A Highly Accurate MCTM for Direct and Inverse Problems of Biharmonic Equation in Arbitrary Plane Domains

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Abstract: Trefftz method (TM) is one of widely used meshless numerical methods in elliptic type boundary value problems, of which the approximate solution is expressed as a linear combination of T-complete bases, and the unknown coefficients are determined from boundary conditions by solving a linear equations system. However, the accuracy of TM is severely limited by its ill-conditioning. This paper is a continuation of the work of Liu (2007a). The collocation TM is modified and applied to the direct and inverse problems of biharmonic equation in a simply connected plane domain. Due to its well-conditioning of the resulting linear equations system, the present modified collocation Trefftz method (MCTM) can effectively solve the inverse problems without needing of overspecified data, iteration, and regularization. So that, the computational cost of MCTM is saving. Numerical examples show the effectiveness of the new method in providing highly accurate numerical solutions even subjecting to large noise of the given boundary data.

Keyword: Inverse problem, Modified Trefftz method, Biharmonic equation, Modified collocation Trefftz method (MCTM)

1 Introduction

Biharmonic equation for its role in the mathematical modeling of Stokes flow and Kirchhoff's elastic plate is well known. Lesnic, Elliott and Ingham (1998) have used the boundary element method to solve the biharmonic equation, pointing out that when noise is added on the boundary data the numerical solution may be unstable and highly oscillatory, if a direct approach is used to solve the resulting linear equations system. Later, Jin (2004) has employed the method of fundamental solutions (MFS) to effectively solve the biharmonic equation under noise by a regularization technique of L-curve. Also, there were some authors used the MFS to solve the biharmonic equation, like as, Karageorghis and Fairweather (1987), Smyrlis and Karageorghis (2003), and Tsangaris, Smyrlis and Karageorghis (2004). Reutskiy (2005) used the MFS to solve the eigenproblem of nonhomogeneous biharmonic equation, showing a high precision in simply and multiply connected domains. Melnikov and Melnikov (2001) have used the Green's functions method for boundary value problems of biharmonic equation over regions of complex configuration in two dimensions.

Tsai, Lin, Young and Atluri (2006) have developed a practical procedure to locate the source points when using the MFS for the operators of Laplacian, Helmholtz, modified Helmholtz, and biharmonic. The procedure is developed through some systematic numerical experiments for relations among the accuracy, condition number, and source positions in different shapes of computational domains. Through some numerical experiments, they found that good accuracy can be achieved when the condition number approaches the limit of equation solver. The technique of collocation of TM was also improved by Jin, Cheung and Zienckiewicz (1993), Jin and Cheung (1999), Herrera and Diaz (1999), Herrera, Yates and Diaz (2002), Diaz and Herrera (2005), and Herrera, Yates and Rubio (2007).

More interestingly, Chen, Wu, Lee and Chen (2007) have derived an equivalent relation between the TM and the MFS when applying them

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on the biharmonic equation. Li, Lu, Huang and Cheng (2007) have given a fairly comprehensive comparison of the Trefftz, collocation and other boundary methods. They concluded that the collocation Trefftz method (CTM) is the simplest algorithm and provides the most accurate solution with the best numerical stability. However, the conventional CTM may have a major drawback that the resulting linear equations system is extremely ill-conditioned. In order to obtain an accurate solution of the linear equations system some special techniques, e.g., preconditioner and truncated SVD, are required.

In order to overcome the difficulties appeared in the conventional CTM, Liu (2007a) first proposed a modified Trefftz method for Laplace equation by taking the characteristic length of problem's domain into the complete bases, such that the condition number of the resulting linear equations system can be greatly reduced. The modified TM is essentially stable and has the exponential rate of convergence. In the present paper we will extend the method of modified TM to the biharmonic equation.

The inverse boundary value problem of elliptic type partial differential equations is difficult to solve, since the solution does not depend continuously on the given data. The errors in measured data may be enlarged in the numerical solution, if we do not take this ill-posedness into account. Therefore, we must treat this type problem with a suitable numerical algorithm, which compromises accuracy and stability.

Li (2005) seems the first directly using the radial basis method to solve the inverse boundary value problem of biharmonic equation by using the collocation technique on the given boundary data. His method does not need iteration and is thus time-saving. However, due to its inherent ill-property of the radial basis method, the numerical results as shown by Li (2005) are not good to avoiding the disturbance of noise, even for smooth boundary value problem. In this paper we extend the modified Trefftz method proposed by Liu (2007a, 2007b, 2008a) to the biharmonic equation, and leave the unknown coefficients determined by partial boundary conditions from the collocation method. The collocation method is useful in engineering computations for direct problems, because the algebraic equations can be easily derived. However, it is seldom used in inverse problems due to inherent ill-posedness.

Ling and Takeuchi (2008) have combined the MFS and boundary control technique to solve the Cauchy problem of Laplace equation. Jin and Zheng (2006) have applied the MFS to solve the inverse problem of Helmholtz equation, and Marin and Lesnic (2005) have applied the MFS to solve the inverse problem associated with a two-dimensional biharmonic equation. In order to tackle of the ill-posedness of MFS and the inherent ill-posed property of inverse problems, those authors proposed new numerical schemes with regularization parameters determined by the L-curve method. Zeb, Elliott, Ingham and Lesnic (2002) have discussed an inverse Stokes problem of biharmonic type by using the overspecified pressure data to recover other boundary conditions.

Our starting point employs a similar meshless method of Trefftz type, and a new modification is required in order to get a non ill-posed linear equations system. Furthermore, the new method does not need of any iteration and regularization, even the input data is noised seriously.

We consider the biharmonic equation:

$$\Delta^2 u = 0, \quad (x,y) \in \Omega, \tag{1}$$

where Ω is an interior domain in the plane.

Typically, two types boundary conditions are imposed on the boundary Γ of Ω:

$$u(\rho,\theta) = h(\theta), \quad u_n(\rho,\theta) = g(\theta), \quad 0 \le \theta \le 2\pi \tag{2}$$

or

$$u(\rho,\theta) = h(\theta), \quad \Delta u(\rho,\theta) = g(\theta), \quad 0 \le \theta \le 2\pi, \tag{3}$$

where $h(\theta)$ and $g(\theta)$ are given functions, and $\rho(\theta)$ is a given function describing the boundary shape of an interior domain Ω. The contour Γ in the polar coordinates is described by

$\Gamma = \{(r,\theta)|r = \rho(\theta),\ 0 \le \theta \le 2\pi\}$. Eq. (2) is a clamped boundary condition, while Eq. (3) is a simply supported boundary condition. In this paper we will develop a new meshless numerical method to solve the above direct problem in Eqs. (1) and (2) or Eqs. (1) and (3).

Given the data of u and $\partial u/\partial n$ or Δu at the point $(x,y) \in \mathbb{R}^2$ with unit outward normal $n(x,y)$ on the accessible part Γ_1 of a noncircular contour, we also consider the inverse boundary value problem of biharmonic equation to find some unknown functions on the inaccessible part Γ_2 of $\Gamma = \Gamma_1 \cup \Gamma_2$. The present inverse boundary value problem is given as follows:

Inverse boundary value problem. *To seek unknown functions about u on the unspecified part Γ_2 of the boundary under the following insufficient boundary conditions*:

$$u(\rho,\theta) = h(\theta),\ \ u_n(\rho,\theta) = g(\theta),\ \ (\rho,\theta) \in \Gamma_1. \tag{4}$$

The inverse boundary value problem studied here could be understood in a generalized sense that a problem is called an inverse boundary value problem if the available boundary data for a direct problem is incomplete. For the inverse boundary value problem of Eq. (1), the present study does not need an overspecified data on Γ_1, and is still recoverable of the unknown boundary conditions very well. In the papers by Zeb, Elliott, Ingham and Lesnic (1999), Li (2005), and Marin and Lesnic (2005), the inverse boundary value problems of biharmonic equation are treated under some overspecified data on Γ. Our inverse boundary value problem is much more difficult than that treated by other researchers.

When the contour is circular, Liu (2008b) has applied the modified Trefftz method to recover the unknown boundary data of Laplace equation, but needs to consider a regularization technique to truncate higher mode components in the given data. In this paper we extend this interesting study to arbitrary plane domain for the biharmonic equation, without needing of regularization technique. In Sections 2 and 3 we describe a modified Trefftz method for biharmonic equation and numerical method of collocation. Numerical examples for direct problems are given in Section 4. In Section 5 we give numerical examples for inverse boundary value problems and explain why the present MCTM is workable. Finally, we draw conclusions in Section 6.

2 A modified Trefftz method

For the Laplace equation defined in a simply connected domain, Liu (2007a, 2007b, 2007c) has proposed a modified Trefftz method by supposing that

$$u(r,\theta) = a_0 + \sum_{k=1}^{m} \left[a_k \left(\frac{r}{R_0}\right)^k \cos k\theta + b_k \left(\frac{r}{R_0}\right)^k \sin k\theta \right], \tag{5}$$

where

$$R_0 \ge \rho_{\max} = \max_{\theta\in[0,2\pi]} \rho(\theta) \tag{6}$$

is a number greater than the characteristic length of the problem domain we consider, and m is a positive integer chosen by the user.

For the biharmonic equation it usually expands the solution of u by

$$u(r,\theta) = a_0 + \sum_{k=1}^{m} [a_k r^k \cos k\theta + b_k r^k \sin k\theta] + c_0 r^2 + \sum_{k=1}^{m} [c_k r^{k+2} \cos k\theta + d_k r^{k+2} \sin k\theta]. \tag{7}$$

The reader may refer the paper by Li, Lu, Huang and Cheng (2007) for a review of the Trefftz method.

However, motivated by Eq. (5) we suggest to use

$$u(r,\theta) = a_0 + \sum_{k=1}^{m} \left[a_k \left(\frac{r}{R_0}\right)^k \cos k\theta + b_k \left(\frac{r}{R_0}\right)^k \sin k\theta \right] + c_0 r^2 + \sum_{k=1}^{m} \left[c_k \left(\frac{r}{R_0}\right)^{k+2} \cos k\theta + d_k \left(\frac{r}{R_0}\right)^{k+2} \sin k\theta \right] \tag{8}$$

as a modified Trefftz solution of the biharmonic equation (1). If $R_0 = 1$ the present method recovers to the Trefftz method. As explored by Liu (2007b), for the Trefftz method the numerical instability is an inherent property, which employs the power functions of r^k and r^{k+2} in the bases. It is a main reason to cause the numerical instability of Eq. (7), because r may be greater than 1 for the problem we consider. When the problem domain has a larger size with its largest distance of the boundary points to the origin point being greater than 1, the powers of r^k and and r^{k+2} are divergent. It would be clear later that this minor revision in Eq. (8) indeed provides us an essentially stable numerical method for the biharmonic equation.

By utilizing the following formula:

$$u_n(\rho,\theta) = \eta(\theta)\left[\frac{\partial u(\rho,\theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho,\theta)}{\partial \theta}\right], \quad (9)$$

where

$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta)+[\rho'(\theta)]^2}}, \quad (10)$$

from Eq. (8) it follows that

$$\begin{aligned} u_n(\rho,\theta) &= 2c_0\eta\rho \\ &+\eta\sum_{k=1}^{m}\gamma^k\left[\left\{\frac{ka_k}{\rho}-\frac{kb_k\rho'}{\rho^2}\right\}\cos k\theta\right. \\ &\left.+\left\{\frac{kb_k}{\rho}+\frac{ka_k\rho'}{\rho^2}\right\}\sin k\theta\right] \\ &+\eta\sum_{k=1}^{m}\gamma^{k+2}\left[\left\{\frac{(k+2)c_k}{\rho}-\frac{kd_k\rho'}{\rho^2}\right\}\cos k\theta\right. \\ &\left.+\left\{\frac{(k+2)d_k}{\rho}+\frac{kc_k\rho'}{\rho^2}\right\}\sin k\theta\right], \end{aligned} \quad (11)$$

where

$$\gamma(\theta) := \frac{\rho(\theta)}{R_0}. \quad (12)$$

Similarly, we have

$$\begin{aligned} \Delta u(\rho,\theta) &= 4c_0 \\ &+4\sum_{k=1}^{m}\frac{(k+1)\gamma^{k+2}}{\rho^2}\left[c_k\cos k\theta + d_k\sin k\theta\right]. \end{aligned} \quad (13)$$

By imposing the conditions in Eq. (2) on Eqs. (8) and (11) we obtain

$$a_0 + c_0\rho^2 + \sum_{k=1}^{m}[A_k a_k + B_k b_k + C_k c_k + D_k d_k] = h(\theta), \quad (14)$$

$$2c_0\eta\rho + \sum_{k=1}^{m}[E_k a_k + F_k b_k + G_k c_k + H_k d_k] = g(\theta), \quad (15)$$

where

$$A_k = \gamma^k\cos k\theta, \quad (16)$$

$$B_k = \gamma^k\sin k\theta, \quad (17)$$

$$C_k = \gamma^{k+2}\cos k\theta, \quad (18)$$

$$D_k = \gamma^{k+2}\sin k\theta, \quad (19)$$

$$E_k = \eta\gamma^k\left[\frac{k\cos k\theta}{\rho}+\frac{k\rho'\sin k\theta}{\rho^2}\right], \quad (20)$$

$$F_k = \eta\gamma^k\left[\frac{k\sin k\theta}{\rho}-\frac{k\rho'\cos k\theta}{\rho^2}\right], \quad (21)$$

$$G_k = \eta\gamma^{k+2}\left[\frac{(k+2)\cos k\theta}{\rho}+\frac{k\rho'\sin k\theta}{\rho^2}\right], \quad (22)$$

$$H_k = \eta\gamma^{k+2}\left[\frac{(k+2)\sin k\theta}{\rho}-\frac{k\rho'\cos k\theta}{\rho^2}\right]. \quad (23)$$

By the same token, when imposing the conditions in Eq. (3) on Eqs. (8) and (13), Eq. (14) is unchanged, but Eq. (15) is replaced by

$$4c_0 + \sum_{k=1}^{m}[M_k c_k + N_k d_k] = g(\theta), \quad (24)$$

where

$$M_k = \frac{(4k+4)\gamma^{k+2}}{\rho^2}\cos k\theta, \quad (25)$$

$$N_k = \frac{(4k+4)\gamma^{k+2}}{\rho^2}\sin k\theta. \quad (26)$$

3 Collocation method

There are totally $4m+2$ unknown coefficients, and Eqs. (14) and (15) are imposed at $2m+1$ different collocated points $(\rho(\theta_i),\theta_i)$ on the interval of $0 \le \theta_i \le 2\pi$:

$$a_0 + c_0\rho^2(\theta_i) + \sum_{k=1}^{m}[A_k^i a_k + B_k^i b_k + C_k^i c_k + D_k^i d_k] = h(\theta_i), \quad (27)$$

$$2c_0\eta(\theta_i)\rho(\theta_i) + \sum_{k=1}^{m}[E_k^i a_k + F_k^i b_k + G_k^i c_k + H_k^i d_k] = g(\theta_i), \quad (28)$$

where for simple notations we use $A_k^i = A_k(\theta_i)$, etc.

When the index i in Eqs. (27) and (28) runs from 1 to $2m+1$ we obtain a linear equations system with dimensions $n = 4m+2$:

$$\begin{bmatrix} 1 & \rho^2(\theta_1) & A_1^1 & B_1^1 & C_1^1 & D_1^1 & \cdots & A_m^1 & B_m^1 & C_m^1 & D_m^1 \\ 0 & 2\eta(\theta_1)\rho(\theta_1) & E_1^1 & F_1^1 & G_1^1 & H_1^1 & \cdots & E_m^1 & F_m^1 & G_m^1 & H_m^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \rho^2(\theta_{2m+1}) & A_1^{2m+1} & B_1^{2m+1} & C_1^{2m+1} & D_1^{2m+1} & \cdots & A_m^{2m+1} & B_m^{2m+1} & C_m^{2m+1} & D_m^{2m+1} \\ 0 & 2\eta(\theta_{2m+1})\rho(\theta_{2m+1}) & E_1^{2m+1} & F_1^{2m+1} & G_1^{2m+1} & H_1^{2m+1} & \cdots & E_m^{2m+1} & F_m^{2m+1} & G_m^{2m+1} & H_m^{2m+1} \end{bmatrix} \begin{bmatrix} a_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ \vdots \\ a_m \\ b_m \\ c_m \\ d_m \end{bmatrix} = \begin{bmatrix} h(\theta_1) \\ g(\theta_1) \\ \vdots \\ h(\theta_{2m+1}) \\ g(\theta_{2m+1}) \end{bmatrix}. \quad (29)$$

It can be seen that the idea behind the collocation method is rather simple, and it has a great advantage of flexibility to apply on different geometric shapes, and the simplicity for a computer program.

Eq. (29) is denoted by

$$\mathbf{Re} = \mathbf{b}_1, \quad (30)$$

where $\mathbf{e} = [a_0, b_0, a_1, b_1, c_1, d_1, \cdots, a_m, b_m, c_m, d_m]^{\mathrm{T}}$ is the vector of unknown coefficients. The conjugate gradient method can be used to solve the following normal equation:

$$\mathbf{Ae} = \mathbf{b}, \quad (31)$$

where

$$\mathbf{A} := \mathbf{R}^{\mathrm{T}}\mathbf{R}, \;\; \mathbf{b} := \mathbf{R}^{\mathrm{T}}\mathbf{b}_1. \quad (32)$$

We denote the convergent criterion used in the conjugate gradient method by ε.

Similarly, for the simply supported boundary condition we have the following linear equations system to determine the coefficients:

$$\begin{bmatrix} 1 & \rho^2(\theta_1) & A_1^1 & B_1^1 & C_1^1 & D_1^1 & \cdots & A_m^1 & B_m^1 & C_m^1 & D_m^1 \\ 0 & 4 & 0 & 0 & M_1^1 & N_1^1 & \cdots & 0 & 0 & M_m^1 & N_m^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \rho^2(\theta_{2m+1}) & A_1^{2m+1} & B_1^{2m+1} & C_1^{2m+1} & D_1^{2m+1} & \cdots & A_m^{2m+1} & B_m^{2m+1} & C_m^{2m+1} & D_m^{2m+1} \\ 0 & 4 & 0 & 0 & M_1^{2m+1} & N_1^{2m+1} & \cdots & 0 & 0 & M_m^{2m+1} & N_m^{2m+1} \end{bmatrix} \begin{bmatrix} a_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ \vdots \\ a_m \\ b_m \\ c_m \\ d_m \end{bmatrix}$$

$$= \begin{bmatrix} h(\theta_1) \\ g(\theta_1) \\ \vdots \\ h(\theta_{2m+1}) \\ g(\theta_{2m+1}) \end{bmatrix}. \quad (33)$$

Inserting the calculated $\mathbf{e} = [a_0, c_0, a_1, b_1, c_1, d_1, \cdots, a_m, b_m, c_m, d_m]^{\mathrm{T}}$ into Eq. (8) we have a semi-analytical solution of $u(r,\theta)$:

$$u(r,\theta) = e_1 + e_2 r^2 + \sum_{k=1}^{m} \left[\left(e_{4k-1} \left(\frac{r}{R_0}\right)^k + e_{4k+1} \left(\frac{r}{R_0}\right)^{k+2} \right) \cos k\theta + \left(e_{4k} \left(\frac{r}{R_0}\right)^k + e_{4k+2} \left(\frac{r}{R_0}\right)^{k+2} \right) \sin k\theta \right], \quad (34)$$

where $(e_1, \ldots, e_n)$ are the components of $\mathbf{e}$.

From above solution, it is not difficult to calculate other quantities about u through some elementary operations. For later convenience we may call the present method a modified collocation Trefftz method (MCTM).

4 Numerical examples for direct problem

Before embarking a numerical study of the new method, we are concerned with the stability of MCTM, in the case when the boundary data are contaminated by random noise, which is investigated by adding different levels of random noise on the boundary data. We use the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i)$, which are random numbers in $[-1,1]$. Hence we use the simulated noisy data given by

$$\hat{h}(\theta_i) = h(\theta_i) + |h(\theta_i)|_{max} s R(i), \quad (35)$$

where $\theta_i = 2i\pi/(2m+1)$, $i = 1, \ldots, 2m+1$, and s is the level of noise. Similarly, this is done for the boundary data $g(\theta)$.

4.1 Example 1

We first consider a complex example with the exact solution [Lesnic, Elliott and Ingham (1998); Marin and Lesnic (2005); Jin (2004)]:

$$u = \frac{1}{2}[x \sin x \cosh y - x \cos x \sinh y]. \quad (36)$$

The domain is $\Omega = \{(x,y) | x^2 + y^2 \le 4\}$. The exact boundary data can be derived by inserting the exact solution into Eq. (2) with $\rho = 2$.

Before applying the MCTM on this example we attempt to see what advantage can be gained by the new method. For this we consider the condition number defined by

$$\mathrm{Cond}(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|. \quad (37)$$

The norm used for $\mathbf{A}$ is the Frobenius norm. Therefore, we have

$$\frac{1}{n}\mathrm{Cond}(\mathbf{A}) \le \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})} \le \mathrm{Cond}(\mathbf{A}), \quad (38)$$

where λ is the eigenvalue of $\mathbf{A}$. Conventionally, $\lambda_{\max}(\mathbf{A})/\lambda_{\min}(\mathbf{A})$ is used to define the condition number of $\mathbf{A}$. In the present study we use Eq. (37) to define the condition number of $\mathbf{A}$.

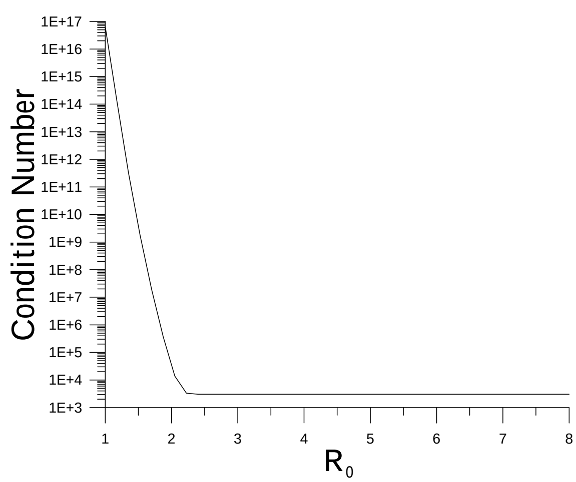

Figure 1: The condition number with respect to R_0 for Example 1.

In Fig. 1 we plot the condition number of $\mathbf{A}$ with respect to R_0, where $m = 20$ was fixed. When $R_0 = 1$ our method reduces to the conventional Trefftz method. It can be seen that the condition number greatly reduces when $R_0 \ge 2$ and tends to a constant value about in the order of 3×10^3. If we fix $R_0 = 1$ the condition number is very large up to 6.7×10^{16}. Hence, it is hopeless to use the original Trefftz method to calculate the solution of this example without resorting to some regularization techniques. Conversely, the new method indeed can improve the ill-condition of the resulting linear equations system by selecting $R_0 > 2$, and it is naturally expected that the new MCTM can deal with the direct problems of biharmonic equation very well.

When there is no noise adding on the boundary data, i.e. $s = 0$, the present method gives very high accurate $v = \Delta u$ and v_n on the boundary with the maximum absolute erros respectively 1.4×10^{-8} and 1.1×10^{-7}. Usually, if the given boundary data is not polluted by random noise, we suggest to use very small ε, like as $\varepsilon = 10^{-15}$, in the solution of unknown coefficients by the conjugate gradient method as described in Section 3. It means that the given exact data must be fitted as accurately as possible. However, when random noise is considered, we must relax the above requirement by loosely fitting the inexact data to use a slightly large ε; for example, $\varepsilon = 10^{-4}$ is used in the following case.

In this case we fixed $m = 20$, $R_0 = 5$, and $s = 0.01$. As shown in Fig. 2(a) by comparing the numerical solution with exact solution of v, and in Fig. 2(b) the numerical solution with exact solution of v_n, the maximum absolute errors are respectively 0.176 and 0.54. It can be seen that the noise disturbs the solutions deviating from the exact ones little. The present method is robust against the noise. In the above cited papers some regularization techniques are required in order to get better numerical results against the noise. Due to its low condition number the present MCTM can calculate the biharmonic equation very well without considering any regularization.

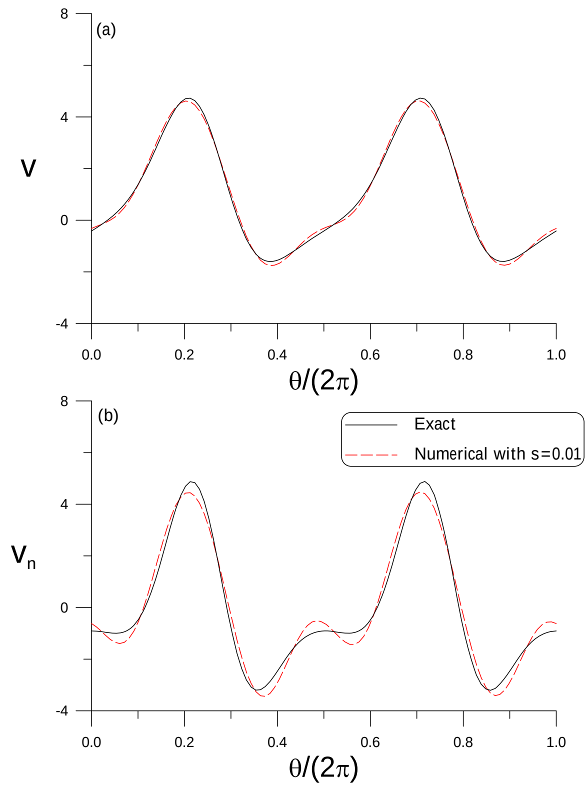

Figure 2: The comparisons of numerical results with the exact ones for Example 1.

4.2 *Example 2*

In this example a complex shape is shown in the inset of Fig. 3:

$$\rho(\theta) = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos(a\theta/b)} \tag{39}$$

with $a = 4$ and $b = 1$. We consider the following analytical solution:

$$u(x,y) = x^3 + y^3. \tag{40}$$

The exact boundary data can be derived by using the exact solution.

Fig. 3 compares the numerical solutions under a noise level $s = 0.01$ with exact solutions for v and v_n. The following parameters are used: $m = 30$ and $R_0 = 15$. It can be seen that the numerical results are good.

Now we turn to the boundary value problem described by Eq. (3). Here we attempt to calculate u_n and v_n on the boundary. Fig. 4 compares the numerical solutions under a noise level $s = 0.01$ with exact solutions for u_n and v_n. The following parameters are used: $m = 30$, $R_0 = 30$ and $\varepsilon = 10^{-6}$. It can be seen that the numerical results are excellent to coincide with the exact solutions.

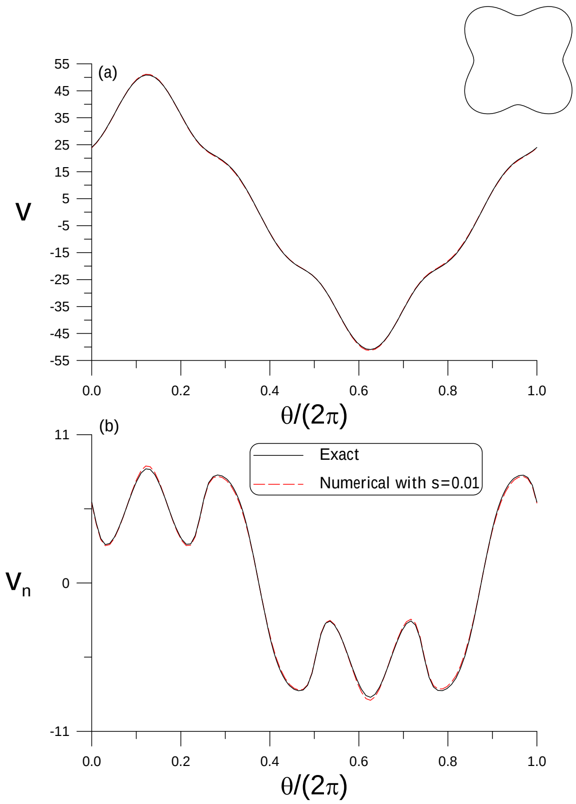

Figure 3: Comparing numerical and exact solutions for Example 2 under the clamped boundary condition.

5 Numerical examples for inverse problem

5.1 Example 3

We revisit Example 2 again. However, now we consider an inverse problem of biharmonic equation under incomplete data given by

$$u(\rho,\theta)=h(\theta),\ \ u_n(\rho,\theta)=g(\theta),\ \ 0\le\theta\le 2\beta\pi. \tag{41}$$

When $\beta=1$ we recover to the direct problem.

In order to test our method we plot the condition number with respect to β in the range of $0.4\le\beta\le 1$ in Fig. 5. When β decreases the condition number increases fast. It means that the available information on the inverse problem is decreased when β decreases.

Under the parameters of $m=20$, $R_0=25$, $\beta=0.55$, $\varepsilon=10^{-6}$ and $s=0.01$, we solve this inverse problem by the method in Section 3, directly collocated the incomplete data, whose numerical results along the remaining boundary are shown in Fig. 6 by the dashed lines. It is almost

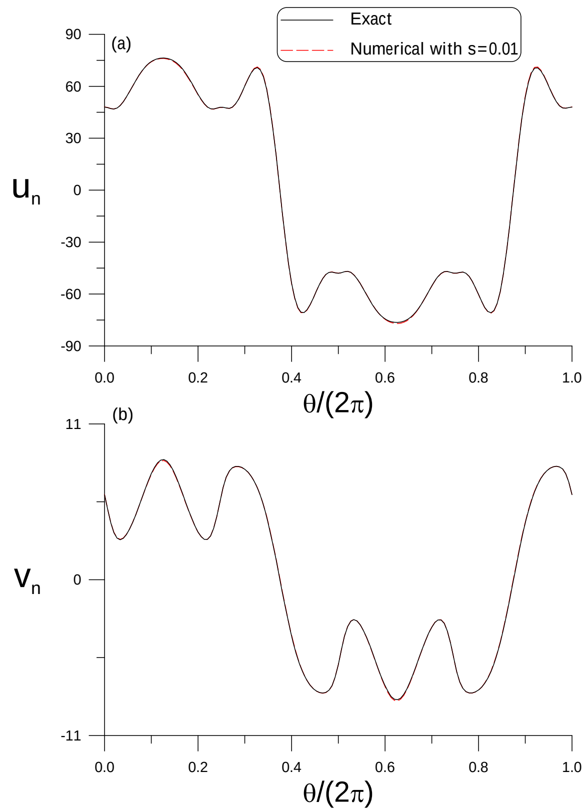

Figure 4: Comparing numerical and exact solutions for Example 2 under the simply supported boundary condition.

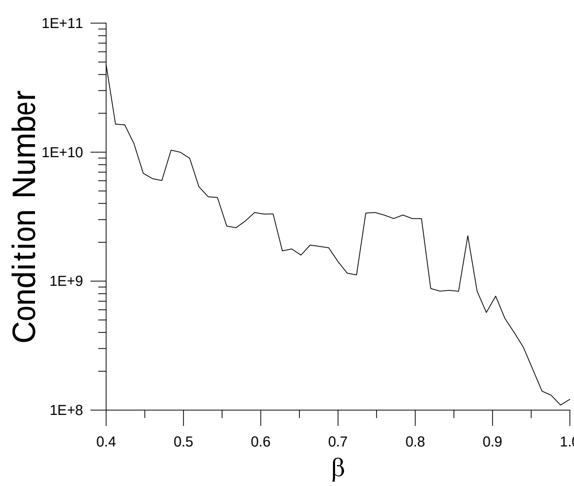

Figure 5: The condition number with respect to β for Example 3.

coincident with the exact ones as shown by the solid lines. Therefore, we can conclude that the present method can recover the unknown functions, even we do not need to use overspecified data and regularization technique. Here we solve the inverse problem by using a same technique as that for solving the direct problem without worrying about the problem of instability. It is generally asserted that the inverse problem is inherently ill-posed and unstable. In the opinion of author, a good numerical method should be equally well to handle direct and inverse problems in a unified manner. Regularization is just a remedy to compensate the shortage and leakage of the ill-posed method. If the method is essentially stable, it can handle some difficult inverse problems in a simple way without resorting on the technique of regularization.

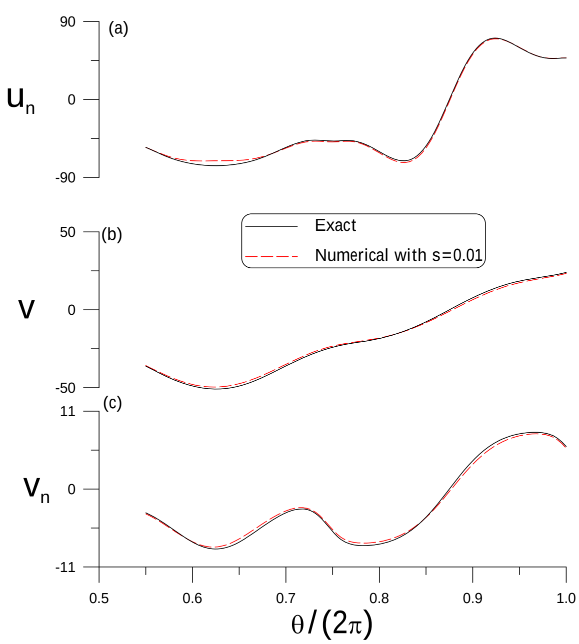

Figure 6: Comparing numerical and exact solutions of inverse problem in Example 3 for a smooth noncircular contour.

5.2 *Example 4*

We revisit Example 1 again. But now we consider an inverse problem on a non-smooth square boundary described by

$$\rho(\theta)=\begin{cases}\frac{1}{\cos\theta} & -\pi/4\le\theta<\pi/4,\\ \frac{1}{\sin\theta} & \pi/4\le\theta<3\pi/4,\\ \frac{-1}{\cos\theta} & 3\pi/4\le\theta<5\pi/4,\\ \frac{-1}{\sin\theta} & 5\pi/4\le\theta<7\pi/4,\end{cases} \tag{42}$$

$$\rho'(\theta)=\begin{cases}\frac{\sin\theta}{\cos^2\theta} & -\pi/4\le\theta<\pi/4,\\ \frac{-\cos\theta}{\sin^2\theta} & \pi/4\le\theta<3\pi/4,\\ \frac{-\sin\theta}{\cos^2\theta} & 3\pi/4\le\theta<5\pi/4,\\ \frac{\cos\theta}{\sin^2\theta} & 5\pi/4\le\theta<7\pi/4.\end{cases} \tag{43}$$

Zeb, Elliott, Ingham and Lesnic (1999) have solved this problem under an overspecified boundary condition, giving u on the whole boundary, and u_n and v on the partial boundary of $\Gamma_1=\{-1\le x\le 1, y=-1\}\cup\{x=1,-1\le y\le 1\}\cup\{-1\le x\le 1, y=1\}$. Even the singular value decomposition technique is used their results are not good to recover the data of v and v_n. In Figs. 10 and 11 of the above cited paper, the data of u_n on $\Gamma_2=\{x=-1,-1\le y\le 1\}$ were plotted under different levels of noise.

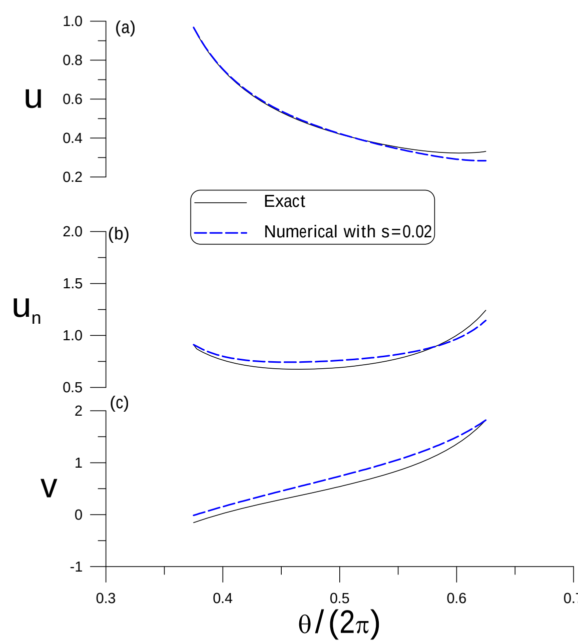

Figure 7: Comparing numerical and exact solutions of inverse problem in Example 4 for a non-smooth contour of square.

Here we only impose the boundary conditions of u and u_n on Γ_1, and leave other data unspecified. Under the following parameters of $m = 10$, $R_0 = 2.7$, $s = 0.02$ and $\varepsilon = 10^{-4}$, we solve this inverse problem by the method in Section 3, directly collocated the incomplete data, whose numerical results along the remaining boundary are shown in Fig. 7 by the dashed lines. They are rather close to the exact ones as shown by the solid lines. This inverse problem possesses several difficulties, including non-smooth boundary, large noise and insufficient boundary conditions. Although we only used a limited data of u and u_n on the boundary Γ_1, the present results are much better than that reported by Zeb, Elliott, Ingham and Lesnic (1999), Marin and Lesnic (2005), and Li (2005) for this benchmark problem.

6 Conclusions

We have employed a new idea to treat the direct and inverse problems of biharmonic equation in arbitrary plane domains by a modified collocation Trefftz method. The new method can provide a semi-analytical solution in terms of a modified T-complete set of bases for biharmonic equation, which renders a rather compendious numerical implementation to solve the direct and inverse problems without needing of any iteration and any regularization. The new method was found accurate, effective and stable. These points are very diiferent from other numerical methods for the inverse problems. This paper demonstrated that a good numerical method is not only applicable to the direct problems but also to the inverse problems. The conventional method is not applicable to the inverse problems is due to its ill-conditioned property. If a generic resolution of ill-condition can be made available, we no more need a remedy of the method by the regularization technique.

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