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A Lie-Group Shooting Method for Simultaneously Estimating the Time-Dependent Damping and Stiffness Coefficients

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Abstract: For the inverse vibration problem, a Lie-group shooting method is proposed to simultaneously estimate the time-dependent damping and stiffness functions by using two sets of displacement as inputs. First, we transform these two ODEs into two parabolic type PDEs. Second, we formulate the inverse vibration problem as a multi-dimensional two-point boundary value problem with unknown coefficients, allowing us to develop the Lie-group shooting method. For the semi-discretizations of PDEs we thus obtain two coupled sets of linear algebraic equations, from which the estimation of damping and stiffness coefficients can be written out explicitly. The present approach is very interesting, which resulting to closed-form estimating equations without needing of any iteration and initial guess of coefficient functions, and more importantly, it does not require to assume a priori the functional forms of unknown coefficients. The estimated results are rather accurate convicing that the new method can be employed in the vibrational engineering to identify viscoelastic property of time-aging materials.

Keyword: Inverse vibration problem, Time-dependent damping and stiffness coefficients, Lie-group shooting method

1 Introduction

One of the major purposes of structural dynamics is to analyze and determine the mechanical parameters and responses of a given structure subjected to various external loading conditions. Based on the results analyzed, structural engineers are able to check whether a proposed structural design meets the requirements of adequate resistance to a combination of loading conditions and, if necessary, to revise a proposed design until all such requirements are satisfied. In the last several decades elastic analysis of structures has been used primarily as the basis for the calculation of forces to obtain a great amount of results in the design of engineering structure. Even, structures may exhibit linearly elastic behavior, there are many structures respond inelastically and exhibit hysteretic behavior [Liu (1997); Liu (2004); Liu and Huang (2004)]. Hysteresis depicts the hereditary and memory nature of an inelastic system, in which the restoring force of the structural member depends not only on current input of loading but also on the past history of loading. Hysteretic models have been used for several vibrational damping isolator made of viscoelastic materials. Since it is important to be informed about the possible dissipation losses, one needs to know their viscoelastic properties in dependence on frequency and temperature. This usually leads to a time-depenent viscoelastic behavior of structures.

The dissipation of energy in a mechanical structure is most frequently described by a viscous damping model. The resulting equation of vibration is attractive because of the ease with which it can be mathematically treated. However, sometimes we may encounter the problem that the viscoelastic properties in structure or the external force are not yet known, and then the resulting problem is an inverse vibration problem. It is concerned with the estimations of these properties such as damping coefficient [Adhikari and Woodhouse (2001a); Adhikari and Woodhouse (2001b); Ingman and Suzdalnitsky (2001); Liang and Feeny (2006)], stiff-

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ness [Huang (2001); Shiguemori, Chiwiacowsky and de Campos Velho (2005)], as well as external force [Huang (2005); Feldman (2007)]. With the aid of measurable vibration data, such as frequency, mode shape, displacement or velocity at different time, the researchers are interesting to estimate these properties.

In the realm of linear inverse vibration problems by estimating constant damping or stiffness coefficients there were many papers, for example, Gladwell (1986), Gladwell and Movahhedy (1995), Lancaster and Maroulas (1987), Starek and Inman (1991, 1995, 1997), and Starek, Inman and Kress (1992). However, when the coefficients are time-dpendent the inverse vibration problems are nonlinear and they are more difficult to solve. Huang (2001) has employed the conjugate gradient method to solve the nonlinear inverse vibration problem for the estimation of time-dependent stiffness coefficient. To the best knowledge of author, in addition the works by Liu (2008a) and Liu, Chang, Chang and Chen (2008), there does not have study to concern with the nonlinear inverse vibration problem for estimating both the time-dependent damping and stiffness coefficients. For this reason we are going to develop an accurate method to solve this nonlinear inverse vibration problem.

Let us consider a second-order ordinary differential equation (ODE) describing the forced vibration of a linear structure with time-dependent parameters $c(t)$ and $k(t)$:

$$\ddot{\phi}+c(t)\dot{\phi}+k(t)\phi=F(t),\ \ 0\leq t\leq t_f, \tag{1}$$

$$\phi(0)=A_0, \tag{2}$$

$$\dot{\phi}(0)=B_0. \tag{3}$$

The direct problem is for the given initial conditions in Eqs. (2) and (3) and the given functions $c(t)$, $k(t)$ and $F(t)$ in Eq. (1) to find the displacement $\phi(t)$ in a time interval of $t\in[0,t_f]$. However, our present inverse vibration problem is to estimate $c(t)$ and $k(t)$ with $t\in[0,t_f]$ by using some measured data of $\phi(t)$ in a time interval of $t\in[0,t_f]$. Because we have only one equation (1), it is difficult to estimate two unknown functions $c(t)$ and $k(t)$. Therefore, in order to supplement another equation we consider

$$\ddot{\psi}+c(t)\dot{\psi}+k(t)\psi=H(t),\ \ 0\leq t\leq t_f, \tag{4}$$

$$\psi(0)=C_0, \tag{5}$$

$$\dot{\psi}(0)=D_0. \tag{6}$$

When we use these two sets of data ϕ and ψ as inputs on our estimation equations, we may estimate $c(t)$ and $k(t)$ simultaneously. Here, for the later convenience we use two different symbols ϕ and ψ in the same equation of motion; however, when either external forces or initial values are different the two functions $\phi(t)$ and $\psi(t)$ are different. In practice, in order to obtain two different functions $\phi(t)$ and $\psi(t)$ we can prepare two specimens made of the same material, and impose them by different external loadings and/or different initial conditions.

The present approach is original. One may appreciate that the present approach is very interesting, which resulting to closed-form estimating equations without needing of any iteration and initial guess of coefficient functions. More importantly, the novel method does not require to assume a priori the functional forms of unknown coefficients.

Recently, Liu (2006a, 2006b, 2006c) has made a breakthrough to extend the method of group preserving scheme (GPS) previously developed by Liu (2001) for ODEs to boundary value problems (BVPs), namely the Lie-group shooting method (LGSM), and the numerical results revealed that the LGSM is a rather promising method to effectively solve the two-point BVPs. In the construction of Lie-group method for the calculations of BVPs, Liu (2006a) has introduced the idea of one-step GPS by utilizing the closure property of Lie group, and hence, the new shooting method has been named the Lie-group shooting method.

On the other hand, in order to effectively solve the backward in time problems of parabolic type PDEs, a past cone structure and a backward group preserving scheme have been successfully developed, such that the one-step Lie-group numerical methods have been used to solve the backward in time Burgers equation by Liu (2006d), and the backward in time heat conduction equation by Liu, Chang and Chang (2006a).

In a series of papers by the author and his coworkers, the Lie-group method reveals its excellent behavior on the numerical solutions of different problems, for example, Chang, Liu and Chang (2005) to calculate the sideways heat conduction problem, Chang, Chang and Liu (2006) to treat the boundary layer equation in fluid mechanics, and Liu (2004), Liu, Chang and Chang (2006a), and Chang, Liu and Chang (2007a, 2007b) to treat the backward heat conduction equation, and Liu, Chang and Chang (2006b) to treat the Burgers equation.

It needs to stress that the one-step Lie-group property is usually not shared by other numerical methods, because those methods do not belong to the Lie-group types. This important property as first pointed out by Liu (2006d) was employed to solve the backward in time Burgers equation. After that, Liu (2006e) has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended to estimate the thermophysical properties of heat conductivity and heat capacity by Liu (2006f, 2007) and Liu, Liu and Hong (2007). Recently, Liu (2008b,2008c) has explored its superiority by using the LGSM to estimate parameters in parabolic type PDEs. The Lie-group method possesses a great advantage than other numerical methods due to its group structure, and it is a powerful technique to solve the inverse problems of parameters identification.

In the paper by Liu, Chang, Chang and Chen (2008), the estimation equations are based on the data of displacement and velocity of one motion. It appears that the accuracy and stability are not so good. This paper will extend this parameters identification technique to the inverse vibration problems but using a different technique of LGSM based on two different displacement sets, which is arranged as follows. We introduce a novel approach of inverse vibration problem in Section 2 by transforming it into an identification problem of parabolic type PDEs, and then the discretizations of PDEs into a system of ODEs at the discretized times are derived. Here we explain why a multi-dimensional two-point boundary value problem appears naturally. In Section 3 we give a brief sketch of the GPS for ODEs for a self-content reason. Due to the good property of Lie-group, we will propose an integration technique, such that the one-step GPS can be used to identify the parameters appeared in the introduced PDEs. The resulting algebraic equations are derived in Section 4 when we apply the one-step GPS to identify $c(t)$ and $k(t)$. In Section 5 numerical examples are examined to test the Lie-group shooting method (LGSM). Finally, we give some conclusions in Section 6.

2 Two transformations

Basically the set of Eqs. (1)-(3) and the set of Eqs. (4)-(6) have the same form. So we only consider the mathematical derivations for the first set of Eqs. (1)-(3), and after deriving the required equations, we can directly apply them to Eqs. (4)-(6).

2.1 Transformation into a PDE

In the solutions of linear PDE, a common technique is the seperation of variables, from which the PDE is transformed into some ODEs. In this study we reverse this process by considering

$$u(x,t) = (1+x)\phi(t), \tag{7}$$

such that Eqs. (1)-(3) can be changed to a parabolic type PDE:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial t^2} + c(t)\frac{\partial u(x,t)}{\partial t} + k(t)u(x,t) + \phi(t) - (1+x)F(t), \tag{8}$$

$$u(0,t) = \phi(t), \tag{9}$$

$$u(x,0) = A_0(1+x), \tag{10}$$

$$u(x,t_f) = \phi(t_f)(1+x), \tag{11}$$

where $\phi(t_f)$ is a measured displacement at a final time t_f. In Eq. (8) $c(t)$ and $k(t)$ are time-dependent functions to be identified, where the domain we consider is $0 \le t \le t_f,\ 0 < x \le x_f$. The coordinate x is a fictitious one; however, from it together with t we can work in a two-dimensional domain and is therefore more easy to view the variations of $c(t)$ and $k(t)$ from the x-direction.

The above idea by transforming ODE into PDE is first proposed by Liu (2008d) to treat an inverse Sturm-Liouville problem. There is maybe another selection of Eq. (7) by using for example $u(x,t) = q(x)\phi(t)$, where we require that $q(0) = 0$. However, when $q(x)$ is more complex than $1+x$ the resulting PDE is more complex than Eq. (8), and there seems no good reason to select a complex $q(x)$.

2.2 Transformation into a set of ODEs

Applying a semi-discrete procedure to PDE yields a coupled system of ODEs. For Eq. (8), we adopt the following discretizations:

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=i\Delta t} = \frac{u_{i+1}(x) - u_i(x)}{\Delta t}, \tag{12}$$

$$\left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{t=i\Delta t} = \frac{u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)}{(\Delta t)^2}, \tag{13}$$

where $\Delta t = t_f/(n+1)$ is a uniform time increment, and $u_i(x) = u(x, i\Delta t)$ for a simple notation. Hence, Eq. (8) can be approximated by

$$u_i'(x) = \frac{1}{(\Delta t)^2}[u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)] + \frac{c_i}{\Delta t}[u_{i+1}(x) - u_i(x)] + k_i u_i(x) + h_i(x), \tag{14}$$

where $c_i = c(t_i)$, $k_i = k(t_i)$, and $h_i(x) = \phi_i - (1+x)F_i$ with $\phi_i = \phi(t_i)$ and $F_i = F(t_i)$, $i = 1,\ldots,n$.

When $i = 1$ the term $u_0(x)$ in Eq. (14) is replaced by the boundary condition (10) with $u_0(x) = A_0(1+x)$. Similarly, when $i = n$ the term $u_{n+1}(x)$ is replaced by the boundary condition (11) with $u_{n+1}(x) = \phi_{n+1}(1+x) = \phi(t_f)(1+x)$. Eq. (14) has totally n coupled linear ODEs for the n variables $u_i(x), i = 1,\ldots,n$.

In this section we have transformed the inverse vibration problem of the second-order ODE in Eq. (1) to an inverse problem for the PDE in Eq. (8), and this is also true for Eq. (4). Therefore we come to an estimation of $2n$ coefficients c_i and k_i in a $2n$-dimensional ODEs system.

Now the problem becomes a two-point boundary value problem with Eq. (14) not only subjecting to an initial condition $u_i(0) = \phi_i$ and also subjecting to a final condition $u_i(x_f) = (1+x_f)\phi_i$ obtained from Eq. (7) by inserting $x = x_f$, where x_f is a new constant chosen by the user. Therefore, we have overspecified conditions for the $2n$-dimensional ODEs system (14) and its counterpart for Eq. (4); however, because c_i and k_i are unknown, we can use this two-point boundary value problem formulation to find c_i and k_i. Below, we will develop a Lie-group shooting method to solve this problem.

3 GPS for differential equations system

3.1 Group-preserving scheme

Upon letting $\mathbf{u} = (u_1,\ldots,u_n)^{\mathrm{T}}$ and denoting by $\mathbf{f}$ the right-hand side of Eq. (14) we can write it as a vector form:

$$\mathbf{u}' = \mathbf{f}(\mathbf{u},x), \;\; \mathbf{u} \in \mathbb{R}^n, \;\; x \in \mathbb{R}. \tag{15}$$

Liu (2001) has embedded Eq. (15) into an augmented differential equations system as follows:

$$\frac{d}{dx}\begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{u},x)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},x)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix}. \tag{16}$$

It is obvious that the first row in Eq. (16) is the same as the original equation (15), but the inclusion of the second row in Eq. (16) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{u}^{\mathrm{T}}, \|\mathbf{u}\|)^{\mathrm{T}}$, which satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \tag{17}$$

where

$$\mathbf{g} := \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{18}$$

is a Minkowski metric, $\mathbf{I}_n$ is the identity matrix of order n, and the superscript T stands for the transpose. In terms of $(\mathbf{u}, \|\mathbf{u}\|)$, Eq. (17) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{u}\cdot\mathbf{u} - \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{u}\|^2 = 0, \tag{19}$$

where the dot between two vectors denotes their Euclidean inner product.

Consequently, we have an $n+1$-dimensional augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \tag{20}$$

with a constraint (17), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{u},x)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},x)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \tag{21}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$, because of $\mathbf{A}$ satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}. \tag{22}$$

This fact prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{23}$$

$$\det \mathbf{G} = 1, \tag{24}$$

$$G_0^0 > 0, \tag{25}$$

where G_0^0 is the 00th component of $\mathbf{G}$.

Although the dimension of the new system is raising one more, it has been shown that the new system permits a GPS given as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell)\mathbf{X}_{\ell}, \tag{26}$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at x_ℓ, and $\mathbf{G}(\ell) \in SO_o(n,1)$ is the group value of $\mathbf{G}$ at x_ℓ. If $\mathbf{G}(\ell)$ satisfies the properties in Eqs. (23)-(25), then $\mathbf{X}_\ell$ satisfies the cone condition in Eq. (17).

The Lie group can be generated from $\mathbf{A} \in so(n,1)$ by an exponential mapping,

$$\mathbf{G}(\ell) = \exp[\Delta x \mathbf{A}(\ell)] = \begin{bmatrix} \mathbf{I}_n + \frac{(a_\ell - 1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell\mathbf{f}_\ell^{\mathrm{T}} & \frac{b_\ell \mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell \mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{27}$$

where

$$a_\ell := \cosh\left(\frac{\Delta x\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right), \tag{28}$$

$$b_\ell := \sinh\left(\frac{\Delta x\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right). \tag{29}$$

Substituting Eq. (27) for $\mathbf{G}(\ell)$ into Eq. (26), we obtain

$$\mathbf{u}_{\ell+1} = \mathbf{u}_\ell + \eta_\ell \mathbf{f}_\ell, \tag{30}$$

$$\|\mathbf{u}_{\ell+1}\| = a_\ell\|\mathbf{u}_\ell\| + \frac{b_\ell}{\|\mathbf{f}_\ell\|}\mathbf{f}_\ell \cdot \mathbf{u}_\ell, \tag{31}$$

where

$$\eta_\ell := \frac{b_\ell\|\mathbf{u}_\ell\|\|\mathbf{f}_\ell\| + (a_\ell - 1)\mathbf{f}_\ell \cdot \mathbf{u}_\ell}{\|\mathbf{f}_\ell\|^2}. \tag{32}$$

3.2 One-step GPS

Throughout this paper the superscript f denotes the value at $x = x_f$, while the superscript 0 denotes the value at $x = 0$. Assume that the total length x_f is divided by K steps, that is, the stepsize we use in the GPS is $\Delta x = x_f/K$.

Starting from $\mathbf{X}^0 = \mathbf{X}(0)$ we want to calculate the value $\mathbf{X}(x_f)$ at $x = x_f$. By Eq. (26) we can obtain

$$\mathbf{X}^f = \mathbf{G}_K(\Delta x)\cdots\mathbf{G}_1(\Delta x)\mathbf{X}^0. \tag{33}$$

However, let us recall that each $\mathbf{G}_i$, $i = 1,\ldots,K$, is an element of the Lie group $SO_o(n,1)$, and by the closure property of Lie group, $\mathbf{G}_K(\Delta x)\cdots\mathbf{G}_1(\Delta x)$ is also a Lie group denoted by $\mathbf{G}$. Hence, we have

$$\mathbf{X}^f = \mathbf{G}\mathbf{X}^0. \tag{34}$$

This is a one-step Lie-group transformation from $\mathbf{X}^0$ to $\mathbf{X}^f$.

3.2.1 A generalized mid-point rule

We can calculate $\mathbf{G}$ by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie group generated from such an $\mathbf{A} \in so(n,1)$ is known as a proper orthochronous Lorentz group, which admits a closed-form representation as follows:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^{\mathrm{T}} & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^{\mathrm{T}}}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{35}$$

where

$$\hat{\mathbf{u}} = r\mathbf{u}^0 + (1-r)\mathbf{u}^f, \tag{36}$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{\mathbf{u}}, \hat{x}), \tag{37}$$

$$a = \cosh\left(\frac{x_f\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right), \tag{38}$$

$$b = \sinh\left(\frac{x_f\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right). \tag{39}$$

Here, we use the initial $\mathbf{u}^0$ and the final $\mathbf{u}^f$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $0 < r < 1$ is a parameter and $\hat{x} = (1-r)x_f$. The above method was applied a generalized mid-point rule on the calculation of $\mathbf{G}$, and the resultant is a single-parameter Lie group element $\mathbf{G}(r)$. After developing the LGSM, we can determine the best r by matching the given final condition.

3.2.2 A Lie group mapping between two points on the cone

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{u}}\|}, \tag{40}$$

such that Eqs. (35), (38) and (39) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{41}$$

$$a = \cosh(x_f\|\mathbf{F}\|), \tag{42}$$

$$b = \sinh(x_f\|\mathbf{F}\|). \tag{43}$$

From Eqs. (34) and (41) it follows that

$$\mathbf{u}^f = \mathbf{u}^0 + \eta\mathbf{F}, \tag{44}$$

$$\|\mathbf{u}^f\| = a\|\mathbf{u}^0\| + b\frac{\mathbf{F}\cdot\mathbf{u}^0}{\|\mathbf{F}\|}, \tag{45}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{u}^0 + b\|\mathbf{u}^0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{46}$$

Substituting

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}^f - \mathbf{u}^0) \tag{47}$$

into Eq. (45) and dividing both the sides by $\|\mathbf{u}^0\|$ we can obtain

$$\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} = a + b\frac{(\mathbf{u}^f - \mathbf{u}^0)\cdot\mathbf{u}^0}{\|\mathbf{u}^f - \mathbf{u}^0\|\|\mathbf{u}^0\|}, \tag{48}$$

where

$$a = \cosh\left(\frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\eta}\right), \tag{49}$$

$$b = \sinh\left(\frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\eta}\right) \tag{50}$$

are obtained by inserting Eq. (47) for $\mathbf{F}$ into Eqs. (42) and (43).

Let

$$\cos\theta := \frac{[\mathbf{u}^f - \mathbf{u}^0]\cdot\mathbf{u}^0}{\|\mathbf{u}^f - \mathbf{u}^0\|\|\mathbf{u}^0\|}, \tag{51}$$

$$S := x_f\|\mathbf{u}^f - \mathbf{u}^0\|, \tag{52}$$

and from Eqs. (48)-(50) it follows that

$$\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos\theta\sinh\left(\frac{S}{\eta}\right). \tag{53}$$

By defining

$$Z := \exp\left(\frac{S}{\eta}\right), \tag{54}$$

we obtain a quadratic equation for Z from Eq. (53):

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|}Z + 1 - \cos\theta = 0. \tag{55}$$

The solution is found to be

$$Z = \frac{\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} + \sqrt{\left(\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|}\right)^2 - 1 + \cos^2\theta}}{1+\cos\theta}, \tag{56}$$

and then from Eqs. (54) and (52) we can obtain

$$\eta = \frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\ln Z}. \tag{57}$$

Therefore, between any two points $(\mathbf{u}^0, \|\mathbf{u}^0\|)$ and $(\mathbf{u}^f, \|\mathbf{u}^f\|)$ on the cone, there exists a Lie group element $\mathbf{G} \in SO_o(n,1)$ mapping $(\mathbf{u}^0, \|\mathbf{u}^0\|)$ onto $(\mathbf{u}^f, \|\mathbf{u}^f\|)$, which is given by

$$\begin{bmatrix} \mathbf{u}^f \\ \|\mathbf{u}^f\| \end{bmatrix} = \mathbf{G}\begin{bmatrix} \mathbf{u}^0 \\ \|\mathbf{u}^0\| \end{bmatrix}, \tag{58}$$

where $\mathbf{G}$ is uniquely determined by $\mathbf{u}^0$ and $\mathbf{u}^f$ through the following equations:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{59}$$

$$a = \cosh(x_f\|\mathbf{F}\|), \tag{60}$$

$$b = \sinh(x_f\|\mathbf{F}\|), \tag{61}$$

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}^f - \mathbf{u}^0). \tag{62}$$

4 Identifying $c(t)$ and $k(t)$ by the LGSM

In this section we start to estimate the time-dependent coefficient functions $c(t)$ and $k(t)$. From Eqs. (40) and (44) it follows a useful equation:

$$\mathbf{u}^f = \mathbf{u}^0 + \eta\frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{u}}\|}. \tag{63}$$

By using Eq. (7) we have

$$u_i^f = (1+x_f)u_i^0 = (1+x_f)\phi_i, \tag{64}$$

and thus the vector $\mathbf{u}^f$ with components u_i^f is proportional to $\mathbf{u}^0$ with components u_i^0 by a multiplier being $1+x_f$ larger than 1. Under this condition we have $\cos\theta = 1$ by Eq. (51), and from Eqs. (55) and (64) it follows that

$$Z = 1 + x_f. \tag{65}$$

Hence from Eqs. (57) and (64) we have

$$\eta = \frac{x_f^2\|\mathbf{u}^0\|}{\ln(1+x_f)}. \tag{66}$$

Moreover, by using Eqs. (36) and (64) we have

$$\|\hat{\mathbf{u}}\| = x_r\|\mathbf{u}^0\|, \tag{67}$$

where

$$x_r := 1 + \hat{x} = r + (1-r)(1+x_f). \tag{68}$$

Substituting Eqs. (66) and (67) into Eq. (63) we have

$$\mathbf{u}^f = \mathbf{u}^0 + \eta_0\hat{\mathbf{f}}, \tag{69}$$

where

$$\eta_0 = \frac{x_f^2}{x_r\ln(1+x_f)}. \tag{70}$$

By applying Eq. (69) to Eq. (14) we obtain

$$u_i^f = u_i^0 + \frac{\eta_0}{(\Delta t)^2}(\hat{u}_{i+1} - 2\hat{u}_i + \hat{u}_{i-1}) + \frac{\eta_0 c_i}{\Delta t}(\hat{u}_{i+1} - \hat{u}_i) + \eta_0 k_i\hat{u}_i + \eta_0\phi_i - \eta_0(1+\hat{x})F_i, \tag{71}$$

where

$$\hat{u}_i = x_r\phi_i, \quad i = 1,\ldots,n. \tag{72}$$

After inserting Eq. (72) for $\hat{u}_i$ and Eq. (70) for η_0, it is not difficult to rewrite Eq. (71) as

$$k_i\phi_i + \frac{c_i}{\Delta t}(\phi_{i+1} - \phi_i) = \frac{\phi_i\ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) - \frac{\phi_i}{x_r} + F_i. \tag{73}$$

Similarly for Eq. (4) we can derive

$$k_i\psi_i + \frac{c_i}{\Delta t}(\psi_{i+1} - \psi_i) = \frac{\psi_i\ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\psi_{i+1} - 2\psi_i + \psi_{i-1}) - \frac{\psi_i}{x_r} + H_i, \tag{74}$$

where $\psi_i = \psi(t_i)$ and $H_i = H(t_i)$.
Denoting Eq. (73) by

$$A_1^i k_i + B_1^i c_i = C_1^i, \tag{75}$$

we have

$$A_1^i = \phi_i, \tag{76}$$

$$B_1^i = \frac{\phi_{i+1} - \phi_i}{\Delta t}, \tag{77}$$

$$C_1^i = \frac{\phi_i\ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) - \frac{\phi_i}{x_r} + F_i. \tag{78}$$

On the other hand, from Eq. (74) we have

$$A_2^i k_i + B_2^i c_i = C_2^i, \tag{79}$$

where

$$A_2^i = \psi_i, \tag{80}$$

$$B_2^i = \frac{\psi_{i+1} - \psi_i}{\Delta t}, \tag{81}$$

$$C_2^i = \frac{\psi_i \ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\psi_{i+1} - 2\psi_i + \psi_{i-1}) - \frac{\psi_i}{x_r} + H_i. \tag{82}$$

From Eqs. (75) and (79) we can solve

$$k_i = \frac{B_2^i C_1^i - B_1^i C_2^i}{A_1^i B_2^i - A_2^i B_1^i}, \tag{83}$$

$$c_i = \frac{A_1^i C_2^i - A_2^i C_1^i}{A_1^i B_2^i - A_2^i B_1^i}. \tag{84}$$

Because of Eq. (68), the above estimating equations depend on r. Now, the problem is how to choose a suitable r. The numerical procedures for determining r are described as follows. In the range of $r \in (0,1)$ we insert each r into the above equations to obtain c_i and k_i, and we can integrate Eq. (14) from $x = 0$ to $x = x_f$ by noting Eq. (7). Then, u_i^f is given by

$$u_i^f = (1+x_f)\phi_i + \frac{1}{2}x_f(2+x_f) \left[\frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + \frac{c_i}{\Delta t}(\phi_{i+1} - \phi_i) + k_i\phi_i - F_i\right]. \tag{85}$$

By the same token we also have

$$v_i^f = (1+x_f)\psi_i + \frac{1}{2}x_f(2+x_f) \left[\frac{1}{(\Delta t)^2}(\psi_{i+1} - 2\psi_i + \psi_{i-1}) + \frac{c_i}{\Delta t}(\psi_{i+1} - \psi_i) + k_i\psi_i - H_i\right], \tag{86}$$

by defining v by $v = (1+x)\psi(t)$ as that defining u by Eq. (7). By comparing the above u_i^f and v_i^f with the targets given exactly by Eq. (64) and $(1+x_f)\psi_i$, we can pick up the best r by satisfying

$$\min_{r\in(0,1)} \sqrt{\sum_{i=1}^{n}[u_i^f - (1+x_f)\phi_i]^2 + [v_i^f - (1+x_f)\psi_i]^2}. \tag{87}$$

When r is selected we can insert it into Eqs. (83) and (84) to calculate k_i and c_i.

In Eqs. (83) and (84) there appears a common denominator $A_1^i B_2^i - A_2^i B_1^i := D_i$, which in view of Eqs. (76), (77), (80) and (81) can be seen as a discretized approximation of

$$D(t) := \phi(t)\dot{\psi}(t) - \psi(t)\dot{\phi}(t). \tag{88}$$

With the help of Eqs. (1) and (4) it is easy to derive

$$\dot{D}(t) + c(t)D(t) = \phi(t)H(t) - \psi(t)F(t). \tag{89}$$

From it we have

$$D(t) = \exp\left[\int_0^t c(\xi)d\xi\right]D(0) + \int_0^t \exp\left[\int_\xi^t c(\zeta)d\zeta\right][\phi(\xi)H(\xi) - \psi(\xi)F(\xi)]d\xi. \tag{90}$$

If we can choose the external forces $F(t)$ and $H(t)$ as such that $\phi(t)H(t) - \psi(t)F(t)$ has the same sign as that of the initial value of $D(0) = A_0D_0 - B_0C_0$ for all time of $0 < t \le t_f$ then $D(t)$ would be nonzero, and thus Eqs. (83) and (84) can be well defined without worrying that dividing by a zero value.

In the present method, the key points hinge on the formulation of two-point boundary value problems, the construction of two one-step GPS for the estimation of parameters, and the full use of the $n+1$ equations (44) and (45). To distinguish the present method by a combining use of the one-step GPS and the closed-form solution with the aid of the above equations, we may call the present method a Lie-group shooting method (LGSM).

5 Numerical examples

5.1 Example 1

Let us consider

$$c(t) = 3 + 2\cos(2\pi t), \tag{91}$$

$$k(t) = 20 + 2\sin(2\pi t), \tag{92}$$

$$F(t) = F_0 + F_1 t, \tag{93}$$

$$H(t) = H_0 + H_1 t. \tag{94}$$

In order to obtain the data of $\phi(t)$ and $\psi(t)$ we have applied the fourth-order Runge-Kutta method (RK4) to Eqs. (1)-(3) and to Eqs. (4)-(6) by using the initial conditions of $A_0 = 1$, $B_0 = 3$, $C_0 = 1.5$ and $D_0 = 6$.

We use the vibration data of displacements ϕ_i and ψ_i as the inputs to estimate c_i and k_i. In this calculation we have fixed $\Delta t = 1/200$, $F_0 = 40$, $F_1 = 0$, $H_0 = 70$, $H_1 = 10$, and $x_f = 0.01$. First we plot the error of mismatching with respect to r in Fig. 1(a), where the minimum is occurred at $r = 0.5$. The profile of $c(t)$ is plotted in Fig. 1(b) by the dashed line, which is compared with the exact one plotted by the solid line. The maximum estimation error of c is about 2.03×10^{-2}. Then, the profile of $k(t)$ is plotted in Fig. 1(c) by the dashed line, which is compared with the exact one plotted by the solid line, and the maximum estimation error of k is about 1.67×10^{-2}.

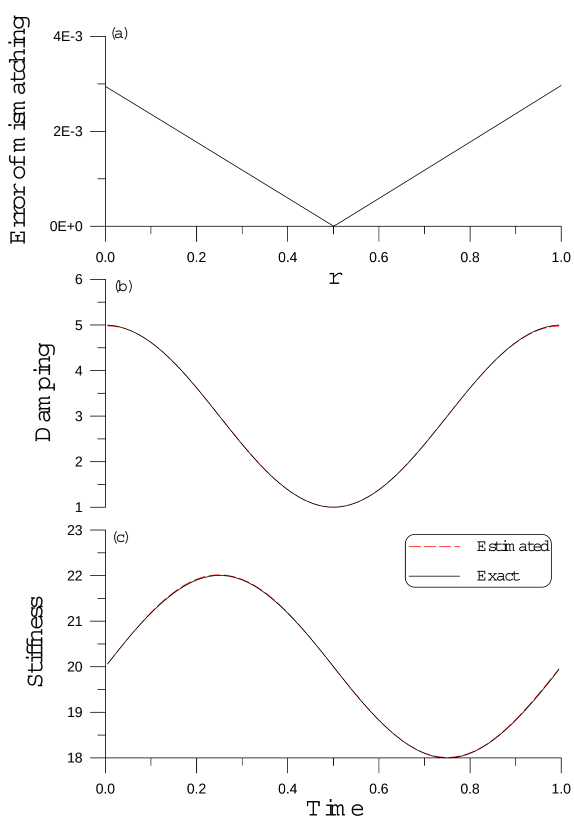

Figure 1: For Example 1: (a) showing the error of mismatching, (b) comparing the estimated and exact damping functions, and (c) comparing the estimated and exact stiffness functions.

5.2 Example 2

Then, we consider

$$c(t) = 3 + t^2, \tag{95}$$

$$k(t) = 20 + t. \tag{96}$$

For this example we use the following parameters $\Delta t = 1/150$, $F_0 = 50$, $F_1 = 20$, $H_0 = 50$, $H_1 = 0$, $A_0 = 1$, $B_0 = 5$, $C_0 = 1.5$, $D_0 = 3$ and $x_f = 0.01$ to estimate c and k. The error of mismatching with respect to r is plotted in Fig. 2(a). The maximum estimation error of c is about 4.14×10^{-2} as shown in Fig. 2(b), and the maximum estimation error of k is about 7.02×10^{-2} as shown in Fig. 2(c).

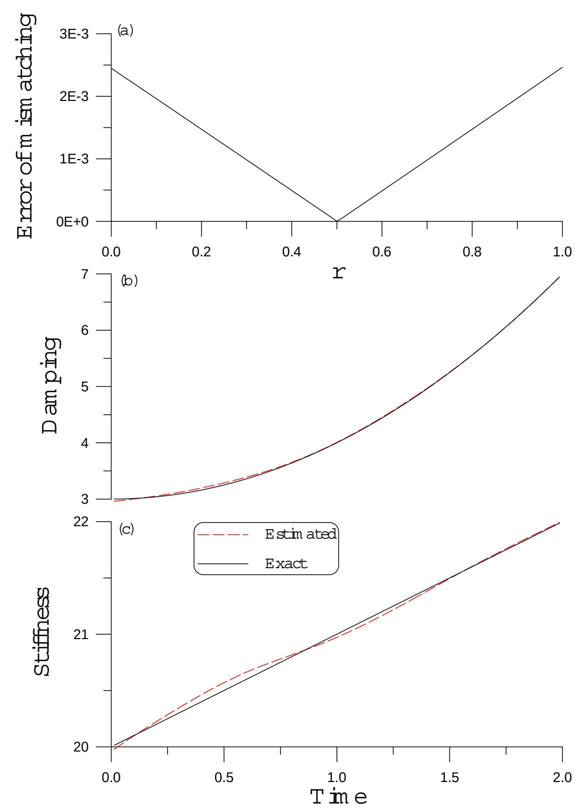

Figure 2: For Example 2: (a) showing the error of mismatching, (b) comparing the estimated and exact damping functions, and (c) comparing the estimated and exact stiffness functions.

5.3 Example 3

Let us consider discontinuous and oscillatory parameters:

$$c(t) = \begin{cases} 2 & t \in [0, 0.1], \\ 10 & t \in (0.1, 0.3), \\ 8 & t \in (0.3, 0.6], \\ 5 + \sin(10\pi t) & t \in (0.6, 1], \end{cases} \tag{97}$$

$$k(t) = \begin{cases} 20 & t \in [0, 0.3], \\ 30 & t \in (0.3, 0.6], \\ 20 + \sin(10\pi t) & t \in (0.6, 1]. \end{cases} \tag{98}$$

For this example we use the following parameters $\Delta t = 1/250$, $F_0 = 40$, $F_1 = 0$, $H_0 = 80$, $H_1 = 5$, $A_0 = 1$, $B_0 = 3$, $C_0 = 0$, $D_0 = 8$ and $x_f = 0.1$ to estimate c and k. The error of mismatching with respect to r is plotted in Fig. 3(a). Exact and estimated value of c is compared in Fig. 3(b), while k is shown in Fig. 3(c). Even for the discontinuous and oscillatory case the estimation accuracy is still better.

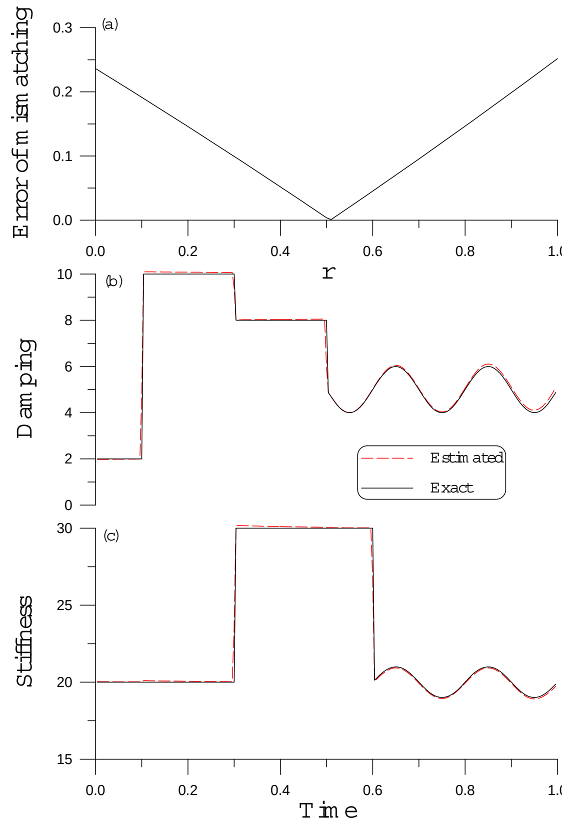

Figure 3: For Example 3: (a) showing the error of mismatching, (b) comparing the estimated and exact damping functions, and (c) comparing the estimated and exact stiffness functions.

5.4 Example 4

In the above three examples the data of ϕ_i and ψ_i used in Eqs. (84) and (85) to estimate k_i and c_i are obtained through numerical integrations by RK4, which means that the data are maybe not the exact ones. In this example we use the following exact data:

$$\phi(t) = t^2 + 1, \tag{99}$$

$$\psi(t) = \frac{t^3}{3} + 5t + 1, \tag{100}$$

and the functions of $c(t)$ and $k(t)$ to be estimated are

$$c(t) = 3 + 2\cos(2\pi t), \tag{101}$$

$$k(t) = 40 + t^3. \tag{102}$$

To obtain this ϕ and ψ the external forces are given by

$$F(t) = 2 + 2t[3 + 2\cos(2\pi t)] + (40 + t^3)(t^2 + 1), \tag{103}$$

$$H(t) = 2t + (t^2 + 5)[3 + 2\cos(2\pi t)] + (40 + t^3)\left(\frac{t^3}{3} + 5t + 1\right). \tag{104}$$

We use the vibration data of displacements at discretized time by inserting t_i into the given functions $\phi_i = \phi(t_i)$ and $\psi_i = \psi(t_i)$ and the forcing functions given by $F_i = F(t_i)$ and $H_i = H(t_i)$ as the inputs in Eqs. (84) and (85) to estimate k_i and c_i. In this calculation we have fixed $x_f = 0.2$, $r = 0.5$ and $\Delta t = 0.001$. The estimation errors of c and k are plotted in Figs. 4(a) and 4(b) with respect to time, which are smaller than 3×10^{-3}. As compared with the accuracy obtained in Examples 1 and 2, the present accuracy is increased one order.

6 Conclusions

The inverse vibration problem of simultaneous estimation of both the damping and stiffness coefficients is rather difficult. To overcome this difficulty we have used two sets of displacement

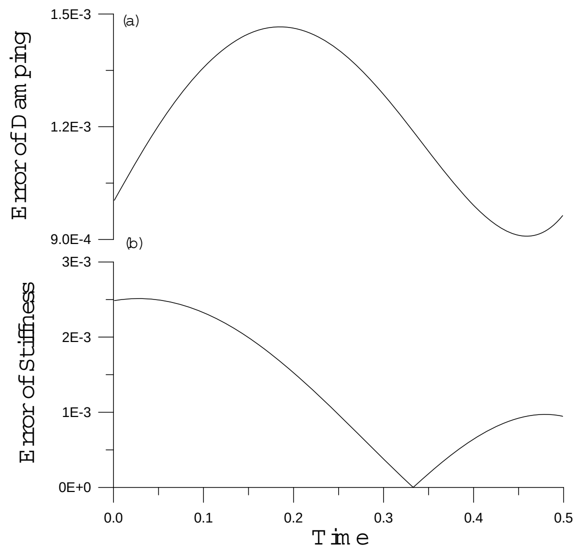

Figure 4: For Example 4: (a) showing the estimation error of damping, and (b) the estimation error of stiffness.

data generated by two different inputs on equation of motion as our formulation variables. In the present paper we offer a rather accurate and simple method without any iteration to estimate both the damping and stiffness coefficients simultaneously. The key points hinge on two type transformations, a two-point boundary value problem formulation as well as an establishment of the Lie-group shooting method. In order to avoid the appearance of zero denominator in the estimation equations, we also provided a criterion to choose the inputing forces in our equations. When two displacement sets are chosen as inputs, the estimation accuracy assessed by using the absolute error can be controlled within the second decimal point or to third decimal point by using exact data. Especially, for the discontinuous and oscillatory case the estimation accuracy is still better.

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References

Adhikari, S.; Woodhouse, J. (2001a): Identification of damping: part 1, viscous damping. *Journal of Sound and Vibration*, vol. 243, pp. 43-61.

Adhikari, S.; Woodhouse, J. (2001b): Identification of damping: part 2, non-viscous damping. *Journal of Sound and Vibration*, vol. 243, pp. 63-88.

Chang, C. W.; Chang, J. R.; Liu, C.-S. (2006): The Lie-group shooting method for boundary layer equations in fluid mechanics. *Journal of Hydrodynamics*, vol. 18, Issue 3, Supplement 1, pp. 103-108.

Chang, C. W.; Liu, C.-S.; Chang, J. R. (2005): A group preserving scheme for inverse heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 10, pp. 13-38.

Chang, J. R.; Liu, C.-S.; Chang, C. W. (2007a): A new shooting method for quasi-boundary regularization of backward heat conduction problems. *International Journal of Heat and Mass Transfer*, vol. 50, pp. 2325-2332.

Chang, C. W.; Liu, C.-S.; Chang, J. R. (2007b): The Lie-group shooting method for quasi-boundary regularization of backward heat conduction problems. *ICCES on line Journal*, vol. 3, pp. 69-79.

Feldman, M. (2007): Consider high harmonics for identification of non-linear systems by Hilbert transform. *Mechanical Systems and Signal Processing*, vol. 21, pp. 943-958.

Gladwell, G. M. L. (1986): Inverse problem in vibration. Kluwer Academic Publishers, Netherlands.

Gladwell, G. M. L.; Movahhedy, M. (1995): Reconstruction of a mass-spring system from spectral data I: Theory. *Inverse Problems in Engineering*, vol. 1, pp. 179-189.

Huang, C. H. (2001): A non-linear inverse vibration problem of estimating the time-dependent stiffness coefficients by conjugate gradient method. *International Journal for Numerical Methods in Engineering*, vol. 50, pp. 1545-1558.

Huang, C. H. (2005): A generalized inverse force vibration vibration problem for simultaneously estimating the time-dependent external forces. *Applied Mathematical Modelling*, vol. 29, pp. 1022-1039.

Ingman, D.; Suzdalnitsky, J. (2001): Iteration method for equation of viscoelastic motion with fractional differential operator of damping. *Computer Methods in Applied Mechanics and Engineering*, vol. 190, pp. 5027-5036.

Lancaster P.; Maroulas J. (1987): Inverse eigenvalue problems for damped vibrating systems. *Journal of Mathematical Analysis and Applications*, vol. 123, pp. 238-261.

Liang, J. W.; Feeny, B. F. (2006): Balancing energy to estimate damping parameters in forced oscillator. *Journal of Sound and Vibration*, vol. 295, pp. 988-998.

Liu, C.-S. (1997): Exact solutions and dynamic responses of SDOF bilinear elastoplastic structures. *Journal of the Chinese Institute of Engineers*, vol. 20, pp. 511-525.

Liu, C.-S. (2001): Cone of non-linear dynamical system and group preserving schemes. *International Journal of Non-Linear Mechanics*, vol. 36, pp. 1047-1068.

Liu, C.-S. (2003): Two-dimensional bilinear oscillator: group-preserving scheme and steady state motion under harmonic loading. *International Journal of Non-Linear Mechanics*, vol. 38, pp. 1581-1602.

Liu, C.-S. (2004): Group preserving scheme for backward heat conduction problems. *International Journal of Heat and Mass Transfer*, vol. 47, pp. 2567-2576.

Liu, C.-S. (2006a): The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, pp. 149-163.

Liu, C.-S. (2006b): Efficient shooting methods for the second order ordinary differential equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, pp. 69-86.

Liu, C.-S. (2006c): The Lie-group shooting method for singularly perturbed two-point boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, pp. 179-196.

Liu, C.-S. (2006d): An efficient backward group preserving scheme for the backward in time Burgers equation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 12, pp. 55-65.

Liu, C.-S. (2006e): One-step GPS for the estimation of temperature-dependent thermal conductivity. *International Journal of Heat and Mass Transfer*, vol. 49, pp. 3084-3093.

Liu, C.-S. (2006f): An efficient simultaneous estimation of temperature-dependent thermophysical properties. *CMES: Computer Modeling in Engineering & Sciences*, vol. 14, pp. 77-90.

Liu, C.-S. (2007): Identification of temperature-dependent thermophysical properties in a partial differential equation subject to extra final measurement data. *Numerical Methods for Partial Differential Equations*, vol. 23, pp. 1083-1109.

Liu, C.-S. (2008a): Identifying time-dependent damping and stiffness functions by a simple and yet accurate method. *Journal of Sound and Vibration*, in press.

Liu, C.-S. (2008b): An LGSM to identify nonhomogeneous heat conductivity functions by an extra measurement of temperature. *International Journal of Heat and Mass Transfer*, vol. 51, pp. 2603-2613.

Liu, C.-S. (2008c): An LGEM to identify time-dependent heat conductivity function by an extra measurement of temperature gradient. *CMC: Computers, Materials & Continua*, in press.

Liu, C.-S. (2008d): Solving an inverse Sturm-Liouville problem by a Lie-group method. *Boundary Value Problems*, vol. 2008, Article ID 749865.

Liu, C.-S.; Chang, C. W.; Chang, J. R. (2006a): Past cone dynamics and backward group preserving schemes for backward heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 12, pp. 67-81.

Liu, C.-S.; Chang, C. W.; Chang, J. R. (2006b): The Lie-group shooting method for steady-state Burgers equation with high Reynolds number. *Journal of Hydrodynamics*, vol. 18, Issue 3, Supplement 1, pp. 367-372.

Liu, C.-S.; Huang, Z. M. (2004): The steady-state responses of SDOF viscous-elastoplastic oscillator under sinusoidal loadings. *Journal of Sound and Vibration*, vol. 273, pp. 149-173.

Liu, C.-S.; Liu, L. W.; Hong, H. K. (2007): Highly accurate computation of spatial-dependent heat conductivity and heat capacity in inverse thermal problem. *CMES: Computer Modeling in Engineering & Sciences*, vol. 17, pp. 1-18.

Liu, C.-S.; Chang, J. R.; Chang, K. H.; Chen, Y. W. (2008): Simultaneously estimating the time-dependent damping and stiffness coefficients with the aid of vibrational data. *CMC: Computers, Materials & Continua*, in press.

Shiguemori, E. H.; Chiwiacowsky, L. D.; de Campos Velho, H. F. (2005): An inverse vibration problem solved by an artificial neural network. *TEMA Tend. Mat. Apl. Comput.*, vol. 6, pp. 163-175.

Starek L.; Inman, D. J. (1991): On the inverse vibration problem with rigid-body modes. *Journal of Applied Mechanics, ASME*, vol. 58, pp. 1101-1104.

Starek L.; Inman, D. J. (1995): A symmetric inverse vibration problem with overdamped modes. *Journal of Sound and Vibration*, vol. 181, pp. 893-903.

Starek L.; Inman, D. J. (1997): A symmetric inverse vibration problem for nonproportional underdamped systems. *Journal of Applied Mechanics, ASME*, vol. 64, pp. 601-605.

Starek L.; Inman, D. J.; Kress, A. A. (1992): symmetric inverse vibration problem. *Journal of Vibration and Acoustics, ASME*, vol. 114, pp. 565-568.