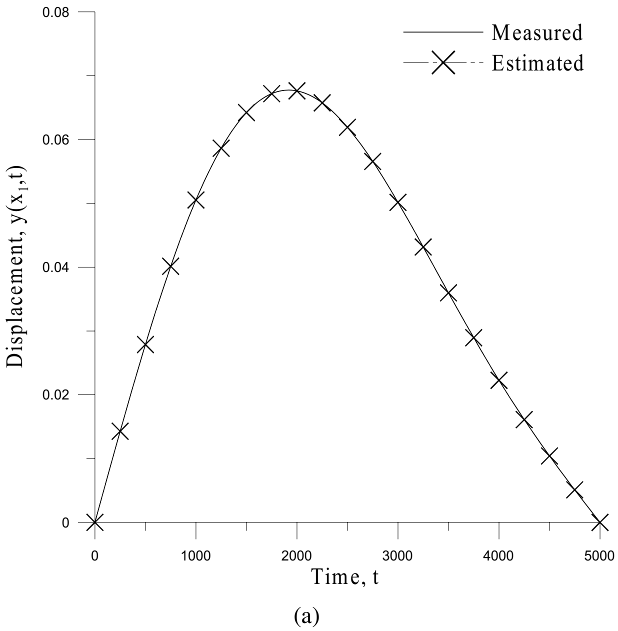


Figure 2: The measured and estimated displacements at (a) $x_1 = 0.5$ and (b) $x_2 = 0.95$ with time using $\sigma = 0$ in test case 1

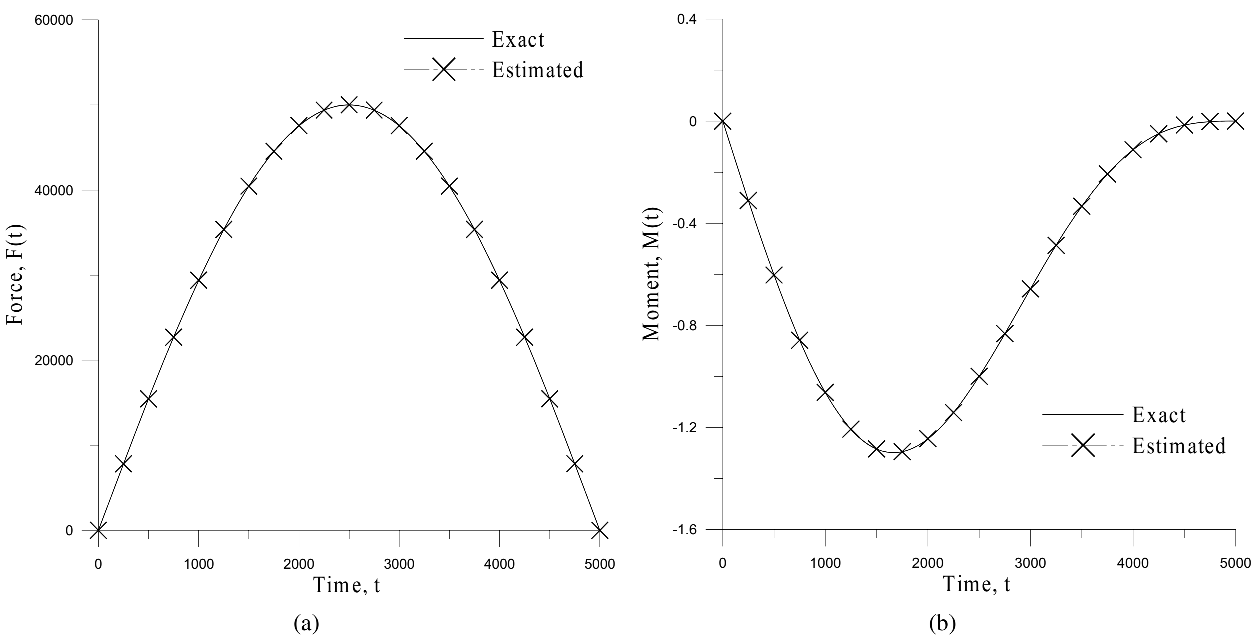


Figure 3: The exact and estimated (a) $F(t)$ and (b) $M(t)$ with $x_1 = 0.5$ and $x_2 = 0.95$ using $\sigma = 0$ in test case 1

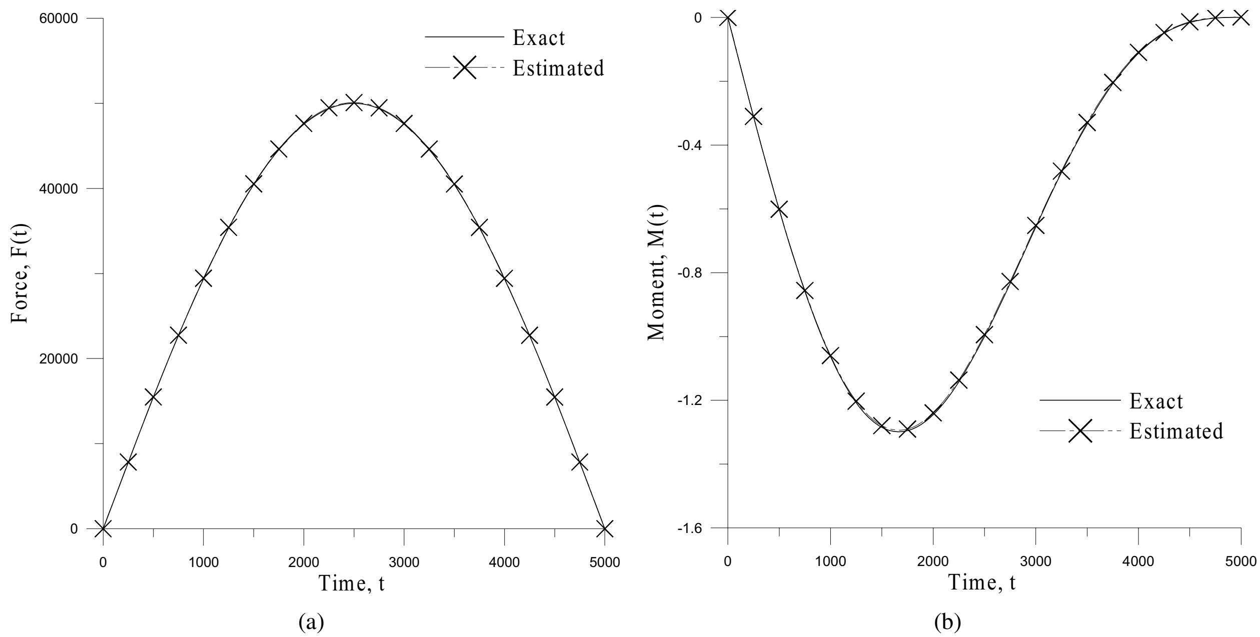


Figure 4: The exact and estimated (a) $F(t)$ and (b) $M(t)$ with $x_1 = 0.4$ and $x_2 = 0.9$ using $\sigma = 0$ in test case 1

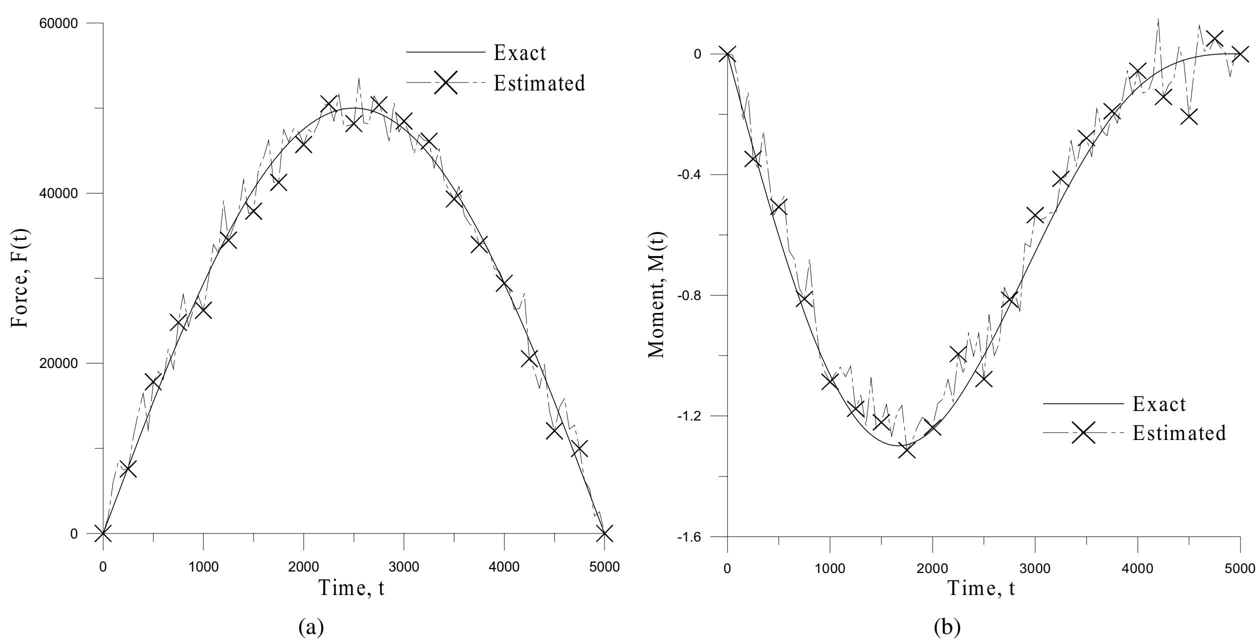
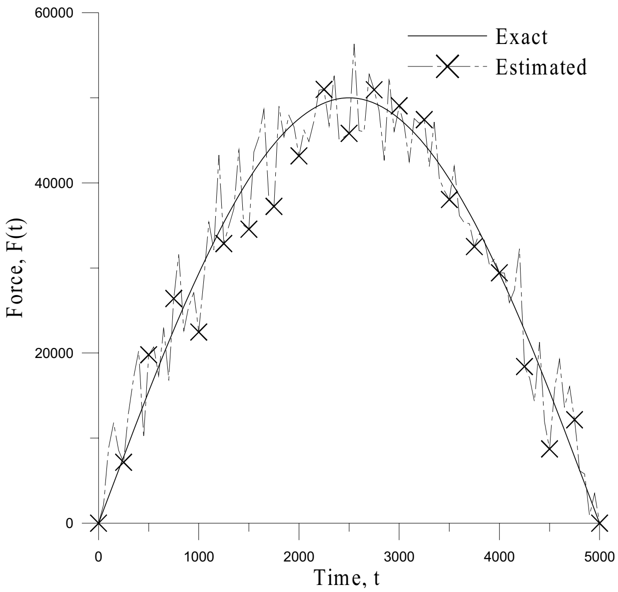
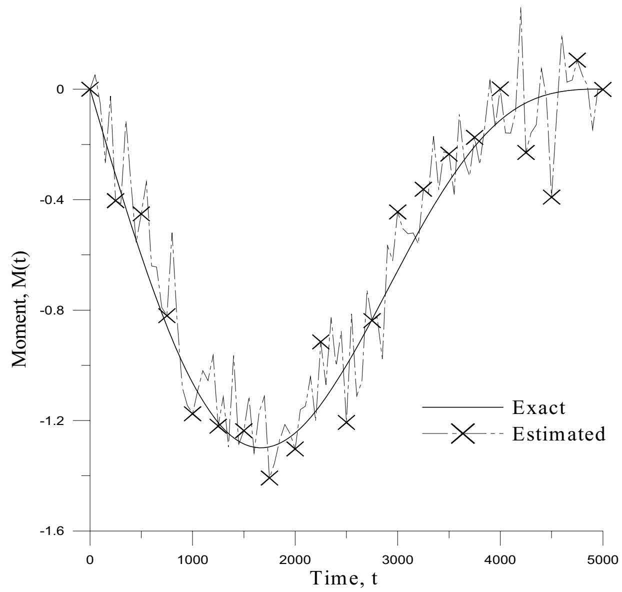


Figure 5: The exact and estimated (a) F(t) and (b) $M(t)$ with $x_1 = 0.5$ and $x_2 = 0.95$ using $\sigma = 2.6 \times 10^{-3}$ in test case 1

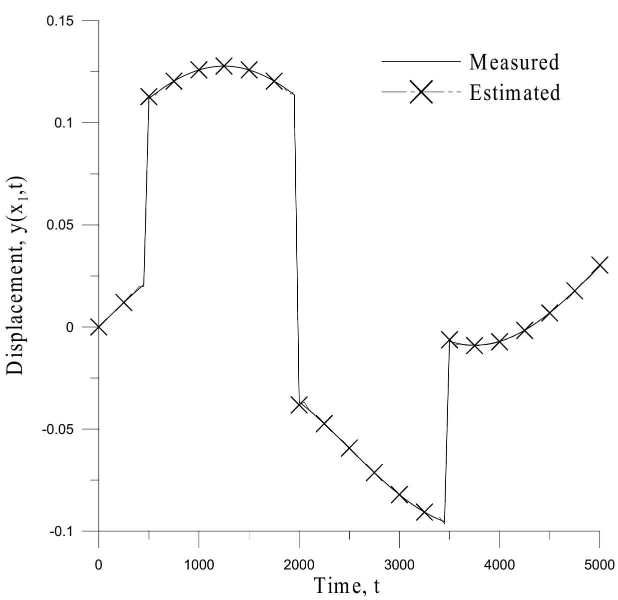


(a)

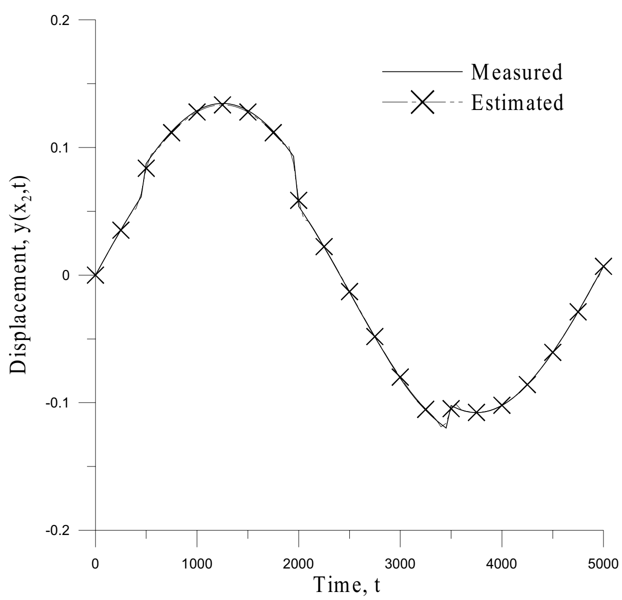


(b)

Figure 6: The exact and estimated (a) $F(t)$ and (b) $M(t)$ with $x_1 = 0.5$ and $x_2 = 0.95$ using $\sigma = 5.2 \times 10^{-3}$ in test case 1

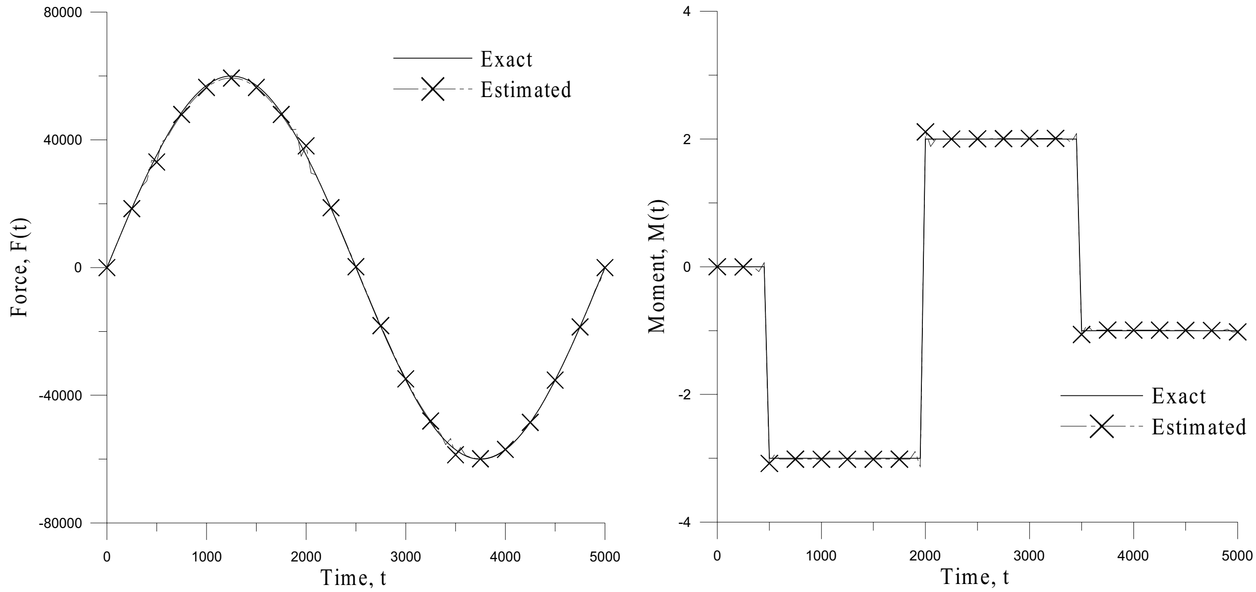


(a)



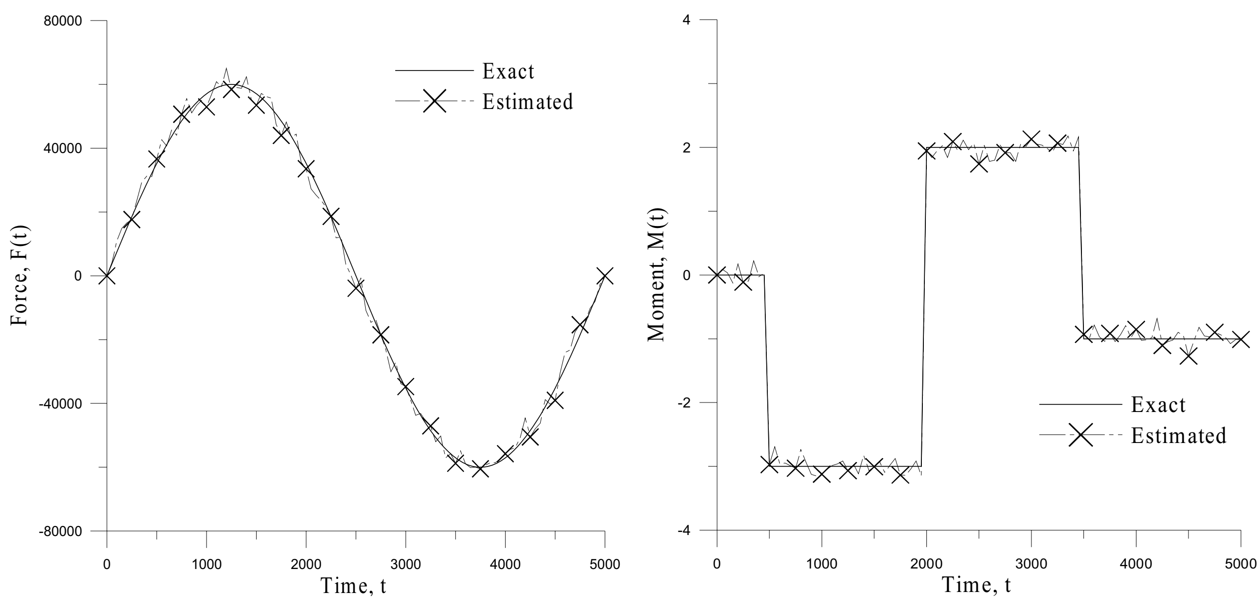
(b)

Figure 7: The measured and estimated displacements at (a) $x_1 = 0.5$ and (b) $x_2 = 0.95$ with time using $\sigma = 0$ in test case 2



(a) (b)

Figure 8: The exact and estimated (a) $F(t)$ and (b) $M(t)$ with $x_1 = 0.5$ and $x_2 = 0.95$ using $\sigma = 0$ in test case 2



(a) (b)

Figure 9: The exact and estimated (a) $F(t)$ and (b) $M(t)$ with $x_1 = 0.5$ and $x_2 = 0.95$ using $\sigma = 3.5 \times 10^{-3}$ in test case 2

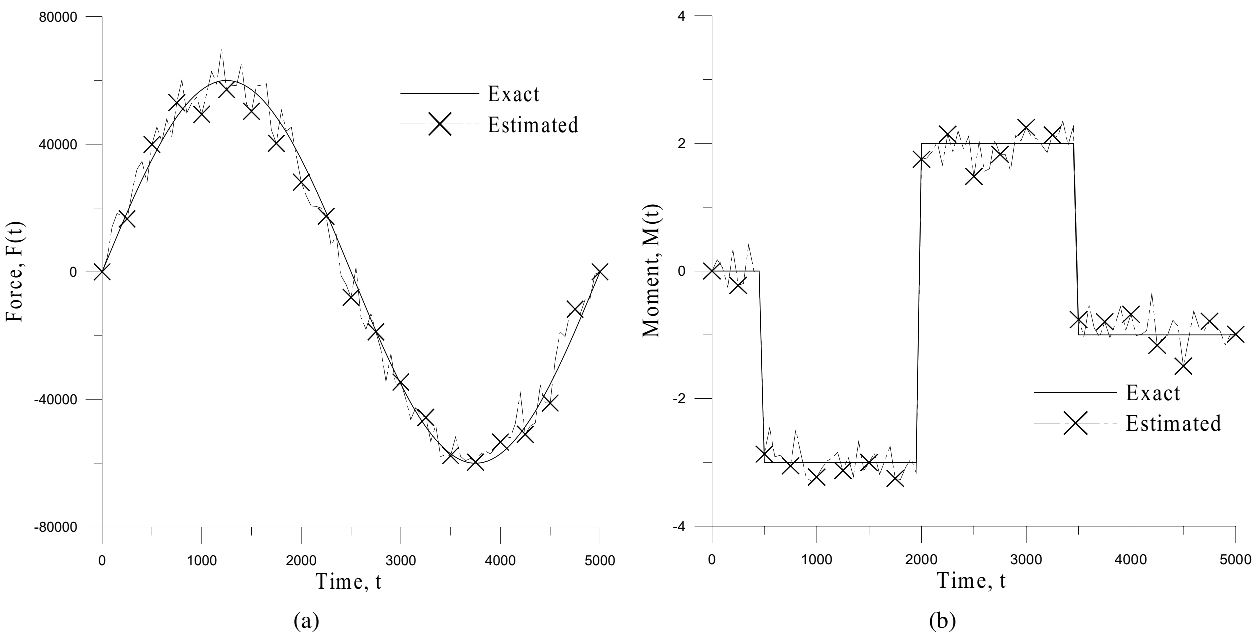


Figure 10: The exact and estimated (a) $F(t)$ and (b) $M(t)$ with x_1 = 0.5 and x_2 = 0.95 using $\sigma = 7.0 \times 10^{-3}$ in test case 2

1.85 % , ERR2 = 7.47×10^{-1} %, ERR3 = 1.29 % and ERR4 = 1.51 %, respectively.

It should be noted that the estimations for $M(t)$ should be not accurate when time approaches to the final time t_f. The reason for this is that the gradient J'_2 at final time $t = t_f$ is always equal to zero since $\lambda_1(x,t_f)$ = 0. If the initial guess values of $M^0(t)$ can not be predicted correctly before the inverse calculations, the estimated values of $M(t)$ will deviate from exact values near the final time conditions. This is the case for $M(t)$ in the present study since $M(t_f)$ does not equal to zero but $M^0(t) = 0$.

However, if we let $\lambda_1(x,t_f) = \lambda_1(x,t_f - \Delta t)$, where Δt denotes the time increment used in the finite difference calculation, and apply this expression to the gradient equation (17), the singularity at $t = t_f$ can be avoided in the present study and reliable inverse solutions can be obtained. It is obvious from Figures 7 and 8 and errors for ERR1 to ERR4 that the CGM has been applied successfully to estimate simultaneously the time-dependent applied force and moment.

Next, it is of interest to discuss the influence of the measurement errors on the inverse solutions. First the measurement error for the displacements measured by sensors is taken as $\sigma = 3.5 \times 10^{-3}$ (about 5 % of the average measured displacements at x_1 and x_2) with x_1 = 0.5 and x_2 = 0.95. After 16 iterations the estimations can be obtained and are plotted in Figures 9(a) and 9(b) for the estimated applied force and moment, respectively. The relative errors ERR1 to ERR4 are calculated as ERR1 = 5.54 % , ERR2 = 4.47 %, ERR3 = 5.12 % and ERR4 = 4.87 %, respectively. Second, the measurement error for the displacements measured by sensors is increased to $\sigma = 7.0 \times 10^{-3}$ (about 10 % of the average measured displacements at x_1 and x_2). After only 6 iterations the applied force and moment can be obtained and plotted in Figures 10(a) and 10(b), respectively. The relative errors ERR1 to ERR4 are calculated as ERR1 = 8.02 % , ERR2 = 6.81 %, ERR3 = 5.03 % and ERR4 = 7.03%, respectively. From those results we learned that the reliable inverse solutions can still be obtained when measurement errors are considered.

From the above two test cases it is learned that an inverse forced vibration problem in estimating simultaneously the unknown applied force and moment is now completed. Reliable estimations can be obtained when using either exact or error measurements.

7 Conclusions

The Conjugate Gradient Method (CGM) was successfully applied for the solution of the inverse forced vibration problem to determine simultaneously the unknown time-dependent applied force and moment for an Euler-Bernoulli beam by utilizing displacement readings obtained from two sensors with time. Several test cases involving different functional forms for applied force and moment, measurement positions and measurement errors were considered. The results show that the inverse solutions obtained by the CGM remain stable and regular as the measurement errors are large. Moreover CPU time needed in the inverse calculations is very short and the initial guesses for external forces can be arbitrarily chosen as zero.

Acknowledgement: This work was supported in part through the National Science Council, R. O. C., Grant number, NSC-96-2221-E-006-065.

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