

Investigation of Multi Geometric Uncertainties by Different Polynomial Chaos Methodologies Using a Fictitious Domain Solver

L. Parussini¹ and V. Pediroda²

Abstract: In this paper different Polynomial Chaos methods coupled to Fictitious Domain approach have been applied to one- and two-dimensional elliptic problems with multi uncertain variables in order to compare the accuracy and convergence of the methodologies. Both intrusive and non-intrusive methods have been considered, with particular attention to their employment for quantification of geometric uncertainties. A Fictitious Domain approach with Least-Squares Spectral Element approximation has been employed for the analysis of differential problems with uncertain boundary domains. Its main advantage lies in the fact that only a Cartesian mesh, that represents the enclosure, needs to be generated. Excellent accuracy properties of considered methods are demonstrated by numerical experiments.

Keyword: Chaos Polynomial, Chaos Collocation, Tensorial-expanded Chaos Collocation, multi geometric uncertainties, Fictitious Domain, Least-Squares Spectral Element Method.

1 Introduction

In most engineering applications, to solve physical problems deterministic mathematical models are adopted. It is evident these models are rough simplifications of reality. Actually, many physical input parameters are not deterministic entities, but stochastic processes which certainly influence the behaviour of solution. In order to obtain reliable results uncertainty quantification is neces-

sary and the influence of inherent physical and geometric uncertain parameters must not be neglected. Thanks to the increasing of computer facilities and the advance of algorithms, which leads to more accurate solutions, the current state of technology allows to include these inherent uncertainties in mathematical models. Thereby an increasing interest in uncertainty analysis applied to computational physics has occurred and probabilistic methods have been developed.

In literature there are several examples of numerical methods to face problems with uncertain input parameters and in Ref. [Loeven, Witteveen, and Bijl (2007); Schoutens (2000); Wiener (1958); Xiu and Karniadakis (2002, 2003b)] we find applications of these methodologies to Thermo-Fluid Dynamics. In these works the effort is focused on exploring random material properties or random boundary conditions, whereas the topology of domain boundaries are described in deterministic terms, without taking into account of their stochastic nature (figure 1).

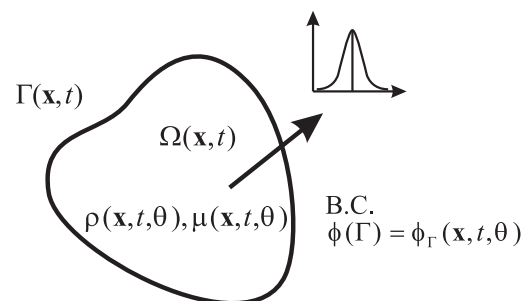


Figure 1: Differential problem with stochastic material properties and stochastic boundary conditions, where θ is the uncertainty.

On the contrary there are really few examples of numerical methods to analyse geometric uncer-

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tainty, that is to solve deterministic problems, in terms of material properties and boundary constraints, on random domains (figure 2), with geometric uncertainty given by shape tolerance.

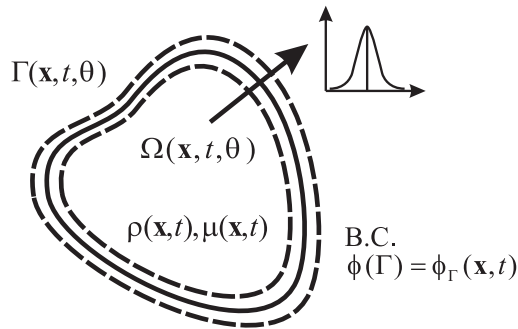


Figure 2: Differential problem with stochastic definition domain, where θ is the uncertainty.

In Ref. [Parussini and Pediroda (2007)] a method has been presented to face geometric tolerance problems based on Chaos Collocation and Fictitious Domain with least-squares spectral element approximation. This method consents to avoid both the difficulty of mapping space variables into a deterministic domain, as in Ref. [Xiu and Tartakovsky (2006)], and the need to remesh the geometry of domain, as in Ref. [Hosder, Walters, and Perez (2006); Lin, Su, and Karniadakis (2006)].

In fact, Fictitious Domain method allows problems formulated on an intricate domain to be solved on a simpler fictitious domain containing the original one. In this way the computational domain of state problem is independent by small variations of original domain boundaries subject to uncertainty, now immersed into computational domain. Being the computational domain independent by random geometric parameters, the remeshing has not to be performed when the domain geometry changes. To solve the Fictitious Domain problem Least Squares Spectral Element Method is employed [Pontaza and Reddy (2003); Proot and Gerritsma (2002)].

In order to improve the previous study [Parussini and Pediroda (2007)] in this work we compare different Polynomial Chaos methodologies [Xiu

and Karniadakis (2003b); Xiu and Tartakovsky (2006); Blith and Pozrikidis (2003); Loeven, Witteveen, and Bijl (2006); Mathelin and Hussaini (2003)] for solving problems with multi uncertain parameters. There exist several methods for uncertainty quantification, which can be divided into two main categories: non-intrusive, or statistical, and intrusive, or non-statistical. Monte Carlo [Blith and Pozrikidis (2003)], Stochastic Collocation [Mathelin and Hussaini (2003)], Chaos Collocation [Loeven, Witteveen, and Bijl (2006)] are examples of non-intrusive approaches, Chaos Polynomials [Xiu and Karniadakis (2002, 2003a)] are examples of intrusive approaches. Non-intrusive methods allow the use of existing deterministic solvers, whereas intrusive approaches need to modify the solver obtaining an efficient tool but limited to solve just a set of problems. In particular our aim in this work is the comprehension of advantages and disadvantages of these methods for description of stochastic phenomena.

We consider a one-dimensional elliptic problem with both random material properties and random boundary conditions on stochastic domain, with geometric uncertainty given by shape tolerance. The excellent accuracy of proposed Chaos methodologies, i.e. Chaos Polynomial, Chaos Collocation and Tensorial-expanded Chaos Collocation, coupled to Fictitious Domain, is demonstrated by numerical experiments.

In consequence of the considerations arisen by previous tests on multi uncertain problems, we employ Chaos Collocation and Tensorial-expanded Chaos Collocation methods with Fictitious Domain approach to solve the stationary heat conduction problem in an electronic chip [Xiu and Karniadakis (2003a)]. In particular multi geometric uncertainties have been considered, in order to determine the differences between these methodologies, both non-intrusive.

The paper is organized as follows. In Section 2 the theory of uncertainty quantification methods is presented. In Section 3 the concept and the analysis of geometric uncertainty are discussed and argued. In Section 4 there is a comparison among Monte Carlo, Chaos Polynomial, Chaos Collocation and Tensorial-expanded Chaos Collocation

methods referring to definition of mean and variance of analytical functions. Section 5 illustrates the Fictitious Domain method. In Section 6 the formulation of Polinomyal Chaos methods with Fictitious Domain approach is expositied and in Section 7 some numerical examples to validate and compare the Polinomyal Chaos methodologies with Fictitious Domain approach are shown. In Section 8 we give some concluding remarks.

2 Uncertainty quantification methods

2.1 Setting of the problem: stochastic differential equation

Let us consider the following stochastic differential equation:

$$L(\mathbf{x}, t, \theta; \phi) = f(\mathbf{x}, t, \theta) \quad (1)$$

where L is a differential operator which contains space and time differentiation and can be non linear and depended on random parameters θ ; $\phi(\mathbf{x}, t, \theta)$ is the solution and function of the space $\mathbf{x} \in \mathfrak{X}^d$, time t and random parameters θ ; $f(\mathbf{x}, t, \theta)$ is a space, time and random parameters dependent source term.

2.2 The Generalized Polynomial Chaos

Under specific conditions [Schoutens (2000)], a stochastic process can be expressed as a spectral expansion based on suitable orthogonal polynomial with weights associated with a particular density. The first study in this field is the Wiener process [Wiener (1958, 1938)], which can be written as a spectral expansion in terms of Hermite polynomials with normal distributed input parameters.

The basic idea is to project the variables of the problem onto a stochastic space spanned by a set of complete orthogonal polynomials Ψ that are functions of random variables $\xi(\theta)$, where θ is a random event. For example, the variable ϕ has the following spectral finite dimensional representation:

$$\phi(\mathbf{x}, t, \theta) = \sum_{i=0}^{\infty} \phi_i(\mathbf{x}, t) \Psi_i(\xi(\theta)) \quad (2)$$

In practical terms the Eq. (2) divides the random variable $\phi(\mathbf{x}, t, \theta)$ into a deterministic part, the coefficient $\phi_i(\mathbf{x}, t)$ and a stochastic part, the polynomial chaos $\Psi_i(\xi(\theta))$. The basis $\{\Psi_i\}$ is a set of orthogonal polynomials with respect to the probability density function of the input parameters. Following the Askey scheme [Askey and Wilson (1985)], it is possible to introduce the Generalized Polynomial Chaos [Xiu and Karniadakis (2003a)]. Thanks to this theory, known also as *Askey-chaos*, for certain input parameter distribution there exists the best representation in terms of convergence rate. For example, for Gaussian random input, we have the Hermite Polynomial Chaos representation, for Gamma distribution the Laguerre representation, for Beta distribution the Jacoby representation, for Uniform distribution the Legendre representation, etc.

In this paper we focus mainly on the Gaussian random input, so we represent the variable $\phi(\mathbf{x}, t, \theta)$ in terms of Hermite spectral representation, following the Askey scheme:

$$\begin{aligned} \phi(\mathbf{x}, t, \theta) = & \phi_0(\mathbf{x}, t) H_0 + \\ & \sum_{i_1=1}^{\infty} \phi_{i_1}(\mathbf{x}, t) H_1(\xi_{i_1}(\theta)) + \\ & \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \phi_{i_1 i_2}(\mathbf{x}, t) H_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) + \\ & \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \phi_{i_1 i_2 i_3}(\mathbf{x}, t) H_3(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)) + \\ & \dots \quad (3) \end{aligned}$$

where $H_p(\xi_{i_1}, \dots, \xi_{i_p})$ is the Hermite polynomial of order p in terms of a n -dimensional Gaussian random variable $\xi = (\xi_1, \dots, \xi_n)$ distributed as $N(0, 1)$ [Xiu and Karniadakis (2003a)]. The Hermite polynomial is expressed in general form by:

$$H_p(\xi_{i_1}, \dots, \xi_{i_p}) = e^{\frac{1}{2}\xi^T \xi} (-1)^p \frac{\partial^p}{\partial \xi_{i_1} \dots \partial \xi_{i_p}} e^{-\frac{1}{2}\xi^T \xi} \quad (4)$$

and for one-dimensional case:

$$H_0 = 1, H_1 = \xi, H_2 = \xi^2 - 1, H_3 = \xi^3 - 3\xi, \dots \quad (5)$$

The polynomial basis $\{\Psi_j\}$ of Hermite-Chaos forms a complete orthogonal basis, i.e.

$$\langle \Psi_i, \Psi_j \rangle = \langle \Psi_i^2 \rangle \delta_{ij} \quad (6)$$

where δ_{ij} is the Kronecker delta and $\langle \cdot, \cdot \rangle$ denotes the ensemble average. This is the inner product in the Hilbert space determined by the support of the Gaussian variables

$$\langle f(\xi)g(\xi) \rangle = \int f(\xi)g(\xi)w(\xi)d\xi \quad (7)$$

with weighting function

$$w(\xi) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\xi^T \xi}. \quad (8)$$

What distinguishes the Hermite-Chaos expansion from other possible expansions is that the basis polynomials are Hermite polynomials in terms of Gaussian variables and are orthogonal with respect to the weighting function $w(\xi)$ that has the form of n -dimensional independent Gaussian probability density function.

For practical cases, the series in Eq. (2) has to be truncated to a finite numbers of terms, here denoted with N . So the form Eq. (2), using the one-to-one correspondence between the function $H_p(\xi_{i_1}, \dots, \xi_{i_p})$ and $\Psi_p(\xi)$, as demonstrated in Ref. [Xiu and Karniadakis (2002)] for Gaussian random input, becomes:

$$\phi(\mathbf{x}, t, \theta) = \sum_{i=0}^N \phi_i(\mathbf{x}, t) H_i(\xi) \quad (9)$$

The number of total terms of the series is determined by:

$$N+1 = \frac{(n+p)!}{n! p!} \quad (10)$$

where n is the uncertainties dimensionality and p is the order of the expansion.

As an example, for a second order two-dimensional Hermite polynomial expression, we get the following form:

$$\begin{aligned} \phi(\mathbf{x}, t, \theta) = & \phi_0(\mathbf{x}, t) + \\ & \phi_1(\mathbf{x}, t) \xi_1(\theta) + \phi_2(\mathbf{x}, t) \xi_2(\theta) + \\ & \phi_3(\mathbf{x}, t) (\xi_1^2(\theta) - 1) + \phi_4(\mathbf{x}, t) (\xi_2^2(\theta) - 1) + \\ & \phi_5(\mathbf{x}, t) \xi_1(\theta) \xi_2(\theta) \end{aligned} \quad (11)$$

where $\xi_1(\theta)$ and $\xi_2(\theta)$ are the two random independent variables.

2.3 Intrusive and non-intrusive Polynomial Chaos methodologies

Substituting the Polynomial Chaos series, given in Eq. (6) for Gaussian random input, into the stochastic differential Eq. (1) we obtain:

$$L \left(\mathbf{x}, t, \theta; \sum_{i=0}^N \phi_i(\mathbf{x}, t) \Psi_i(\xi(\theta)) \right) \cong f(\mathbf{x}, t, \theta). \quad (12)$$

The method of Weighted Residuals is adopted to solve this equation. The coefficients $\phi_i(\mathbf{x}, t)$ are obtained imposing the inner product of the residual with respect to a weight function is equal to zero.

If the weight functions are chosen to be the same as the expansion functions Ψ_i we produce Galerkin method. Performing the Galerkin projection on both sides of the equation, the form becomes:

$$\left\langle L \left(\mathbf{x}, t, \theta; \sum_{i=0}^N \phi_i(\mathbf{x}, t) \Psi_i \right), \Psi_j \right\rangle = \langle f(\mathbf{x}, t, \theta), \Psi_j \rangle \quad j = 0, \dots, N. \quad (13)$$

If the operator L is non linear, the deterministic set of $N+1$ equation is coupled and this form is called *Chaos Polynomial* [Lin, Wan, Su, and Karniadakis (2007)].

If we employ Dirac delta function as weight function we produce Collocation method. Using a collocation projection on both sides of Eq. (12), we obtain:

$$L(\mathbf{x}, t, \theta_j; \phi_j) = f(\mathbf{x}, t, \theta_j) \quad j = 0, \dots, N. \quad (14)$$

This formulation is a linear system equivalent to solving a deterministic problem at each grid point; this form is called *Chaos Collocation* [Loeven, Witteveen, and Bijl (2007)].

To reconstruct the stochastic solution $\phi(\mathbf{x}, t, \theta)$, the Eq. (2) is used:

$$E_{PC}(\phi) = \mu_\phi = \phi_0(\mathbf{x}, t, \theta) \quad (15)$$

$$Var_{PC}(\phi) = \sigma_\phi^2 = \sum_{i=1}^N [\phi_i^2(\mathbf{x}, t, \theta) \langle \Psi_i^2 \rangle]. \quad (16)$$

Here for one-dimensional case

$$\langle \Psi_i, \Psi_j \rangle = \langle \Psi_i^2 \rangle \delta_{ij} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2/2} \Psi_i(\xi) \Psi_j(\xi) d\xi = 2^i i! \delta_{ij} \quad (17)$$

where δ_{ij} is the Kronecker operator.

The two approaches, Chaos Polynomial and Chaos Collocation, are based on the same theory, but gives different numerical representations. In practice the intrusive method consists in resolution of a coupled system of deterministic equations, the non-intrusive method consists in resolution of a decoupled system of deterministic equations. It is evident the difficulty to design an efficient intrusive solver, both because of computational cost and because of the obvious handicap to imply an internal modification of the deterministic solver. The non-intrusive methodology has a simpler computational management. A remarkable advantage of this approach is the deterministic solver represents a black-box and there is no need to modify it. This means the non-intrusive method is more versatile than intrusive method.

A still open problem of Chaos Collocation approach is the difficulty to select collocation points: with multi dimensional uncertainties the choice is not unique [Hosder, Walters, and Perez (2006)]. This problem does not exist for one stochastic parameter, because collocation points are the roots of polynomial of order $p + 1$, the same polynomial employed for the representation of random variable. So to overcome this difficulty of Chaos Collocation method, the stochastic process will be expanded into polynomials obtained by tensorial product of one-dimensional polynomials. We obtain the *Tensorial-expanded Chaos Collocation* formulation:

$$L(\mathbf{x}, t, \theta_j; \phi_j) = f(\mathbf{x}, t, \theta_j) \quad j = 0, \dots, (p+1)^n - 1 \quad (18)$$

with n number of uncertain variables and p expansion polynomial order respect to i -th uncertain variable, if the same expansion order is chosen for all variables. We still have a linear system equivalent to solving a deterministic problem at each $(p+1)^n$ grid point. The methodology is non-intrusive but computationally more

expensive than multidimensional Chaos Collocation. The advantage is to avoid an arbitrary choice of collocation points.

In this paper we focus on geometric tolerances using Fictitious Domain approach. We will consider multi uncertain parameters with Gaussian distribution.

3 The concept and analysis of geometric uncertainty

In engineering design a particular attention must be paid to geometric tolerance and its influence on the performance of designed component. So there is a great interest in developing a methodology to face problems where the geometry of definition domain is a stochastic phenomenon. As defined in Ref. [Xiu and Tartakovsky (2006)], the problem under study writes:

Let $\theta \in \Theta$ be a random realization drawn from a complete probability space (Θ, A, P) , whose event space Θ generates its σ -algebra $A \subset 2^\Theta$ and is characterized by a probability measure P . For all $\theta \in \Theta$, let $\Omega(\theta) \subset \mathfrak{R}^d$ be a d -dimensional random domain bounded by boundary $\partial\Omega(\theta)$. We consider the following stochastic boundary value problem: for P -almost everywhere in Θ , given $f : \Omega(\theta) \rightarrow \mathfrak{R}$ and $g : \partial\Omega(\theta) \rightarrow \mathfrak{R}$, find a stochastic solution $u : \overline{\Omega}(\theta) \rightarrow \mathfrak{R}$ such that:

$$A(\mathbf{x}; u) = f(\mathbf{x}) \quad \text{in } \Omega(\theta) \quad (19)$$

$$B(\mathbf{x}; u) = g(\mathbf{x}) \quad \text{on } \partial\Omega(\theta) \quad (20)$$

where $\mathbf{x} = (x_1, \dots, x_d)$, A is a differential operator and B is a boundary operator.

Except for a few studies, random domain problems have not been systematically analyzed. The most complete work on these topics is presented in Ref. [Xiu and Tartakovsky (2006)], where a mapping methodology is introduced to transform the original problem defined in a random domain into a stochastic problem defined in a deterministic domain. The new stochastic problem defined on a deterministic domain, thanks to this mapping, can be solved by already existing Polynomial Chaos techniques. The methodology illustrated above has been efficiently implemented to solve two diffusion problems: in a channel with

rough surface and in double-connected domains with rough exclusion. The Polynomial Chaos methodology is demonstrated to be more accurate than Monte Carlo method and with lower computational cost. The drawback of the presented method is the difficulty of mapping. In fact this process is simple for connected domains, but it is computationally challenging for complex non-connected domains.

To ride over this problem, i.e. the mapping of complex domains, in this work we present the coupling of Fictitious Domain approach with Polynomial Chaos methodologies for geometric uncertainties. The idea is to avoid the mapping of stochastic domain onto a deterministic domain and to use absolute coordinates.

Let us consider this one-dimensional problem as example:

$$\frac{d^2\phi}{dx^2} = 0 \quad \text{in } [0, L]$$

with $\phi|_{x=0} = \phi_0$, $\phi|_{x=L} = \phi_L$ (21)
and $L = N(L_{Mean}, \sigma_L)$

where ϕ_0 and ϕ_L are constants and L is a random parameter with Gaussian distribution. In figure 3 is shown how we mean the stochastic domain problem of Eq. (21).

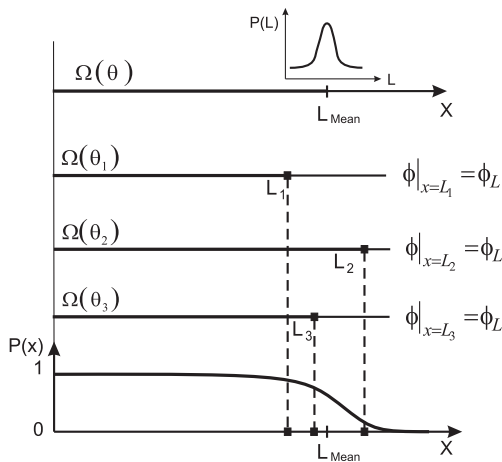


Figure 3: Representation of stochastic domain problem Eq.(21) in absolute coordinates with normal distribution of length L .

Geometric uncertainty, represented by probabilis-

tic distribution $P(L)$ of domain length L , becomes an uncertainty on the position of boundary condition. As every point of domain is studied in absolute coordinates, there is no need of mapping the stochastic domain onto a deterministic domain. The solution of the problem has a probability distribution $pdf(\phi)$ associated to each point of domain in absolute coordinates. This probability distribution of the solution depends on the position of boundary condition in $x = L$, which is a stochastic phenomenon. In figure 3 it is shown the probability $P(x)$ of a point of belonging to domain, which depends on the probabilistic distribution of L in problem Eq. (21). To solve this problem Polynomial Chaos methodologies can be used. If Chaos Polynomial is employed, we have to solve N coupled problems, whereas if Chaos Collocation is employed, we have to solve N distinct deterministic problems defined on different lengths of domain.

4 Comparison of Polynomial Chaos methodologies and analytical solution

To compare the Polynomial Chaos methodologies, we consider the problem of thermal diffusion in heat sink profiles with negligible thickness:

$$\frac{d^2T}{dx^2} = \frac{2h}{s\lambda}T \quad (22)$$

with

$$-\lambda \frac{dT}{dx} \Big|_{x=0} = q_0 \quad \text{and} \quad -\lambda \frac{dT}{dx} \Big|_{x=L} = hT \quad (23)$$

where s is the thickness profile, λ the conduction coefficient, h the convection coefficient and q_0 is the inlet thermal flux.

The analytic solution of this problem, if the parameters h , λ , q_0 and L are deterministic variables, writes as:

$$T(x) = \frac{q_0 \left(\frac{h}{\lambda} + m \right) e^{mL} e^{-mx} - \left(\frac{h}{\lambda} - m \right) e^{-mL} e^{mx}}{\lambda m \left(\left(\frac{h}{\lambda} - m \right) e^{-mL} + \left(\frac{h}{\lambda} + m \right) e^{mL} \right)} \quad (24)$$

where $m^2 = \frac{2h}{s\lambda}$.

As we know the analytic function $T(x)$ given the parameters h , λ , q_0 and L , we can compute the

analytic probability distribution of T if these parameters are uncertain. To compute the expected value $E(T)$ and the variance $Var(T)$ we refer to formulation presented in Ref. [Rotondi, Pedroni, and Pievatolo (2001)] for a function $f(\mathbf{X})$:

$$\begin{aligned} Z &= f(X_1, X_2, \dots, X_n) \equiv f(\mathbf{X}) \\ \text{with joint pdf } p_{\mathbf{X}}(x_1, x_2, \dots, x_n) \\ E(Z) &= \int f(x_1, x_2, \dots, x_n) p_Z(z) dz \\ Var(Z) &= \int [f(x_1, x_2, \dots, x_n) - E(Z)]^2 p_Z(z) dz \end{aligned} \quad (25)$$

$$\begin{aligned} p_Z(z) &= \int \dots \int p_{\mathbf{X}}(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n = \\ &= \int \dots \int p_{\mathbf{X}}(f^{-1}(z, x_2, \dots, x_n), x_2, \dots, x_n) \left| \frac{\partial f^{-1}}{\partial z} \right| \\ &\quad dx_2 \dots dx_n \end{aligned}$$

where $p_{\mathbf{X}}(x_1, x_2, \dots, x_n)$ is the probability density of n random variables \mathbf{X} and $p_Z(z)$ is the probability density of random output variable Z .

A remark needs to be done: to compute the mean and the variance employing Eq. (25) we have to solve integrals and the method we use for their solution is Gauss Quadrature. So, as mean and variance are computed employing a numerical method, they are affected by numerical errors.

In figure 4 the mean function $E(T)$ and the uncertainty bars $E(T) \pm Std(T)$ of function:

$$\begin{aligned} T(x, h, \lambda, q_0, L) &= \\ \frac{q_0}{\lambda m} \frac{\left(\frac{h}{\lambda} + m\right) e^{mL} e^{-mx} - \left(\frac{h}{\lambda} - m\right) e^{-mL} e^{mx}}{\left(\frac{h}{\lambda} - m\right) e^{-mL} + \left(\frac{h}{\lambda} + m\right) e^{mL}} \end{aligned}$$

with $m^2 = \frac{2h}{s\lambda}$, $s = 2mm$, $h = N(8, 0.1)W/m^2K$,
 $\lambda = N(65, 0.1)W/mK$, $q_0 = N(10000, 50)W/m^2$
and $L = N(20, 0.07)mm$ (26)

are shown.

The expected value $E(T)$ and the standard deviation $Std(T)$ obtained according to Eq. (25), which we will refer as *analytical* values, will be compared to those ones obtained by means of Monte Carlo method, Chaos Polynomial, Chaos Collocation and Tensorial-expanded Chaos Collocation, for the problem given in Eq. (26).

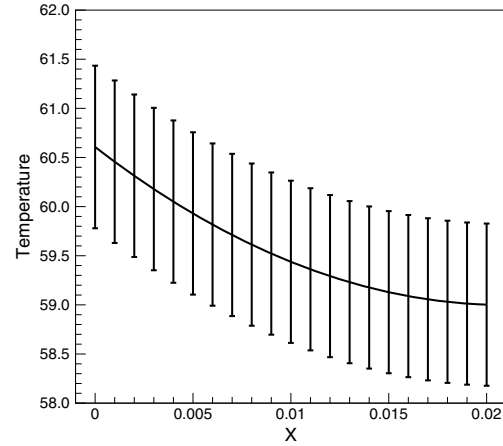


Figure 4: Mean function $E(T)$ and uncertainty bars $E(T) \pm Std(T)$ of problem Eq. (26) according to definition of mean and variance given in Eq. (25) and computed by means of Gauss quadrature formulae.

Figure 5 shows the absolute error of mean $E(T)$ and standard deviation $Std(T)$ obtained by Monte Carlo respect to analytical solution. The Monte Carlo $E(T)$ and $Std(T)$ have been computed with different number of realizations, i.e. 1000 and 10000.

In figure 6 the absolute errors of $E(T)$ and $Std(T)$ obtained by means of Chaos Polynomial are shown. Different expansion polynomial orders ($p = 1$, $p = 3$ and $p = 5$) have been considered.

In the end figure 7 shows the absolute error of $E(T)$ and $Std(T)$ obtained by means of Chaos Collocation respect to analytical solutions and figure 8 shows the absolute error of $E(T)$ and $Std(T)$ obtained by means of Tensorial-expanded Chaos Collocation.

Chaos Collocation method and Tensorial-expanded Chaos Collocation method differ just for the selection of collocation points. The collocation points, employed in Tensorial-expanded Chaos Collocation approach, are obtained by a full factorial of roots of order $p + 1$ one-dimensional Hermite polynomial with

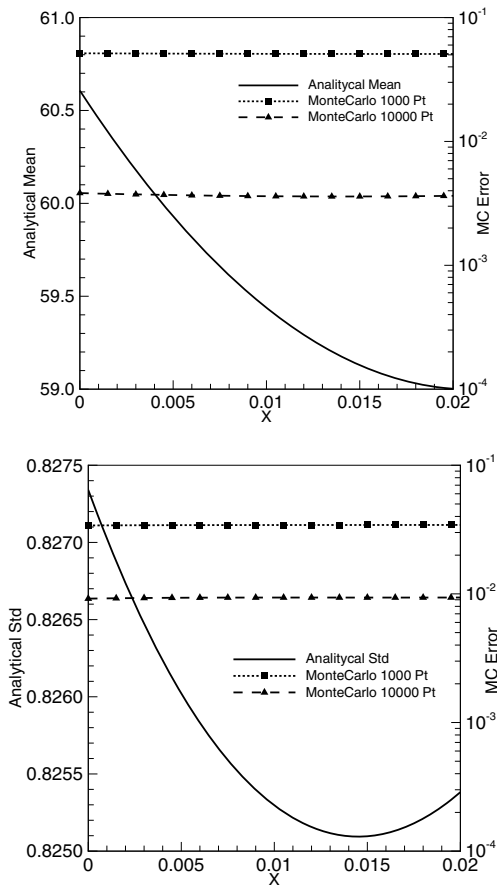


Figure 5: Mean $E(T)$ (above) and standard deviation $Std(T)$ (below) of problem Eq. (26): absolute error of solution obtained by Monte Carlo respect to analytical solution. The Monte Carlo $E(T)$ and $Std(T)$ have been computed with different number of realizations: 1000 and 10000.

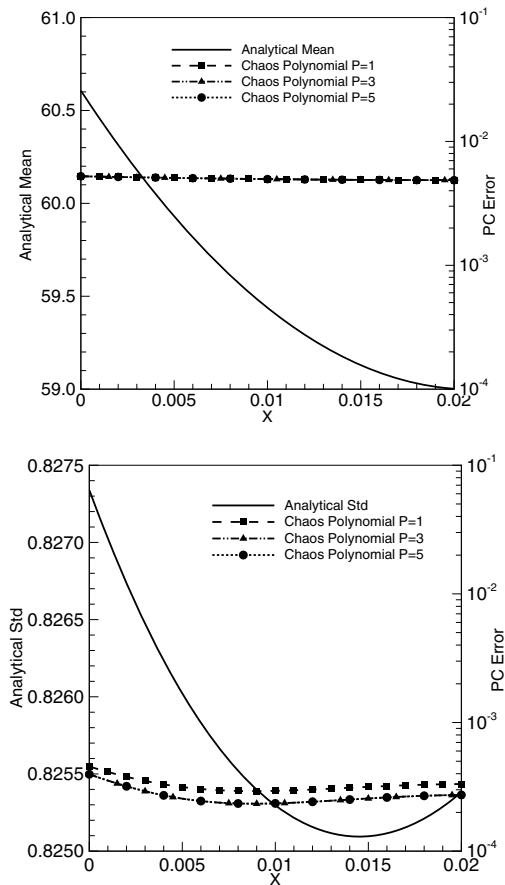


Figure 6: Mean $E(T)$ (above) and standard deviation $Std(T)$ (below) of problem Eq. (26): absolute error of solution obtained by Chaos Polynomial respect to analytical solution. The Chaos Polynomial $E(T)$ and $Std(T)$ have been computed with different expansion polynomial orders: $P = 1$, $P = 3$ and $P = 5$.

n factors, where n is the number of uncertain variables. The collocation points, employed in Chaos Collocation approach, are the set of $N + 1$ ($< (p + 1)^n$) points with the highest associated probability chosen among full-factorial points. As random parameters are independent, the probability associated to the point belonging to full-factorial is given by the product of the probabilities associated to each single root.

The results, illustrated above, suggest several comments.

Monte Carlo, which is the methodology mainly used for uncertainty quantification [Xiu and Tar-

takovsky (2006); Xiu and Karniadakis (2003a)], is strongly dependent on the number of realizations, as demonstrated in Ref. [Parussini and Pediroda (2007); Xiu and Karniadakis (2003a)]. Figure 5 shows an higher number of realizations increases significantly the accuracy, in particular of standard deviation and besides it is evident, taking in account both of accuracy and computational cost, Monte Carlo method is not comparable with Polynomial Chaos methods.

Polynomial Chaos methodologies, both intrusive and non-intrusive, have played up much the same

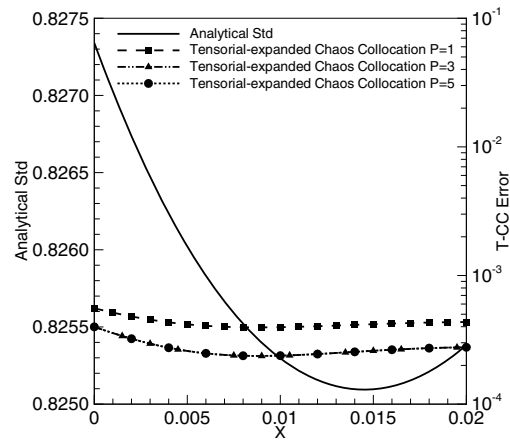
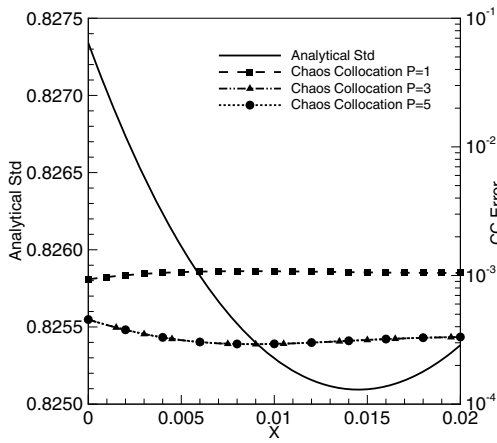
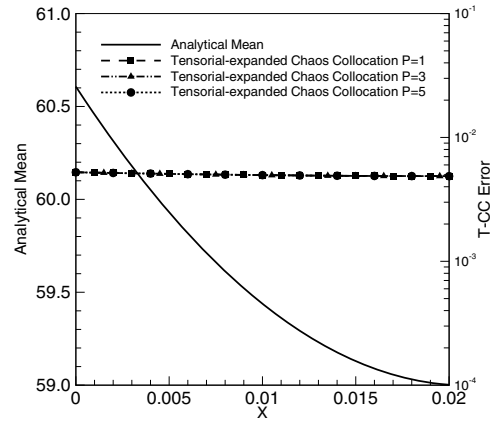
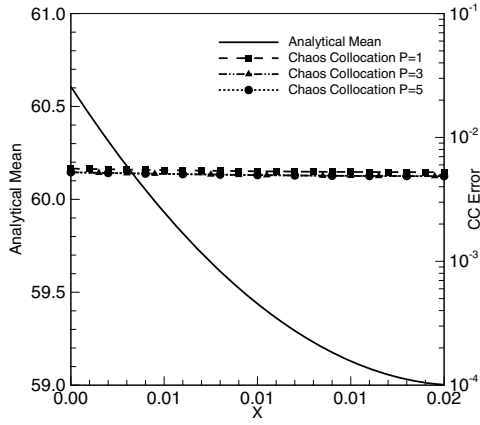


Figure 7: Mean $E(T)$ (above) and standard deviation $Std(T)$ (below) of problem Eq. (26): absolute error of solution obtained by Chaos Collocation respect to analytical solution. The Chaos Collocation $E(T)$ and $Std(T)$ have been computed with different expansion polynomial orders: $P = 1$, $P = 3$ and $P = 5$.

Figure 8: Mean $E(T)$ (above) and standard deviation $Std(T)$ (below) of problem Eq. (26): absolute error of solution obtained by Tensorial-expanded Chaos Collocation respect to analytical solution. The Tensorial-expanded Chaos Collocation $E(T)$ and $Std(T)$ have been computed with different expansion polynomial orders: $P = 1$, $P = 3$ and $P = 5$.

accuracy for the problem under study and it has been verified that increasing the expansion polynomial order further on $p = 3$ there is not a significantly improvement of accuracy. The reason why there is not a decrease of error with increasing p from 3 to 5 is related to two different aspects: the first is the analytical solution is affected by numerical integration errors, as explained above, the second is the use of single precision number representation.

Let us notice that employing Polynomial Chaos methodologies, if the expansion polynomial has

order p and n is the number of uncertain variables, Chaos Polynomial method needs to solve a system of $\frac{(n+p)!}{n!p!}$ coupled equations, Chaos Collocation needs to solve $\frac{(n+p)!}{n!p!}$ decoupled equations and Tensorial-expanded Chaos Collocation needs to solve $(p+1)^n$ decoupled equations. This means, if $p = 3$ and $n = 4$, Chaos Polynomial method needs to solve a system of 35 coupled equations, Chaos Collocation needs to solve 35 decoupled equations and Tensorial-expanded Chaos Collocation needs to solve 256 decoupled

equations. This demonstrates as the results we obtain by Polynomial Chaos are better than those ones obtained by Monte Carlo method with a considerable higher number of simulations.

About a comparison among Polynomial Chaos methodologies, the accuracy we get by the different approaches is pretty the same, but the non-intrusive Polynomial Chaos methods have the not negligible advantage respect to Chaos Polynomial they do not require an internal modification of differential problem solver.

As last remark, in the problem under study we have computed the mean and standard deviation of an analytical function, so that the implementation of Polynomial Chaos methodologies is quite simple, but usually we have to compute the mean and standard deviation of the solution of a differential equation. It appears clear the drawback of these methodologies is the need to modify the computational domain for every different simulation. If we can not solve analytically the differential equation, we have to remesh the computational domain for each new simulation and it is well-known to find an appropriate parameterization of partitions of domain, which is good for all geometries, is a difficult task.

To overcome this problem we introduce the Fictitious Domain methodology and exploit it to solve differential problems with uncertain parameters. In this way the stochastic domain does not coincide with the computational domain, which is the same for all simulations, and for every new geometry the trace of Lagrange multipliers, which enforce the boundary conditions immersed in the computational domain, has just to be modified.

5 Fictitious Domain via Lagrange Multipliers with Least-Squares Spectral Element Method

5.1 Fictitious Domain approach

Fictitious domain approach rises to solve differential problems defined on domain changing in time and space, i.e. in general structural elastic problems, fluid dynamics problems with moving rigid bodies, shape optimization problems, and so on. This means the same problem is solved on differ-

ent domains. Therefore such a method is suitable to solve differential problems defined on stochastic geometries.

Unlike the usual approach, based on the boundary variation technique where a sequence of domains is considered (figure 9), in Fictitious Domain approach (figure 10) the computational domain is not the same of the definition domain of problem, but it contains that one. Hence when the definition domain changes the computational domain does not change with evident advantages.

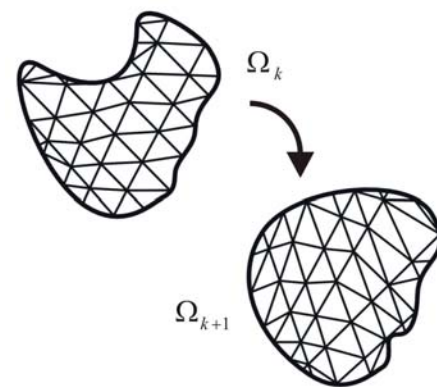


Figure 9: Classical approach based on the boundary variation technique to solve differential problems defined on domain changing in time and space.

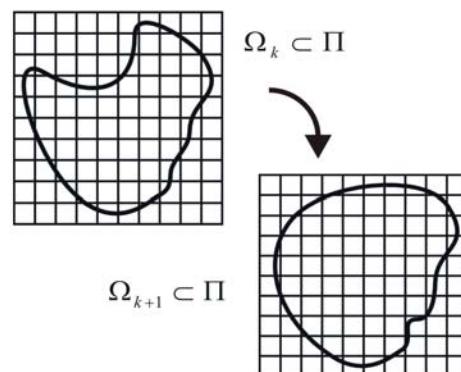


Figure 10: Fictitious Domain approach to solve differential problems defined on domain changing in time and space.

Several variants of fictitious domain method exist: the basic idea is to extend the operator and the domain into a larger simple shaped domain. The most important ways to do this are algebraic [Makinen, Rossi, and Toivanen (2000); Rossi and Toivanen (1999)] and functional analytic approaches [Glowinski, Pan, Hesla, Joseph, and Periaux (2000); Glowinski, Pan, and Periaux (1994); Ramiere, Angot, and Belliard (2007)]. More flexibility and better efficiency can be obtained by using a functional analytic approach where the use of constraints ensures that the solution of extended problem coincides with the solution of original problem. In our implementation we enforce constraints by Lagrange multipliers [Parussini (2007)].

The physical aspects of the problem are stated in a variational principle form, which specifies a scalar quantity, the functional J , defined by an integral form

$$J = \int_{\Omega} F \left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \dots, x, y, \dots \right) d\Omega + \int_{\Gamma} E \left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \dots, x, y, \dots \right) d\Gamma \quad (27)$$

where $\Gamma = \partial\Omega$, ϕ is the unknown function and F and E are specified operators. The solution to the continuum problem is a function ϕ which make J stationary with respect to small changes $\delta\phi$; thus, for a solution to the continuum problem, the variation is $\delta J = 0$.

To implement the Fictitious Domain approach we have to extend the operator F and the domain Ω into a larger simple shaped domain Π and to constrain the functional on $\Gamma = \partial\Omega$ (figure 11). To treat such problems Lagrangian multipliers are introduced, so that the problem is now equivalent to find the stationary point of J' , where

$$J' = \int_{\Pi} F \left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \dots, x, y, \dots \right) d\Omega + \int_{\Gamma} \lambda(\mathbf{x}) E \left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \dots, x, y, \dots \right) d\Gamma. \quad (28)$$

Here $\lambda(\mathbf{x})$ is an undetermined multiplier which is in general a function of position, because the local condition must be satisfied at every point of Γ , rather than being a global restriction.

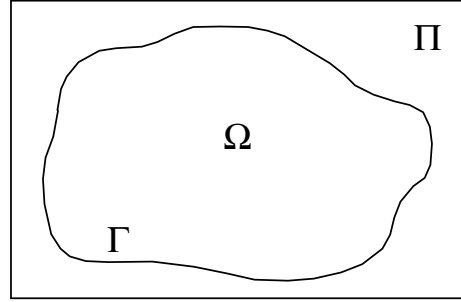


Figure 11: Example of a fictitious rectangular domain Π containing the original domain Ω .

5.2 Least Squares variational principle and Spectral Element approximation

The Fictitious Domain solver, we employ in this work, is based on a high order method. To discretize the problem under study we use the Least Squares Spectral Element Method, based on higher order functions, locally defined over finite size parts of domain. The Least Squares Spectral Element Method (LSQSEM) [Pontaza and Reddy (2003); Proot and Gerritsma (2002); Pontaza and Reddy (2006); Proot and Gerritsma (2005)] combines the least squares formulation with a spectral element approximation. This provides several advantages. The method produces symmetric positive definite linear systems for every type of partial differential equation, i.e. elliptic, parabolic and hyperbolic equations. The method converges just as fast with hp refinement than conventional Galerkin methods. Furthermore, no stabilization is required for convection dominated flows.

Let us consider a model problem stated as follows:

Find $\phi(\mathbf{x})$ such that

$$-\Delta\phi = f \quad \text{in } \Omega \quad (29)$$

$$\phi = \phi^s \quad \text{on } \Gamma \quad (30)$$

where Ω in \mathfrak{R}^d with d number of space dimensions, $\partial\Omega = \Gamma$ is the boundary of Ω , f is the source term and ϕ^s is the prescribed value of ϕ on boundary Γ . This problem is chosen only for notational simplicity; our statements are also valid for every type of deterministic differential prob-

lem, i.e. elliptic, parabolic and hyperbolic equations, and every type of boundary condition, i.e. Dirichlet, Neumann and Robin.

The Fictitious Domain Least-Squares functional associated with first-order equivalent system of Eqs. (29-30) will be:

$$J(\phi, \mathbf{q}, \lambda; f, \phi^s) = \frac{1}{2} \|\nabla \cdot \mathbf{q} - f\|_{0,\Pi}^2 + \frac{1}{2} \|\nabla \phi - \mathbf{q}\|_{0,\Pi}^2 + \frac{1}{2} \|\nabla \times \mathbf{q}\|_{0,\Pi}^2 + \|\lambda(\phi - \phi^s)\|_{0,\Gamma} \quad (31)$$

where $\Pi \supset \Omega$ and the Lagrange multiplier defined on Γ is denoted by λ , with μ associated weight function.

Therefore the least squares principle for functional Eq. (31) can be stated as:

Find $(\phi, \mathbf{q}, \lambda) \in \mathbf{X} \times M$ such that for all $(\psi, \mathbf{p}, \mu) \in \mathbf{X} \times M$

$$J(\phi, \mathbf{q}, \lambda; f, \phi^s) \leq J(\psi, \mathbf{p}, \mu; f, \phi^s), \quad (32)$$

where we use the spaces $\mathbf{X} = \{(\phi, \mathbf{q}) \in H^1(\Pi) \times \mathbf{H}^1(\Pi)\}$ and $M = \{\lambda \in H^{-1/2}(\Gamma)\}$.

The solution of problem, Eq. (29-30), will be the restriction to Ω of the minimum, defined on domain Π , of functional Eq. (32).

To get approximated solution of minimization problem of least-squares functional the spectral *hp* element method is employed [Karniadakis and Sherwin (1999); Gerritsma and Maerschalck (2006); Wu, Liu, Scarpas, and Ge (2006); Wu, Al-Khoury, Kasbergen, Liu, and Scarpas (2007); Mitra and Gopalakrishnan (2006); Komatitsch and Vilotte (1998); Komatitsch, Vilotte, Vai, Castillo-Covarrubias, and Sánchez-Sesma (1999); Lin (1998)].

For more details on the implementation of Fictitious Domain and Least-Squares Spectral Element method see Ref.[Parussini and Pediroda (2007)].

5.3 Numerical example

To verify the accuracy of numerical algorithm based on Fictitious Domain Method and Least-Squares Spectral Element approximation we

solve the equation of thermal diffusion in cooling fins with negligible thickness. The problem writes as:

$$\frac{d^2 T}{dx^2} = \frac{2h}{s\lambda} T \quad \text{with} \\ -\lambda \frac{dT}{dx} \Big|_{x=0} = q_0 \quad \text{and} \quad -\lambda \frac{dT}{dx} \Big|_{x=L} = hT \quad (33)$$

where we set $s = 2mm$, $h = 8W/m^2K$, $\lambda = 65W/mK$, $q_0 = 10000W/m^2$ and $L = 20mm$. The analytical solution of the problem is plotted in figure 12 (see Ref. [Bonacina, Cavallini, and Matarolo (1989)] for more details about cooling fins with negligible thickness).

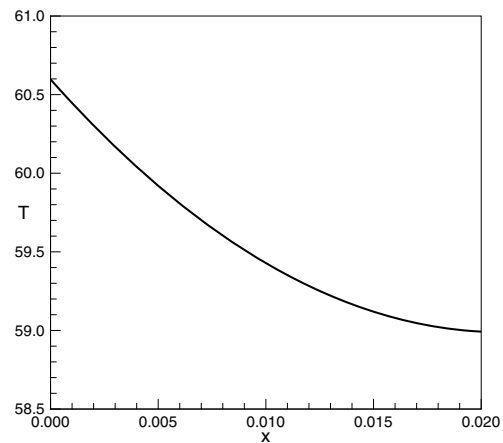


Figure 12: Analytical solution of stationary diffusion problem Eq. (33) in a cooling fin with negligible thickness.

To verify the accuracy of our algorithm, the numerical solution has been compared with the analytical one. The numerical solution is obtained considering the fictitious domain $\Pi = [-5.0mm, 25.0mm]$, larger than the original one $\Omega = [0.0mm, 20.0mm]$. The Neumann and Robin constraints, which are now *immersed* in the domain, have been imposed by Lagrange multipliers. Let us notice in one-dimensional problems the Lagrange multiplier is just a constant defined on the constrained point. To get the solution of

the equivalent problem least squares spectral elements have been employed. The domain Π has been meshed uniformly. To understand how the accuracy of solution improves varying the grid size, we have tested different grids, that means Π has been divided into 2, 5 and 8 equal elements.

In figure 13 we plot the relative energy norm of temperature T as function of expansion order P , for different grid steps, where the relative energy norm is so defined

$$\eta = \sqrt{\frac{\int_{\Omega} (\theta_{numeric} - \theta_{analytic})^2 d\Omega}{\int_{\Omega} \theta_{analytic}^2 d\Omega}}. \quad (34)$$

We can observe that the results are significantly accurate for $P \geq 5$. In the convergence plots spectral convergence is evident because on the linear-logarithmic plot the convergence line is linear. It can be remarked that η has an asymptotic behaviour and it can not be improved, beyond its asymptotic value, increasing the number of spectral elements or the polynomial order of trial functions. The asymptotic value, corresponding to machine round-off, is reached quickly, indicating a very high accuracy of the method. We can observe that an increase on number of grid elements does not improve remarkably the accuracy.

More examples and tests on Fictitious Domain and Least-Squares Spectral Element method can be found in Ref. [Parussini (2007); Parussini and Pediroda (2007)].

6 Formulation of stochastic Fictitious Domain problems

Let $\theta \in \Theta$ be a random realization drawn from a complete probability space (Θ, A, P) , whose event space Θ generates its σ -algebra $A \subset 2^{\Theta}$ and is characterized by a probability measure P . For all $\theta \in \Theta$, let $\Omega(\theta) \subset \mathfrak{R}^d$ be a d -dimensional random domain bounded by boundary $\Gamma(\theta)$. We consider the following stochastic boundary value problem: for P -almost everywhere in Θ , find a stochastic solution $\phi : \overline{\Omega}(\theta) \rightarrow \mathfrak{R}$ such that:

Find $\phi(\mathbf{x}, \theta)$ such that

$$-\Delta \phi = f \quad \text{in } \Omega(\theta) \quad (35)$$

$$\phi = \phi^s \quad \text{on } \Gamma(\theta) \quad (36)$$

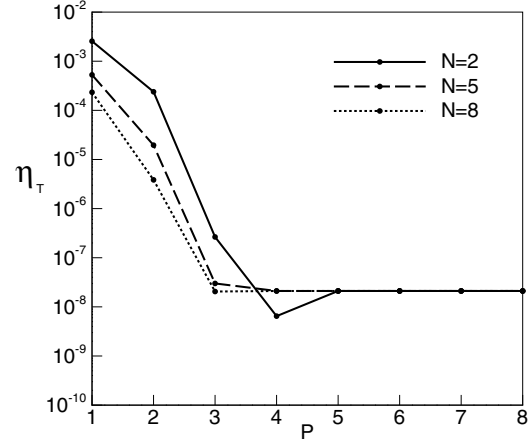


Figure 13: Relative energy norm η of temperature T versus the expansion order p of spectral elements for thermal diffusion problem Eq. (33). Several discretization h have been considered: fictitious domain has been divided in 2, 5 and 8 elements.

where f is the source term and ϕ^s is the prescribed value of ϕ on stochastic boundary $\Gamma(\theta)$. This problem is chosen only for notational simplicity.

We proceed by replacing the problem, Eq. (35-36), with its first-order equivalent system:

Find $\phi(\mathbf{x}, \theta)$ and $\mathbf{q}(\mathbf{x}, \theta)$ such that

$$-\nabla \cdot \mathbf{q} = f \quad \text{in } \Omega(\theta) \quad (37)$$

$$\nabla \phi - \mathbf{q} = 0 \quad \text{in } \Omega(\theta) \quad (38)$$

$$\nabla \times \mathbf{q} = 0 \quad \text{in } \Omega(\theta) \quad (39)$$

$$\phi = \phi^s \quad \text{on } \Gamma(\theta) \quad (40)$$

where \mathbf{q} is the flux of scalar function ϕ .

The L^2 least-squares functional associated with first-order equivalent system formulation is given by

$$J(\phi, \mathbf{q}; f) = \frac{1}{2} \|\nabla \cdot \mathbf{q} - f\|_{0, \Omega(\theta)}^2 + \frac{1}{2} \|\nabla \phi - \mathbf{q}\|_{0, \Omega(\theta)}^2 + \frac{1}{2} \|\nabla \times \mathbf{q}\|_{0, \Omega(\theta)}^2. \quad (41)$$

and its Fictitious Domain implementation will be:

$$J(\phi, \mathbf{q}, \lambda; f, \phi^s) = \frac{1}{2} \|\nabla \cdot \mathbf{q} - f\|_{0,\Pi}^2 + \frac{1}{2} \|\nabla \phi - \mathbf{q}\|_{0,\Pi}^2 + \frac{1}{2} \|\nabla \times \mathbf{q}\|_{0,\Pi}^2 + \|\lambda(\phi - \phi^s)\|_{0,\Gamma(\theta)} \quad (42)$$

where the Lagrange multiplier defined on Γ is denoted by λ , with μ the associated weight function.

The least squares principles for functional Eq. (42) can be stated as:

Find $(\phi, \mathbf{q}, \lambda) \in \mathbf{X} \times M(\theta)$ such that for all $(\psi, \mathbf{p}, \mu) \in \mathbf{X} \times M(\theta)$

$$J(\phi, \mathbf{q}, \lambda; f, \phi^s) \leq J(\psi, \mathbf{p}, \mu; f, \phi^s), \quad (43)$$

where we use the spaces $\mathbf{X} = \{(\phi, \mathbf{q}) \in H^1(\Pi) \times \mathbf{H}^1(\Pi)\}$ and $M(\theta) = \{\lambda \in H^{-1/2}(\Gamma(\theta))\}$.

This yields:

$$\left\{ \begin{array}{l} \text{Find } (\phi, \mathbf{q}, \lambda) \in \mathbf{X} \times M(\theta) \text{ such that} \\ a((\phi, \mathbf{q}), (\psi, \mathbf{p})) + b((\psi, \mathbf{p}), \lambda) = \\ l((\psi, \mathbf{p})) \quad \forall (\psi, \mathbf{p}) \in \mathbf{X} \\ b((\phi, \mathbf{q}), \mu) = g(\mu) \quad \forall \mu \in M(\theta) \end{array} \right. \quad (44)$$

where

$$a((\phi, \mathbf{q}), (\psi, \mathbf{p})) = \int_{\Pi} (-\nabla \cdot \mathbf{q})(-\nabla \cdot \mathbf{p}) d\Pi + \int_{\Pi} (\nabla \phi - \mathbf{q}) \cdot (\nabla \psi - \mathbf{p}) d\Pi + \int_{\Pi} (\nabla \times \mathbf{q}) \cdot (\nabla \times \mathbf{p}) d\Pi \quad (45)$$

$$b((\psi, \mathbf{p}), \lambda) = \int_{\Gamma(\theta)} \psi \lambda d\Gamma(\theta) \quad (46)$$

$$l((\psi, \mathbf{p})) = \int_{\Pi} f(-\nabla \cdot \mathbf{p}) d\Pi \quad (47)$$

$$g(\mu) = \int_{\Gamma(\theta)} \phi^s \mu d\Gamma(\theta). \quad (48)$$

The solution of problem, Eq. (35-36), will be the restriction to $\Omega(\theta)$ of the minimum, defined on domain Π , of functional Eq. (43).

The saddle point problem Eq. (44) has a stochastic formulation. We assume that the boundary $\Gamma(\mathbf{x}, \theta)$ of $\Omega(\mathbf{x}, \theta) \subset \Pi(\mathbf{x})$ depends on θ via n mutually independent real random variables $\xi(\theta)$ with zero mean and unit variance with respect to a density function ρ defined on some interval $I \in \mathfrak{R}$, so that $\mathbf{I} = I^n$. Referring to Eq. (6) we can write the stochastic process as

$$\Gamma(\mathbf{x}, \theta) \cong \Gamma^*(\mathbf{x}, \theta) = \sum_{i=0}^N \Gamma_i(\mathbf{x}) H_i(\xi). \quad (49)$$

Substituting the polynomial Chaos series into Eq. (44) we obtain

$$\left\{ \begin{array}{l} \text{Find } (\phi^*, \mathbf{q}^*, \lambda^*) \in L_{\rho}^2(\mathbf{I}; \mathbf{X}) \times L_{\rho}^2(\mathbf{I}; M^*) \\ \text{such that} \\ a((\phi^*, \mathbf{q}^*), (\psi^*, \mathbf{p}^*)) + b((\psi^*, \mathbf{p}^*), \lambda^*) = \\ l((\psi^*, \mathbf{p}^*)) \quad \forall (\psi^*, \mathbf{p}^*) \in L_{\rho}^2(\mathbf{I}; \mathbf{X}) \\ b((\phi^*, \mathbf{q}^*), \mu^*) = g(\mu^*) \quad \forall \mu^* \in L_{\rho}^2(\mathbf{I}; M^*) \end{array} \right. \quad (50)$$

where

$$\phi^*(\mathbf{x}, \theta) = \sum_{i=0}^N \phi_i(\mathbf{x}) H_i(\xi) \quad (51)$$

$$\mathbf{q}^*(\mathbf{x}, \theta) = \sum_{i=0}^N \mathbf{q}_i(\mathbf{x}) H_i(\xi) \quad (52)$$

$$\lambda^*(\mathbf{x}, \theta) = \sum_{i=0}^N \lambda_i(\mathbf{x}) H_i(\xi) \quad (53)$$

and $M^* = \{\lambda^* \in H^{-1/2}(\Gamma^*)\}$. In this way we divide the random process into a deterministic part and a stochastic part. To solve Eq. (48) the method of Weighted Residuals is adopted.

Performing the Galerkin projection, the fomulation gives a system of coupled equations:

$$\left\{ \begin{array}{l} \text{Find } (\phi^*, \mathbf{q}^*, \lambda^*) \in L_{\rho}^2(\mathbf{I}; \mathbf{X}) \times L_{\rho}^2(\mathbf{I}; M^*) \\ \text{such that} \\ \langle (a^*(\cdot, \cdot) + b^*(\cdot, \cdot)), H_j \rangle = \\ \langle l^*(\cdot), H_j \rangle \quad \forall (\psi^*, \mathbf{p}^*) \in L_{\rho}^2(\mathbf{I}; \mathbf{X}) \\ \langle (b^*(\cdot, \cdot)), H_j \rangle = \\ \langle g^*(\cdot), H_j \rangle \quad \forall \mu^* \in L_{\rho}^2(\mathbf{I}; M^*) \end{array} \right. \quad (54)$$

for $j = 0, \dots, N$ with obvious notation.

Performing the Collocation projection, the formulation gives a linear system of decoupled equations equivalent to solving a deterministic problem at each grid point:

$$\begin{cases} \text{Find } (\phi_i, \mathbf{q}_i, \lambda_i) \in \mathbf{X} \times M_i \text{ such that} \\ a_i(\cdot, \cdot) + b_i(\cdot, \cdot) = l_i(\cdot) \quad \forall (\boldsymbol{\psi}_i, \mathbf{p}_i) \in \mathbf{X} \\ b_i(\cdot, \cdot) = g_i(\cdot) \quad \forall \boldsymbol{\mu}_i \in M_i \end{cases} \quad (55)$$

with $i = 0, \dots, N$ where $M_i = \{\lambda_i \in H^{-1/2}(\Gamma_i)\}$. To reconstruct the stochastic solution $\phi(\mathbf{x}, \boldsymbol{\theta})$ the equations Eq. (15-16) are used.

7 Applications of Polynomial Chaos methods with Fictitious Domain solver

7.1 One-dimensional problem

To verify the accuracy of Polynomial Chaos methodologies coupled to Fictitious Domain approach let us consider the heat conduction problem of Eqs. (22-23) with uncertainties defined in Eq. 26 once again. We will compare the numerical results to the analytical solution shown in figure 4.

In Section 4 it has been already demonstrated the Monte Carlo method is not as accurate as Polynomial Chaos approximations on equal computational cost, so we will not analyze the results of Monte Carlo method coupled to Fictitious Domain.

Figure 14 shows the accuracy of $E(T)$ and $Std(T)$ obtained by Chaos Polynomial with Fictitious Domain solver, figure 15 shows that one obtained by Chaos Collocation with Fictitious Domain solver and figure 16 shows that one obtained by Tensorial-expanded Chaos Collocation with Fictitious Domain solver.

All the tests have been computed employing the Fictitious Domain approach. For the Chaos Polynomial method the Fictitious domain solver has been modified, so as to implement internally the uncertainty quantification methodology. Differently for the Chaos Collocation and Tensorial-expanded Chaos Collocation methods the original Fictitious Domain solver, developed to solve

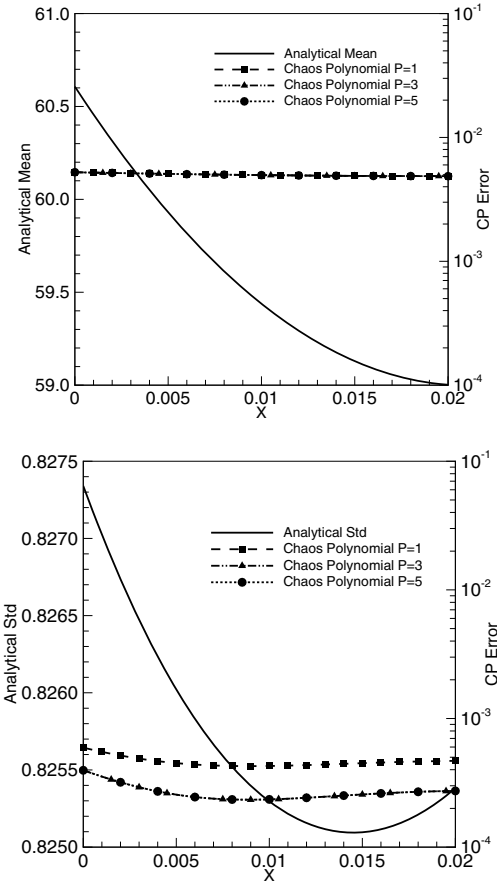


Figure 14: Mean $E(T)$ (above) and standard deviation $Std(T)$ (below) of problem Eq. (26): absolute error of solution obtained by Chaos Polynomial with Fictitious Domain solver respect to analytical solution. The Chaos Polynomial $E(T)$ and $Std(T)$ have been computed with different expansion polynomial orders: $P = 1$, $P = 3$ and $P = 5$.

deterministic differential problems, has been exploited.

The fictitious domain we have considered is $\Pi = [-5.0mm, 25.0mm]$ and it has been divided into 2 elements with expansion polynomial order 7.

By the figures, it is evident the introduction of Fictitious Domain approach into uncertainty quantification methodologies to solve the differential equation does not imply a loss of accuracy. The error plots illustrated above and those shown in

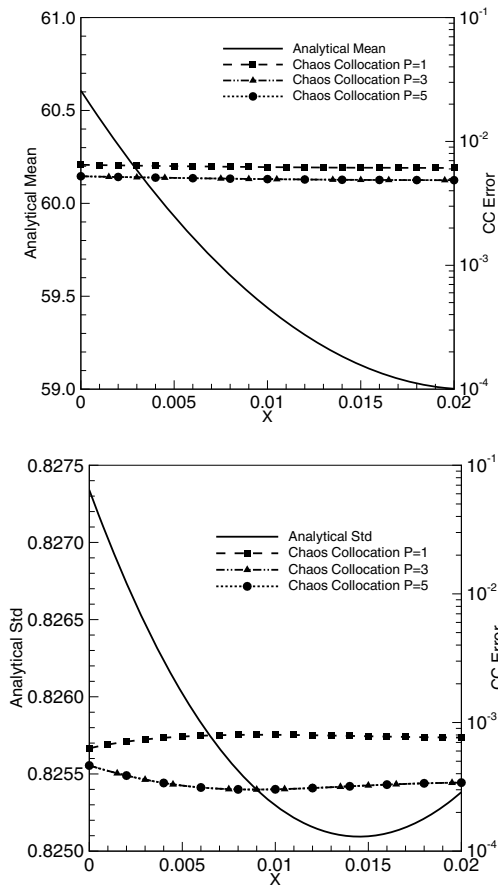


Figure 15: Mean $E(T)$ (above) and standard deviation $Std(T)$ (below) of problem Eq. (26): absolute error of solution obtained by Chaos Collocation with Fictitious Domain solver respect to analytical solution. The Chaos Collocation $E(T)$ and $Std(T)$ have been computed with different expansion polynomial orders: $P = 1$, $P = 3$ and $P = 5$.

Section 4, where the probability distribution of the analytical solution has been studied, are pretty the same and unvaried remarks about accuracy and computational cost can be done.

Once again, this example demonstrates that a non-intrusive methodology has the same capability of an intrusive approach, without the need of an internal modification of the differential problem solver. Thereby, that being so, we can use Chaos Collocation method and Tensorial-expanded Chaos Collocation to study a two-dimensional elliptic problem with multi geomet-

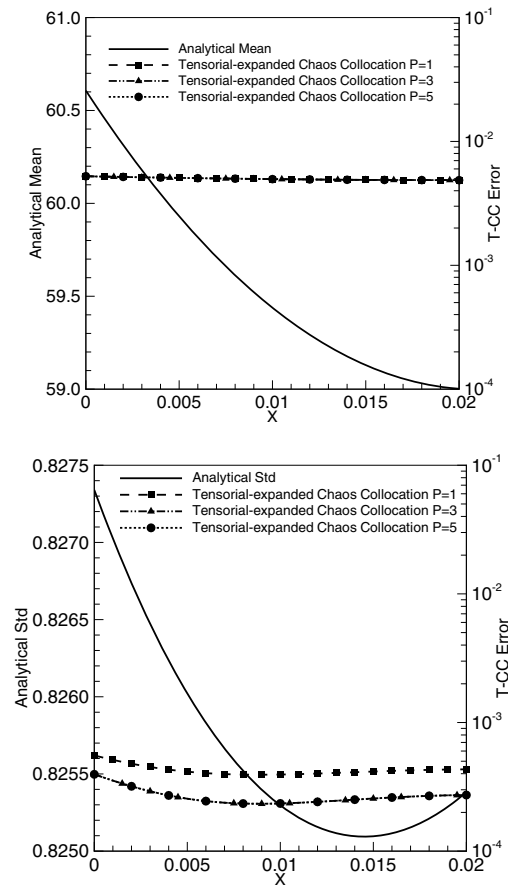


Figure 16: Mean $E(T)$ (above) and standard deviation $Std(T)$ (below) of problem Eq. (26): absolute error of solution obtained by Tensorial-expanded Chaos Collocation with Fictitious Domain solver respect to analytical solution. The Tensorial-expanded Chaos Collocation $E(T)$ and $Std(T)$ have been computed with different expansion polynomial orders: $P = 1$, $P = 3$ and $P = 5$.

ric uncertainties.

Another consideration can be appended. In figure 15 it can be notice the error plot of mean and standard deviation obtained by means of Chaos Collocation is more influenced by polynomial order than error plot of mean and standard deviation obtained by means of Tensorial-expanded Chaos Collocation. It is possible this behaviour is due to the choice of collocation points which is not unique for Chaos Collocation method. In the next section the consequences of this arbitrary choice

will be illustrated, as for a two-dimensional problem they will be more evident.

7.2 Two-dimensional problem

In this section we consider the stationary heat conduction in an electronic chip [Xiu and Karniadakis (2003a)], subject to geometric tolerances:

$$-\nabla \cdot (k\nabla T) = f \quad \text{in } \Omega(\theta) \quad (56)$$

with $k = 1$ and $f = 0$. The stochastic domain is shown in figure 17. The domain dimensions are deterministic parameters except thickness of cavity L_1 which has a normal distribution $N(0.6, 0.01)$ and the lengths $L_2 = N(2.0, 0.01)$ and $L_3 = N(2.0, 0.01)$. The boundary of domain consists of four segments: the top Γ_T , the bottom Γ_B , the two sides Γ_S and the boundaries of the cavity Γ_C . Adiabatic boundary conditions are prescribed on Γ_B and Γ_S . The cavity boundary Γ_C is exposed to heat flux $q_b|_{\Gamma_C} = 1$. On the top Γ_T is maintained at constant temperature $T = 0$.

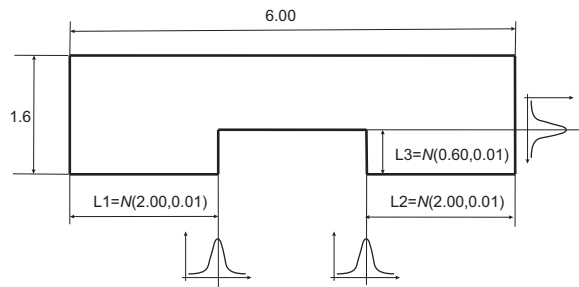


Figure 17: Stochastic domain of stationary heat conduction problem Eq. (56).

Figure 18 shows the computational domain which differs from chip geometry and in particular contains it, according to Fictitious Domain approach. The fictitious domain has been discretized into 15 spectral elements of order 8.

We are interested in the stochastic solution at the points of domain. Actually we solve the heat conduction problem on all the points of fictitious domain and we associate to each point the probability to belong to chip, as shown in figure 19 for the top left corner of the cavity. In this way we

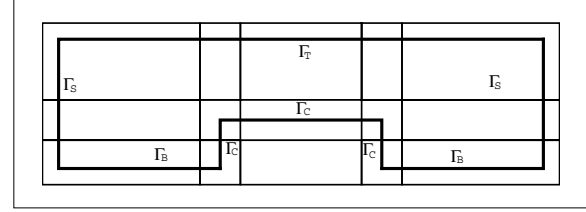


Figure 18: Schematic of the computational fictitious domain for stationary heat conduction problem of Eq. (56).

have the mean and the standard deviation of temperature due to geometric tolerance even in points which do not really belong to chip, but, as we associate a probability of belonging, the results are still significant.

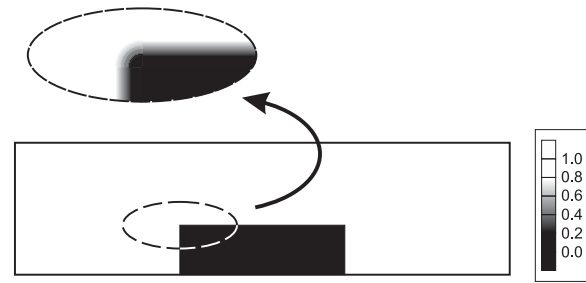


Figure 19: Probability of belonging to electronic chip under study with stochastic geometry $\Omega(\theta)$ for points of fictitious domain shown in figure 18.

Let us consider the solution along the axis of symmetry $A - A$ shown in figure 20, which corresponds to section $x = 0$.

Figure 21 shows the mean $E(T)$ along section $A - A$ obtained by means of Chaos Collocation and Tensorial-expanded Chaos Collocation with different polynomial orders: $p = 1$, $p = 2$, $p = 3$ and $p = 4$.

Figure 22 shows the standard deviation $Std(T)$ along section $A - A$ obtained by means of Chaos Collocation and Tensorial-expanded Chaos Collocation with the same polynomial orders.

The values of mean we compute by Chaos Collocation or Tensorial-expanded Chaos Collocation are the same, whereas the curves of standard devi-

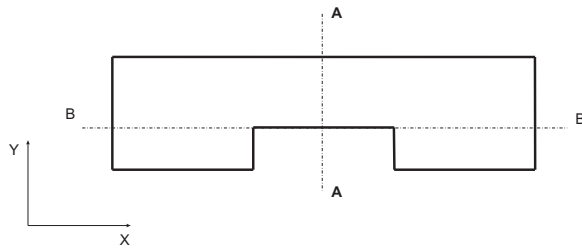


Figure 20: Section A – A and section B – B identify the slices along which Chaos Collocation and Tensorial-expanded Chaos Collocation methods have been compared.

ation are different on equal polynomial expansion order. This difference is more evident comparing figure 23 and figure 24.

Figure 23 shows the contours of mean field and the contours of standard deviation of temperature computed by means of Chaos Collocation with order $p = 3$.

Figure 24 shows the contours of mean field and the contours of standard deviation of temperature computed by means of Tensorial-expanded Chaos Collocation with order $p = 3$.

These differences are due to the choice of collocation points, which is different for Chaos Collocation and Tensorial-expanded Chaos Collocation, as already emphasized. Taking a look to solutions obtained by Chaos Collocation and Tensorial-expanded Chaos Collocation, the risk of computing not accurate standard deviation solutions employing the Chaos Collocation is plainly observable. For the problem under study the contours of standard deviation of temperature should be symmetric respect to section A – A, but in figure 23 they are not really symmetric. This wrong behaviour of Chaos Collocation standard deviation is more evident in figure 25, which shows the comparison of standard deviation along the axis B – B ($y = 0$) obtained by Chaos Collocation and Tensorial-expanded Chaos Collocation with polynomial order $p = 3$.

This example has highlighted the difficulty for a Chaos Collocation approach to select a good set of collocation points when multi dimensional un-

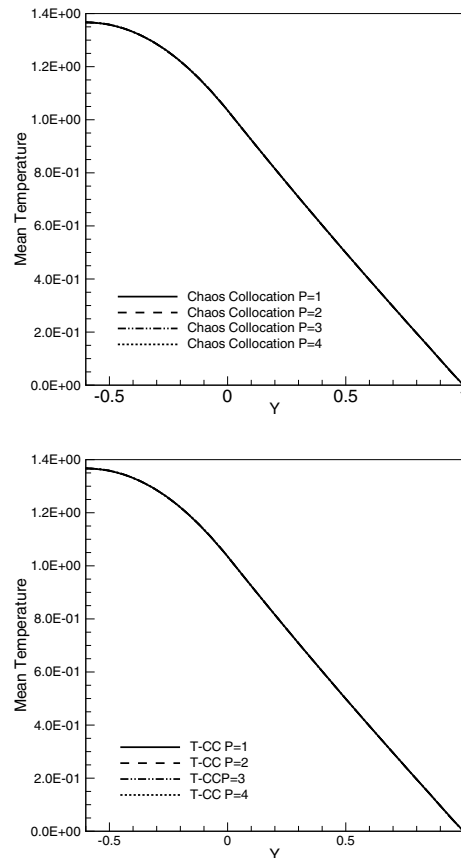


Figure 21: Expected value of solution of problem Eq. (56) along section A – A obtained by means of: Chaos Collocation coupled to Fictitious Domain solver (above) and Tensorial-expanded Chaos Collocation coupled to Fictitious Domain solver (below), with expansion polynomial order $P = 1$, $P = 2$, $P = 3$ and $P = 4$.

certainties are present. The choice is not unique and can bring to unphysical results. This problem does not exist for Tensorial-expanded Chaos Collocation formulation. It is true the methodology is computationally more expensive than multidimensional Chaos Collocation, but the advantage is to avoid an arbitrary choice of collocation points in behalf of an higher accuracy.

As final remark, on the base of implemented tests and applications, in our opinion Tensorial-expanded Chaos Collocation method owns several advantages respect other Polynomial Chaos

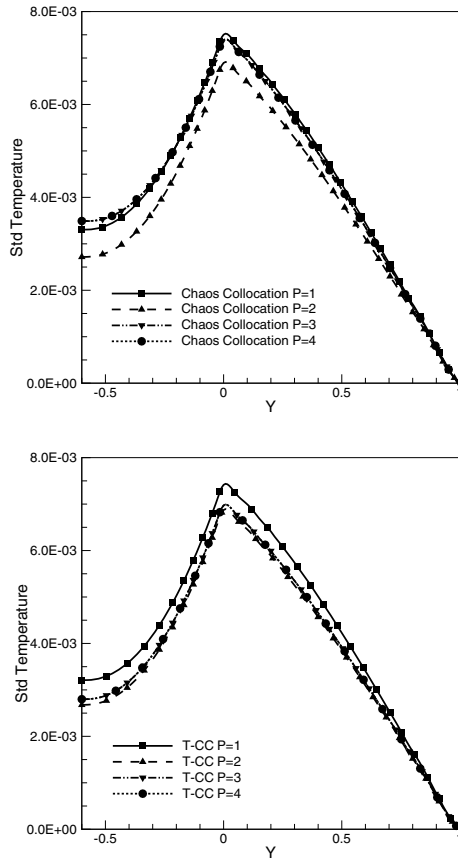


Figure 22: Standard deviation of solution of problem Eq. (56) along section A – A obtained by means of: Chaos Collocation coupled to Fictitious Domain solver (above) and Tensorial-expanded Chaos Collocation coupled to Fictitious Domain solver (below), with expansion polynomial order $P = 1, P = 2, P = 3$ and $P = 4$.

methodologies, because it is non-intrusive and independent by an arbitrary choice of collocation points.

8 Conclusions

In this paper a comparison among different Polynomial Chaos methodologies has been presented, in order to find the best method for facing multi uncertain problems, with particular attention to geometric uncertainties given by shape tolerances.

In literature there are not systematic studies and

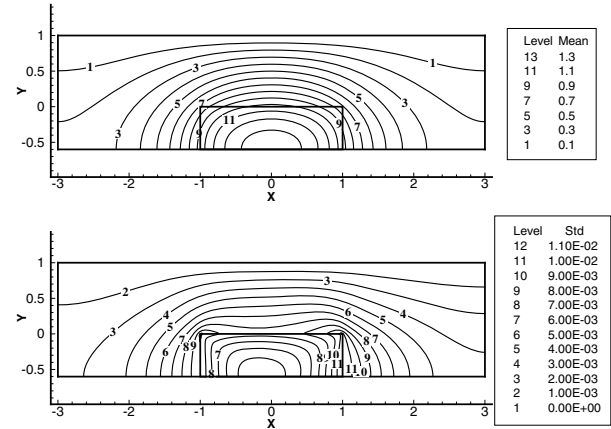


Figure 23: Contours of temperature distribution in the electronic chip under study: mean field (above) and standard deviation (below) field. Mean and standard deviation have been obtained by means of Chaos Collocation method with expansion polynomial order $P = 3$ coupled to Fictitious Domain solver.

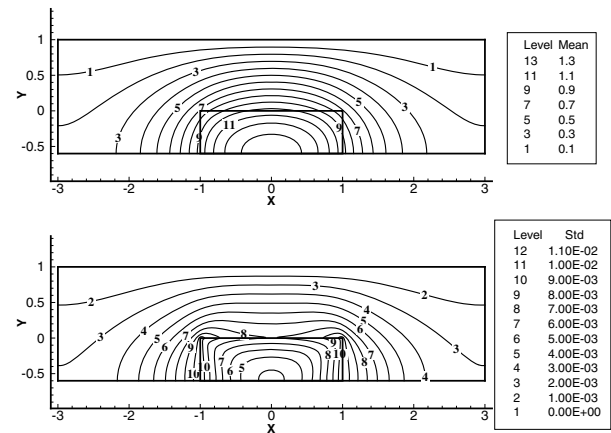


Figure 24: Contours of temperature distribution in the electronic chip under study: mean field (above) and standard deviation (below) field. Mean and standard deviation have been obtained by means of Tensorial-expanded Chaos Collocation method with expansion polynomial order $P = 3$ coupled to Fictitious Domain solver.

comparisons among different methodologies for uncertainty quantification. Moreover the

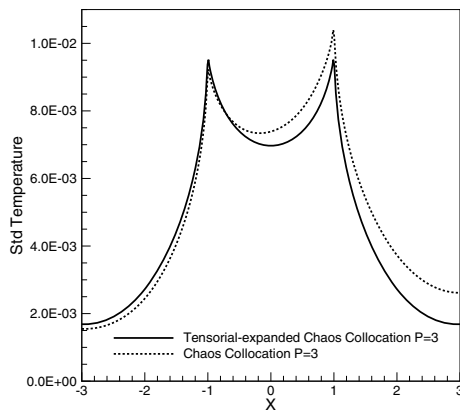


Figure 25: Comparison of standard deviation of problem Eq. (56) along section $B - B$ obtained by Chaos Collocation and Tensorial-expanded Chaos Collocation with polynomial order $P = 3$ coupled to Fictitious Domain solver.

sis of stochastic geometries is still an unprobed field, especially when multi geometric uncertainties are present. This work is an attempt to fill in this blank.

The tests have been accomplished sequentially from the simplest to the most complex cases, in order to cast away the least suitable methodologies to study differential problems with multi geometric uncertainties. The methods we have compared are Monte Carlo, Chaos Polynomial, Chaos Collocation and Tensorial-expanded Chaos Collocation.

Initial numerical experiments have been performed on analytic functions and have demonstrated Polynomial Chaos methodologies are better than Monte Carlo Method for accuracy on equal computational cost.

Pointed out the superiority of Polynomial Chaos methods, these methodologies have been coupled to Fictitious Domain approach and Least-Squares Spectral Element method. This formulation is of particular interest to study problems on stochastic domain, as Fictitious Domain approach allows to avoid the remeshing of computational domain in the presence of geometric uncertainties. Its main advantage lies in the fact that only one Cartesian

mesh, that represents the enclosure, needs to be generated.

In order to demonstrate the capabilities and the drawbacks of Polynomial Chaos methodologies coupled with a Fictitious Domain solver, they have been employed to solve a one-dimensional elliptic problem with multi uncertainties, both on geometry and material properties. Excellent accuracy properties of non-intrusive methods, i.e. Chaos Collocation and Tensorial-expanded Chaos Collocation, have been demonstrated. Their performance is comparable to that one of intrusive Chaos Polynomial method, with the advantage of avoiding to modify the solver of partial differential equation. The solver is just a black-box and in this way we get a simplification of computational process management.

On the consequence of these observations, Chaos Collocation and Tensorial-expanded Chaos Collocation methods with Fictitious Domain approach have been employed to solve a two-dimensional elliptic problem: the stationary heat conduction in an electronic chip [Xiu and Karniadakis (2003a)]. In particular multi geometric uncertainties have been considered, in order to determine the differences between these non-intrusive methodologies. This example highlights the difficulty to select collocation points in presence of multi geometric uncertainties employing the Chaos Collocation approach, because the choice is not unique and the results we obtain should be unphysical. Certainly physical solution will be obtained by the Tensorial-expanded Chaos Collocation method, where the employed set of collocation point is not arbitrary. Nevertheless the computational cost of tensorial approach is higher compared to Chaos Collocation method.

In conclusion, it has been observed as a non-intrusive tensorial method coupled to a Fictitious Domain solver represents an efficient methodology to solve multi uncertain problems, in particular when stochastic.

Several issues need to be addressed:

- Tensorial-expanded Chaos Collocation method with different expansion orders for each uncertain variable, in order to determine

the minimum number of collocation points and decrease the computational cost;

- investigation of Fictitious Domain method for solving Navier-Stokes equations and the examination of accuracy of Polynomial Chaos methods coupled to Fictitious Domain solver for Fluid Dynamic problems with geometric uncertainties.

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