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A Lie-Group Shooting Method for Post Buckling Calculations of Elastica

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Abstract: In this paper we propose a new numerical integration method of second-order boundary value problems (BVPs) resulting from the elastica of slender rods under different loading conditions and boundary conditions. We construct a compact space shooting method for finding unknown initial conditions. The key point is based on the construction of a one-step Lie group element $\mathbf{G}(T)$ and the establishment of a generalized mid-point Lie group element $\mathbf{G}(r)$ by using the mean value theorem. Then, by imposing $\mathbf{G}(T) = \mathbf{G}(r)$ we can search the missing initial condition through a closed-form solution in terms of the weighting factor $r \in (0,1)$. The Lie-group shooting method is very effective for large deflection problems of elastica even exhibiting multiple solutions.

Keyword: Elastica, Post-Buckling, Multiple solutions, Lie-Group shooting method

1 Introduction

The beam is one of the most common structural elements used in a variety of aerospace, civil and mechanical engineering structures. In dealing with the nonlinear deformable behavior of beams the relation between extensional strain and displacement is taken to be nonlinear. This type of nonlinearity is most commonly found in structures.

If a naturally straight rod is subjecting to a small compressive load at its ends it remains straight. If the load is slowly increased beyond a certain critical value, called the buckling load, the rod assumes a configuration called a buckled state, that is not straight. This process is called buckling. The buckling analyses of structural members are important; see, e.g., Baiz and Aliabadi (2006), Sapountzakis and Tsiatas (2007), Li, Xiang and Xue (2005), Yoda and Kodama (2006), Kim, Kim and Lee (2004), and Lin and Hsiao (2003).

Analysis of finite deflections of prismatic elastic columns after buckling was initiated by Euler in the 18th century; he used elliptic integrals and elliptic functions, and the relevant deflection curve was called Euler's elastica. This direction was later widely developed; classical results were presented in the books by Popov (1948) and Frisch-Fay (1962). In most cases the behavior of force during buckling was assumed as follows: the point of application fixed to the matter, and the direction fixed in space. This type of behavior is being called the Eulerian behavior of loading. In many cases, however, the loading behavior does not conform to this scheme.

Probably the first exact analysis of finite deflections of columns under non-Eulerian loading behavior is due to Stern (1979). He considered elastica for a cantilever column loaded via a rigid rod with sliding upper end and determined the regions of stable and unstable post-buckling path. The direction of force loading the column is here no longer constant in space. Later results concerning with the non-Eulerian forces based on elastica were made by Wilson and Snyder (1988) and Kandakis (1992).

The advantages of Euler's approach are connected with exactness of the analysis and with applicability within the whole range of loadings and deflections. However, the results are not so perspicuous and the stability of postbuckling behavior is difficult to be estimated. Koiter (1945) proposed an alternative approach based on energy criterion of stability combined with perturbations, namely, expansions into power series of a certain small pa-

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rameter ε characterizing deformations. This parameter should be defined as a suitable norm for functions describing deformation, monotonically increasing during the buckling process. In the simplest case of a cantilever column usually the deflection at the free end is chosen as that parameter: it may be regarded as a Chebyshev's norm for deflections. From among other proposals defining ε we mention that by Riks (1979) with the current length of the equilibrium path, obviously increasing during the process, and by Privalova and Seyranian (1997) with the Gaussian norm for the deflection angle.

The signs of coefficients of expansion of the loading parameter P into power series of ε determine stability of post-buckling behavior. Such expansions can also be derived directly from the governing nonlinear equation of the problem without energy considerations. Then the manoeuvre proposed by Croll (1971), orthogonalization of the expanded equation with a suitably chosen function and integration by parts, makes it possible to evaluate a higher-order expansion coefficient for P by using just a lower-order deflection function.

For some applications it is important to know relations of the type $\varepsilon = \varepsilon(P)$, inverse with respect to those described above. Such relations may be obtained for stable post-buckling behavior. For a prismatic column under Eulerian force it was derived by Zyczkowski (1991), who used the inversion of the relevant power series for $P = P(\varepsilon)$ [Zyczkowski (1965)]. However, the convergence of such a series is rather poor; it can be improved by a suitable extrapolation procedure, e.g., Zyczkowski (1998).

Koiter's theory and related approaches were extensively developed, mainly in connection with shell buckling problems since initial post-buckling behavior of shells is in most cases unstable. Moreover, for imperfect shells it determines upper critical loading in terms of the imperfection parameter. Many results were presented in the survey papers by Hutchinson and Koiter (1970), Budiansky (1974), Potier-Ferry (1987), and in the books by Thompson and Hunt (1973), Dym (1974), Huseyin (1975), Bazant and Cedolin (1991), Troger and Steindl (1991), Nguyen (1995), and Atanackovic (1997). More papers devoted to the post-buckling behavior of prismatic elastic columns were Plaut (1979) and Kolkka (1984) for multiparameter loading, Plaut (1978), Kounadis (1991), Rao and Rao (1991) for non-conservative loading, Haslach (1985), Kounadis and Mallis (1987), Szymczak and Mikulski (1990) for effects of material nonlinearity, Tauchert and Lu (1987), Hui (1988), Lee and Waas (1996), Wu and Zhong (1999) for columns on elastic foundation, Damil and Potier-Ferry (1990) for higher-order expansion terms, Luongo and Pignataro (1992) for nearly symmetric structures, Godoy (1996) for the dependence on certain parameters, e.g. responsible for deformation of a cross-section, Beda (1996) and Wu (1998) for secondary buckling, Kuo and Yang (1991a) for curved beam, and Kuo and Yang (1991b) for torsional loading.

Most of the research that deals with large deflection problems has used four different approaches. The first approach is based on elliptic integral formulation, e.g., Barten (1945), Bisshop and Drucker (1945), Timoshenko and Gere (1961), Lau (1985), Mattiasson (1981), Chucheepsakul, Buncharoen and Wang (1994), Bona and Zelenika (1997), Wang, Lam, He and Chucheepsakul (1997), Chucheepsakul, Wang, He and Monprapussorn (1999), Coffin and Bloom (1999), and Ohtsuki and Ellyin (2001). This approach is tedious and only suitable for simple loading cases. For example, it cannot solve any non-prismatic or prismatic beam with simple uniformly distributed load in the vertical or horizontal directions. The second approach uses numerical integration with iterative shooting techniques, e.g., Freeman (1946), Conway (1947), Holden (1972), Wang and Watson (1980, 1982), Wang (1981), Watson and Wang (1981, 1983), Mau (1990), Wang and Kitipornchai (1992), Lee and Oh (2000), Lee (2001), and Magnusson, Ristinmaa and Ljung (2001). However, it is suitable only for beams subjected to loads producing moderate deflections and it fails in cases incorporating very large deflection. The third approach utilizes incremental finite element method in connection with Newton-Rhapson iteration techniques for solving elastic problems, e.g., Schmidt (1977), Golley (1984, 1997), and Kooi (1985). This method requires the use of expensive commercial packages and the generation of a very fine mesh requiring huge computational time. In addition, the method may experience divergence problems in very large deflection cases and it requires special types of numerical techniques such as arc-length continuation method. Moreover, an experience user of this program is needed to setup the model and the solution method in the proper form. In the fourth approach, the incremental finite differences method in connection with Newton-Rhapson iteration techniques is used e.g., Kooi and Kuipers (1984), Saje and Srpcic (1985), and Srpcic and Saje (1986). This method requires a very large number of nodes for accurate results and it is prone to divergence in very large deflection cases.

The problem of buckling of a thin rod subjecting to compressive load is one of the most important and yet simple problems in nonlinear structural mechanics, e.g., Yang and Kuo (1994). Assume a rod with length L, fixed at one end at $x=0$ and compressed by the load P at the other end at $x=L$. When P is small the rod only reveals a compression along the x-direction, remaining a straight rod. However, when P is increased over a critical value, the rod experiences a drastical change from a straight state to a buckled state with deflection u in the y-direction. The arc length of the deflected curve is assumed to be s and the tangential angle of the deflected curve with the x-axis is assumed to be θ; hence, the curvature of the deflected curve is $\kappa = d\theta/ds$. Under a compressive load P, there exists a bending moment Pu, which is proportional with the curvature κ by

$$\kappa(s)+p^2u(s)=0,\;\; u(0)=u(L)=0,\;\; p=\sqrt{\frac{P}{EI}}, \tag{1}$$

where E is the Youngs modulus of the rod and I is the inertial moment [Ramachandra and Roy (2001); Vaz and Silva (2003); Vaz and Mascaro (2005)].

In the case of small deflection with $u_x \sim 0$, the curvature is approximated by $\kappa = u''$, and Eq. (1) reduces to a linear differential equation $EIu''+Pu=0$. The solution is $u=C\sin(k_nx)$ by imposing the boundary conditions of $u(0)=u(L)=0$, where $k_n=n\pi/L$, $n=1,2,\ldots$, and C is a constant not yet uniquely determined. There are several critical loads with $P=P_n=EI(n\pi/L)^2$, of which the smallest one is called the first critical load denoted by $P_{cr}=EI(\pi/L)^2$. It can be seen that the linear theory is insufficient to uniquely determine the height C of deflection curve and to give solutions under these loads with $P>P_{cr}$ and P not equal to the critical load.

To remedy these two defects of the linear theory we must consider the nonlinear equation (1) again, but with a further differential of it with respect to s, resulting in

$$\theta''(s)+p^2\sin\theta(s)=0,\;\; \theta'(0)=\theta'(L)=0, \tag{2}$$

where $d\kappa/ds=d^2\theta/ds^2$ and $u'(s)=\sin\theta$ were used. The boundary conditions are obtained by imposing the simply support conditions at two ends.

On the other hand, we can also derive the governing equation of the buckling problem of a cantilever beam under a transversal loading [Ramachandra and Roy (2001); Kumar, Ramachandra and Roy (2004)]:

$$\theta''(s)+p^2\cos\theta(s)=0,\;\; \theta(0)=\theta'(L)=0. \tag{3}$$

Depending on the precise mode of loading and the nature of the rod, the transition to a buckled state can be very rapid. If the loading is further increased the deflection of the rod from its straight state is likewise increased. If this entire process is repeated, the rod may well buckle into another configuration such as the reflection of the first state through a plane of symmetry. The performance of a whole series of such experiments on different rods would lead to the observation that the buckling loads and the nature of buckled states depend upon the material and shape of the rod and upon the manner in which it is supported at its ends. It can also be observed that the experimental results are highly sensitive to slight deviations of the rod from perfect straightness or of the load from perfect symmetry. The study of buckling for different bodies is one of the richest sources of important problems in nonlinear solid mechanics, e.g., Antman (1995).

The present objective is to appropriately integrate the governing nonlinear ODEs, such that the solution satisfies all the boundary conditions. Since the governing nonlinear ODEs are generally non-integrable, the most efficient and accurate way to integrate the nonlinear vector fields is to employ some direct integration schemes. However, in order to effectively employ one of these techniques, sufficient information on initial conditions (at one of the two boundaries) is needed a priori, i.e., boundary value problem (BVP) needs to be posed as a conditional initial value problem (IVP). To begin with, some of the initial conditions of this conditional IVP are not known and they would have to be optimally selected via a genetic search so as to satisfy all the known boundary conditions. In this paper we propose a new method for the computation of the following second-order nonlinear two-point BVPs:

$$w'' = H(x,w,w'), \;\; a < x < b, \tag{4}$$

under one of the following boundary conditions:

$$w(a) = \alpha, \;\; w'(b) = \beta, \tag{5}$$

$$w'(a) = \alpha, \;\; w'(b) = \beta, \tag{6}$$

which include Eqs. (3) and (2) as special cases. After developing the Lie-group shooting method for these problems in the following sections, one would find that the present method is very effective to find the unknown missing initial condition, for example, $\theta(0)$ for Eq. (2) or $\theta'(0)$ for Eq. (3).

Eq. (4) together with Eq. (5) is called the Cauchy-Neumann type BVP, while Eq. (4) together with Eq. (6) is called the Neumann type BVP. The last problem will be discussed until Section 7.

2 Transforming the BVP into a canonical one

For Eqs. (4) and (5) we consider the following transformations:

$$x = b + (b-a)t, \tag{7}$$

$$y(t) = w(x) - \beta(b-a)(1+t) + c - \alpha, \tag{8}$$

and they can be reduced to

$$\ddot{y} = F(t,y,\dot{y}), \tag{9}$$

$$y(-1) = c, \;\; \dot{y}(0) = 0, \tag{10}$$

where

$$\begin{aligned} F(t,y,\dot{y}) :=& (b-a)^2 H(b+(b-a)t, \\ & y+\beta(b-a)(1+t)+\alpha-c, \\ & \dot{y}/(b-a)+\beta). \end{aligned} \tag{11}$$

Through a symmetric extension into the interval of $t \in (0,1]$, we can write Eqs. (9) and (10) to be

$$\ddot{y} = f(t,y,\dot{y}), \tag{12}$$

$$y(-1) = c, \;\; y(1) = c, \tag{13}$$

where

$$f(t,y,\dot{y}) = \begin{cases} F(-t,y,-\dot{y}) & \text{if } \; 0 < t \le 1, \\ F(t,y,\dot{y}) & \text{if } \; -1 \le t \le 0. \end{cases} \tag{14}$$

If the initial value of $y_2(t_0) = \dot{y}(t_0) = A$ is available together with the known initial value of $y_1(t_0) = y(t_0) = c$, then we can numerically integrate the following IVP step-by-step in a forward direction from $t = t_0 = -1$ to $t = 1$:

$$\dot{y}_1 = y_2, \tag{15}$$

$$\dot{y}_2 = f(t,y_1,y_2), \tag{16}$$

$$y_1(t_0) = c, \tag{17}$$

$$y_2(t_0) = A. \tag{18}$$

Eqs. (15)-(18) are called the $(\mathbf{y},t)$-IVP, where $\mathbf{y}(t) = (y_1(t), y_2(t))$ denotes the system variables in the t-domain. We are going to develop a Lie-group shooting method to solve A.

3 One-step GPS

Our approach of the above second order BVP is based on the group preserving scheme (GPS) developed by Liu (2001) for the integration of IVP. The GPS method is very effective to deal with ODEs with special structures as shown by Liu (2005, 2006a) for stiff equations and ODEs with constraints. Previously, Liu (2006b, 2006c, 2006d) has developed the Lie-group shooting method for second-order BVPs, and it is not yet applied to the solution of elastica's BVPs. Our method can be applied to the elastica BVPs, since we are able to search the missing initial condition through a closed-form solution in terms of r in a compact space of $r \in (0,1)$, where the factor r is used in a generalized mid-point rule for the Lie group of one-step GPS.

3.1 The GPS

Let us write Eqs. (15) and (16) in a vector form:

$$\dot{\mathbf{y}} = \mathbf{f}(t,\mathbf{y}), \tag{19}$$

where

$$\mathbf{y} := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ \ \mathbf{f} := \begin{bmatrix} y_2 \\ f(t,y_1,y_2) \end{bmatrix}. \tag{20}$$

Liu (2001) has embedded Eq. (19) into an augmented system:

$$\dot{\mathbf{X}} := \frac{d}{dt}\begin{bmatrix} \mathbf{y} \\ \|\mathbf{y}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{2\times 2} & \frac{\mathbf{f}(t,\mathbf{y})}{\|\mathbf{y}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(t,\mathbf{y})}{\|\mathbf{y}\|} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{y} \\ \|\mathbf{y}\| \end{bmatrix} := \mathbf{A}\mathbf{X}, \tag{21}$$

where $\mathbf{A}$ is an element of the Lie algebra $so(2,1)$ satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0} \tag{22}$$

with

$$\mathbf{g} := \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & -1 \end{bmatrix} \tag{23}$$

a Minkowski metric. Here, $\mathbf{I}_2$ is the identity matrix, and the superscript T stands for the transpose.

The augmented variable $\mathbf{X}$ satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{y}\cdot\mathbf{y} - \|\mathbf{y}\|^2 = 0. \tag{24}$$

Accordingly, Liu (2001) has developed a group-preserving scheme (GPS) as follows:

$$\mathbf{X}_{k+1} = \mathbf{G}(k)\mathbf{X}_k, \tag{25}$$

where $\mathbf{X}_k$ denotes the numerical value of $\mathbf{X}$ at the discrete t_k, and $\mathbf{G}(k) \in SO_o(2,1)$ satisfies

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{26}$$

$$\det \mathbf{G} = 1, \tag{27}$$

$$G_0^0 > 0, \tag{28}$$

where G_0^0 is the 00th component of $\mathbf{G}$. In Section 6.1 we will write a GPS explicitly.

3.2 Generalized mid-point rule

Applying scheme (25) on Eq. (21) with a specified initial condition $\mathbf{X}(t_0) = \mathbf{X}_0$ we can compute the solution $\mathbf{X}(t)$ by GPS. Assuming that the stepsize used in GPS is $h = (1-t_0)/K$, and starting from an initial augmented condition $\mathbf{X}_0 = \mathbf{X}(t_0) = (\mathbf{y}_0^{\mathrm{T}}, \|\mathbf{y}_0\|)^{\mathrm{T}}$ we will calculate the value $\mathbf{X}(1) = (\mathbf{y}^{\mathrm{T}}(1), \|\mathbf{y}(1)\|)^{\mathrm{T}}$ at $t = 1$.

By applying Eq. (25) step-by-step we can obtain

$$\mathbf{X}_f = \mathbf{G}_K(h)\cdots\mathbf{G}_1(h)\mathbf{X}_0, \tag{29}$$

where $\mathbf{X}_f$ approximates the exact $\mathbf{X}(1)$ with a certain accuracy depending on h. However, let us recall that each $\mathbf{G}_i,\ i = 1,\ldots,K$, is an element of the Lie group $SO_o(2,1)$, and by the closure property of the Lie group, $\mathbf{G}_K(h)\cdots\mathbf{G}_1(h)$ is also a Lie group denoted by $\mathbf{G}$. Hence, we have

$$\mathbf{X}_f = \mathbf{G}\mathbf{X}_0. \tag{30}$$

This is a one-step Lie-group transformation from $\mathbf{X}_0$ to $\mathbf{X}_f$.

We can calculate $\mathbf{G}$ by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie group generated from such an $\mathbf{A} \in so(2,1)$ is

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_2 + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^{\mathrm{T}} & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^{\mathrm{T}}}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{31}$$

where

$$\hat{\mathbf{y}} = r\mathbf{y}_0 + (1-r)\mathbf{y}_f, \tag{32}$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{t}, \hat{\mathbf{y}}), \tag{33}$$

$$a = \cosh\left((1-t_0)\frac{\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right), \tag{34}$$

$$b = \sinh\left((1-t_0)\frac{\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right). \tag{35}$$

Here, we use the initial $\mathbf{y}_0$ and the final $\mathbf{y}_f$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $\hat{t} = t_0 + r(1-t_0)$ and $0 < r < 1$ is a parameter to be determined. The above method applied a generalized mid-point rule on the calculation of $\mathbf{G}$, and the resultant is a single-parameter Lie group element $\mathbf{G}(r)$.

3.3 A Lie group mapping between two points

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{y}}\|}, \tag{36}$$

such that Eqs. (31), (34) and (35) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_2 + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{37}$$

$$a = \cosh[(1-t_0)\|\mathbf{F}\|], \tag{38}$$

$$b = \sinh[(1-t_0)\|\mathbf{F}\|]. \tag{39}$$

From Eqs. (30) and (37) it follows that

$$\mathbf{y}_f = \mathbf{y}_0 + \eta\mathbf{F}, \tag{40}$$

$$\|\mathbf{y}_f\| = a\|\mathbf{y}_0\| + b\frac{\mathbf{F}\cdot\mathbf{y}_0}{\|\mathbf{F}\|}, \tag{41}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{y}_0 + b\|\mathbf{y}_0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{42}$$

Substituting

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}_f - \mathbf{y}_0) \tag{43}$$

into Eq. (41) we obtain

$$\frac{\|\mathbf{y}_f\|}{\|\mathbf{y}_0\|} = a + b\frac{(\mathbf{y}_f - \mathbf{y}_0)\cdot\mathbf{y}_0}{\|\mathbf{y}_f - \mathbf{y}_0\|\|\mathbf{y}_0\|}, \tag{44}$$

where

$$a = \cosh\left(\frac{(1-t_0)\|\mathbf{y}_f - \mathbf{y}_0\|}{\eta}\right), \tag{45}$$

$$b = \sinh\left(\frac{(1-t_0)\|\mathbf{y}_f - \mathbf{y}_0\|}{\eta}\right) \tag{46}$$

are obtained by inserting Eq. (43) for $\mathbf{F}$ into Eqs. (38) and (39).

Let

$$\cos\phi := \frac{[\mathbf{y}_f - \mathbf{y}_0]\cdot\mathbf{y}_0}{\|\mathbf{y}_f - \mathbf{y}_0\|\|\mathbf{y}_0\|}, \tag{47}$$

$$S := (1-t_0)\|\mathbf{y}_f - \mathbf{y}_0\|, \tag{48}$$

and from Eqs. (44)-(46) it follows that

$$\frac{\|\mathbf{y}_f\|}{\|\mathbf{y}_0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos\phi\sinh\left(\frac{S}{\eta}\right). \tag{49}$$

Upon defining

$$Z := \exp\left(\frac{S}{\eta}\right), \tag{50}$$

from Eq. (49) we obtain a quadratic equation for Z:

$$(1+\cos\phi)Z^2 - \frac{2\|\mathbf{y}_f\|}{\|\mathbf{y}_0\|}Z + 1 - \cos\phi = 0. \tag{51}$$

The solution is found to be

$$Z = \frac{\frac{\|\mathbf{y}_f\|}{\|\mathbf{y}_0\|} + \sqrt{\left(\frac{\|\mathbf{y}_f\|}{\|\mathbf{y}_0\|}\right)^2 - 1 + \cos^2\phi}}{1+\cos\phi}, \tag{52}$$

and from Eqs. (50) and (48)

$$\eta = \frac{(1-t_0)\|\mathbf{y}_f - \mathbf{y}_0\|}{\ln Z} \tag{53}$$

is uniquely determined by $\mathbf{y}_0$ and $\mathbf{y}_f$.

Therefore, between any two points $(\mathbf{y}_0, \|\mathbf{y}_0\|)$ and $(\mathbf{y}_f, \|\mathbf{y}_f\|)$ on the cone, there exists a Lie group element $\mathbf{G} \in SO_o(2,1)$ mapping $(\mathbf{y}_0, \|\mathbf{y}_0\|)$ onto $(\mathbf{y}_f, \|\mathbf{y}_f\|)$, which is given by

$$\begin{bmatrix} \mathbf{y}_f \\ \|\mathbf{y}_f\| \end{bmatrix} = \mathbf{G}\begin{bmatrix} \mathbf{y}_0 \\ \|\mathbf{y}_0\| \end{bmatrix}, \tag{54}$$

where $\mathbf{G}$ is uniquely determined by $\mathbf{y}_0$ and $\mathbf{y}_f$ through the following equations:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_2 + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{55}$$

$$a = \cosh((1-t_0)\|\mathbf{F}\|), \tag{56}$$

$$b = \sinh((1-t_0)\|\mathbf{F}\|), \tag{57}$$

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}_f - \mathbf{y}_0). \tag{58}$$

4 The Lie-group shooting method for BVP

From Eqs. (15)-(18) it follows that

$$\dot{y}_1 = y_2, \tag{59}$$

$$\dot{y}_2 = f(t, y_1, y_2), \tag{60}$$

$$y_1(t_0) = c, \ \ y_1(1) = c, \tag{61}$$

$$y_2(t_0) = A, \ \ y_2(1) = B, \tag{62}$$

where A and B are two supplemented unknown constants, and c is a given positive constant.

From Eqs. (40), (61) and (62) it follows that

$$\mathbf{F} := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{\eta}\begin{bmatrix} 0 \\ B-A \end{bmatrix}. \tag{63}$$

Starting from an initial guess of (A, B) we use the following equation to calculate η:

$$\eta = \frac{(1-t_0)\sqrt{(A-B)^2}}{\ln Z}, \tag{64}$$

in which Z is calculated by

$$Z = \frac{\frac{\sqrt{c^2+B^2}}{\sqrt{c^2+A^2}} + \sqrt{\frac{c^2+B^2}{c^2+A^2} - 1 + \cos^2\phi}}{1+\cos\phi}, \tag{65}$$

where

$$\cos\phi = \frac{A(B-A)}{\sqrt{(A-B)^2}\sqrt{c^2+A^2}}. \tag{66}$$

The above three equations were obtained from Eqs. (53), (52) and (47) by inserting Eq. (20) for $\mathbf{y}$.

When compare Eq. (63) with Eq. (36), with the aid of Eqs. (32), (33) and (59)-(62) we obtain

$$rA + (1-r)B = 0, \tag{67}$$

$$A - B + \frac{\eta}{\xi}\hat{f} = 0, \tag{68}$$

where

$$\hat{f}(r) := f(t_0 + r(1-t_0), c, 0), \tag{69}$$

$$\xi := \sqrt{c^2 + [rA + (1-r)B]^2}. \tag{70}$$

It can be seen that $\hat{f}$ is a simple function of r. This result is due to the fact of $\hat{y}_1 = rc + (1-r)c = c$ and $\hat{y}_2 = rA + (1-r)B = 0$ by Eqs. (61), (62) and (67).

The above derivation of the governing equations (64)-(70) is based on by equating the two $\mathbf{F}$'s in Eqs. (36) and (58). It also means that the two Lie groups defined by Eqs. (31) and (55) are equal. Under this sense we may call our shooting technique a Lie-group shooting method.

From Eqs. (67) and (70) it follows that

$$\xi = c, \tag{71}$$

where c is a positive constant. Hence, from Eqs. (67)-(69) and (71) we obtain an algebraic equation for A:

$$H(A) = Ac + \eta_0\hat{f} = 0, \tag{72}$$

where

$$Z = \frac{\sqrt{c^2+B^2} + \sqrt{B^2}}{\sqrt{c^2+A^2} - \sqrt{A^2}}, \tag{73}$$

$$\eta_0 = \frac{(1-t_0)\sqrt{A^2}}{\ln Z}. \tag{74}$$

Here $B = rA/(r-1)$ has a different sign with A.

Eq. (72) can be used to solve A for the given r. If A is available, we can integrate Eqs. (15)-(18) by a suitable forward IVP solver.

5 The solution of A

Eq. (72) can be solved analytically for A. Here we just consider the case of $A > 0$. For this case inserting Eq. (74) for η_0 into Eq. (72) we obtain

$$\ln Z = \frac{-(1-t_0)\hat{f}}{c}. \tag{75}$$

Defining

$$f_1 := \exp\left(-\frac{(1-t_0)\hat{f}}{c}\right), \tag{76}$$

and substituting Eq. (73) for Z into Eq. (75) we obtain

$$\frac{\sqrt{c^2+B^2}+\sqrt{B^2}}{\sqrt{c^2+A^2}-\sqrt{A^2}} = f_1. \tag{77}$$

Eq. (77) can be written as

$$f_1 A - B = f_1\sqrt{c^2+A^2} - \sqrt{c^2+B^2} \tag{78}$$

by using $A > 0$ and $B < 0$. Squaring the above equation and cancelling the common terms we can rearrange it to

$$2f_1\sqrt{c^2+B^2}\sqrt{c^2+A^2} = (1+f_1^2)c^2 + 2f_1 AB. \tag{79}$$

Squaring again and cancelling the common term get

$$4f_1^2(A^2+B^2) - 4f_1(1+f_1^2)AB = (1-f_1^2)^2c^2. \tag{80}$$

Inserting $B = rA/(r-1)$ and through some algebraic manipulations we eventually obtain:

$$\frac{4f_1}{(r-1)^2}[f_1-(1-f_1)^2r^2+(1-f_1)^2r]A^2 = (1-f_1^2)^2c^2. \tag{81}$$

If the following condition holds

$$f_2(r) := f_1 - (1-f_1)^2r^2 + (1-f_1)^2r > 0, \tag{82}$$

then A has a positive solution:

$$A = \sqrt{\frac{(r-1)^2(1-f_1^2)^2c^2}{4f_1f_2}}. \tag{83}$$

The condition (82) can be used to detect the range where r is permitted.

6 Adjusting the slope A

6.1 The GPS

A closed-form solution to calculate the slope A for each r in its admissible range is derived, and we can integrate the $(\mathbf{y},t)$-IVP in Eqs. (15)-(18) by the following GPS method [Liu (2005)]:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{\sinh\left(\frac{h\|\mathbf{f}_n\|}{\|\mathbf{y}_n\|}\right)\|\mathbf{y}_n\|\|\mathbf{f}_n\| + \left[\cosh\left(\frac{h\|\mathbf{f}_n\|}{\|\mathbf{y}_n\|}\right)-1\right]\mathbf{f}_n\cdot\mathbf{y}_n}{\|\mathbf{f}_n\|^2}\mathbf{f}_n, \tag{84}$$

where

$$\mathbf{f}_n = \mathbf{f}(t_n, \mathbf{y}_n). \tag{85}$$

6.2 Adjusting A for the other two type BVPs

For the Cauchy-Neumann type BVPs, we have employed the symmetric extension technique to construct the canonical equations in Section 2. Therefore the target used to adjust the slope A is $\dot{y}(0) = 0$.

For a trial r in the admissible range, we calculate A and then numerically integrate Eqs. (15)-(18) from $t = -1$ to $t = 0$, and compare the end value of $y_2^r(0)$ with the exact one $y_2(0) = 0$. If $|y_2^r(0)|$ is smaller than a given tolerance error ε, then the process of finding solution is finished. Otherwise, we need to calculate the end values of $y_2(0)$ corresponding to a different $r_1 < r$ or $r_2 > r$, which are denoted by $y_2^{r_1}(0)$ and $y_2^{r_2}(0)$, respectively. If $y_2^{r_1}(0)y_2^r(0) < 0$, then there exists one root between r_1 and r; otherwise, the root is located between (r, r_2). Then, we apply the half-interval method to find a suitable r, which requires to calculate Eqs. (15)-(18) at each of the calculation of $y_2^r(0)$, until $|y_2^r(0)|$ is smaller enough to satisfy the criterion of $|y_2^r(0)| \le \varepsilon$.

In principle, we can increase the accuracy by imposing a smaller ε on the shooting error, which however requires more iterations. Since the numerical method is very stable we can quickly pick up the correct value of r through some trials and modifications. Therefore, in the following calculations we do not use the above half-interval method to pick up the weighting factor r.

7 The Neumann problem

For Eqs. (4) and (6), in addition Eq. (7), we use the following transformation:

$$\begin{aligned} y(t) &= w(x) + \frac{1}{2}k_1 t^2 + k_2 t \\ &= w(x) + \frac{1}{2}(\beta - \alpha)(a-b)t^2 + \alpha(a-b)t, \end{aligned} \quad (86)$$

and then, Eqs. (4) and (6) can be reduced to

$$\ddot{y} = f(t, y, \dot{y}), \quad (87)$$

$$\dot{y}(0) = 0, \ \ \dot{y}(1) = 0, \quad (88)$$

where

$$\begin{aligned} f(t,y,\dot{y}) :=& (b-a)^2 H(a + (b-a)t, \\ & y - k_1 t^2/2 - k_2 t, \\ & (\dot{y} - k_1 t - k_2)/(b-a)) + k_1. \end{aligned} \quad (89)$$

The equation, required to determine the unknown $y(0) = C$, can be obtained by a similar argument as that in Section 3. For this purpose let us write

$$\dot{y}_1 = y_2, \quad (90)$$

$$\dot{y}_2 = f(t, y_1, y_2), \quad (91)$$

$$y_1(t_0) = C, \ \ y_1(1) = D, \quad (92)$$

$$y_2(t_0) = A, \ \ y_2(1) = B. \quad (93)$$

In above $t_0 = 0$ and $A = B = 0$.

From Eq. (47) it follows that

$$\cos\phi = \frac{C(D-C) + A(B-A)}{\sqrt{(C-D)^2 + (A-B)^2}\sqrt{C^2+A^2}}. \quad (94)$$

Because of $A = B = 0$, $\cos\phi$ may be -1 or +1. Let us first consider the case of $\cos\phi = -1$, of which $C(D-C) < 0$ is deduced. Under this condition from Eq. (51) we obtain

$$Z = \frac{\sqrt{C^2}}{\sqrt{D^2}}. \quad (95)$$

If $C < 0$ then $D - C > 0$ and from Eqs. (48) and (50) we have

$$S = D - C, \ \ \eta = \frac{D-C}{\ln \frac{\sqrt{C^2}}{\sqrt{D^2}}}, \quad (96)$$

due to $A = B = 0$.

From Eqs. (43), (92), (93) and (36) it follows that

$$\mathbf{F} := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{\eta}\begin{bmatrix} D-C \\ B-A \end{bmatrix} = \frac{1}{\|\hat{\mathbf{y}}\|}\begin{bmatrix} \hat{y}_2 \\ \hat{f} \end{bmatrix}, \quad (97)$$

and that $\hat{y}_1 = rC + (1-r)D$ and $\hat{y}_2 = rA + (1-r)B = 0$, where $\|\hat{\mathbf{y}}\| = \sqrt{\hat{y}_1^2 + \hat{y}_2^2} = \sqrt{[rC + (1-r)D]^2} \neq 0$.

From the second equation in Eq. (97) it follows that

$$\hat{f} = f(r, \hat{y}_1, \hat{y}_2) = f(r, rC + (1-r)D, 0) = 0, \quad (98)$$

due to $A = B = 0$.

On the other hand, from the first equation in Eq. (97) we have

$$\frac{1}{\eta}(D - C) = \frac{\hat{y}_2}{\|\hat{\mathbf{y}}\|} = 0, \quad (99)$$

because of $\hat{y}_2 = 0$. Substituting Eq. (96) for η into the above equation we obtain

$$\ln\frac{\sqrt{C^2}}{\sqrt{D^2}} = 0. \quad (100)$$

Therefore, we have $D = -C$, and Eq. (98) can be used to solve C for the given r.

Now, suppose that $C > 0$, and then $D - C < 0$ follows from the inequality $C(D-C) < 0$. Under this condition from Eqs. (51), (48) and (50) we obtain

$$Z = \frac{\sqrt{D^2}}{\sqrt{C^2}}, \quad (101)$$

$$S = C - D, \ \ \eta = \frac{C-D}{\ln \frac{\sqrt{D^2}}{\sqrt{C^2}}}. \quad (102)$$

A similar argument as that in the above leads to $D = -C$.

The case of $\cos\phi = 1$ implies that $C(D-C) > 0$ by Eq. (94). However, the same argument as that in the above leads to $D = -C$. This results in $C(D-C) = -2C^2 < 0$, which contradicts to $C(D-C) > 0$. It means that there exists no such case that $\cos\phi = 1$.

Therefore, by inserting $D = -C$ into Eq. (98) we have the following equation to solve C:

$$f(r, (2r-1)C, 0) = 0, \quad (103)$$

no matter C is positive or negative. If C is available, we can integrate Eqs. (90)-(93) by a suitable IVP solver with $y_1(0) = C$ and $y_2(0) = A$.

8 Numerical examples

8.1 Example 1

Let us consider the Euler problem of a slender rod under a simple support subjecting to a compressive load as shown by Eq. (2) with $L = \pi$. The above equation can be simply transformed by $t = s/\pi$ and $y = \theta$ into Eqs. (87) and (88) with

$$f(t, y, \dot{y}) = -p^2\pi^2 \sin y. \tag{104}$$

Substituting Eq. (104) into Eq. (103) we obtain

$$f(r, (2r-1)C, 0) = -p^2\pi^2 \sin[(2r-1)C] = 0, \tag{105}$$

which means that

$$C = \frac{k\pi}{2r-1} \tag{106}$$

for some integer k.

Then we use the GPS method to integrate the following equations:

$$\dot{y}_1 = y_2, \tag{107}$$

$$\dot{y}_2 = -p^2 \sin y_1, \tag{108}$$

$$y_1(0) = C, \tag{109}$$

$$y_2(0) = 0, \tag{110}$$

with C given by Eq. (106) with $k = 1$. Here, y_1 and y_2 represent respectively θ and θ'. Let r run in the interval of $(0, 0.5)$ and we plot the $\theta(\pi)$ in Fig. 1 under the load $p = 3.1$. It can be seen that there are many intersection points of the curve with the zero line $\theta(\pi) = 0$, which means that the Euler problem has multiple solutions.

In Figs. 2-7 we plot these solutions in the planes of (s, θ) and (x, u), where

$$u = \int_0^s \sin\theta(\xi)d\xi = \frac{-y_2}{\pi p^2}, \tag{111}$$

$$x = \int_0^s \cos\theta(\xi)d\xi. \tag{112}$$

The corresponding r is marked in each figure.

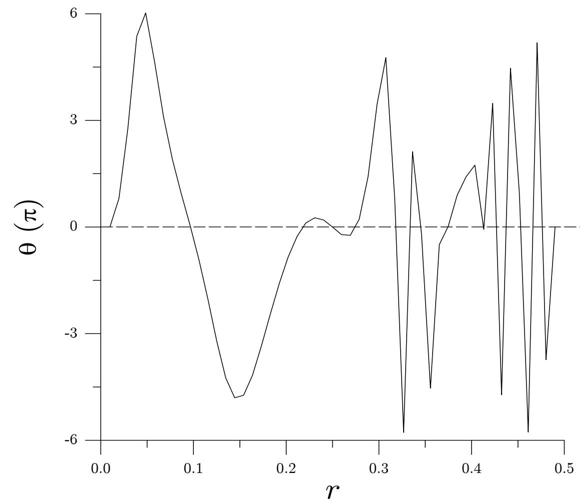

Figure 1: For Example 1 the roots of $\theta(\pi) = 0$ are plotted as the intersection points.

8.2 Example 2

Now, let us consider the Euler problem of a slender rod under a transversal loading as shown by Eq. (3) with $L = \pi$. The above equation is simply transformed by $t = s/\pi$ and $y = \theta + c$ into Eqs. (12) and (13) with

$$f(t, y, \dot{y}) = -p^2\pi^2 \cos(y - c). \tag{113}$$

In Fig. 8(a) by subjecting to $p = 0.5$ and with $c = 10$ we plot the variation of $y_2 = \dot{y}$ with respect to r in the range of $r \in (0, 1)$. It can be seen that there exists only one root of $\dot{y} = 0$. Therefore, by fixing $r = 0.608$ we plot the curve of deflection angle θ with respect to s in Fig. 8(b), of which the boundary conditions of $\theta(0) = \theta'(\pi) = 0$ are matched very well. We found that the Euler elastica problem under these boundary conditions has no multiple solutions.

9 Conclusions

In this paper we have developed a new Lie-group shooting method for the elastica problems by subjecting to different boundary conditions. These problems can be solved in a closed-form of the unknown initial conditions in terms of r in a compact space of $r \in (0, 1)$, which is without needing of any iterations. This method is better than

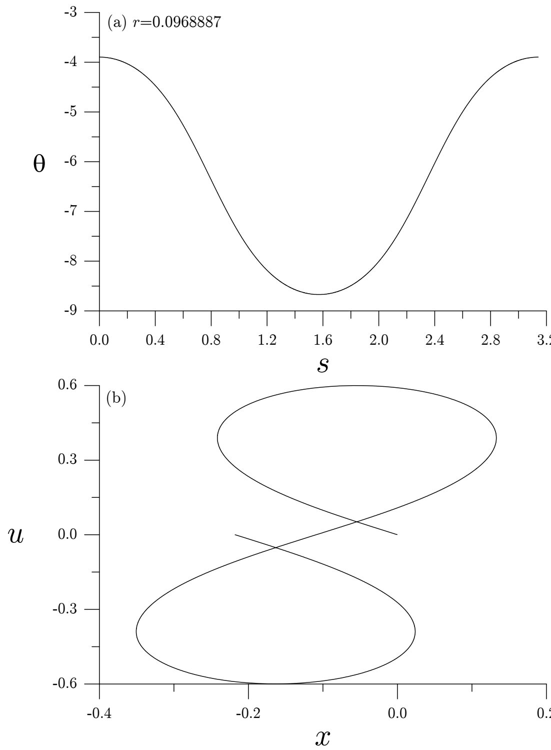

Figure 2: For Example 1 the (θ, s) and (u, x) curves are plotted with r=0.0968887.

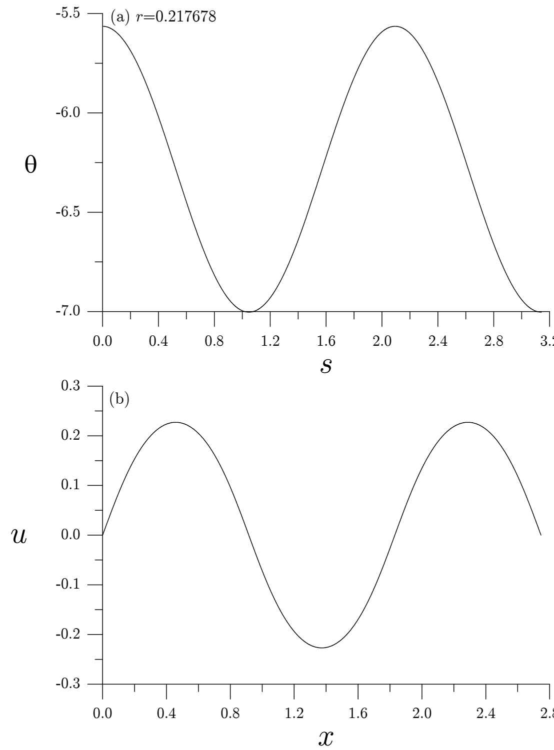

Figure 3: For Example 1 the (θ, s) and (u, x) curves are plotted with r=0.217678.

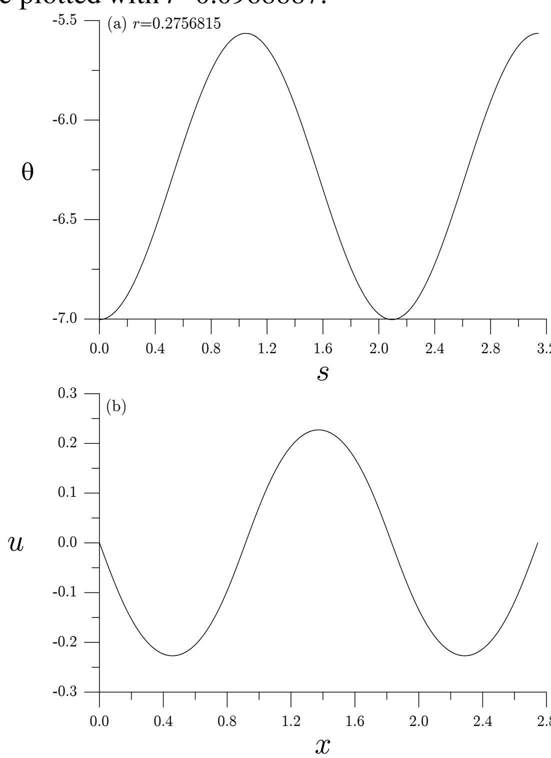

Figure 4: For Example 1 the (θ, s) and (u, x) curves are plotted with r=0.2756815.

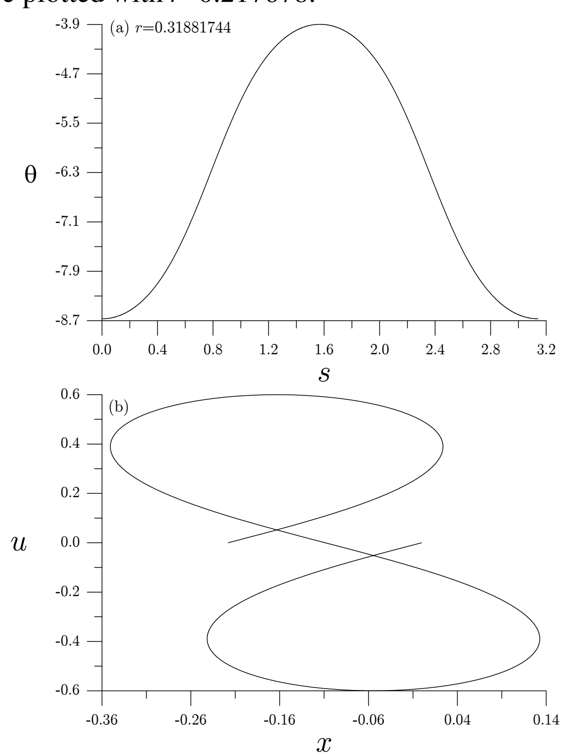

Figure 5: For Example 1 the (θ, s) and (u, x) curves are plotted with r=0.31881744.

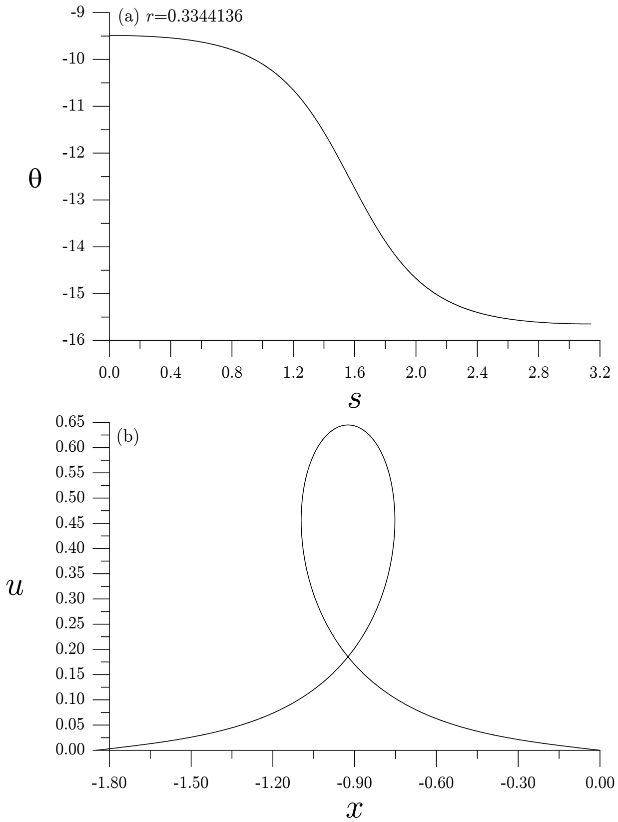

Figure 6: For Example 1 the (θ, s) and (u, x) curves are plotted with *r*=0.3344136.

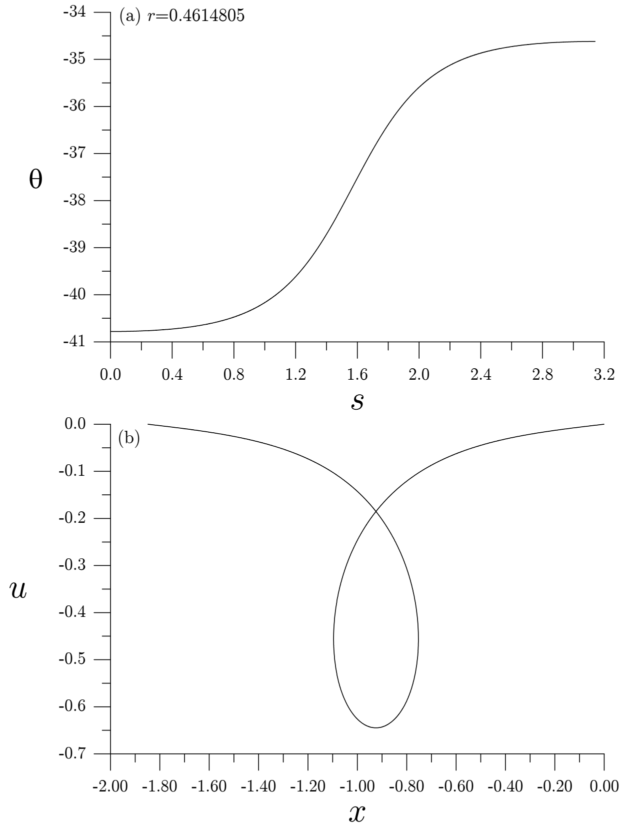

Figure 7: For Example 1 the (θ, s) and (u, x) curves are plotted with *r*=0.4614805.

that used in the previous literature by using the Newton-Rhapson method. More importantly, we can use the new method to detect the number of multiple solutions. Numerical examples were examined to ensure that the Lie-group shooting method can calculate the solutions of elastica. The numerical solutions could match the specified boundary conditions with high accuracy. We can conclude that the Lie-group shooting method is accurate and effective, and its numerical implementation is simple and the computation cost is low to find all possible solutions.

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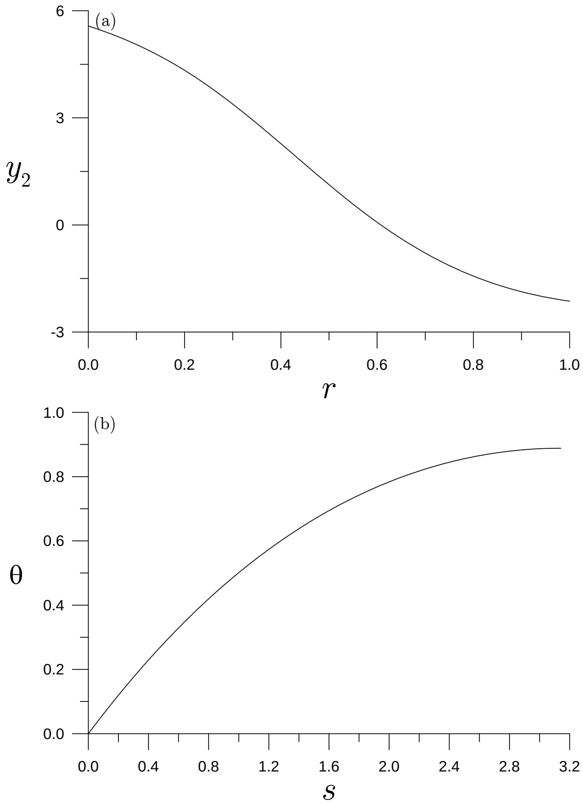

Figure 8: For Example 2 the (y_2, r) and (θ, s) curves are plotted.

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