Boundary Control for Inverse Cauchy Problems of the Laplace Equations

Leevan Ling[1] **and Tomoya Takeuchi**[2]

Abstract: The method of fundamental solutions is coupled with the boundary control technique to solve the Cauchy problems of the Laplace Equations. The main idea of the proposed method is to solve a sequence of direct problems instead of solving the inverse problem directly. In particular, we use a boundary control technique to obtain an approximation of the missing Dirichlet boundary data; the Tikhonov regularization technique and the L-curve method are employed to achieve such goal stably. Once the boundary data on the whole boundary are known, the numerical solution to the Cauchy problem can be obtained by solving a direct problem. Numerical examples are provided for verifications of the proposed method on the steady-state heat conduction problems.

Keyword: Method of fundamental solution, method of particular solution, collocation method, Tikhonov regularization, L-curve.

1 Introduction

The Cauchy problem for an elliptic equation is a typical ill-posed problem whose solution does not depend continuously on the boundary data. That is, a small error in the specified data may result in an enormous error in the numerical solution. This problem appears in many applications for example in the cardiography, the non-destructive testing, and etc. Stable and efficient numerical methods are of high importance. However, it is well-known that the Cauchy problem for an elliptic equation is ill-posed without any *a priori* bounds of u (e.g., Tikhonov and Arsenin Tikhonov and Arsenin (1977)). However, given some *a priori* bounds of u, we can restore the stability and, for stable numerical reconstructions of solutions, we can use regularization techniques. There are a large number of works devoting to stable numerical methods. We cannot list all works completely and the following is a partial list: Cheng, Hon, Wei and Yamamoto Cheng, Hon, Wei, and Yamamoto (2001), Hào and Lesnic Hào and Lesnic (2000), Hon and Wei Hon and Wei (2001), Klibanov and Santosa Klibanov and Santosa (1991), Lattes and Lions Lattès and Lions (1969), Reinhardt, Han et al. Han (1982); Reinhardt, Han, and Hào (1999). For inverse heat conduction problem, readers are referred to Chang, Liu, and Chang (2005); Hon and Wei (2005); Ling and Atluri (2006). For Euler-Bernoulli Beam, see Huang and Shih (2007). See Marin, Power, Bowtell, Sanchez, Becker, Glover, and Jones (2008) for Magnetic Resonance Imaging Gradient Coils.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Let Γ be a relatively open subset of $\partial\Omega$ on which Cauchy data is available and the remaining boundary (with no data) is denoted by $\Sigma := \partial\Omega \setminus \Gamma$. Denote $\nu = \nu(x)$ to be the unit outward normal vector to $\partial\Omega$ at x and

$$\partial_\nu u = \sum_{i,j=1}^{n} (\partial_j u)\nu_i.$$

For any kernel $G(x,\xi)$, the differential operator in $\partial_\nu G$ always acts upon the first variable.

We consider the classical ill-posed Cauchy problem for the Laplace equations: Given h, g_1 and g_2, find u in Ω or on Σ where

$$\begin{cases} \Delta u = h, & x \in \Omega, \\ u|_\Gamma = f, \\ \partial_\nu u|_\Gamma = g, \end{cases} \qquad (1)$$

[1] Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong.
[2] Graduate School of Mathematical Sciences, University of Tokyo, Tokyo, Japan.

We proposed a numerical procedure based on the method of fundamental solutions (MFS) and a boundary control technique. The MFS has recently been used extensively for solving various types of linear partial differential equations (PDEs) and is successfully applied to many engineering problems, for example Chen (1995); Fairweather and Karageorghis (1998); Golberg and Chen (2001); Jin (2004); Seyrafian, Gatmiri, and Noorzad (2006); Tsai, Young, and Cheng (2002); Young, Chen, Fan, and Tsai (2006); Young, Chen, Chen, and Kao (2007). These problems are well-posed direct problems in which the Dirichlet or Neumann data on the whole boundary are known. Detailed reviews of the MFS have been given independently by Fairweather and Karageorghis Fairweather and Karageorghis (1998) and Golberg and Chen Golberg and Chen (1999). In the studies of inverse problems, the boundary data are given with noises on part of the accessible boundary. This usually poses difficulty on most of the traditional numerical methods to obtaining acceptable numerical approximation to the solution. The truly meshless MFS is an excellent candidate for solving these kinds of inverse problems; in particular, the Cauchy problems of elliptic operators have been studied in Jin and Zheng (2005); Wei, Hon, and Ling (2007).

In Section 2, we present the boundary control technique in a general setting. The technique is, then, coupled with the MFS and the implementation is given in Section 3. The fourth section is devoted to the presentation of numerical examples. We focus on the 2D steady-state heat conduction equation. Accuracy for various noise levels and different Cauchy boundaries Γ are shown to demonstrate the feasibility and efficiency of the proposed method.

2 Boundary Control

To begin, we have the following procedure to obtain an approximation of the missing Dirichlet boundary data to (1) on Σ. Consider the following two direct problems

$$\begin{cases} \Delta u_1 = h, & x \in \Omega, \\ u_1|_\Sigma = 0, \\ \partial_\nu u_1|_\Gamma = g, \end{cases} \tag{2}$$

and

$$\begin{cases} \Delta u_2 = 0, & x \in \Omega, \\ u_2|_\Sigma = \varphi, \\ \partial_\nu u_2|_\Gamma = 0. \end{cases} \tag{3}$$

In (3), we consider φ as an unknown and we want to control φ on Γ with the target

$$u_2|_\Gamma = f - u_1|_\Gamma. \tag{4}$$

Once the Dirichlet boundary data on Σ is obtained, we look for the solution to our Cauchy problem (1) by solving the following direct problem

$$\begin{cases} \Delta u_3 = h, & x \in \Omega, \\ u_3|_\Sigma = \varphi, \\ u_3|_\Gamma = f. \end{cases} \tag{5}$$

Suppose all data f and g are known exactly with no noise. It is easy to see that the solution to (5) coincides with the desired solution to the Cauchy problem (1).

Suppose that we control φ such that (4) is satisfied exactly. Let $w = u_1 + u_2$. Then w is the unique solution to both (1) and (5). To see this, note that

$$\Delta w = \Delta(u_1 + u_2) = h + 0 = h$$

for all $x \in \Omega$. First,

$$w|_\Gamma = u_1|_\Gamma + u_2|_\Gamma = u_1|_\Gamma + (f - u_1|_\Gamma) = f.$$

To show that w is the solution to (1), we consider the Neumann data of w at Γ. Moreover,

$$\partial_\nu w|_\Gamma = \partial_\nu u_1|_\Gamma + \partial_\nu u_2|_\Gamma = g + 0 = g.$$

Lastly, combining (2) and (3) gives that

$$w|_\Sigma = u_1|_\Sigma + u_2|_\Sigma = \varphi + 0 = \varphi.$$

Therefore, w is the solution to (5). By the uniqueness result, the solution u_3 to (5) is also the solution to (1).

In the studies of inverse problems, we want to consider the case when the boundary data are given with noise. Throughout the paper, we use the notation u_j^δ $(j = 1,2,3)$ to denote the solutions to corresponding PDEs in order to emphasize the presence of noise. Although our ill-posed problem is broken down into a sequence of direct problems, the ill-posed nature is now reflected in the control problem (4). One may notice that (2) and (3) solves the noise-free Cauchy problem already. The presence of (5), however, allows us the make better use of the noise data f^δ and ensure a solution with better quality.

3 Numerical Implementation with MFS

We now present a numerical procedure for solving the Cauchy problem (1) using the MFS. Denote $G(x,\xi)$ to be the fundamental solution of the Laplacian. When the source point ξ is located outside the domain $\overline{\Omega}$, the fundamental solution satisfies the elliptic equation in domain Ω. The basic idea of the MFS is to approximate the solution in term of a series of fundamental solutions. By construction, any numerical solution automatically satisfies the differential equation. The unknown coefficients are then chosen to match the boundary conditions.

To solve for u_1 and u_3 in PDE (2) and (5), one can first employ the method of particular solution. A particular solution is defined as a solution satisfies the nonhomogeneous equation but does not necessarily satisfy the boundary conditions. For simple nonhomogeneous term h, it is possible to obtain a particular solution analytically. In most cases, we can apply the dual reciprocity method Partridge, Brebbia, and Wrobel (1992) in which we evaluate the particular solution by a series of approximate particular solutions. More recent progress in deriving the close-form particular solutions can be found in Chen and Rashed (1998); Golberg, Muleshkov, Chen, and Cheng (2003); Muleshkov, Golberg, and Chen (1999). By splitting the solution in to particular solution and its associated homogeneous solution, the nonhomogeneous PDE can be transformed to a homogenous PDE that can be solved by the standard MFS. The details of such methods are not the scope of this paper. Readers are also referred to Golberg and Chen (1999) and the references there within.

To solve the control problem (4), we need to solve many direct problems. The number of PDEs to be solved here depends on the number of terms used to expand φ. This makes the MFS an attractive alternative in comparison to other domain-based methods. Let $\Xi := \{\xi_i\}_{i=1}^m$ be a set of source points in $\mathbb{R}\backslash\overline{\Omega}$. We represent the missing boundary data by the linear sum of the trace of fundamental solutions with singularity outside of our domain and control the coefficients by "control method" originated by Lions Lattès and Lions (1969). Suppose the function φ in (3) is expanded by the fundamental solution

$$\varphi = \sum_{j=1}^{m} \beta_j G(\cdot - \xi_j) =: \sum_{i=1}^{m} \beta_j G_j. \tag{6}$$

By decomposing $u_2 = \beta_1 u_{2,1} + \ldots + \beta_m u_{2,m}$, we decompose (3) into a sequence of elliptic problems

$$\begin{cases} Au_{2,j} = 0, & x \in \Omega, \\ u_{2,j}|_\Sigma = G_j, & \\ \partial_\nu u_{2,j}|_\Gamma = 0. & \end{cases} \tag{7}$$

Using the same set of source points, we can expand each $u_{2,j}$ for $j = 1,\ldots,m$ as

$$u_{2,j} = \sum_{i=1}^{m} \lambda_{i,j} G(\cdot - \xi_i).$$

Let $X = (X_\Sigma, X_\Gamma) := \{x_i\}_{i=1}^M \subset \Sigma \cup \Gamma$ with $M \geq m$ be the set of collocation points. We can solve the whole sequence (7) in one step by solving the following matrix system for $\lambda_{i,j}$

$$[\lambda_{i,j}]_{1\leq i,j\leq m} =: \Lambda = \begin{bmatrix} G_{X_\Sigma,\Xi} \\ (\partial_\nu G)_{X_\Gamma,\Xi} \end{bmatrix}^\dagger \begin{bmatrix} G_{X_\Sigma,\Xi} \\ O \end{bmatrix},$$

where $\dagger$ denotes the pseudoinverse, O denotes the zero matrix with size $|X_\Gamma| \times m$,

$$G_{X,Y} := [G(x,y)]_{x\in X, y\in Y} \text{ and}$$
$$(\partial_\nu G)_{X,Y} := [\partial_\nu G(x,y)]_{x\in X, y\in Y}.$$

Our MFS based approach is completely meshless and is very fast to compute with no numerical integration involved.

In practice, we only have an approximation to $u_2^\delta|_\Gamma$; we denote by

$$\Phi\beta := K_m(\varphi) = [G_{X_\Gamma,\Xi}]\,[\Lambda]\,[\beta_j]_{1\leq j\leq m}$$

that is the trace of the approximate solution of (3) at X_Γ for all φ.As mention above, the control problem is ill-posed. For that matter, the minimization problem we solve will be of the form

$$\min_{\beta \in \mathbb{R}^m} \left\| \Phi\beta - (f^\delta - u_1^\delta|_\Gamma)|_{X_\Gamma} \right\|^2 + \alpha \left\| \beta \right\|^2. \qquad (8)$$

The determination of a suitable value for the regularization parameter α is crucial to the accuracy Tikhonov and Arsenin (1977). In our computation, we employ the L-curve criterion by Hansen and O'Leary Hansen and O'Leary (1993).

Once the coefficients β are determined, the control can be evaluated everywhere on Σ by using (6). The set up of PDE (5) is now completed. It, again, is an elliptic differential equation with Dirichlet boundary condition. Employing the standard MFS approach to solve (5) provides the final approximation to our Cauchy problem.

4 Numerical Examples

In this section, we verify the numerical accuracy of the proposed method for the Cauchy problem (1). In particular, we are interested in the two-dimensional steady-state solution of heat conduction. Applications to 3D problems or even other elliptic operators (e.g. the Helmholtz operator) are straightforward.

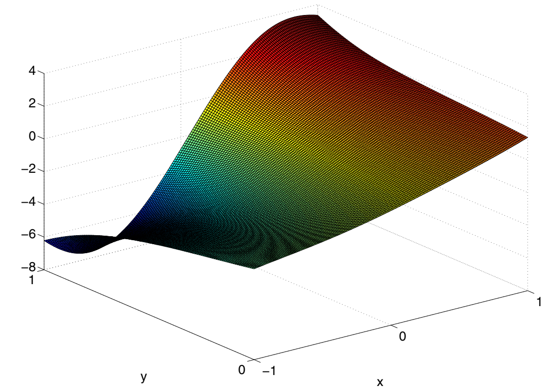
Consider the 2D case where $\Omega = [-1,1] \times [0,1]$ with two kinds of Cauchy boundaries. That is, the Cauchy data $\{f^\delta, g^\delta\}$ are given on either one side $\Gamma_1 = [-1,1] \times \{0\}$ or three sides such that $\Omega \backslash \Gamma_2 = [-1,1] \times \{1\}$ of the rectangular domain $\Omega = [-1,1] \times [0,1]$.

We choose the following functions as test examples:

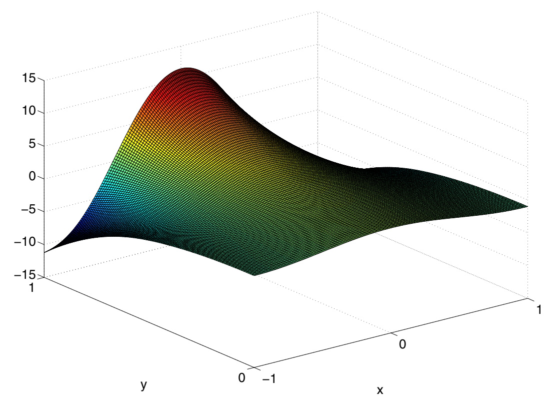
Example 1 $u(x,y) = x^3 - 3xy^2 + e^{2y}\sin 2x - e^y \cos x$.

Example 2 $u(x,y) = \cos \pi x \cosh \pi y$.

These exact solutions of both examples are shown in Figure 1. Note that in Example 2, u is relatively flat on $y = 0$ where our Cauchy data is imposed in comparison to that on $y = 1$. This makes Example 2 more difficult to solve accurately than Example 1.



Example 1



Example 2

Figure 1: Exact solutions u in both examples 1–2.

Noisy data $\{f^\delta, g^\delta\}$ is obtained by adding random numbers to the exact data $\{f, g\} = \{u|_\Gamma, \partial_\nu u|_\Gamma\}$ by

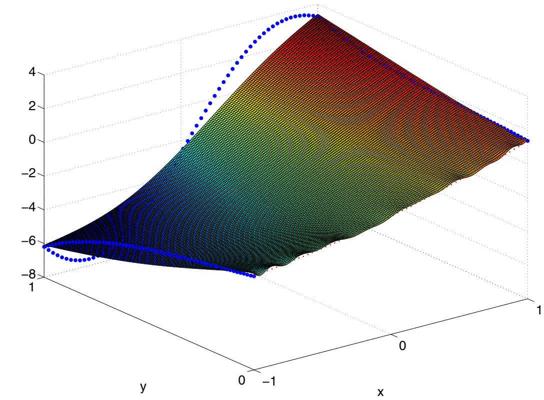
$$\chi^\delta(\xi) = \chi(\xi) + \frac{\delta}{100} \max_{z \in \Gamma} |\chi(z)| \text{rand}(\xi), \qquad \chi \in \{f, g\}$$

for $\xi \in \Gamma$, where rand(ξ) is a random number between $[-1,1]$ and $\delta\% \in \{0\%, 1\%, 5\%, 10\%\}$ is noise level. For all given noisy data $\{f^\delta, g^\delta\}$ with various noisy levels, we apply the proposed method to obtain an approximate solution to u in each example. We denote by U the approximate solution obtained. For the numerical error estimations, we compute the relative error of U over the whole domain Ω:

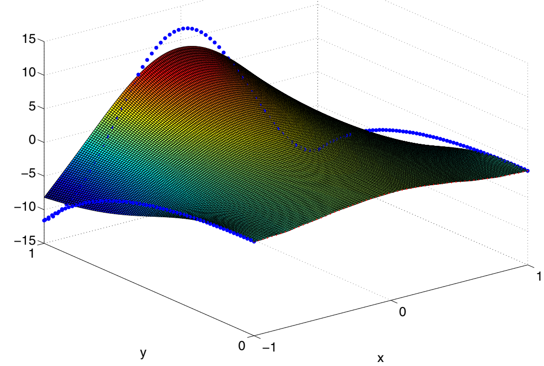
$$E_r(U) := \frac{\|U-u\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}.$$

Since the numerical results are not sensitive to the value of M, we uniformly distribute $M = |X| = 250$ collocation points on $\partial\Omega$ for all computations throughout the section. There are $m = |\Xi| = 72$ source points placed evenly outside $\overline{\Omega}$ on $[-1-R, 1+R] \times [-R, 1+R]$ for $R = 0.20,\ 0.25,$ and 0.30.

Intuitively, more data results in better solution. This is exactly what we observe in Table 1 in which the relative errors of all runs are listed. Under the same noise level, the numerical solution obtained from Γ_2 is more accurate than that from Γ_1 in both examples. As pointed out earlier, the reconstruction accuracies of all cases in Examples 1 are better than that of Example 2. Based on the numbers in Table 1, the numerical reconstructions from 10% noisy on Γ_1 may seem inferior. From Figure 2, however, we see that these numerical reconstructions still give reasonable approximations to the overall shapes of u's. Note that the direct MFS approaches Jin and Marin (2007); Wei, Hon, and Ling (2007), in which the unknown solution is expanded by MFS and the unknown coefficients are identified by direct collocation on Γ with some regularization techniques, can provide reasonable numerical approximations for noise levels up to around 2%. On the other hand, it seems to be universally true that all MFS based methods are capable in providing high accuracy approximations when the data is noise free. Furthermore, we point out that some finite element based reconstruction schemes are also able to handle a 10% noise level, see Bourgeois (2005); Chakib and Nachaoui (2006) for example.



Example 1



Example 2

Figure 2: Numerical reconstructions (surfaces) and the exact boundary data (dotted lines) of u in both examples 1–2 from 10% noisy data on Γ_1.

To get a better insight about the error profiles, we focus on the case of $R = 0.25$ for the rest of the section. The log of relative errors for the noise free reconstructions of u in Example 1 from both Γ_1 and Γ_2 are shown in Figure 3; Example 2 gives similar error profiles and we omit these plots here. For the case of Γ_1, we observe that the error propagates radially from the origin. Hence, the maximum error will almost always occur at the two corners $[\pm 1, 1]$. For that of Γ_2, the error propagates outwards from the point $[0, 1]$, where it is farthest away from boundary data with maximum error, as if there is a heat source. In Figure 4–5, we show the error contour for Example 2 with Γ_1 for various noise levels; again, Example 1 shows similar patterns and is omitted here. Although the magnitudes of error increase with the noise levels, we see an overall similar error distribution for all tested cases.

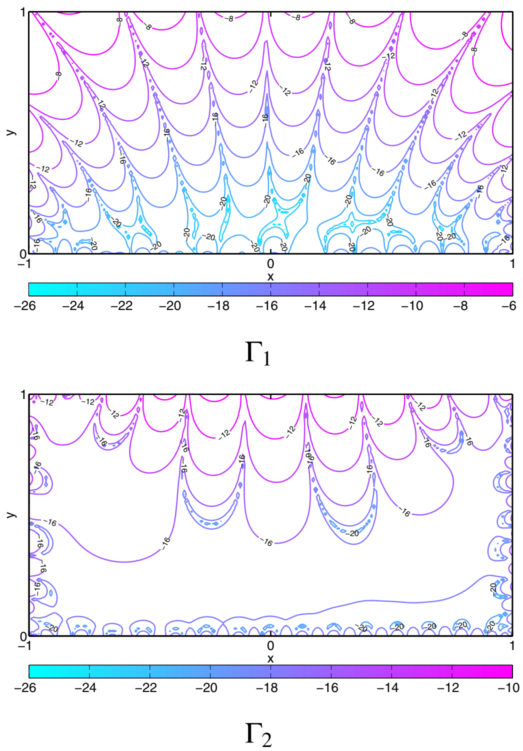


Figure 3: Contour lines of the log of relative errors in the numerical reconstructions of u in Example 1 from noise-free data on different Cauchy boundaries.

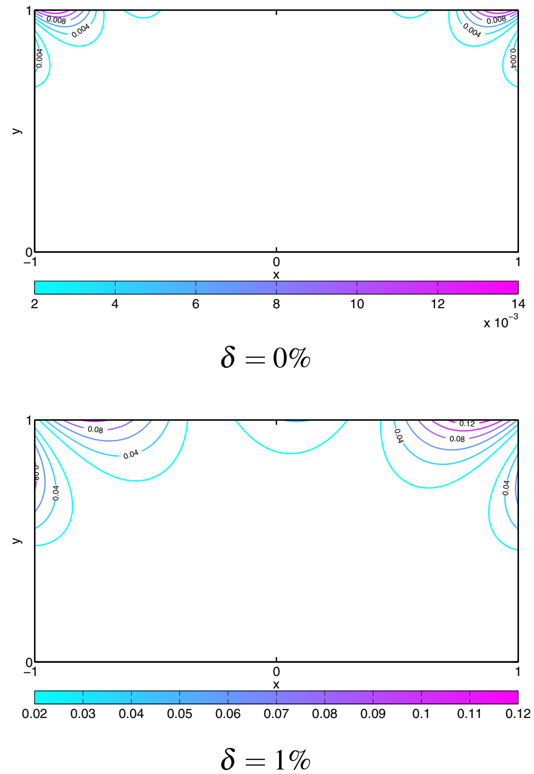


Figure 4: Contour lines of the relative errors in the numerical reconstructions of u in Example 2 from noise-free data on Γ_1.

4.1 Singular Boundary Conditions

Although the MFS is designed to work for smooth function, in this supplementary section, the proposed method is applied to solve an inverse Cauchy problem with discontinuous boundary condition on the unit circle Ω.

We consider the Cauchy problem (1) with exact solution given by

Example 3 $u(x,y) = \frac{2}{\pi}\arctan\left(\frac{2y}{1-x^2-y^2}\right)$.

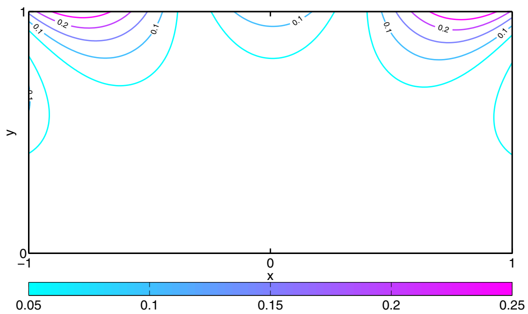
The Cauchy data is imposed on Γ that is taken to be the upper half of the unit circle. It is easy to see that f and g in (1) are given as

$$f(\theta) = 1,$$
$$g(\theta) = \frac{2}{\pi\sin\theta}, \quad 0 < \theta < \pi.$$

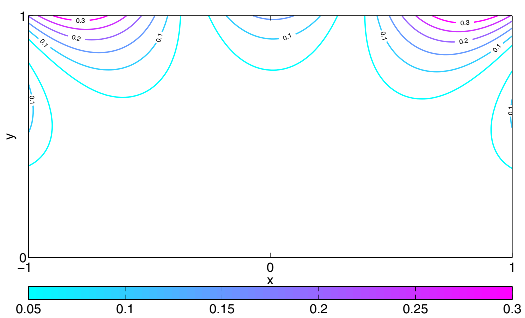
Note that the exact solution u is discontinuous on $\partial\Omega$; it equals to 1 and -1, respectively, on the upper and lower circles. Moreover, g is singular when $\theta = 0$ and $\theta = 2\pi$.

On the Cauchy boundary Γ and the unknown boundary Σ, we uniformly place in 152 collocation points, respectively. For the source points, $m = 600$ uniformly spaced points are placed on the circle with radius $1+R$ for $R = \{0.7, 0.75, 0.8\}$.

Example 3 was solved in Liu (2008) using the collocation Trefftz method. In the same paper, the discontinuity is smoothed by the truncated Fourier expansions. In this section, no special treatment is taken to smooth the discontinuities. The proposed MFS boundary control is *directly* applied to Example 3. We want to point out that, in comparison to the previous examples, the numerical re-



$\delta = 5\%$



$\delta = 10\%$

Figure 5: Contour lines of the relative errors in the numerical reconstructions of u in Example 2 from noise-free data on Γ_1.

sults are much more sensitive to parameters' values.

Firstly, the relative errors under different noise levels are shown in Table 2. When the data is noise-free, the accuracy is not as good as that in Example 1 and Example 2. More importantly, different source distances R results in very different relative errors. From Figure 6, we can observe the numerical approximation for $R = 0.75$ shows the Gibbs phenomenon (over- and undershooting) near the boundary discontinuities of u.

When the noise level increases to 10%, the accuracy for $R = 0.70$ and $R = 0.75$ remains in the same order of magnitude as in the noise-free case. The solution profiles under 10% noise are similar for all R; see Figure 6 for that of $R = 0.75$. The main source of error in this case is again near the boundary discontinuities of u.

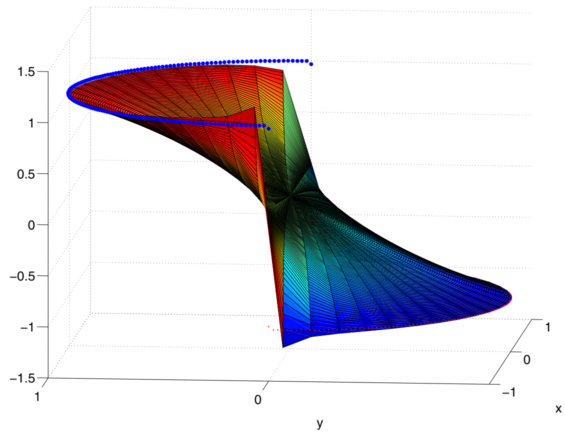
$R = 0.30$				
	Example1		Example2	
Noise	Γ_1	Γ_2	Γ_1	Γ_2
0%	8.6E-4	1.6E-5	0.0029	1.5E-4
1%	0.0755	0.0053	0.1241	0.0081
5%	0.1674	0.0167	0.2817	0.0368
10%	0.1948	0.0332	0.3271	0.1798
$R = 0.25$				
	Example1		Example2	
Noise	Γ_1	Γ_2	Γ_1	Γ_2
0%	0.0015	1.8E-4	0.0141	8.8E-4
1%	0.0704	0.0067	0.1324	0.0074
5%	0.1023	0.0164	0.2963	0.0365
10%	0.1877	0.0322	0.3476	0.0843
$R = 0.20$				
	Example1		Example2	
Noise	Γ_1	Γ_2	Γ_1	Γ_2
0%	0.0074	1.1E-4	0.0324	1.9E-4
1%	0.0696	0.0078	0.1496	0.0076
5%	0.1147	0.0162	0.3272	0.0369
10%	0.1869	0.0325	0.3828	0.0831

Table 1: The relative errors $E_r(U)$ on the whole domain Ω when the Cauchy data are given on the boundaries Γ_1 and Γ_2 for different source point distributions.

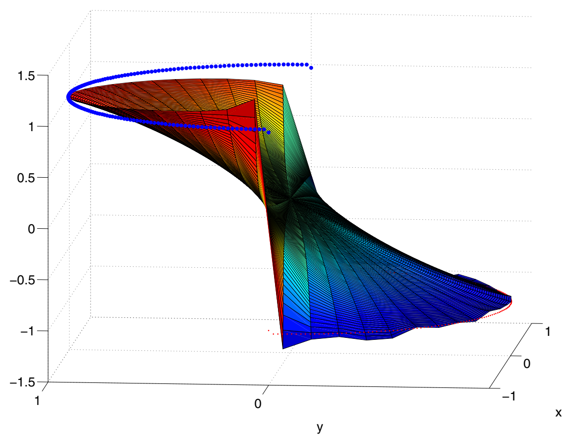
Example 3			
	Error		
Noise	$R = 0.70$	$R = 0.75$	$R = 0.80$
0%	0.0393	0.0258	0.1143
1%	0.0277	0.0372	0.1395
5%	0.0470	0.0467	0.0488
10%	0.0698	0.0679	0.0644

Table 2: The relative errors $E_r(U)$ on the whole domain Ω when the Cauchy data is singular.

For $R = 0.80$, interestingly, accuracy improves after noise is added. With the presence of noise, Figure 7 shows that the numerical solution behaves more like a best-fit approximation, *instead*



$\delta = 0\%$



$\delta = 10\%$

Figure 6: Numerical reconstructions (surfaces) and the exact boundary data (dotted lines) of u in Examples 3 with $R = 0.75$, $\delta = 0\%$ and 10% noisy data.

of interpolant, to the data on Γ. Hence, the Gibbs effect is reduced and the instability for $R = 0.8$ is removed. All tested R's show similar accuracy under 10% (so as 5%) of noise.

This preliminary test makes us believe that (1) the proposed method is, without a doubt, not optimal for Cauchy problem with singular data, but (2) the MFS boundary control approach along with other appropriate techniques will be able to solve such problem accurately. In particular, existing techniques that prevent Gibbs phenomenons from

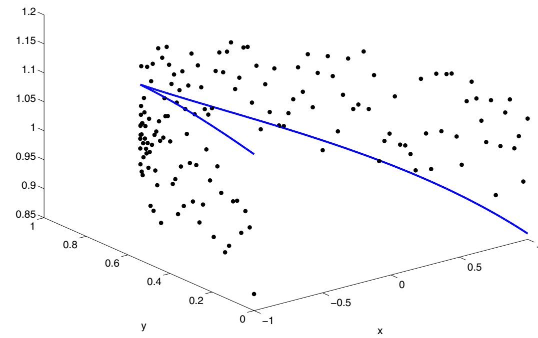


Figure 7: Numerical reconstructions (solid line) in Examples 3 and the $\delta = 10\%$ noisy boundary data (dot).

the MFS should be coupled with the proposed method. We leave this to our future work.

5 Conclusion

We propose a reconstruction method for solving the Cauchy problems of elliptic operators. The method decomposes the inverse problem into a sequence of direct problems and a control problem. They are solved by the mean of the method of fundamental solutions. Numerical examples demonstrate that the method is robust against data noises and reasonably accurate as a solver for the ill-posed problems. The method is efficient, non-iterative and mesh-free. Furthermore, the method is applicable to other inverse problem that makes the method practical to handle real-life problems. In particular, the proposed method is readily applicable to the Cauchy problems of Helmholtz operators.

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