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Applications of Meta-Models in Finite Element Based Reliability Analysis of Engineering Structures

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Abstract: The problem of reliability analysis of randomly parametered, linear (or) nonlinear, structures subjected to static and (or) dynamic loads is considered. A deterministic finite element model for the structure to analyze sample realization of the structure is assumed to be available. The reliability analysis is carried out within the framework of response surface methods which involves the construction of surrogate models for performance functions to be employed in reliability calculations. This construction, in the present study, has involved combining space filling optimal Latin hypercube sampling, kriging models and methods from data-based asymptotic extreme value modeling of sequence of random variables. Illustrative examples on numerical prediction of reliability of a ten-bay truss, a W-seal in an aircraft structure, and a nonlinear randomly parametered dynamical system are presented. Limited Monte Carlo simulations are used to validate the approximate procedures developed.

1 Introduction

The problem of predicting structural reliability using computational models is a challenging problem. Comprehensive overviews of the progress achieved so far in this area of research have been documented in the works of Rackwitz (2001), Sudret and Der Kiureghian (2002), Manohar and Ibrahim (1998), Manohar and Gupta (2006), Schueller (1997), Sundararajan (1995), Melchers (1999), Haldar and Mahadevan (2000), and Schueller and Pradlwarter (2007). The problem of evaluation of reliability of an engineering structure, with respect to a specified performance criterion, can be formulated as follows. Let the source of uncertainties in the given problem be represented by a $l \times 1$ vector of random variables X with a joint probability density function (pdf) $p_X(x)$. Here we use the lower case x to denote the states realized by the random vector X. The vector X collectively and exhaustively models all the sources of uncertainties in the problem that includes aleatory and epistemic uncertainties associated with specification of loads, structural properties and models for structural behavior. This prescription results in an ensemble of realization of nominally identical structures and any member of this ensemble can be viewed as a point in a l-dimensional space of these random variables. We define a performance function $g(X)$, such that, any point x^* in the l-dimensional space can be classified as belonging to a safe or an unsafe region depending on whether $g(x^*) > 0$ or $g(x^*) <$ 0respectively. The surface $g(x) = 0$ is called the limit surface or the failure surface associated with the performance function $g(X)$. The problem of reliability estimation consists of determining the probability measure $P[g(X) > 0]$. Alternatively, the complement of reliability, namely, the probability of failure, given by $P_f = 1 - P[g(X) > 0]$ needs to be determined. Also of interest is the determination of a design point at which the probability of failure reaches its maximum value. The evaluation of P_f clearly involves a l-dimensional quadrature of $p_X(x)$ over a hyper-volume defined by $g(x) < 0$. The implementation of this quadrature scheme in practice is seldom possible because of a wide variety of reasons the important ones being the difficulties arising out of large value of l, lack of adequate knowledge of $p_X(x)$, and highly nonlinear nature of $g(X)$.

While dealing with large scale structures there would be further difficulties in obtaining an ex-

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plicit expression for the function $g(X)$. The limit surface $g(x) = 0$ in this case would be defined implicitly through a long running computer code that often is a finite element code. This type of problems could be tackled, at least in principle, by using Monte Carlo simulations and its variants involving various variance reduction schemes. The Monte Carlo simulation procedures have found wide applicability in solving problems structural mechanics (see, for instance, the recent papers by Xu *et al,* 2008, Liou and Fang 2000, Lee *et al,* 2006). However, such methods could become computationally infeasible, especially, while analyzing large scale structures, wherein a single run of the finite element code typically could take tens of minutes of CPU even on fast computers. One of the alternative approaches employs response surface models to replace the long running computer codes by functional representations that offer acceptable fits to the failure surface near the design points. Such models, which serve as models for models, are called meta-models. Here the performance function $g(X)$ is approximated by a response surface, that is typically of the form $\tilde{g}(X) = A + BX^t + X^tCX$, where A is a scalar constant, B is a $l \times 1$ array and C is a $l \times l$square symmetric matrix. The problem of determination of $\tilde{g}(X)$consists of two steps: (a) design of an appropriate experiment that helps to locate the points in X-space at which $g(X)$ needs to be evaluated, and (b) estimation of undetermined parameters A, B and C in $\tilde{g}(X)$ using known values of $g(X)$ at the chosen points. Once a satisfactory procedure is evolved to determine $\tilde{g}(X)$, the analysis of reliability could be subsequently carried out using standard reliability index based methods or by using Monte Carlo simulation possibly augmented with variance reduction techniques. The success of the method essentially hinges on how efficiently quantities A, B and C are computed and how well the response surface represents the failure surface in regions that make notable contributions to the failure probability. Also, for the response surface method to be a meaningful substitute for Monte Carlo simulations, clearly, the computational effort expended in finding the constants A, B and C must be notably smaller than what an acceptable Monte Carlo simulation study would demand.

From the study of literature on response surface modeling for reliability calculations, the present authors perceive that there exist two broad categories of approaches. The first, as represented by works of Wong (1984, 1985), Faravelli (1989, 1992) and Breitung and Faravelli (1996), adheres to response surface modeling that is rooted in statistical sampling techniques as described in the mathematical statistics literature (see, for example, the books by Khuri and Cornell 1997 and Myers and Montgomery 1995). On the other hand, in the studies conducted by Bucher and Bourgund (1990), Rajashekhar and Ellingwood (1993), Liu and Moses (1994) and Kim and Na (1997), the response surface modeling combines concepts of structural reliability analysis (such as the first order reliability method) with statistical sampling methods. Huh and Haldar (2002) have developed response surface models for reliability analysis of randomly parametered nonlinear frames subject to recorded earthquake ground motions. The recent studies by Kaymaz (2005) and Kaymaz and McMahon (2005) combine concepts from kriging with techniques developed by Bucher and Bourgund. Gupta and Manohar (2004 a,b) outlined methods for treating multiple design points and functions with multiple regions of comparable importance in the parameter space. These authors have also studied the application of response surface techniques for analyzing time variant reliability problems associated with nonlinear structural system subjected to nonstationary random excitations.

In the present study, we explore the application of techniques developed in the field of design and analysis of computer experiments (DACE) for structural reliability analysis. A comprehensive treatment of DACE, along with a review of the class of research goals that can be realized using these methods, can be found in the work of Santner *et al.,* (2003). The review paper by Sacks *et al.,* (1989) contains information on early work in this area of research. The report by Lophaven *et al.,* (2002), and the accompanying suite of software, provides a computational toolkit for implementing the relevant techniques. In these ap-

proaches the essential idea is to represent the output of a computer code as a non-stationary Gaussian random field with a postulated functional form for the non-stationary mean term and a homogeneous random term having a covariance with an assumed functional form and with unknown parameters. The output of the computer program is next computed at a set of input points, and, based on this data, the unknown parameters of the model are estimated using method of maximum likelihood estimation (or other statistical methods). Subsequently, the output of the computer program at any input point, at which the output is unknown, is estimated using the criterion of minimization of the mean square error of the prediction. In the context of the present study we take the estimate so obtained to be the response surface for the performance function that surrogates for the given computer program. The computation of the probability of failure subsequently is based on either first order reliability methods or Monte Carlo simulations. The present study also investigates the use of space-filling optimal Latin hypercube sampling (LHS) techniques to obtain global fits to the performance function. This, in turn, is shown to lead to acceptable estimates of failure probability, with fewer samples, for performance functions with multiple design points and (or) performance functions with multiple regions of comparable importance. The study also proposes the application of the above method for analyzing time variant reliability of randomly driven oscillators with uncertain parameters. This has involved a two step procedure: the first step involves an objective selection of the form of the extreme value distribution, and the second, consists of fitting kriging models for the parameters of this distribution. A set of illustrative examples on reliability analysis of linear/nonlinear static/ dynamic problems are presented to bring out the scope of the proposed procedure. Limited Monte Carlo simulation studies are performed to evaluate the acceptability of these results.

2 Meta-models based on computer experiments

We begin by considering a computational code (typically based on finite element analysis) that models the behavior of a given structural system. This code itself could represent linear or nonlinear system behavior and it could model static or dynamic system response. Let $g(x)$ be the performance function (as specified in the previous section) defined with respect to the l-dimensional input vector x. It may be noted that, at this stage, we do not consider the input vector to be random in nature. Let $\{x_1 \quad x_2 \quad \dots \quad x_n\}^t$ be such that $x_i \in R^l$ $(i = 1,2,\dots,n)$ denote the $n \times l$ matrix of input points at which the output $g(x_i)$, $i = 1,2,\dots,n$ are computed using the computer code. We model the computer output using the meta-model

$$g(x_i) = \sum_{j=1}^{p} f_j(x_i)\beta_j + Z(x_i) = f^t(x_i)\beta + Z(x_i); \quad i = 1,2,\dots n \quad (1)$$

Here the superscript t denotes matrix transposition, $\{f_j(x_i)\}_{j=1}^{p}$ is a set of known functions, $\{\beta_j\}_{j=1}^{p}$ a set of unknown constants to be determined and $Z(x)$ is a homogenous, zero mean, Gaussian random field evolving in x with an unknown covariance function. Thus, for example, with l=3 and quadratic basis functions, at the i^{th} sampling point we get

$$\begin{aligned} & g\left[x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right] \\ =& \beta_1 + \beta_2 x_i^{(1)} + \beta_3 x_i^{(2)} + \beta_4 x_i^{(3)} + \beta_5 \left(x_i^{(1)}\right)^2 \\ & + \beta_6 \left(x_i^{(1)} x_i^{(2)}\right) + \beta_7 \left(x_i^{(1)} x_i^{(3)}\right) + \beta_8 \left(x_i^{(2)}\right)^2 \\ & + \beta_9 \left(x_i^{(2)} x_i^{(3)}\right) + \beta_{10} \left(x_i^{(3)}\right)^2 \\ & + Z\left[\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right)^t\right] \end{aligned} \quad (2)$$

so that, in this approximation, $p = 1 + l + \frac{l(l+1)}{2} = 10$. The covariance function is taken to be of the

form

$$Cov[Z(x_r)Z(x_s)] = \sigma^2 R(x_r - x_s, \theta) \tag{3}$$

Here σ^2 is the unknown variance of the random field $Z(x_i)$ and $R(x_r - x_s, \theta)$ is the $(r,s)^{th}$ element of the $n \times n$ matrix of correlation coefficients with θ being a $q \times 1$ vector of unknown parameters in the covariance model. The functional form of the correlation matrix $R(x_r - x_s, \theta)$ is taken to be known but the function itself is considered to be dependent on the unknown parameters θ. Some of the choices for the functional form of $R(x_r - x_s, \theta)$ employed in the existing literature include the following (Sacks, *et al*., 1989, Lophaven *et al.,* 2003)

$$\begin{aligned}
&R(\theta_{rs}, x_r - x_s) = \exp\left[-\theta_{rs}|x_r - x_s|\right] \\
&R(\theta_{rs}, x_r - x_s) = \exp\left[-\theta_{rs}|x_r - x_s|^{\theta_{n+1}}\right]; \\
&\qquad 0 < \theta_{n+1} < 2 \\
&R(\theta_{rs}, x_r - x_s) = \exp\left[-\theta_{rs}(x_r - x_s)^2\right] \\
&R(\theta_{rs}, x_r - x_s) = \max\{0, 1 - \theta_{rs}|x_r - x_s|\} \\
&R(\theta_{rs}, x_r - x_s) = 1 - 1.5\xi_{rs} + 0.5\xi_{rs}^3; \\
&\xi_{rs} = \min(1, \theta_{rs}|x_r - x_s|) \\
&R(\theta_{rs}, x_r - x_s) = 1 - 3\xi_{rs}^2 + 2\xi_{rs}^3; \\
&\xi_{rs} = \min(1, \theta_{rs}|x_r - x_s|) \\
&R(\theta_{rs}, x_r - x_s) = \varsigma(\xi_{rs}); \\
&\varsigma(\xi_{rs}) = \begin{cases} 1 - 15\xi_{rs}^2 + 30\xi_{rs}^3 & \text{for } 0 \le \xi_{rs} \le 0.2 \\ 1.25(1 - \xi_{rs})^3 & \text{for } 0.2 \le \xi_{rs} < 1 \\ 0 & \text{for } \xi_{rs} \ge 1 \end{cases}
\end{aligned} \tag{4}$$

The work of Santner *et al.,* (2003) discusses the issues related to smoothness of the functional representation in equation (1) *vis-à-vis* the choice of the form of $R(x_r - x_s, \theta)$ and values of parameters θ. In the present study we employ the third of the above correlation coefficient model. It may be noted that the function $g(\bullet)$, as in equation 1, constitutes a non-homogeneous Gaussian random field with mean $E[g(x)] = f^t(x)\beta$ and covariance $E[\{g(x_r) - f^t(x_r)\beta\}\{g(x_s) - f^t(x_s)\beta\}] = \sigma^2 R(x_r - x_s, \theta)$. The non-homogeneity of the random field here arises due to the dependence of mean value of $g(x)$ on the parameter x. Thus, the set of unknowns to be determined in equation (1) are $\{\beta_j\}_{j=1}^p$, σ^2 and $\{\theta_i\}_{i=1}^q$. Corresponding to the choice of the training points $x = \{x_1 \quad x_2 \quad \ldots \quad x_n\}^t$, we can re-write equation (1) as

$$g(x) = F(x)\beta + Z(x) \tag{5}$$

Here $g(x)$ is a $n \times 1$vector of random variables, $F(x)$ is a $n \times p$ matrix of deterministic functions with $(i,j)^{th}$ element given by $F_{ij}(x) = f_j(x_i)$ and $Z(x)$ is a $n \times 1$ vector of zero mean Gaussian random variables. Thus the probability distribution function (PDF) of $g(x)$ is given by

$$g(x) = N\left[F(x)\beta, \sigma^2 R(x_r - x_s, \theta)\right] \tag{6}$$

where N denotes the normal PDF with mean vector $F(x)\beta$ and covariance matrix $\sigma^2 R(x_r - x_s, \theta)$. In order to determine the unknown model parameters $\{\beta_j\}_{j=1}^p$, σ^2 and θ , we invoke the method of maximum likelihood estimation. Accordingly, we construct the negative log-likelihood function

$$\begin{aligned}
L(\beta, \sigma^2, \theta) = \frac{1}{2}\Big\{&n\ln\sigma^2 + \ln|R| \\
&+ (g - F(x)\beta)^t \frac{1}{\sigma^2} R^{-1}(g - F(x)\beta) + n\ln 2\pi\Big\}
\end{aligned} \tag{7}$$

and minimize this function with respect to the parameters $\{\beta_j\}_{j=1}^p$, σ^2 and $\{\theta_i\}_{i=1}^q$. The minimization with respect to $\{\beta_j\}_{j=1}^p$ can be shown to lead to the condition

$$\hat{\beta} = \left(F^T R^{-1} F\right)^{-1} F^T R^{-1} g \tag{8}$$

Similarly, the minimization with respect to σ^2 leads to

$$\hat{\sigma}^2 = \frac{1}{n}(g - F\beta)^t R^{-1}(g - F\beta) \tag{9}$$

Now by substituting 8 and 9 into equation 7, we get the objective function only in terms of parameter vector θ as

$$\begin{aligned}
&L\left(\hat{\beta}, \hat{\sigma}^2, \theta\right) = \\
&\qquad \frac{1}{2}\left\{n\ln\hat{\sigma}^2(\theta) + \ln|R(\theta)| + n\ln 2\pi\right\}
\end{aligned} \tag{10}$$

The minimization of this function, with respect to the remaining variables $\{\theta_i\}_{i=1}^{q}$, can now be carried out using numerical optimization methods. The conditions $\theta_i > 0; i = 1, 2, \ldots, q$ are imposed as constraints in this optimization problem. These constraints are needed given that the correlation function that we are using is as given by the third of the models listed in equation 4. In the present study we use the genetic algorithm to solve this constrained nonlinear optimization problem. For this purpose we utilize the *ga* solver that is available on the Matlab computational platform. A penalty function to artificially enforce very high value of the given function is used to avoid negative values of θ. The resulting predictor $g(x)$ with $\beta = \hat{\beta}\left(\hat{\theta}\right)$, $\sigma^2 = \hat{\sigma}^2\left(\hat{\theta}\right)$ and $\theta = \hat{\theta}$ is called the maximum likelihood empirical best linear unbiased predictor. At this stage we have determined the values of all the unknowns in the model in equation 6. Now we consider the question of predicting the output at a point x_0 at which the computer program has not been run.

3 Response prediction at an unmeasured point

Let x_0 be a point in the input space at which we now aim to estimate $g(x_0)$ and from equation 1, we get $g(x_0) = f^t(x_0)\beta + Z(x_0)$. It is clear that $g(x_0)$ is a Gaussian random variable and the vector $\left\{g(x_0) \quad g(x)\right\}^t$ forms a $(n+1)\times 1$ vector of Gaussian random variables with PDF given by

$$\begin{Bmatrix} g(x_0) \\ g(x) \end{Bmatrix} = N\left(\begin{Bmatrix} f^t(x_0) \\ F(x) \end{Bmatrix}\hat{\beta}(\hat{\theta}), \hat{\sigma}^2(\hat{\theta})\begin{bmatrix} 1 & r_0^t(\hat{\theta}) \\ r_0(\hat{\theta}) & Rr_0(\hat{\theta}) \end{bmatrix}\right) \tag{11}$$

Here $r_0 = \left[R(x_0 - x_1) \quad \ldots \quad R(x_0 - x_n)\right]^t$. The estimate $\hat{g}(x_0)$ of $g(x_0)$ that minimizes the mean square error $E\left\{\left[g(x_0) - \hat{g}(x_0)\right]^2\right\}$ is well known to be given by (Papoulis and Pillai 2001)

$$\begin{aligned} &\hat{g}(x_0) = E\left[g(x_0)\,|g(x)\right] \\ &= f^t(x_0)\hat{\beta}\left(\hat{\theta}\right) + r_0^t\left(\hat{\theta}\right)R^{-1}\left(\hat{\theta}\right)\left[g(x) - F\hat{\beta}\left(\hat{\theta}\right)\right] \end{aligned} \tag{12}$$

This expression constitutes the kriging model and this forms the response surface model that we use as the surrogate in the subsequent reliability calculation. In implementing this model we still need to choose the functions $\left\{f_j(x)\right\}_{j=1}^{p}$ and, in the present study, we take quadratic functions (as illustrated in equation 2 for l=3) so that $p = 1 + l + \frac{l(l+1)}{2}$. If cross terms are ignored, one gets $p = 2l + 1$. Figure 2 summarizes the steps involved in the construction of the response surface based on the procedure outlined in this and the preceding section.

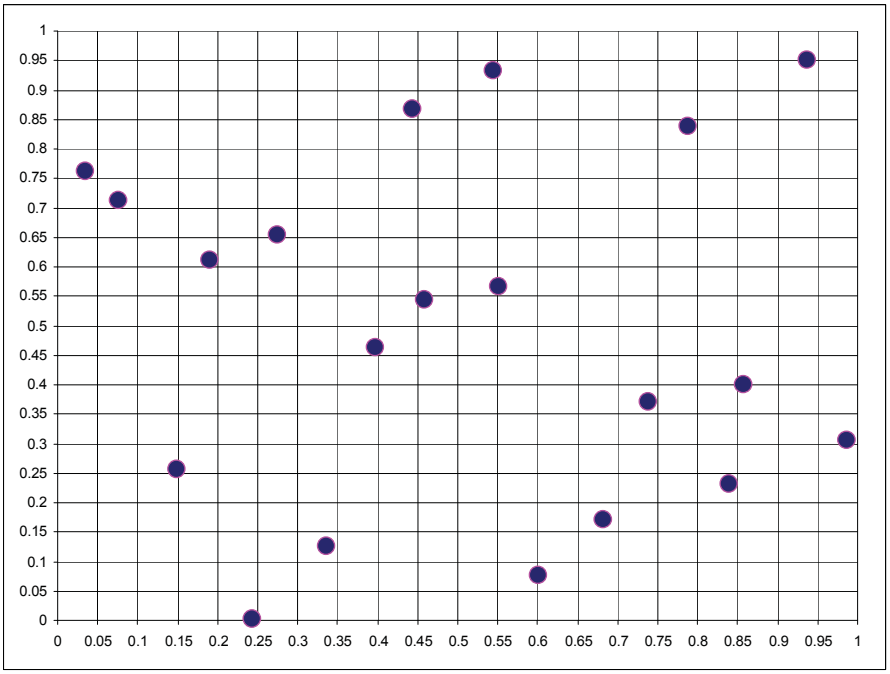
4 Selection of sampling points

A step that precedes the modeling described in the previous sections consists of selecting the $n \times l$ matrix of inputs $\left\{x_1 \quad x_2 \quad \ldots \quad x_n\right\}^t$ and this step is called the experimental design. We employ in our study a version of Latin hypercube sampling that has the potential for filling up the input space in a uniform fashion. A Latin square is a square grid containing sample positions in which each row and column has one (and only one) sample. A Latin hypercube is a generalization of a Latin square in multiple dimensions. If there are n samples to be chosen from p random variables, then, it can be represented by $n \times p$ matrix L, in which each column consists of a permutation of the real numbers from n equally divided intervals from 0 to 1. We refer to each row of L as a sample point in p dimensions and use the notation (Liefvendahl and Stocki, 2006)

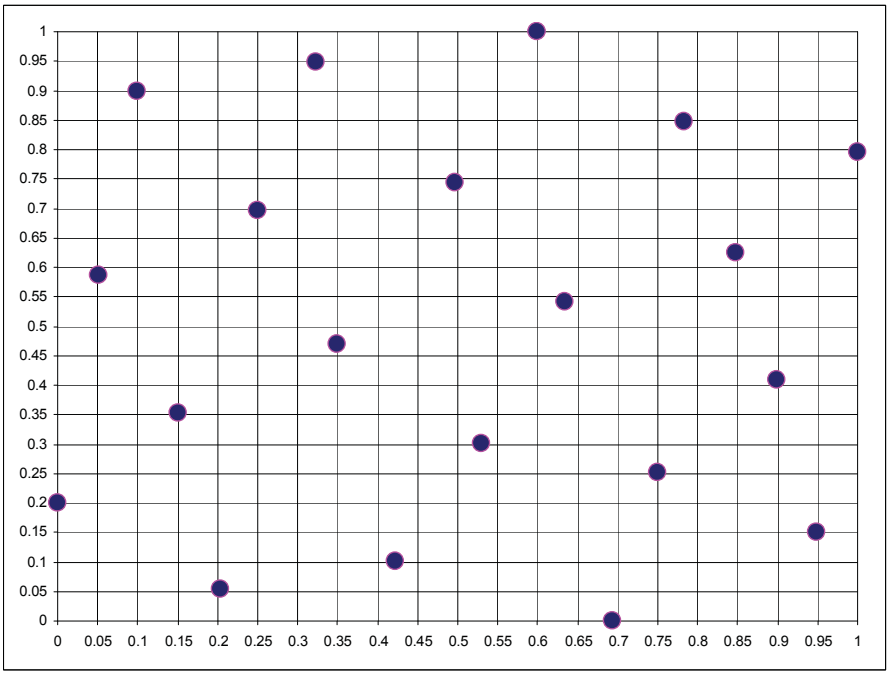
$$L = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} x_{11} & . & . & x_{1p} \\ x_{21} & . & . & x_{2p} \\ . & . & . & . \\ . & . & . & . \\ x_{n1} & . & . & x_{np} \end{bmatrix}$$

where x_i, $1 \le i \le n$, is the i-th sample point.

Figure 1(a) illustrates 20 samples chosen as per the Latin hypercube design. It is clear that Latin hypercube sampling can lead to several admissible sampling scenarios. The origins of this sampling method lie in the work of Mackay *et al.,* (1979) who discussed the variance of estimators of mean and empirical PDF based on three sampling strategies that included random sampling,



(a)



(b)

Figure 1: Latin hypercube designs for 20 samples of two random variables; (a) a typical random Latin hypercube sample; (b) the optimized Latin hypercube samples.

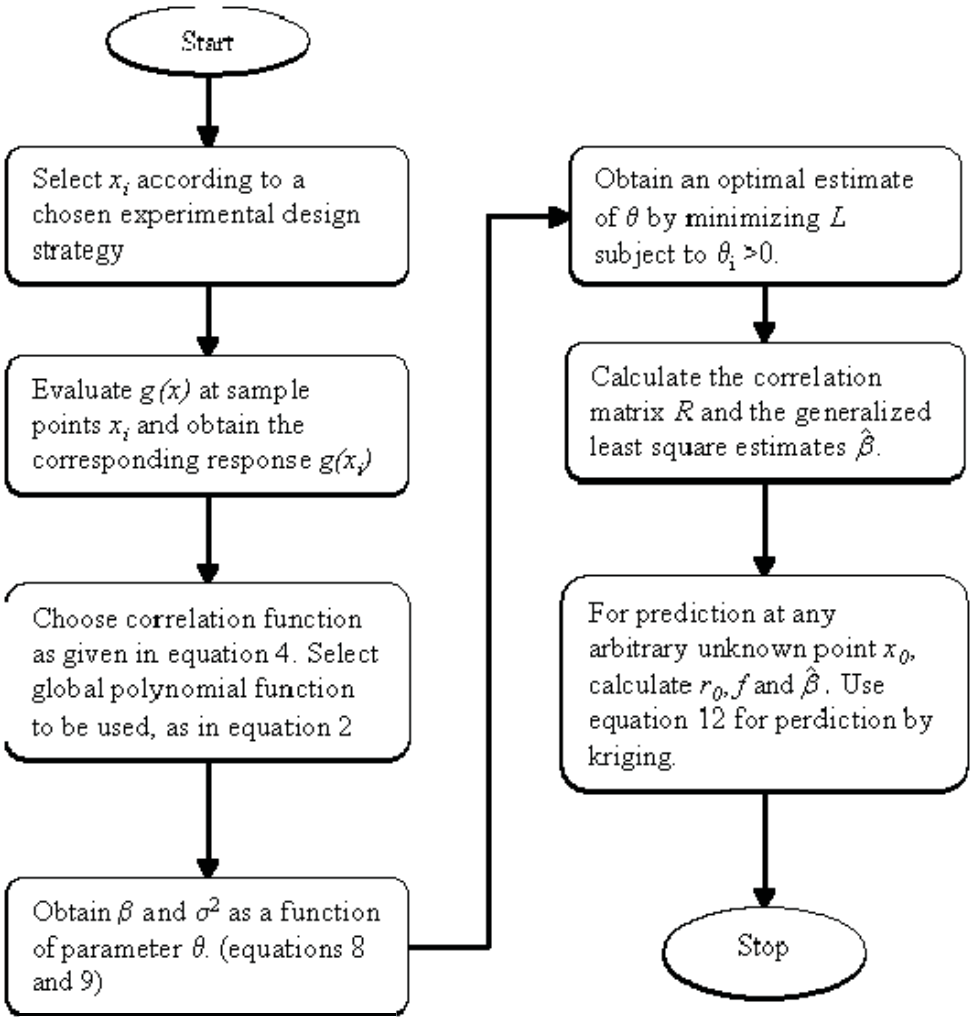


Figure 2: Flow chart for the construction of kriging based response surface (sections 2 and 3).

stratified sampling and Latin hypercube sampling. The study investigated the condition under which the Latin hypercube sampling reduces the variance in comparison to the simple random sampling. Subsequently, there have been several papers that have developed this method further and the study by Helton *et al.*, (2006) provides a contemporary perspective in the area of experimental design. In our work, we follow the Audze-Eglais uniform Latin hypercube design (Bates *et al.,* 2003). This method introduces the notion of potential energy of points in a design of experiments. The analogy is based on a physical model in which the sample points are perceived as unit point masses that repel each other with forces that are inversely proportional to the distance between the masses. A quantity termed as potential energy is defined as

$$U = \sum_{p=1}^{P} \sum_{q=p+1}^{P} \frac{1}{L_{pq}} \tag{13}$$

where L_{pq} is the distance between the points p and q (note: clearly $p \neq q$). According to the method due to Audze-Eglais, we select the Latin hypercube sample that minimizes the quantity U as given above. This is expected to distribute the experiment points as uniformly distributed as possible within the design variable domain. Figures 1 (a) and (b) illustrate a typical Latin hypercube design along with the optimized Latin hypercube design. A flowchart for drawing samples based on the optimal Latin hypercube sampling technique

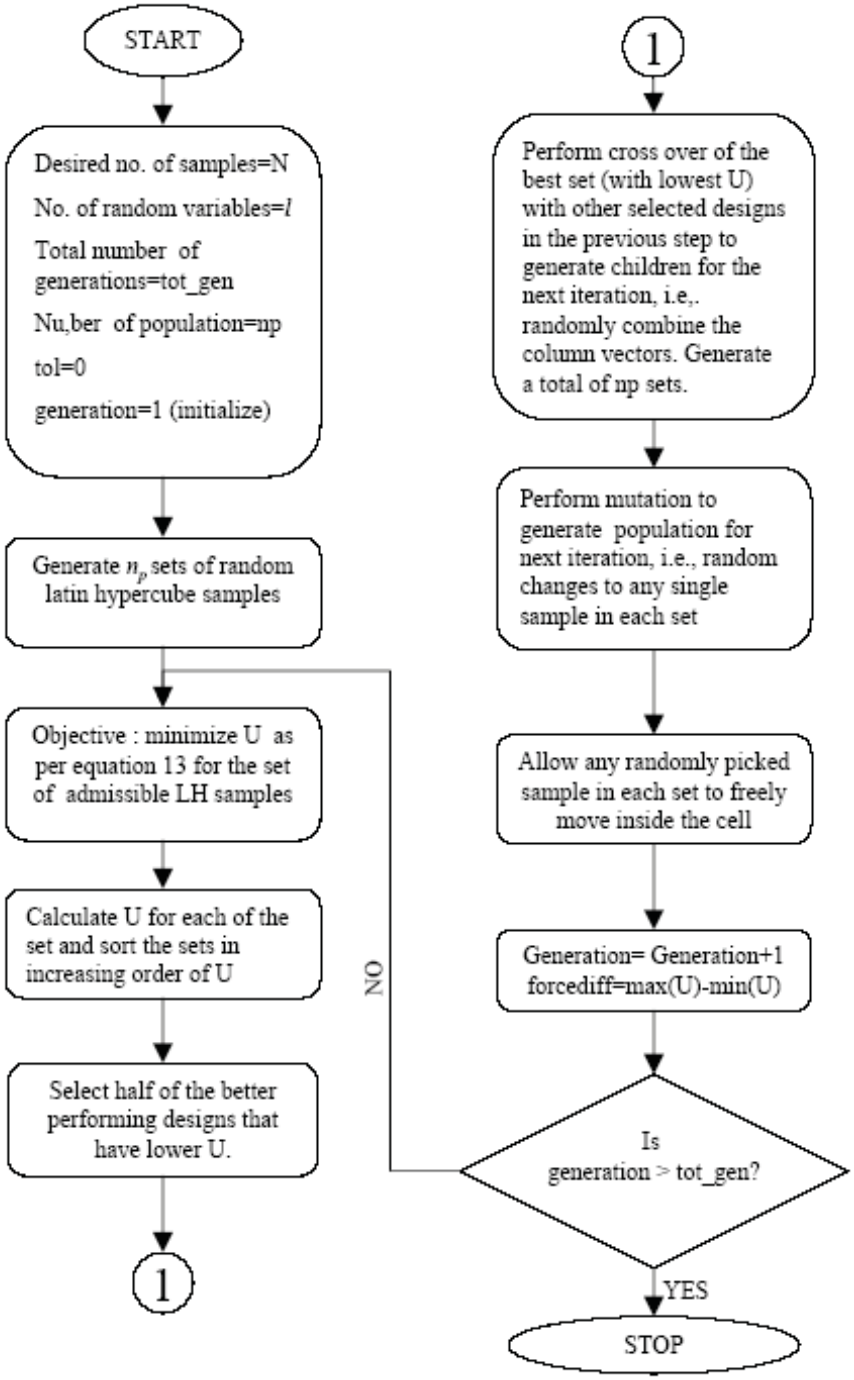


Figure 3: Flow chart for the generation of optimal Latin hypercube samples using genetic algorithm (Section 4).

is presented in Figure 3. The integration of this procedure into the reliability calculation is illustrated in Figure 4.

5 Sampling design based on a variation of Bucher-Bourgund method

Bucher and Bourgund (1990) have proposed an adaptive interpolation scheme to arrive at the response surface parameters. These authors assume the response surface to be of the form $\tilde{g}(X) = A + BX^t + X^tCX$ and take C to be a diagonal matrix so that the cross quadratic terms are not included. The authors adopt a fully saturated experimental design and evaluate $g(X)$ at the sampling points given by $2l+1$ combinations $\mu_l \pm h\sigma_i$ where μ_i and σ_i are, respectively, the mean and standard deviation of X_i. Using this information, an estimate of the undetermined response surface parameters is obtained. Corresponding to the surface $\tilde{g}(X)$ so obtained, the Hasofer-Lind reliability index and the associated design point x_D are determined. In doing so, it is assumed that the random variables $\{X\}_{i=1}^{l}$ are uncorrelated and Gaussian distributed. An update on the location of the experimental design point is now obtained as

$$x_m = x_D + [x_D - \mu]\frac{g(\mu)}{g(\mu) - g(x_D)} \tag{14}$$

This helps to locate the center point closer to the limit surface $g(X) = 0$. A new surface using $x_m$as the center point is obtained and this is used as the final estimate of the response surface. Thus, this procedure requires $4l+3$ number of evaluations of the performance function $g(X)$. Rajashekhar and Ellingwood (1994, 1995) examined issues related to the choice of experimental points and suggested modifications to the approach used by Bucher and Bourgund (1990). They questioned if a single cycle of updating, as was proposed by Bucher and Bourgund, is always adequate. In answer to this question, these authors detailed how subsequent updating could be satisfactorily carried out. Other issues examined by these authors include a discussion on selecting design points near tails of probability distributions of the basic random variables and also on including cross terms in the response surface fit. One of the common features of the procedures suggested by both Bucher & Bourgund and Rajashekhar & Ellingwood, is that, at different stages of response surface modeling, only a part of available information on the functional evaluations of $g(x)$ is directly used. Given that, computationally, the most involved step in response surface modeling lies in evaluation of $g(X)$, it is desirable that the modeling procedure should utilize all the information available on $g(X)$ at every stage of response surface fitting. The idea of kriging modeling for response surface, as being studied in the present paper, can be combined with the earlier procedures of Bucher and Bourgund and Rajashekhar and Ellingwood. Thus, at any stage of response surface construction, the kriging model is made utilize at least a part (if not all) of the information available on evaluation of $g(x)$. In the numerical work it was found that, after the first iteration of finding the design point using $2l+1$ sample points (that is, using the points $\mu_l \pm h\sigma_i; i = 1, 2, \ldots, l$), in the subsequent fitting of kriging surfaces, only the newly found centre point be included in response surface modeling along with the earlier $2l+1$ points. This was found to give solutions whose accuracy compared well with the results from refinements proposed by Rajashekhar and Ellingwood. Figure 5 summarizes the steps involved in the proposed calculation steps.

6 Numerical examples

For the purpose of discussion, we adopt the following nomenclature for alternative methods of reliability assessment:

Method I Direct Monte Carlo simulation with a large number of samples.

Method II Response surface modeling based on the Bucher-Bourgund approach (1990) possibly with refinements as proposed by Rajashekhar and Ellingwood (1993).

Method III The approach developed in the present study in sections 2-5. The steps are summarized in Figure 4.

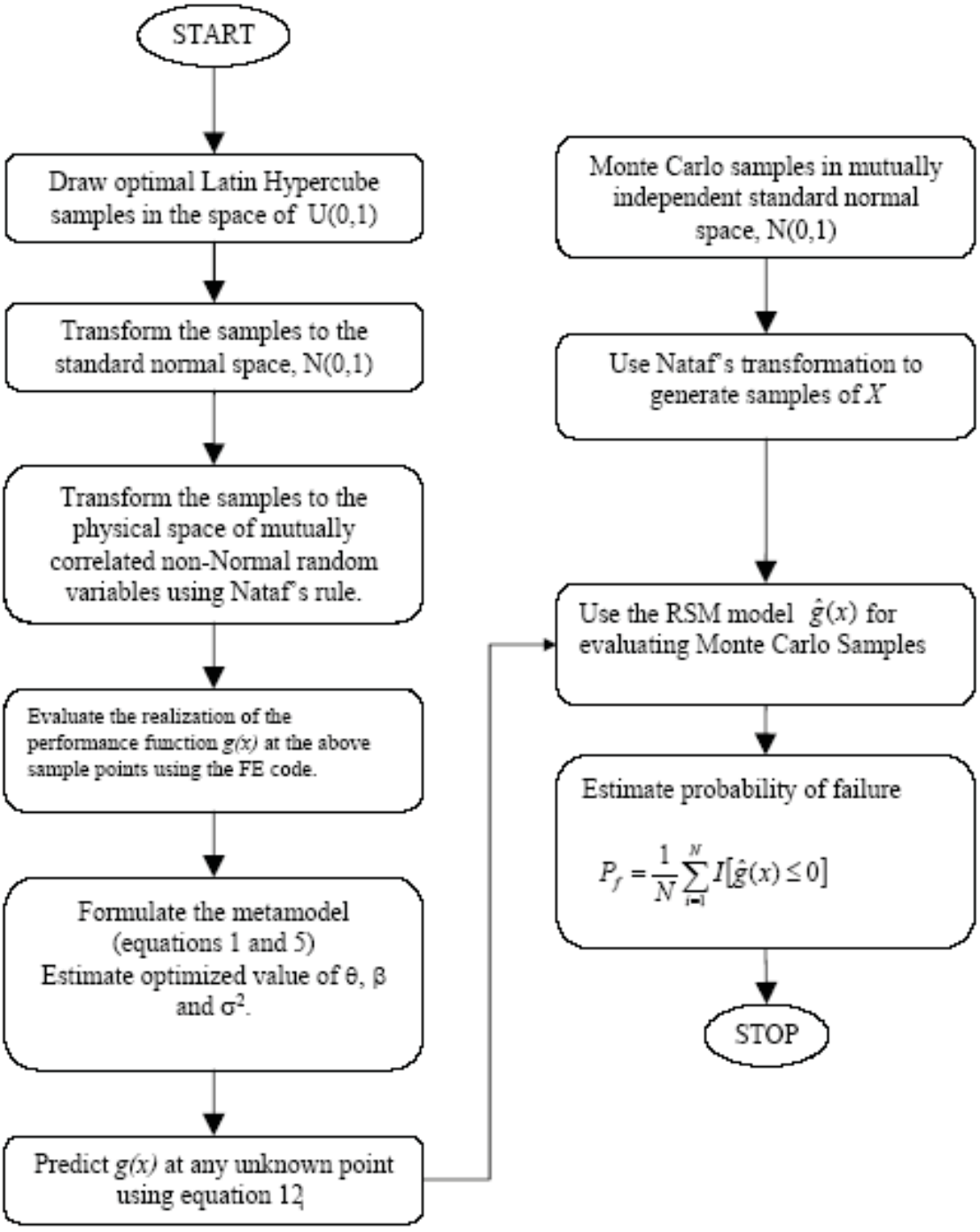


Figure 4: Flowchart for method III for reliability estimation (sections 2-4).

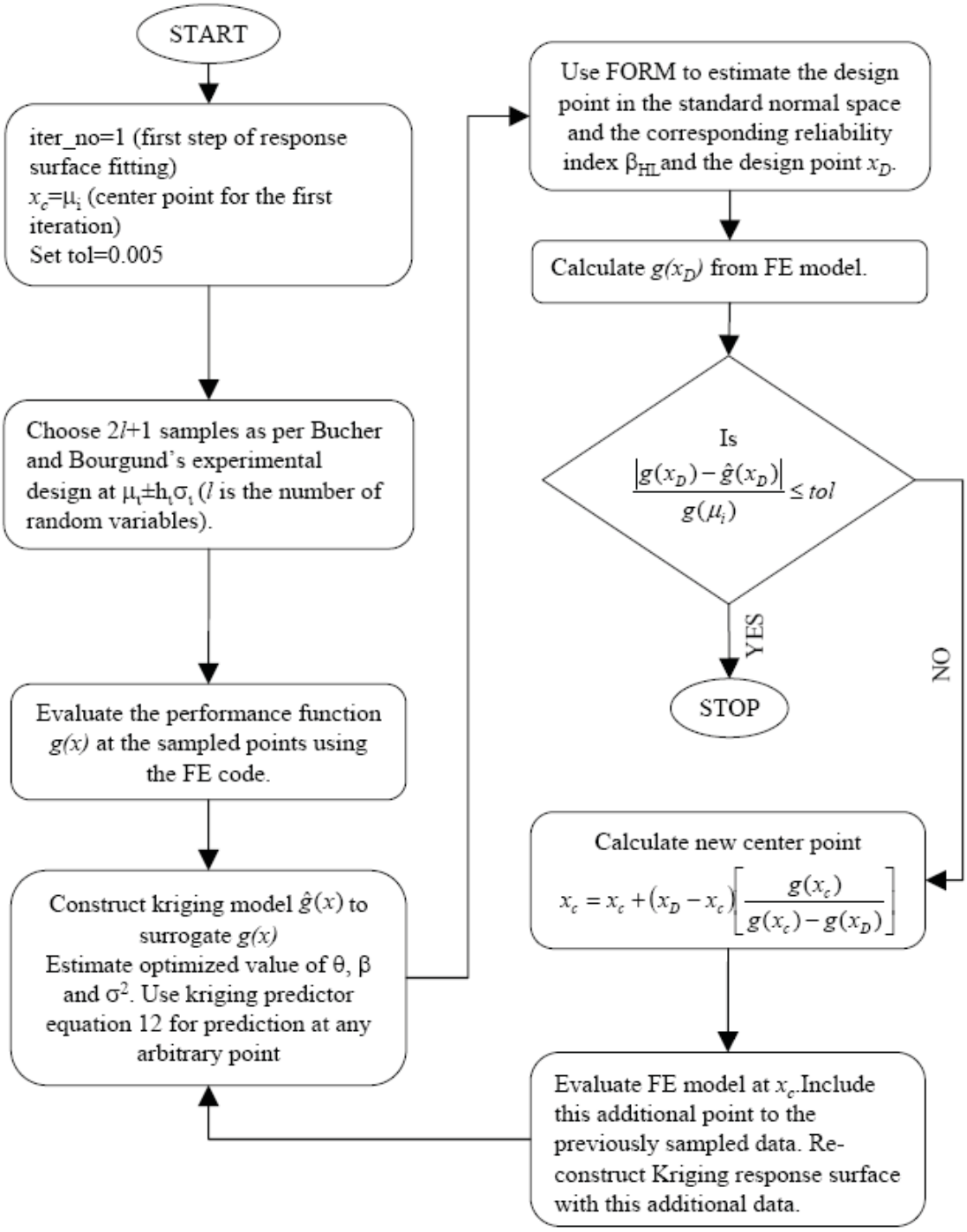


Figure 5: Flowchart for Method IV for reliability estimation (Section 5).

Method IV The approach outlined in section 2-4. The steps are as summarized in Figures 2-4.

It may be noted that in the present discussion we take the results from Method I as the benchmark against which other methods can be compared. Although this result itself is subject to sampling fluctuations, given the large size of samples used, these fluctuations are considered to be negligible.

6.1 Preliminary examples

Here we begin by considering a set of problems in which performance function is explicitly known and apply the response surface modeling approach developed in this study to these problems. This enables the evaluation of proposed methods with respect to problems that can be easily treated by alternative methods.

First, we consider the performance function $g(X_1,X_2) = 7 - X_1^2 - X_2$ with X_1 and X_2 being mutually independent standard normal random variables (Gupta and Manohar 2004a). Figure 6(a) shows the performance function. The probability of failure estimated using 10^6 samples (Method I) leads to an estimate of P_f =0.0097. From Figure 6(a) it can be observed that the performance function has two design points and they can be shown to be given by (2.5500,0.4975) and (-2.5500,0.4975). Based on the first order reliability method (FORM), it can also be shown that the associated Hasofer-Lind reliability index is β_{HL}=2.5981 and the associated nominal probability of failure to be 0.0047. Clearly, the FORM would perform poorly in this example since it can take into account only one design point. In fact, the estimated nominal failure probability 0.0047 compares poorly with the estimate of P_f =0.0097 (Method I). Methods II and IV provide estimates of β_{HL}, which compare well with the FORM solution and, consequently, have the same weakness. The response surface model based on Method III is shown in Figure 6(b) and the estimate of failure probability (with 10^6 samples simulation) turns out to be 0.0097 which agrees very well with results obtained using Method I.

Next, the performance function $g(X_1,X_2) = a^2 - \left(X_1^2 + X_2^2\right)$ with X_1 and X_2 being mutually independent standard normal random variables and a being a deterministic constant is considered. The performance function here clearly constitutes a circle and, consequently, there exist infinite number of design points with every point on the limit surface being a design point. This function is clearly the most difficult to treat using FORM. The results from Method I (with 10^6 samples) are obtained as 0.0110 for a=3 and the FORM estimate for the nominal probability of failure is 0.0013. While methods II and IV provide comparable estimates as FORM, method III provides an answer that is very close to estimate from Method I with less than 0.001% accuracy.

Here we investigate the static behavior of a beam and take the performance function to be given by $g(Y,Z,M) = YZ - M$, where Y= yield strength, Z=section modulus and M= applied bending moment (Ang and Tang, 1984). It is assumed that Y and Z are lognormal distributed and M is type I asymptotic extreme value random variable. The mean values of these quantities are taken to be $\mu_Y = 2.7579\times10^2$ N/mm^2,$\mu_Z = 8.1935\times10^5$ mm^{3}and $\mu_M = 1.1298\times10^8$Nmm respectively with the associated coefficient of variations being Ω_Y=0.125; Ω_Z=0.050; Ω_M=0.200 respectively. Furthermore, the matrix of correlation coefficients for these random variables is taken to be given by

$$\rho = \begin{bmatrix} 1 & 0.4 & 0 \\ 0.4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The FORM estimate of the nominal failure probability based on β_{HL}=2.658 is 0.0039 (Ang and Tang, 1984). The failure probability estimated using 10^6 samples Monte Carlo simulations (Method I) is 0.0038. The results obtained using Method II, III and IV are 0.0039, 0.0039 and 0.0038 respectively. In this example it can be concluded that all the alternative methods (methods II, III, IV) lead to acceptable results. This points towards the possible existence of only one design point and essentially one region that contributes significantly to the failure probability. It is of interest to note that Method II converged in

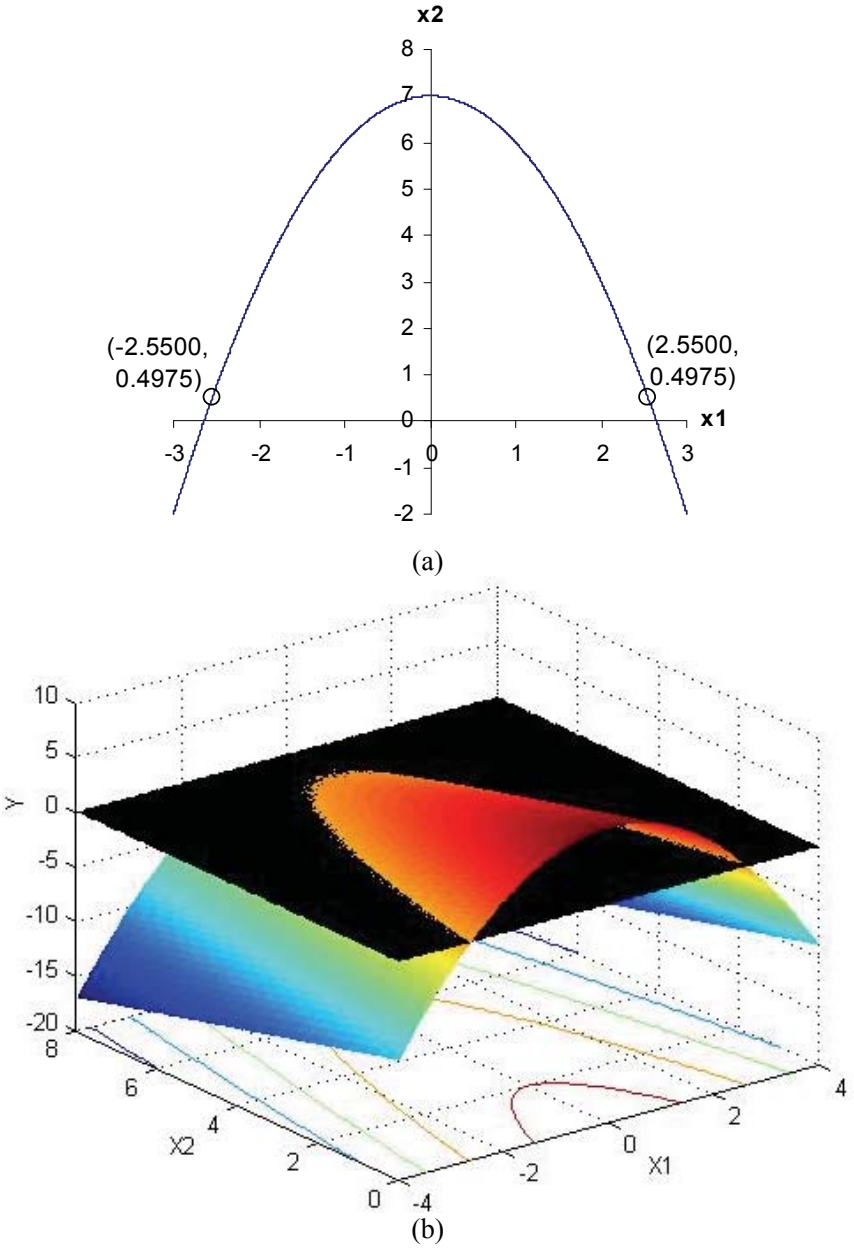


Figure 6: Example 6.1a; Performance function with two design points; (a) the performance function and the design points; (b) kriging response surface approximation to the performance function and the failure surface.

two iterations, requiring a total of 16 limit state evaluations; a total of 24 samples were used in Method III. Method IV converged in three steps requiring totally 12 limit state evaluations. Table 1 shows the results on estimation of probability of failure as a function of μ_Z obtained using methods I-IV. The response surface based methods (II-IV) are observed to perform very well in this example with the number performance function evaluations being the smallest for Method IV.

Here we consider a nonlinear single degree of freedom (sdof) system subjected to a box-input as shown in Figure 7. The governing equation of motion for the given system can be given by

$$m\ddot{u} + K_{eqv}u = F_1(t); \quad u(0) = 0; \quad \dot{u}(0) = 0$$

Table 1: Example 6.1c; reliability of a beam in terms of moment carrying capacity.

μ_Z (in mm^3)	Probability of failure			
	Method I	Method II	Method III	Method IV
4.9161E+05	0.2248	0.2178	0.2261	0.2176
5.7355E+05	0.0890	0.0857	0.0891	0.0855
6.5548E+05	0.0325	0.0312	0.0323	0.0310
7.3742E+05	0.0112	0.0110	0.0114	0.0109
8.1935E+05	0.0038	0.0039	0.0039	0.0038
9.0129E+05	0.0014	0.0014	0.0015	0.0013
9.8322E+05	0.0005	0.0005	0.0005	0.0005

Table 2: Example 6.1d; reliability of a sdof system (figure 7).

Sl. No.	Random Variables	Distribution Type	Mean	Std. Deviation	Units
1	m	Log-Normal	1.0	0.05	kg
2	$K1$	Log-Normal	1.0	0.10	N/mm
3	$K2$	Log-Normal	0.1	0.01	N/mm
4	r	Log-Normal	0.5	0.05	mm
5	$F1$	Log-Normal	1.0	0.20	N
6	$t1$	Log-Normal	1.0	0.20	s

$$K_{eqv} = \begin{cases} (K_1+K_2) & \forall\{(u(t)<r)\&(u(t)<u(t+\Delta t))\} \\ (K_1) & \forall\{(u(t)>r)\&(u(t)<u(t+\Delta t)\} \\ (K_1+K_2) & \forall(u(t)>u(t+\Delta t)) \end{cases} \tag{15}$$

The simulation was carried out for a duration of 10 s. The parameters m, K_1, K_2, r, F_1 and t_1 are considered to be independent lognormal random variables with their properties as summarized in Table 2. The limit state function is defined as $g = 3r - |Z_{\max}|$, where Z_{max} is the maximum displacement response of the system. The failure probability estimated using Method I is 0.0323 using 10^6 samples. The estimate obtained using methods II, III and IV are 0.0334, 0.0322 and 0.0329 respectively. A total of 42 samples, *i.e.*, 6 samples per random variable, were used for the estimation in method III. Methods II and IV converged in 3 and 2 iterations respectively requiring a total of 42 and 16 evaluations respectively.

6.2 Reliability of a ten-bar truss

Here we consider the reliability of a pin jointed truss loaded as shown in Figure 8a. Table 3 lists the details of the probabilistic models adopted for the truss parameters. This problem has been analyzed using Methods I- IV. The structural behavior has been modeled using the finite element method with one 2-noded truss element per member and under the assumption that the joints are perfect pins. The performance of the structure has been defined with respect to mid-span deflection with an allowable limit of 110 mm, leading to the performance function $g(x) = 110 - D(x)$, with $D(x)$ measured in mm. In implementing Method I, a sample size of 10^6 has been used. The failure probability estimated is 0.0088 using Method I, while results obtained using Methods II, III and IV are 0.0050, 0.0079 and 0.0049 respectively. To investigate the robustness of this conclusion, the probability of failure is plotted as a function of one of the system parameter, namely E_m, in Figure 7b. Clearly, Method III is observed to outperform Methods II and IV, in terms of comparison with results from brute force Monte Carlo samples. Table 4 lists the coefficients $\{\beta_i\}_{i=0}^{20}$ and the parameters $\{\theta_i\}_{i=1}^{10}$ that appear in the response surface model (equation 12). In the computational work, it was observed that the results from Method II converged in 3 iterations with a 66 $g-$function evaluations while Method IV converged in 2 it-

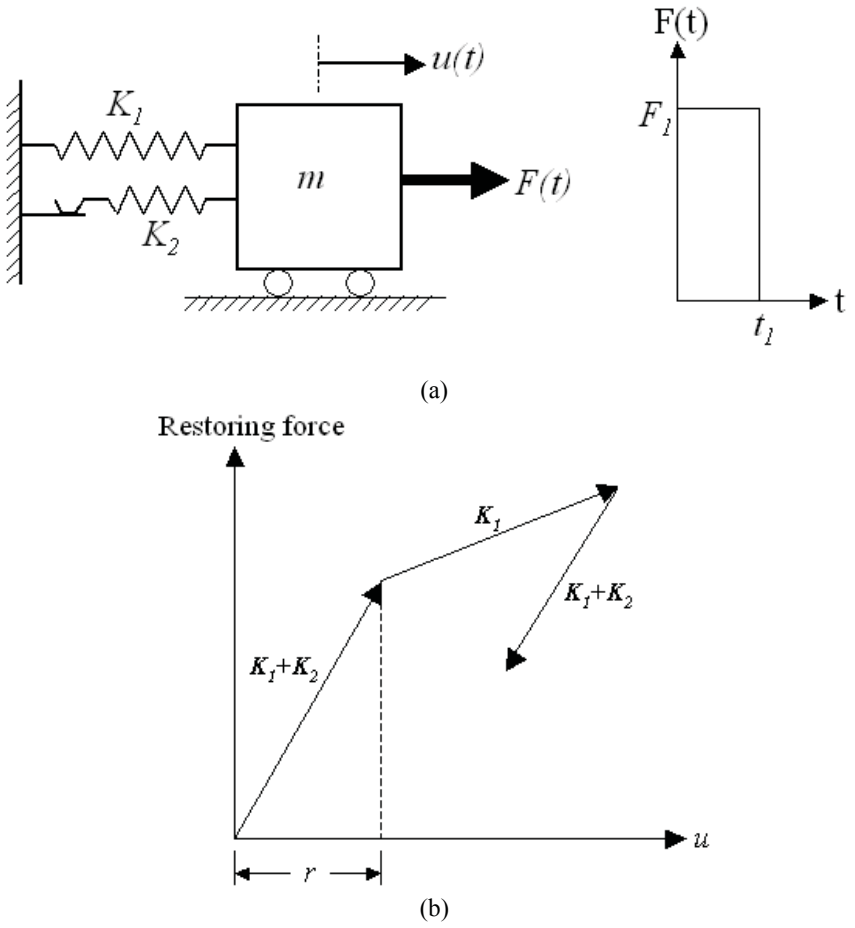
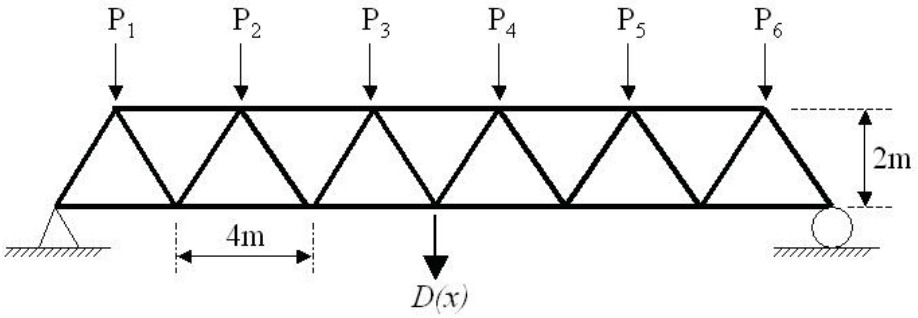


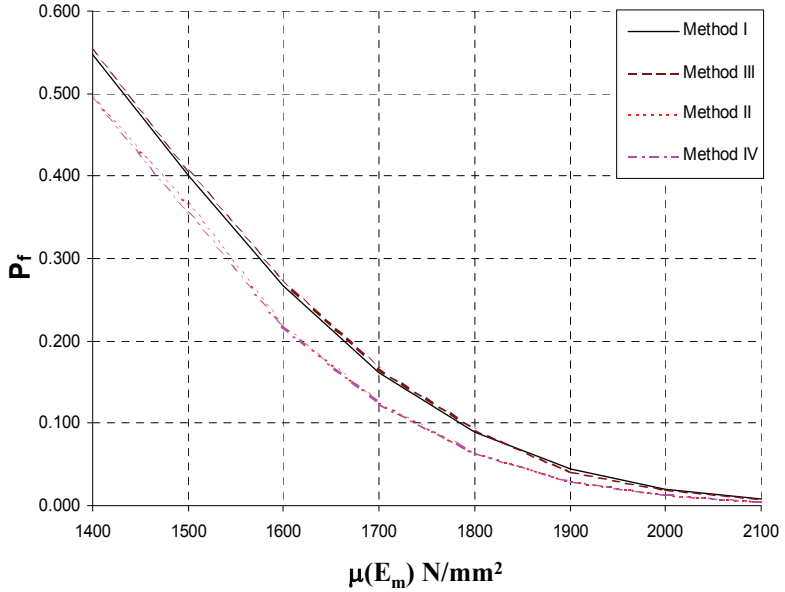
Figure 7: Example 6.1d; (a) Nonlinear oscillator subjected to step input; (b) the restoring force displacement characteristics.

erations requiring 26 g-function evaluations. The accuracy of predictions using method III is expected to depend upon the number of divisions used in constructing the optimal Latin hypercube samples. Figure 8c shows the variation of the failure probability as a function of the number of divisions and it is seen that with about 7 or 8 divisions, one gets satisfactory results. Similar results were obtained for some of the other examples studied in this paper.

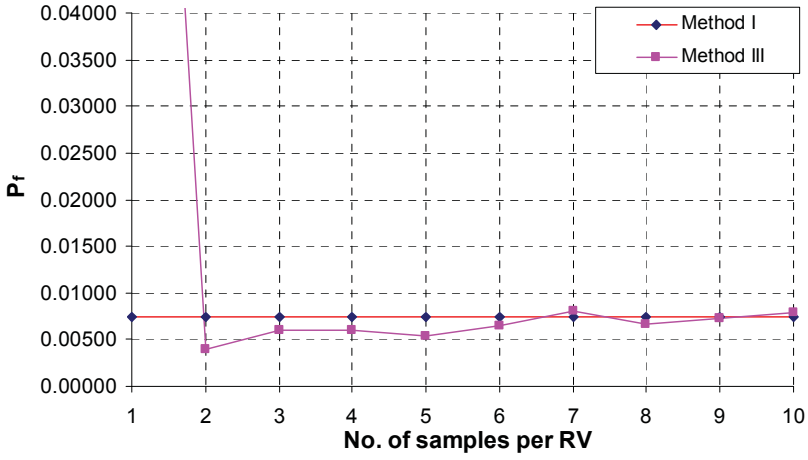
As a further extension of the same problem, we now consider a set of 12 additional performance functions, defined in terms of axial strains in the diagonal members. The members are considered to have failed, if, the axial strain exceeds a threshold value of 0.014. Accordingly, the performance function with respect to mid-span deflection and the 12 performance functions with respect to axial strains are taken to constitute a series system. The failure probability can be represented as $P\big[\{11.0-D(x)<0\}\cap\{.014-\varepsilon_1<0\}\ \ldots\cap\{.014-\varepsilon_{12}<0\}\big]$ where $\varepsilon_i i=1,2,\ldots,12$ represents the strains in each of the 12 diagonal members. We employ Method III to solve the problem on hand. A set of 13 response surfaces i.e. $\hat{g}(x_1),\hat{g}(x_2),\ldots,\hat{g}(x_{13})$are generated corresponding each of the performance functions. In doing so, the same set of samples from optimal Latin hypercubes is used in fitting the distinct response surfaces. Using a sample size of 10^6, the series system failure probability is subsequently computed using the 13 surrogate functions obtained and this probability is obtained as 0.0084. This number compares reasonably well with an estimate of P_f =0.0096, obtained using brute force



(a)



(b)



(c)

Figure 8: Example 6.2 A ten bar truss; the performance function is defined in terms of the admissible mid-span deflection (a) the truss structure and the applied loads; (b) probability of failure as a function of expected Young's modulus of the truss members. (c) variation of the failure probability estimate with number of samples.

Table 3: Example 6.2; reliability analysis of a ten bar truss (figure 8); details of probabilistic models adopted for the underlying random variables.

Sl. No.	Random Variables	Distribution Type	Mean	Std. Deviation	Units
1	Young's Modulus (Main member) – $\mathbf{E}_m$	Log-Normal	2100.00	210.00	N/mm^2
2	Cross-Sectional Area (Main member) – $\mathbf{C}_m$	Log-Normal	2000.00	200.00	mm^2
3	Young's Modulus (Diagonal member) – $\mathbf{E}_d$	Log-Normal	2100.00	210.00	N/mm^2
4	Cross-Sectional Area (Diagonal member) – $\mathbf{C}_d$	Log-Normal	1000.00	100.00	mm^2
5	P_1	Type-1-Extreme	500.00	75.00	N
6	P_2	Type-1-Extreme	500.00	75.00	N
7	P_3	Type-1-Extreme	500.00	75.00	N
8	P_4	Type-1-Extreme	500.00	75.00	N
9	P_5	Type-1-Extreme	500.00	75.00	N
10	P_6	Type-1-Extreme	500.00	75.00	N

Table 4: Example 6.2; reliability analysis of a ten bar truss (figure 7); the parameters of the response surface model (equation 12).

θ_1	0.9587	β_0	-5.7302E+01	β_{11}	-2.7456E-05
θ_2	0.9550	β_1	1.5430E-01	β_{12}	2.7627E-05
θ_3	0.5261	β_2	-9.8300E-02	β_{13}	-4.4755E-06
θ_4	0.7919	β_3	4.7800E-02	β_{14}	-1.5054E-06
θ_5	0.1555	β_4	6.5000E-03	β_{15}	2.8095E-05
θ_6	0.8891	β_5	-2.5500E-02	β_{16}	1.4844E-05
θ_7	0.4855	β_6	-4.3400E-02	β_{17}	1.6814E-05
θ_8	0.3964	β_7	-6.9300E-02	β_{18}	4.3275E-05
θ_9	0.8790	β_8	-9.6000E-02	β_{19}	4.9629E-05
θ_{10}	0.7246	β_9	-8.8800E-02	β_{20}	-3.7224E-05
		β_{10}	2.5300E-02		

Table 5: Example 6.3; reliability analysis of a w-seal (figure 8); details of probabilistic models adopted for the underlying random variables.

Sl. No.	Random variables	Distribution type	Mean	Std. Deviation	Units
1	Temperature	Log-Normal	4.0000E+02	6.0000E+01	deg C
2	Pressure (cavity I)	Type-1-Extreme	6.2000E-01	1.3500E-01	N/mm^2
3	Pressure (cavity II)	Type-1-Extreme	5.0000E-01	1.2000E-01	N/mm^2
4	Displacement	Normal	2.0000E-01	4.0000E-02	mm
5	Thickness	Normal	1.5000E-01	1.5000E-02	mm
6	Young's Modulus	Log-Normal	2.1000E+05	2.1000E+04	N/mm^2
7	Yield Stress	Normal	9.8000E+02	5.0000E+01	N/mm^2

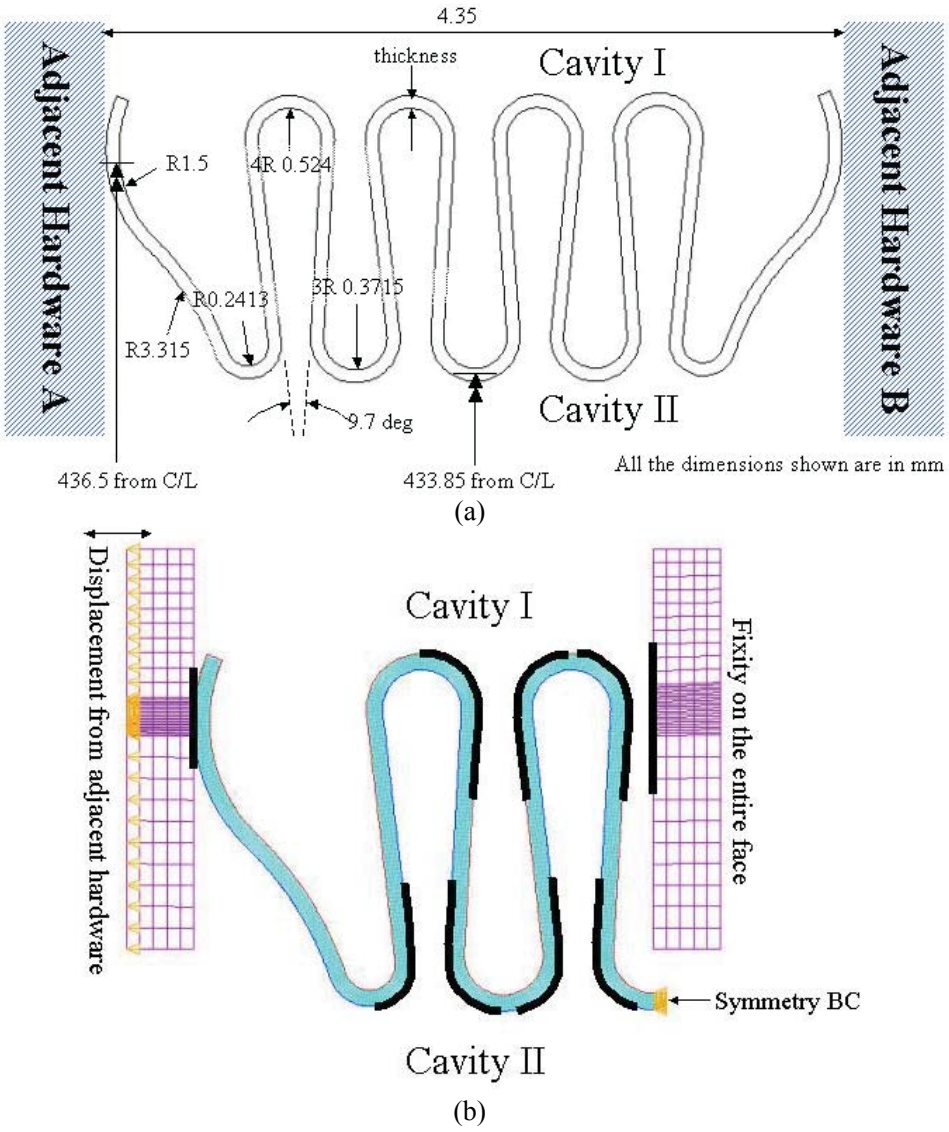


Figure 9: Example 6.3; reliability analysis of a W-seal; (a) geometry of the W-seal; (b) finite element model used in the reliability study; gap elements at the interfaces are shown with thick lines.

Monte Carlo simulation on the full FE model with 10^6 runs of the finite element code.

6.3 Reliability of a W-seal

In this example, we consider the problem of reliability analysis of a W-seal that is commonly used in aircraft engines. Such seals serve to prevent leakage between two pressure cavities within an aircraft engine. Figure 9 shows a typical W-seal that prevents leakage between two cavities I and II. The seal deformation is governed by the relative displacements between the adjacent hardware A and B, pressure fields from cavities I and II and due to the body temperature of the seal. A total of 7 random variables are considered, leading to X={body temperature of the seal, pressures in cavities I and II, applied relative displacements between the adjacent hardware, seal thickness, Young's modulus, and yield stress of the seal material}. The statistical properties of these random variables are provided in Table 5. The correlation

coefficient matrix is taken to be

$$\rho = \begin{bmatrix} 1.0 & 0.5 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.5 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 1.0 \end{bmatrix}$$

Thus, in this example, the random variables are non-normal and mutually correlated. The seal is considered to have failed if the maximum von Mises' stress anywhere in the seal exceeds the yield stress during the operation. Hence, the limit state can be defined as $g(x) = \max(\sigma_{vonmises}) - \sigma_{yield}$. The seal geometry and the associated dimensions are shown in Figure 9(a). Figure 9(b) shows the finite element model of the W-seal, created on the ANSYS version 8.1 platform. Taking advantage of the symmetry of the structure and the loads, only half of the seal is considered for the finite element modeling. The model is meshed with SOLID 42 elements with axi-symmetric option. The wall, modeled on either side, represents adjacent hardware. The relative axial displacement between adjacent hardware A and B is modeled as applied displacements on the wall A, as shown in Figure 9(b). Standard surface to surface contact elements are used to model the interface between the seal and the adjacent hardware and between its own surfaces. A high contact stiffness of 2.1×10^6 N/mm^2 is used to reduce contact penetration and a pinball radius of 0.5 mm is used for contact identification. W-seals are made up of sheet metal components and are highly flexible; consequently, the analysis is taken to include the effect of nonlinear strain-displacement relations. There is an interference fit between the seal and the adjacent hardware. To model this, the analysis is carried out in two steps. The first step is to run for an interference of 0.15mm, modeled as applied displacement from adjacent hardware A. This analysis is performed at an ambient condition of 25 degC. The second step is to simulate the operating condition of the hardware, under the action of body temperature, relative displacements between the hardware and pressure in cavities I and II. Hence, the second step is carried out under the action of the all the loads. Coefficient of thermal expansion used is 6.5×10^{-6} per deg C. One run of the FE calculation requires about 8 minutes on a Pentium 2.66GHz 1GB RAM machine. For such problems, it is practically infeasible to carry out a large number of FE simulations (of the order of 10^6 as has been done in previous examples) and hence method I was not considered suitable in this example. Instead, the failure probability was estimated using methods II, III and IV. Results obtained using methods II, III and IV predicted a failure probability of 0.0020, 0.0020 and 0.0019 respectively. It required 7 iterations for the Raj Shekhar & Ellingwood's procedure to converge at the design point requiring a total of 112 function evaluations, while method IV converged in 6 iterations thus requiring a total of 26 evaluation of the FE model to arrive at the design point. A total of 56 samples were used in method III.

6.4 Reliability of randomly parametered dynamical systems excited by random excitations

The response surface method developed in sections 2 and 3 can be modified to tackle problems of time variant reliability analysis of randomly excited structural systems whose parameters are also random in nature. We illustrate this possibility by considering an example of a randomly driven sdof Duffing's oscillator governed by the equation

$$m\ddot{x} + c\dot{x} + kx + \varepsilon x^3 = f(t);\ x(0) = x_0;\ \dot{x}(0) = \dot{x}_0 \tag{16}$$

Here $f(t)$ is taken to be a stationary Gaussian random process with a specified power spectral density function. In the present study we take

$$S_{ff}(\omega) = \frac{\lambda_0}{\lambda_0^2 + (\omega - \beta_0)^2} + \frac{\lambda_0}{\lambda_0^2 + (\omega + \beta_0)^2}; \quad -\infty < \omega < \infty \tag{17}$$

with λ_0 =0.5 and $\beta_0 = 2$. We also consider the parameters m, c, k, and ε to constitute a vector of random variables with prescribed joint probability density function. For the purpose of reliability

analysis, we consider performance functions defined as functions of the state variables $x(t)$ and $\dot{x}(t)$. Thus, for instance, if interest is focused on reaction transferred to the support, we obtain the reaction as $R(t) = c\dot{x} + kx + \varepsilon x^3$ and define probability of failure through the relation

$$\begin{aligned} 1 - P_f &= P[R(t) < R_0 \forall t \in (t_0, t_0 + T)] \\ &= P\left[\max_{t_0<t<t_0+T} R(t) < R_0\right] \\ &= P\left[\max_{t_0<t<t_0+T} R(t) - R_0 < 0\right] \\ &= \int\limits_{r-r_0<0} \int p_{R_m R_0}(r, r_0)\, dr dr_0 \end{aligned} \tag{18}$$

where $R_m = \max\limits_{t_0<t<t_0+T} R(t)$ is the extreme value of $R(t)$ and R_0 is the threshold value of the permissible reaction. If R_m and R_0 are taken to be mutually independent one gets

$$1 - P_f = \int\limits_{r-r_0<0} \int p_{R_m}(r) p_{R_0}(r_0)\, dr dr_0 \tag{19}$$

The problem of time variant reliability thus reduces to the problem of determination of extreme value of the random process $R(t)$ over the time interval $(t_0, t_0 + T)$. The problem of determination of pdf of extremes of sequence of random variables and extremes of random processes over specified parameter intervals is a widely studied topic (see, for example, Castillo 1988, Kotz and Nadarajah 2000, and Coles 2001). One of the main results of the extreme value theory of sequence of independent identical distributed (iid) random variables is that the limiting form of the extreme value distribution assumes one of the three limiting forms, namely, the Gumbel, Weibull or Frechet distributions. Consequently, it is possible to envisage three basins of attractions so that a given iid sequence of random variables has extremes that belong to one of three extreme value distribution types. Techniques for identifying the basin of attraction, to which a given iid sequence belongs to, are available so that the basin could be identified even when the knowledge of the underlying random variable is limited to a sample (Castillo 1988). Recently, Radhika *et al.,* (2007) have proposed that this technique could be extended to study reliability of randomly excited dynamical systems. In the present section, we extend this formulation to combine methods from extreme value analysis with response surface models. We restrict our attention to situations in which the extremes of random processes are modeled using Gumbel distributions. Techniques for ascertaining whether or not Gumbel model could be used for modeling extremes based on limited data are available in the book by Castillo (1988) and are discussed in detail in the works of Radhika *et al.,* (2007) and these details are not provided here. Also, we restrict our attention here to study the extremes of the displacement response$X(t)$ in the steady state. Accordingly, the starting point in the present study is the model for the PDF of $X_m = \max\limits_{t_0<t<t_0+T} X(t)$ conditioned on the random variables m, c, k, and ε taken to be of the form

$$P_{X_m}(x|\psi) = \exp\left[-\exp\left\{\frac{-(x-\lambda(\psi)))}{\delta(\psi)}\right\}\right], \quad -\infty < x < \infty \tag{20}$$

where $\psi = \begin{pmatrix} m & c & k & \varepsilon \end{pmatrix}^t$ is the vector of random variables denoting uncertainties in the system characteristics. We propose in this study that we model the quantities $\lambda(\psi)$ and $\delta(\psi)$ using the kriging based response surface models. Accordingly, we develop surrogate models for $\lambda(\psi)$ and $\delta(\psi)$ using method III outlined in sections 2 and 3. Following this we determine the unconditional PDF of $X_m = \max\limits_{t_0<t<t_0+T} X(t)$ using

$$\begin{aligned} & P_{X_m}(x) \\ &= \int \exp\left[-\exp\left\{\frac{-(x-\lambda(\psi)))}{\delta(\psi)}\right\}\right] p_\psi(\psi) d\psi \\ &\approx \frac{1}{N*} \sum_{s=1}^{N*} \exp\left[-\exp\left\{\frac{-(x-\lambda(\psi_i)))}{\delta(\psi_i)}\right\}\right] \end{aligned} \tag{21}$$

where $\{\psi_i\}_{i=1}^{N*}$ are samples drawn from the known pdf $p_\psi(\psi)$.

For the purpose of illustration, we consider the parameters m, c, k to be mutually independent and

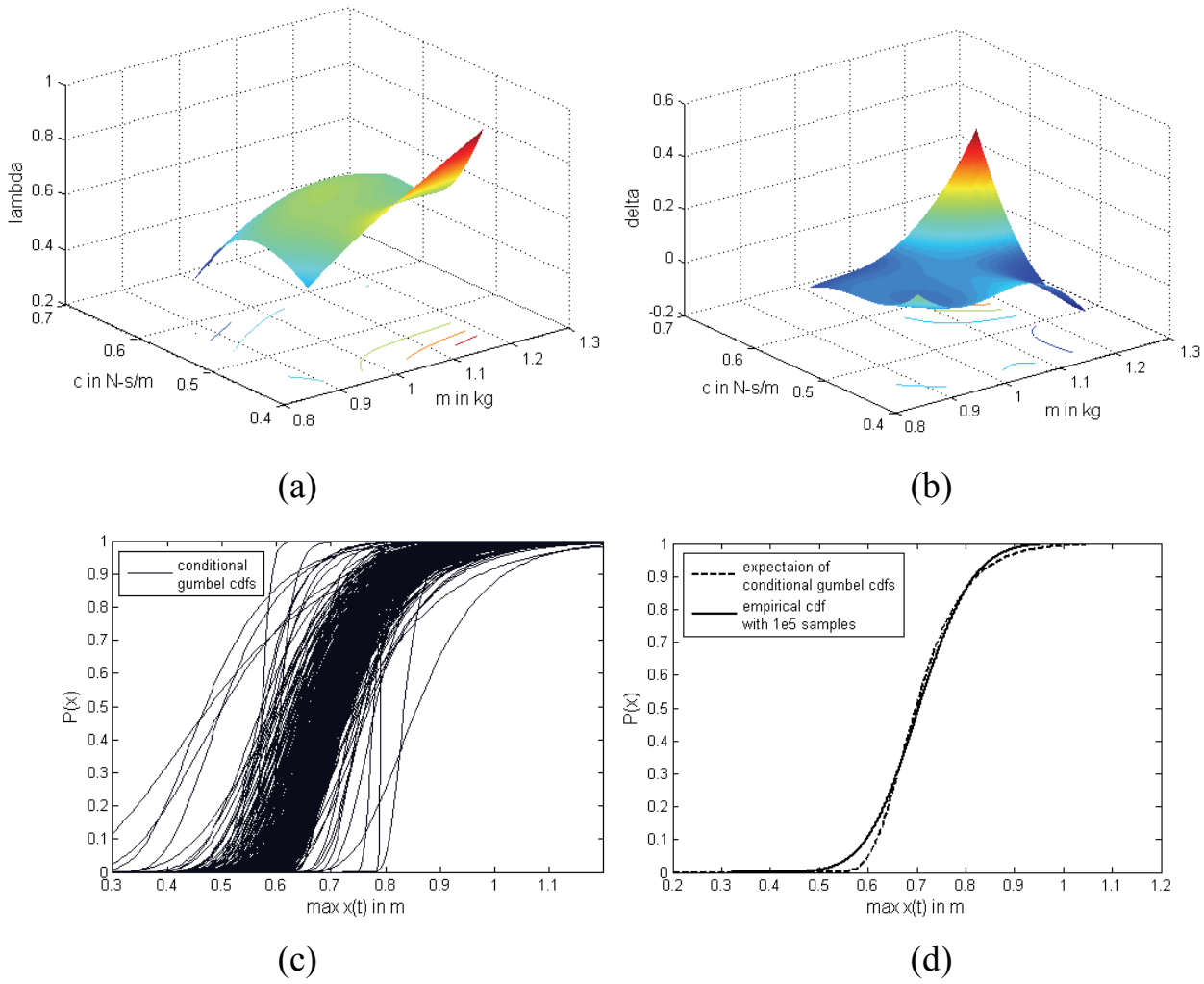


Figure 10: Example 6.4; reliability analysis of a randomly parametered Duffing's oscillator subjected to random excitations; (a) kriging surface for the parameter λ in equation (19); here k and ε are fixed at their respective mean values; (b) kriging surface for the parameter δ in equation (19); here k and ε are fixed at their respective mean values; (c) sample realization of the Gumbel PDF; (d) models for the PDF of extreme displacement response.

log-normally distributed random variables and ε to be a type 1 asymptotic random variable with mean values of 1 kg, 0.500Ns/m, 4N/m, 10N/m^3 and standard deviations 0.050kg, 0.025 Ns/m, 0.200 N/m, and 1 N/m^3 respectively. The excitation model parameters (equation 16) are selected to be $\alpha_0 = 0.5$ and β_0=2. Samples of excitation time histories compatible with the psd function as in equation 16 are simulated by using a Fourier representation with random variable coefficients (Papoulis and Pillay 2001). For specific realizations of the excitation $f(t)$ and the model parameters m, c, k, and ε the sample solution of equation 15 is obtained using the 4^{th} order Runge Kutta fourth order algorithm. In finding extreme responses we take $x(0) = \dot{x}(0) = 0$, $t_0 = 5$s and $T = 45$s. Figure 10 (a) and (b) show the sections of kriging surfaces for the parameters λ and δ as functions of c and m with the other parameters k and ε fixed at their respective mean values. An ensemble of realizations of the conditional PDF $P_{X_m}(x|\psi)$ is shown in figure 10 (c) and the unconditional PDF $P_{X_m}(x)$ obtained by averaging this ensemble is compared with results from 10^5 direct

Monte Carlo simulations solution of equation 15 in figure 10(d). As can be observed from figure 10(d) the performance of the proposed response surface method is observed to compare reasonably well with direct Monte Carlo simulation results.

7 Discussion

Clearly Method I (that is, the brute force Monte Carlo simulations) is uniformly applicable, at least, in principle, to all the problems considered in this study. However, for problems of practical interest, such as the analysis of the W-seal, in which, a single rtun of Monte Carlo simulation takes about 8 min of CPU (on a Pentium 2.66GHz 1GB RAM machine), the evaluation of P_f of the order of 10^{-3} requires at least about 10^4 simulation runs leading to a CPU time requirement of approximately about 1300 hrs. This clearly prohibits the application of Method I for this type of problems. On the other hand, the response surface based methods do not suffer from such computational demands. Moreover, the computational effort involved in predicting P_f using Method I increases significantly as P_f reduces. The response surface methods, by and large, do not suffer from this limitation. Among Methods II-IV, Method III is the most robust in terms of its ability to handle multiple design points and (or) multiple regions of comparable importance for reliability calculation. This feature of the method is particularly advantageous in real-life situations in which it is difficult (if not impossible) to establish *a priori* whether or not a given performance function possess multiple design points. Between Method II and IV, the latter seems advantageous since in Method II, the number of function evaluations depends upon the number of iterations needed to reach a satisfactory convergence on the reliability. Also, while Methods II and IV are rooted in the classical reliability index methods (with the concomitant need to linearize the performance function, as in the first order reliability method), Method III, on the other hand, is based on statistical principles and does not involve any linearization of performance functions. In terms of computational effort Method III is perhaps the most demanding among Methods II-IV and for problems with single design points, the Methods II and IV are equally well applicable.

8 Closing remarks

The development of computational models for reliability analysis of complex engineering structures is one of the difficult problems that lie at the forefront of structural engineering research. The developments in the area of finite element modeling have ensured that it is possible to perform sample simulations of structural behavior allowing for several complexities such as nonlinearity, transient dynamic behavior and large sized problems. However, when questions on reliability of such structures are to be answered, especially when the probability of failure is very small (of the order of 10^{-5} or less), the problem becomes computationally intractable even with modern computing facilities. This has prompted development of simplified modeling tools that lead to surrogate models for long-running finite element codes. The present study explores the application of developments in the area of design and analysis of computer experiments to problems of structural reliability analysis. Specifically, the study combines the space filling optimal Latin hypercube sampling techniques with kriging models to develop surrogate functions for performance functions associated with reliability of structural systems whose behavior is modeled via finite element models. The range of issues covered includes linear/nonlinear structural models, static/dynamic behavior, Gaussian/non-Gaussian uncertainty models, random variable/random field models, time variant/invariant reliability analyses, single/multiple performance functions, and random excitation and random structural parameter models. The application of kriging based surrogate models for parameters of asymptotic extreme value distributions in the context of response of randomly parametered nonlinear systems subjected to random excitations is also discussed. The performance of the proposed procedures is assessed (where possible) by comparing the results with direct Monte Carlo simulation results obtained using large sample. The procedures developed are demonstrated to perform sat-

isfactorily in terms of accuracy, computational efficiency and ability to handle performance functions with multiple design points or multiple regions of comparable importance. The methods developed herein have promise for further extensions to obtain measures of sensitivity of failure probability with respect to different random variables and also to carry out reliability based optimization studies. These aspects are currently being investigated by the present authors.

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