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Discrete Constitutive Equations over Hexahedral Grids for Eddy-current Problems

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Abstract: In the paper we introduce a methodology to construct discrete constitutive matrices relating magnetic fluxes with magneto motive forces (reluctance matrix) and electro motive forces with currents (conductance matrix) needed for discretizing eddy current problems over hexahedral primal grids by means of the Finite Integration Technique (FIT) and the Cell Method (CM). We prove that, unlike the mass matrices of Finite Elements, the proposed matrices ensure both the stability and the consistency of the discrete equations introduced in FIT and CM.

Keyword: Discrete constitutive equations, discrete geometric approach, eddy-currents.

1 Introduction

In the recent years, the role of geometry and algebraic topology gained a considerable importance in the research on computational electromagnetism. In this respect the fundamental works of T. Weiland with the *Finite Integration Technique* (FIT) [Clemens and Weiland (2001)], E. Tonti with *Cell Method* (CM) [Tonti (1995)], [Tonti (2001)] and A. Bossavit [Bossavit (1998b)], [Bossavit and Kettunen (2000)] reveal a "Discrete Geometric Approach" (DGA) to solving directly Maxwell equations in an alternative way with respect to the classical Galerkin method in Finite Elements, [Castillo, Koning, Rieben, and White (2004)], [Heshmatzadeh and Bridges (2007)]. Several applications of DGA to solving other physical problems have been developed by a number of authors since its introduction, i.e. [Cosmi (2001)], [Ferretti (2003)], [Ferretti (2004b)], [Ferretti (2004a)], [Cosmi (2005)], [Cosmi (2008)].

The DGA allows the construction of an algebraic system of equations by combining both the physical laws of electromagnetism, formulated *exactly* in a purely topological way and the constitutive relations, *approximated* in a geometric way on a specified grid. Even though the DGA is general, in this paper we will focus an eddy-current problem as a working example [Trevisan and Kettunen (2006)].

For the sake of clarity, we will briefly retrace the fundamental steps of the DGA in order to address the reader towards the novelty content of our work: the geometric construction of the discrete constitutive relations on an hexahedra grid complying with precise properties necessary for the solution of a discrete formulation of eddy-current problem.

Firstly a pair of oriented dual grids is introduced in the domain of interest. One grid is denoted as the *primal* grid and the other as the *dual* grid. A grid is a collection of oriented geometric elements like nodes, edges, faces and volumes [Bossavit (1998a)]. The geometric elements of one grid are in a one-to-one correspondence with the geometric elements of the other grid. For example to a face of the primal grid corresponds an edge of the dual grid.

A second step is the unique association of the so called integral or *global* variables describing electromagnetic phenomena to a precise geometric elements of the primal or dual grid, [Tonti (1998)]. For example, the magnetic induction flux is asso-

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ciated with the faces of the primal grid, the electric current is associated with the faces of the dual grid while the magneto motive force is attached to the edges of the dual grid.

As third step, the physical laws of electromagnetism can be written directly in terms of exact algebraic relations involving the global variables associated with the geometric elements of the primal and dual grids. For instance Ampère's law relates the current crossing a dual face with the magneto motive force along the dual edges bounding that face.

In this way, the so called balance equations are formed, which relay on the topology of the grids only. On the contrary the discrete counterparts of the continuous level constitutive relations are finite dimensional linear operators – i.e. matrices – mapping in an *approximate* way global variables associated with the geometric elements of one grid to the global variables associated with the corresponding geometric elements of the other grid. To construct such matrices, we need metric concepts (like lengths, areas and volumes) and material properties; usually an element wise constant material medium property is assumed. For example, in our eddy currents problem, the magnetic induction fluxes – attached to the faces of the primal grid – are transformed into the magneto motive forces along the corresponding edges of the dual grid; this matrix will be denoted as the reluctance matrix; similarly, but at a different geometric level, the conductance matrix transforms the electro motive forces along the edges of the primal grid into the currents crossing the faces of the dual grid.

By combining the balance equations with the constitutive matrices, a final system of discretized equations is deduced. It is a known result [Bossavit and Kettunen (2000)], [Codecasa, Minerva, and Politi (2004)], that to ensure the consistency and the stability of the final system, the constitutive matrices are required to satisfy a pair of fundamental properties: i) a consistency property, ii) a stability property. Since discrete constitutive relations, as it is common, are assumed to be constructed primal volume by primal volume, without loosing generality, we can consider a primal grid over a single primal volume having homogenoeus reluctivity or conductivity according to the case; thence to ensure the consistency property, the reluctance matrix is required to exactly transform the fluxes through primal faces of a *uniform* magnetic induction into the circulations along dual edges of the corresponding *uniform* magnetic field [Codecasa, Specogna, and Trevisan (2007)]. Similarly, but at a different geometric level, the conductance matrix complies with the consistency property when it exactly transforms the circulations along primal edges of a *uniform* electric field into the currents through dual faces of the corresponding *uniform* current density [Codecasa, Specogna, and Trevisan (2007)]. Finally the stability property is guaranteed if the reluctance and conductance matrices are *symmetric and positive definite*.

Discrete constitutive relations, satisfying both the consistency and stability properties, were initially introduced in a straightforward and natural ways for pairs of orthogonal Cartesian dual grids [Clemens and Weiland (2001)]. Recently, also for a pair of dual grids in which the primal grid is made of tetrahedra and the dual grid is obtained by means of the barycentric subdivision of the primal grid, constitutive relations satisfying both the consistency and stability properties have been introduced. In this respect, A. Bossavit showed [Bossavit (1998b)], [Bossavit (1998a)] that the so called *mass matrices* constructed in the Finite Element Method (FEM) by means of Whitney's edge and face vector functions, not only satisfy the stability property but also the consistency property above mentioned; thus such mass matrices for tetrahedral grids can be borrowed as constitutive matrices for the DGA. Besides, also the present authors [Codecasa, Minerva, and Politi (2004)], [Codecasa, Specogna, and Trevisan (2007)] proposed for tetrahedra and prisms with triangular bases a so called *energetic approach* to compute, in a fully geometric way, an independent pair of novel stable and consistent constitutive matrices to be used in the Discrete Geometric Approach.

However for primal grids in which the volumes are generic hexahedra, no constitutive matrices, satisfying both the consistency and the stability properties, have been reported in literature. In this paper, we will try to fill in this gap.

Firstly, we will show, by a counter-example that the mass matrices constructed in the FEM for an hexahedral primal grid, by means of the so called *mixed elements* edge and face vector functions described in [Dular, Hody, Nicolet, Genon, and Legros (1994)], even if they are symmetric and positive definite and thus satisfy the stability property *ii*), do *not* satisfy the consistency property *i*) for any choice of the dual grid in correspondence of the hexahedral primal grid. Thus such mass matrices for the hexahedral grids *cannot* be borrowed as constitutive matrices for the DGA.

Then we will propose novel discrete constitutive matrices, satisfying both the consistency and stability properties, for pairs of dual grids in which the volumes of the primal grid are generic hexahedra and the dual grid is obtained by means of the barycentric subdivision of the *boundaries* of the volumes of the primal grid. Numerical experiments will show that such novel discrete constitutive relations can be constructed at a low computational cost and that they lead to an accurate approximation of the solution to our eddy current problem.

The remainder of this paper is organized as follows. In section 2 the equations obtained by the DGA for eddy-current problems are recalled. Also it is verified that the mass matrices constructed in the FEM do not satisfy the consistency property of discrete constitutive relations. The novel method for constructing the discrete constitutive relation is then presented in successive steps. In sections 4, 5 we prove the main geometric properties needed to construct the discrete constitutive matrices. Sections 6 and 7 are then dedicated to the construction of such matrices and to prove the corresponding properties of consistency and of symmetric positive definiteness they comply with. Section 8 is devoted to the presentation of numerical results. All ancillary results needed in overall the paper are collected in Appendix A:, Appendix B:, Appendix C:.

2 Discrete equations for eddy current problems

We state here a typical eddy current problem. The domain of interest D contains a source region D_s where prescribed currents are present and the conducting region D_c. The insulating region D_a is the complement of D_c and D_s with respect to D. In D we introduce a pair of interlocked primal-dual grids whose interconnections are described by the usual incidence matrices $\mathbf{G}$ between primal edges e and primal nodes n and $\mathbf{C}$ between primal faces f and primal edges e. The reluctivity and conductivity of the media are assumed element-wise constants.

We briefly recall the basic equations of a DGA to solve eddy-current problems in the frequency domain, [Trevisan (2004)], [Specogna and Trevisan (2005)], [Trevisan and Kettunen (2006)]. We search for the array $\mathbf{A}$ of the circulations of the magnetic vector potential along primal edges e of D and for the array $\boldsymbol{\chi}$ of scalar potential χ associated with primal nodes n of D_c such that

$$(\mathbf{C}^T\mathbf{MCA})_e = (\mathbf{I}^s)_e \quad \forall e \in D \backslash D_c$$

$$(\mathbf{C}^T\mathbf{MCA})_e + i\omega(\mathbf{NA}_c)_e + i\omega(\mathbf{NG}\boldsymbol{\chi})_e = 0 \qquad \forall e \in D_c$$

$$i\omega(\mathbf{G}^T\mathbf{NA}_c)_n + i\omega(\mathbf{G}^T\mathbf{NG}\boldsymbol{\chi})_n = 0 \quad \forall n \in D_c,$$

where the array $\mathbf{I}^s$ contains the source currents I^s crossing the dual faces in D_s; $\mathbf{A}_c$ is the sub-array of $\mathbf{A}$, associated with primal edges in D_c; the matrix $\mathbf{G}$ is associated with pairs (e, n) of D_c only. With $(\mathbf{x})_k$ we mean the k-th row of array $\mathbf{x}$, where $k = \{e, n\}$ is the label of edge e or of node n. Finally the reluctance and conductance constitutive matrices are denoted with $\mathbf{M}$, $\mathbf{N}$ respectively such that $\dim(\mathbf{M}) = F$, F being the number of faces in D and $\dim(\mathbf{N}) = L_c$, L_c being the number of edges in D_c. This system of equations is singular and to solve it we rely on CG method without gauge condition [Kameari and Koganezawa (1997)].

As shown in [Bossavit and Kettunen (2000)], [Codecasa and Trevisan (2006)] in order to ensure the consistency of the discrete system obtained by the DGA, the constitutive matrices $\mathbf{M}$, $\mathbf{N}$, are both required to comply with the above mentioned consistency i) and stability ii) properties, [Codecasa, Specogna, and Trevisan (2007)].

The existing technique for constructing the mass matrices in the framework of finite elements over an hexahedral primal grid, does not lead to constitutive matrices complying with the consistency property i). This is demonstrated in Appendix C: by a simple counter-example. Hereafter we will construct in a purely geometric way a pair of novel constitutive matrices **M**, **N** which instead satisfy both the consistency i) and stability ii) properties for *hexahedral* primal grids.

3 Notation

Let $\mathrm{T} = \mathrm{u} \otimes \mathrm{v}$ be the double tensor T obtained by means of the tensor product $\otimes$ of the two vectors u, v. The product $\mathrm{T}\,\mathrm{u}$ between a double tensor T and a vector u is a vector; the inner product $\mathrm{v} \cdot \mathrm{T}\,\mathrm{u}$ is a scalar, v being a vector. Between the tensor $\mathrm{T} = \mathrm{u} \otimes \mathrm{v}$ and a vector a the following relation

$$\mathrm{u} \otimes \mathrm{v}\,\mathrm{a} = (\mathrm{v} \cdot \mathrm{a})\,\mathrm{u}$$

holds. The identity tensor is denoted with I and it is such that $\mathrm{I}\,\mathrm{u} = \mathrm{u}$ holds.

4 Primal and dual grids

In the following sections we will consider a single hexahedron v as primal grid, Fig. 1. Let the conductivity σ and the reluctivity ν within v be homogeneous, symmetric positive definite double tensors.

Let $|v|$ be the measure of the volume v. Let f_i, with $i = 1, \ldots, F = 6$ be the primal faces[1] of v, let e_j with $j = 1, \ldots, L = 12$ be its primal edges and let p_k with $k = 1, \ldots, N = 8$ be its primal nodes.

We denote in *roman type* a position vector r drawn from an origin of a Cartesian reference frame to a generic point r within v. Let p_k be the position vector associated with the primal node p_k. Let g_{f_i} be the position vector of the barycenter of the face f_i defined by

$$\mathrm{g}_{f_i} = \frac{1}{|f_i|} \int_{f_i} \mathrm{r}\, ds,$$

[1] By definition, the faces of an hexahedron are planar faces.

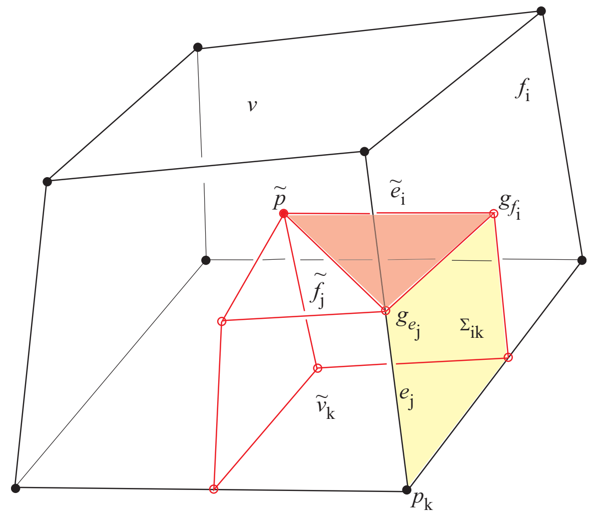

Figure 1: Hexahedron v, primal face f_i, primal edge e_j, primal node p_k; dual volume $\tilde{v}_k$, dual face $\tilde{f}_j$, dual edge $\tilde{e}_i$ and dual node $\tilde{p}$. Moreover the barycenter g_{e_j} of edge e_j and the barycenter g_{f_i} of face f_i are shown.

in which $|f_i|$ is the area of f_i, with $i = 1, \ldots, F$, and let g_{e_j} be the position vector of the barycenter of the edge e_j, with $j = 1, \ldots, L$.

Let $\tilde{p}$ be the dual node in v, as in Fig. 1. This node can be arbitrarily chosen within v; as a particular case it can be the barycenter of v. The segment drawn between $\tilde{p}$ and the barycenter g_{f_i} defines the dual edge $\tilde{e}_i$ and it is in one to one correspondence with the primal face f_i, with $i = 1, \ldots, F$. The dual face $\tilde{f}_j$ is in a one to one correspondence with the primal edge e_j, with $j = 1, \cdots, L$. In general it is *not* a planar face and it is formed by the union of two triangles; each triangle has as nodes $\tilde{p}$, the barycenter g_{e_j} and the barycenter g_{f_i} of one face f_i of the two adjacent to e_j. The dual volume $\tilde{v}_k$ is in one to one correspondence with node p_k, as in Fig. 1.

The primal geometric entities p_k, e_j, f_i and v are endowed with an inner orientation. Similarly the dual geometric entities like $\tilde{p}$, $\tilde{e}_i$, $\tilde{f}_j$ and $\tilde{v}_k$ are endowed with an outer orientation [Tonti (1998)], in such a way that the pairs (p_k, $\tilde{v}_k$), (e_j, $\tilde{f}_j$), (f_i, $\tilde{e}_i$) and (v, $\tilde{p}$) are oriented in a congruent way.

We denote with e_j the edge vector associated with edge e_j. Its amplitude and orientation coincide respectively with the length and orientation of e_j. with $j = 1, \ldots, L$. We denote with f_i the face vector of f_i defined by

$$\mathrm{f}_i = \int_{f_i} \mathrm{n}(\mathrm{r}) ds,$$

$\mathrm{n}(\mathrm{r})$ being the vector normal to and oriented as f_i, with $i = 1, \ldots F$. Similarly $\tilde{\mathrm{e}}_i$ is the edge vector associated with $\tilde{e}_i$, with $i = 1, \ldots, F$, and $\tilde{\mathrm{f}}_j$ is the face vector associated with $\tilde{f}_j$, with $j = 1, \ldots, L$. We have that $\mathrm{e}_j \cdot \tilde{\mathrm{f}}_j > 0$ and $\mathrm{f}_i \cdot \tilde{\mathrm{e}}_i > 0$ hold.

As a consequence of this particular choice of dual grid, constructed by means of the barycenters of the primal edges and primal faces, the following two geometrical properties hold

Property 1 *It results in*

$$|v|\mathrm{I} = \sum_{1}^{L}{}_{j} \mathrm{e}_j \otimes \tilde{\mathrm{f}}_j, \tag{1}$$

Proof. Let a and b be a pair of spatially uniform vectors. It is

$$\int_v \mathrm{a} \cdot \mathrm{b}\, dv = \sum_{1}^{N}{}_{k} \int_{\tilde{v}_k} \mathrm{a} \cdot \mathrm{b}\, dv.$$

Besides, since a is spatially uniform and thus it is $\mathrm{a} = \nabla u(\mathrm{r})$ with $u(\mathrm{r}) = \mathrm{a} \cdot \mathrm{r}$, it results in

$$\int_{\tilde{v}_k} \mathrm{a} \cdot \mathrm{b}\, dv =$$
$$\int_{\tilde{v}_k} \nabla(u(\mathrm{r}) - u(\mathrm{p}_k)) \cdot \mathrm{b}\, dv =$$
$$\int_{\tilde{v}_k} \nabla \cdot (u(\mathrm{r}) - u(\mathrm{p}_k))\, \mathrm{b}\, dv - \int_{\tilde{v}_k} (u(\mathrm{r}) - u(\mathrm{p}_k)) \nabla \cdot \mathrm{b}\, dv =$$
$$\int_{\partial \tilde{v}_k} (u(\mathrm{r}) - u(\mathrm{p}_k))\, \mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds =$$
$$\sum_{1}^{F}{}_{i} \int_{\tilde{v}_k \cap f_i} (u(\mathrm{r}) - u(\mathrm{p}_k))\, \mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds +$$
$$+ \sum_{1}^{L}{}_{j} \int_{\tilde{v}_k \cap \tilde{f}_j} (u(\mathrm{r}) - u(\mathrm{p}_k))\, \mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds,$$

being $\mathrm{n}(\mathrm{r})$ a unit vector normal to and oriented as $\partial \tilde{v}_k$ at r. It is

$$\int_{\tilde{v}_k \cap \tilde{f}_j} (u(\mathrm{r}) - u(\mathrm{p}_k))\, \mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds =$$
$$\int_{\tilde{v}_k \cap \tilde{f}_j} (u(\mathrm{g}_{e_j}) - u(\mathrm{p}_k))\, \mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds +$$
$$+ \int_{\tilde{v}_k \cap \tilde{f}_j} (u(\mathrm{r}) - u(\mathrm{g}_{e_j}))\, \mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds.$$

Besides it results in

$$\sum_{1}^{L}{}_{j} \sum_{1}^{N}{}_{k} \int_{\tilde{v}_k \cap \tilde{f}_j} (u(\mathrm{g}_{e_j}) - u(\mathrm{p}_k))\, \mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds =$$
$$\sum_{1}^{L}{}_{j} (\mathrm{a} \cdot \mathrm{e}_j)(\mathrm{b} \cdot \tilde{\mathrm{f}}_j)$$

and

$$\sum_{1}^{N}{}_{k} \int_{\tilde{v}_k \cap \tilde{f}_j} (u(\mathrm{r}) - u(\mathrm{g}_{e_j}))\, \mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds = 0.$$

Lastly, from (23) in Lemma 2 of Appendix C: it results in

$$\sum_{1}^{N}{}_{k} \int_{\tilde{v}_k \cap f_i} \mathrm{a} \cdot (\mathrm{r} - \mathrm{p}_k) \mathrm{n}(\mathrm{r}) \cdot \mathrm{b}\, ds = 0, \quad i = 1, \ldots, F$$

and the claim follows. ■

Property 2 *It results in*

$$|v|\mathrm{I} = \sum_{1}^{F}{}_{i} \tilde{\mathrm{e}}_i \otimes \mathrm{f}_i \tag{2}$$

Proof. Let a and b be a pair of spatially uniform vectors. Then it is $\mathrm{b} = \nabla u(\mathrm{r})$ with $u(\mathrm{r}) = \mathrm{b} \cdot \mathrm{r}$ and it results in

$$\int_v \mathrm{a} \cdot \mathrm{b}\, dv =$$
$$\int_v \mathrm{a} \cdot \nabla(u(\mathrm{r}) - u(\tilde{\mathrm{p}}))\, dv =$$
$$\int_v \nabla \cdot (u(\mathrm{r}) - u(\tilde{\mathrm{p}}))\, \mathrm{a}\, dv - \int_v (u(\mathrm{r}) - u(\tilde{\mathrm{p}})) \nabla \cdot \mathrm{a}\, dv =$$
$$\int_{\partial v} (u(\mathrm{r}) - u(\tilde{\mathrm{p}}))\, \mathrm{a} \cdot \mathrm{n}(\mathrm{r})\, dv =$$
$$\sum_{1}^{F}{}_{i} \int_{f_i} (u(\mathrm{r}) - u(\mathrm{g}_{f_i}))\, \mathrm{a} \cdot \mathrm{n}(\mathrm{r})\, dv +$$
$$+ \sum_{1}^{F}{}_{i} \int_{f_i} (u(\mathrm{g}_{f_i}) - u(\tilde{\mathrm{p}}))\, \mathrm{a} \cdot \mathrm{n}(\mathrm{r})\, dv,$$

$\mathrm{n}(\mathrm{r})$ being a unit vector oriented as the outward normal to ∂v. It is

$$\sum_{1}^{F}{}_{i} \int_{f_i} (u(\mathrm{g}_{f_i}) - u(\tilde{\mathrm{p}}))\, \mathrm{a} \cdot \mathrm{n}(\mathrm{r})\, dv = \sum_{1}^{F}{}_{i} (\mathrm{a} \cdot \mathrm{f}_i)(\mathrm{b} \cdot \tilde{\mathrm{e}}_i)$$

Besides it is

$$\int_{f_i} \mathrm{b} \cdot (\mathrm{r} - \mathrm{g}_{f_i}) \mathrm{n}(\mathrm{r}) \cdot \mathrm{a}\, dv = 0, \quad i = 1, \ldots, F$$

and the claim follows. ■

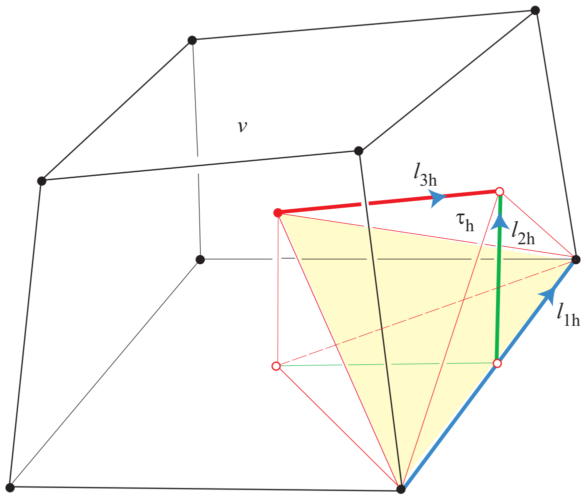

Figure 2: Tetrahedron τ_h, and associated base vectors $(\mathrm{l}_{1h}, \mathrm{l}_{2h}, \mathrm{l}_{3h})$.

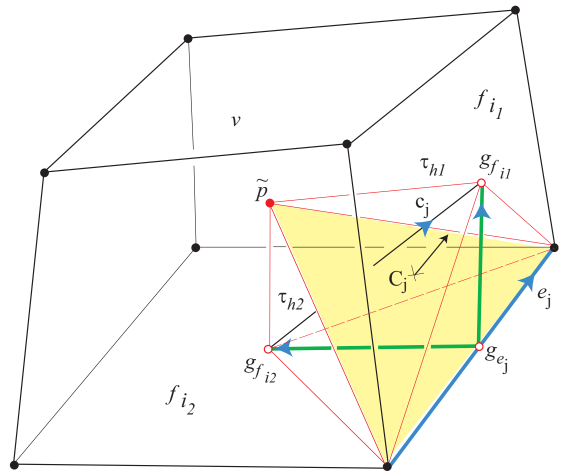

Figure 3: Elements c_j and C_j, with $j = 1 \dots L$.

5 Subdivision of an hexahedron into tetrahedra

An hexahedron v can be thought as the union of $2L$ tetrahedra τ_h, with $h = 1, \dots, 2L$. The vertices of the tetrahedron τ_h are $\tilde{p}$, the pair of nodes bounding an edge e_j and the barycenter g_{f_i} of a face f_i adjacent to e_j, as shown in Fig. 2. We expressly note that this subdivision of an hexahedron into tetrahedron is just introduced for naming geometric entities used in the construction of the discrete constitutive relations. We do not intend to substitute the primal hexahedral grid with a primal tetrahedral grid.

We associate to each tetrahedron τ_h, a triplet of vectors forming a basis, Fig. 2. Precisely, we associate to τ_h the triplet $(\mathrm{l}_{1h}, \mathrm{l}_{2h}, \mathrm{l}_{3h})$ defined as

$$(\mathrm{l}_{1h}, \mathrm{l}_{2h}, \mathrm{l}_{3h}) = \left(\mathrm{e}_j, (\mathrm{g}_{f_i} - \mathrm{g}_{e_j}), (\mathrm{g}_{f_i} - \tilde{\mathrm{p}})\right).$$

We also construct, as defined in Appendix B: formula (16), the basis of face vectors $(\mathrm{s}_{1h}, \mathrm{s}_{2h}, \mathrm{s}_{3h})$ associated with $(\mathrm{l}_{1h}, \mathrm{l}_{2h}, \mathrm{l}_{3h})$.

Let now f_{i_1} and f_{i_2} be the pair of faces adjacent to edge e_j, as shown in Fig. 3. Let c_j be the edge vector of the edge c_j drawn from $g_{f_{i_2}}$ to $g_{f_{i_1}}$. Let C_j be face vector of the triangular face C_j, whose vertices are $\tilde{p}$ and the two extrema of edge e_j, oriented in such a way that $\mathrm{c}_j \cdot \mathrm{C}_j > 0$ holds, with $j = 1, \dots, L$. The following result is now proven, similarly to Properties 1 and 2.

Lemma 1 *It results in*

$$|v|\mathrm{I} = \sum_{1}^{L}{}_j\, \mathrm{c}_j \otimes \mathrm{C}_j \tag{3}$$

Proof. Let a, b be spatially uniform fields, so that $\mathrm{a} = \nabla u(\mathrm{r})$ with $u(\mathrm{r}) = \mathrm{a} \cdot \mathrm{r}$. Let ρ_i be the pyramid whose base is the f_i face and has vertex $\tilde{\mathrm{p}}$, with $i = 1 \dots F$. The lateral faces of these pyramids are the faces C_j with $j = 1 \dots L$. It results in

$$\begin{aligned}
&\int_{\rho_i} \mathrm{a} \cdot \mathrm{b}\, dv = \\
&\int_{\rho_i} \nabla(u(\mathrm{r}) - u(\mathrm{g}_{f_i})) \cdot \mathrm{b}\, dv = \\
&\int_{\rho_i} \nabla \cdot (u(\mathrm{r}) - u(\mathrm{g}_{f_i}))\mathrm{b}\, dv - \\
&\int_{\rho_i} (u(\mathrm{r}) - u(\mathrm{g}_{f_i}))\nabla \cdot \mathrm{b}\, dv = \\
&\int_{\partial\rho_i} (u(\mathrm{r}) - u(\mathrm{g}_{f_i}))\mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds = \\
&\int_{f_i} (u(\mathrm{r}) - u(\mathrm{g}_{f_i}))\mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds + \\
&+ \sum_{1}^{L}{}_j \int_{\partial\rho_i \cap C_j} \big((u(\mathrm{r}) - u(\mathrm{g}_{e_j})) + \\
&+ (u(\mathrm{g}_{e_j}) - u(\mathrm{g}_{f_i}))\big)\mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds.
\end{aligned}$$

Since it is straightforwardly

$$\int_{f_i} (u(\mathrm{r}) - u(\mathrm{g}_{f_i}))\mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds = 0, \quad i = 1, \dots, F$$

and, for each $j = 1 \dots L$, it is

$$\sum_1^F{}_i \int_{\partial\rho_i \cap C_j} (u(\mathrm{r}) - u(\mathrm{g}_{e_j}))\mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds = 0,$$
$$\sum_1^F{}_i \int_{\partial\rho_i \cap C_j} (u(\mathrm{g}_{e_j}) - u(\mathrm{g}_{f_i}))\mathrm{b} \cdot \mathrm{n}(\mathrm{r})\, ds = (\mathrm{a} \cdot \mathrm{c}_j)(\mathrm{b} \cdot \mathrm{C}_j),$$

it results in

$$|v|\mathrm{a} \cdot \mathrm{b} = \sum_1^F{}_i \int_{\rho_i} \mathrm{a} \cdot \mathrm{b}\, dv = \sum_1^L{}_j (\mathrm{a} \cdot \mathrm{c}_j)(\mathrm{b} \cdot \mathrm{C}_j).$$

Because a, b are arbitrary, (3) follows. ∎

Hereafter, using Lemma 1, a geometric property involving the basis vectors introduced for the tetrahedra τ_h with $h = 1, \dots, 2L$, is proven which will turn out to be crucial in sections 6, 7 for the construction of the discrete constitutive relations.

Property 3 *It results in*

$$2\mathrm{I}|v| = \sum_1^{2L}{}_h \mathrm{l}_{2h} \otimes \mathrm{s}_{2h}. \tag{4}$$

Proof. Let τ_{h_1} and τ_{h_2} be the pair of tetrahedra adjacent to the edge e_j, as shown in Fig. 3. It results in

$$\begin{aligned} \mathrm{s}_{2h_1} &= \mathrm{l}_{3h_1} \times \mathrm{l}_{1h_1} \\ &= (\mathrm{l}_{3h_1} - \mathrm{l}_{2h_1}) \times \mathrm{l}_{1h_1} + \mathrm{l}_{2h_1} \times \mathrm{l}_{1h_1} \\ &= 2\mathrm{C}_j - \mathrm{s}_{3h_1}. \end{aligned}$$

Similarly

$$\begin{aligned} \mathrm{s}_{2h_2} &= -\mathrm{l}_{3h_2} \times \mathrm{l}_{1h_2} \\ &= (-\mathrm{l}_{3h_2} + \mathrm{l}_{2h_2}) \times \mathrm{l}_{1h_2} - \mathrm{l}_{2h_2} \times \mathrm{l}_{1h_2} \\ &= -2\mathrm{C}_j - \mathrm{s}_{3h_2}. \end{aligned}$$

Thus

$$\mathrm{l}_{2h_1} \otimes \mathrm{s}_{2h_1} = 2\mathrm{l}_{2h_1} \otimes \mathrm{C}_j - \mathrm{l}_{2h_1} \otimes \mathrm{s}_{3h_1}, \tag{5}$$
$$\mathrm{l}_{2h_2} \otimes \mathrm{s}_{2h_2} = -2\mathrm{l}_{2h_2} \otimes \mathrm{C}_j - \mathrm{l}_{2h_2} \otimes \mathrm{s}_{3h_2}. \tag{6}$$

By summing (5), (6) over all edges e_j and by observing that

$$\mathrm{l}_{2h_1} - \mathrm{l}_{2h_2} = \mathrm{c}_j,$$

then

$$\sum_1^{2L}{}_h \mathrm{l}_{2h} \otimes \mathrm{s}_{2h} = 2\sum_1^{l}{}_j \mathrm{c}_j \otimes \mathrm{C}_j - \sum_1^{2L}{}_h \mathrm{l}_{2h} \otimes \mathrm{s}_{3h}. \tag{7}$$

Summing (22) of Lemma 2 of Appendix C: over all faces f_i, with $i = 1 \dots, F$, it results in

$$\sum_1^{2L}{}_h \mathrm{l}_{2h} \otimes \mathrm{s}_{3h} = 0.$$

Thus, from Lemma 1, the claim follows. ∎

6 Geometric construction of the discrete conductance constitutive relation

Let $\mathbf{u}$ be the array of the circulations u_j of the electric field E along the primal edges e_j, with $j = 1, \dots, L$. Similarly let $\mathbf{U}_h$ be the array of the circulations U_{1h}, U_{2h}, U_{3h} of the electric field E along the edges l_{1h}, l_{2h}, l_{3h}, for $h = 1, \dots, 2L$.

For an electric field E spatially *uniform* in v, by taking the dot product of (1) with E, it is

$$\mathrm{E} = \frac{1}{|v|} \sum_{j=1}^{L} u_j \tilde{\mathrm{f}}_j,$$

and thus the circulations of the array $\mathbf{U}_h$ can be reconstructed from the circulations of the array $\mathbf{u}$ by

$$\mathbf{U}_h = \mathbf{A}_h \mathbf{u}$$

where

$$\mathbf{A}_h = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \mathrm{l}_{2h} \cdot \frac{\tilde{\mathrm{f}}_1}{|v|} & \cdots & \mathrm{l}_{2h} \cdot \frac{\tilde{\mathrm{f}}_j}{|v|} & \cdots & \mathrm{l}_{2h} \cdot \frac{\tilde{\mathrm{f}}_L}{|v|} \\ \mathrm{l}_{3h} \cdot \frac{\tilde{\mathrm{f}}_1}{|v|} & \cdots & \mathrm{l}_{3h} \cdot \frac{\tilde{\mathrm{f}}_j}{|v|} & \cdots & \mathrm{l}_{3h} \cdot \frac{\tilde{\mathrm{f}}_L}{|v|} \end{bmatrix},$$

the first row having all zero elements but in the column corresponding to the primal edge e_j which is adjacent to the tetrahedron τ_h.

Let $\mathbf{N}_h$ be the matrices which transform the circulations of a uniform vector E along the three edges of edge vectors l_{1h}, l_{2h}, l_{3h} into the fluxes of $\mathrm{J} = \sigma\mathrm{E}$ through the three faces of face vectors s_{1h}, s_{2h}, s_{3h}. These matrices are defined as in (19) of Appendix B: by assuming $\mathrm{T} = \sigma$ and $\mathrm{s}_1 = \mathrm{s}_{1h}$, $\mathrm{s}_2 = \mathrm{s}_{2h}$ and $\mathrm{s}_3 = \mathrm{s}_{3h}$, with $h = 1 \dots 2L$. Thus they also are symmetric, positive definite. Now, using Properties 1, 3, we can prove the following main result.

Property 4 *Matrix*

$$\mathbf{N} = \frac{1}{6}\sum_{1}^{2L}{}_h \mathbf{A}_h^T \mathbf{N}_h \mathbf{A}_h \tag{8}$$

satisfies both the consistency and stability properties of a conductance constitutive relation for the DGA.

Proof. For an electric field E, spatially uniform in v, it is

$$\mathbf{A}_h\mathbf{u} = \begin{bmatrix} \mathrm{l}_{1h}\cdot \mathrm{E} \\ \mathrm{l}_{2h}\cdot \mathrm{E} \\ \mathrm{l}_{3h}\cdot \mathrm{E} \end{bmatrix}$$

and

$$\mathbf{N}_h\mathbf{A}_h\mathbf{u} = \begin{bmatrix} \mathrm{s}_{1h}\cdot \mathrm{J} \\ \mathrm{s}_{2h}\cdot \mathrm{J} \\ \mathrm{s}_{3h}\cdot \mathrm{J} \end{bmatrix},$$

being $\mathrm{J} = \sigma\mathrm{E}$. Then

$$\mathbf{Nu} = \frac{1}{6}\sum_{1}^{2L}{}_h \mathbf{A}_h^T \begin{bmatrix} \mathrm{s}_{1h}\cdot \mathrm{J} \\ \mathrm{s}_{2h}\cdot \mathrm{J} \\ \mathrm{s}_{3h}\cdot \mathrm{J} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{6}\frac{\tilde{\mathrm{f}}_1}{|v|}\cdot\left(2|v|\mathrm{I} + \sum_{1}^{2L}{}_h \mathrm{l}_{2h}\otimes \mathrm{s}_{2h} + \sum_{1}^{2L}{}_h \mathrm{l}_{3h}\otimes \mathrm{s}_{3h}\right)\mathrm{J} \\ \vdots \\ \frac{1}{6}\frac{\tilde{\mathrm{f}}_L}{|v|}\cdot\left(2|v|\mathrm{I} + \sum_{1}^{2L}{}_h \mathrm{l}_{2h}\otimes \mathrm{s}_{2h} + \sum_{1}^{2L}{}_h \mathrm{l}_{3h}\otimes \mathrm{s}_{3h}\right)\mathrm{J} \end{bmatrix}.$$

Thus from Property 3, and since from Property 1 it is

$$\sum_{1}^{2L}{}_h \mathrm{l}_{3h}\otimes \mathrm{s}_{3h} = 2\sum_{1}^{F}{}_i \tilde{\mathrm{e}}_i \otimes \mathrm{f}_i = 2|v|\mathrm{I},$$

it results in

$$\mathbf{Nu} = \begin{bmatrix} \tilde{\mathrm{f}}_1\cdot \mathrm{J} \\ \vdots \\ \tilde{\mathrm{f}}_L\cdot \mathrm{J} \end{bmatrix}$$

and $\mathbf{N}$ satisfies the consistency property ii).
Since $\mathbf{N}_h^T = \mathbf{N}_h$, for each $h = 1\ldots 2L$, it results in

$$\mathbf{N}^T = \sum_{1}^{2L}{}_h \mathbf{A}_h^T \mathbf{N}_h^T \mathbf{A}_h = \sum_{1}^{2L}{}_h \mathbf{A}_h^T \mathbf{N}_h \mathbf{A}_h = \mathbf{N}$$

and $\mathbf{N}$ is *symmetric*.

Since $\mathbf{U}_h^T\mathbf{N}_h\mathbf{U}_h \geq 0$, for each $h = 1\ldots 2L$, it results in

$$\begin{aligned} \frac{1}{2}\mathbf{u}^T\mathbf{Nu} =& \frac{1}{12}\sum_{1}^{2L}{}_h \mathbf{u}^T\mathbf{A}_h^T\mathbf{N}_h\mathbf{A}_h\mathbf{u} \\ =& \frac{1}{12}\sum_{1}^{2L}{}_h \mathbf{U}_h^T\mathbf{N}_h\mathbf{U}_h \\ \geq& 0. \end{aligned}$$

Also $\mathbf{u}^T\mathbf{Nu} = 0$ implies $\mathbf{U}_h^T\mathbf{N}_h\mathbf{U}_h = 0$ and thus $\mathbf{U}_h = \mathbf{A}_h\mathbf{u} = \mathbf{0}$ for all $h = 1\ldots 2L$. Then $U_{1h} = 0$ for all $h = 1\ldots 2L$, or equivalently $u_j = 0$ for all $j = 1\ldots L$ that is $\mathbf{u} = \mathbf{0}$. Thus $\mathbf{N}$ is *positive definite*. Thus $\mathbf{N}$ also satisfies the stability property ii). ■

7 Geometric construction of the discrete reluctance constitutive relation

We proceed in a way similar to the previous section 6. Let $\boldsymbol{\phi}$ be the array of the fluxes ϕ_i of the magnetic induction field B though the primal faces f_i, with $i = 1,\ldots,F$. Similarly let $\boldsymbol{\Phi}_h$ be the array of the fluxes Φ_{1h}, Φ_{2h}, Φ_{3h} of the magnetic induction field B through the faces s_{1h}, s_{2h}, s_{3h} corresponding to the tetrahedron τ_h, for $h = 1,\ldots,2L$.
For a magnetic induction field B spatially *uniform* in v, by taking the dot product of (2) with B, it is

$$\mathrm{B} = \frac{1}{|v|}\sum_{i=1}^{F}\phi_i\tilde{\mathrm{e}}_i,$$

and thus fluxes of the array $\boldsymbol{\Phi}_h$ can be reconstructed from the fluxes of the array $\boldsymbol{\phi}$ by

$$\boldsymbol{\Phi}_h = \mathbf{B}_h\boldsymbol{\phi}$$

where

$$\mathbf{B}_h = \begin{bmatrix} \mathrm{s}_{2h}\cdot\frac{\tilde{\mathrm{e}}_1}{|v|} & \cdots & \mathrm{s}_{2h}\cdot\frac{\tilde{\mathrm{e}}_i}{|v|} & \cdots & \mathrm{s}_{2h}\cdot\frac{\tilde{\mathrm{e}}_L}{|v|} \\ \mathrm{s}_{3h}\cdot\frac{\tilde{\mathrm{e}}_1}{|v|} & \cdots & \mathrm{s}_{3h}\cdot\frac{\tilde{\mathrm{e}}_i}{|v|} & \cdots & \mathrm{s}_{3h}\cdot\frac{\tilde{\mathrm{e}}_L}{|v|} \\ 0 & \cdots & \xi_i & \cdots & 0 \end{bmatrix},$$

the third row having all zero elements but in the column corresponding to the primal face f_i which is adjacent to the tetrahedron τ_h and being $\xi_i = \mathrm{s}_{3h}\cdot\mathrm{f}_i/|\mathrm{f}_i|^2$.

Let $\mathbf{M}_h$ be the matrices which transform the fluxes of a uniform vector B through the three faces of face vectors s_{1h}, s_{2h}, s_{3h} into the circulations of $\mathrm{H} = \nu\mathrm{B}$ along the three edges of edge vectors l_{1h}, l_{2h}, l_{3h}. These matrices are defined by (20) of Appendix B: by assuming $\mathrm{T} = \nu$, and $\mathrm{l}_1 = \mathrm{l}_{1h}, \mathrm{l}_2 = \mathrm{l}_{2h}$ and $\mathrm{l}_3 = \mathrm{l}_{3h}$, with $h = 1 \ldots 2L$. Thus they are also symmetric, positive definite. Now, using Properties 2, 3, we can prove the following main result.

Property 5 *Matrix*

$$\mathbf{M} = \frac{1}{6}\sum_{1}^{2L}{}_h \mathbf{B}_h^T \mathbf{M}_h \mathbf{B}_h \tag{9}$$

satisfies both the consistency and stability properties of a reluctance constitutive relation for the DGA.

Proof. For a magnetic induction field B, spatially uniform in v, it is

$$\mathbf{B}_h\boldsymbol{\phi} = \begin{bmatrix} \mathrm{s}_{1h}\cdot\mathrm{B} \\ \mathrm{s}_{2h}\cdot\mathrm{B} \\ \mathrm{s}_{3h}\cdot\mathrm{B} \end{bmatrix}$$

and

$$\mathbf{M}_h\mathbf{B}_h\boldsymbol{\phi} = \begin{bmatrix} \mathrm{l}_{1h}\cdot\mathrm{H} \\ \mathrm{l}_{2h}\cdot\mathrm{H} \\ \mathrm{l}_{3h}\cdot\mathrm{H} \end{bmatrix},$$

being $\mathrm{H} = \nu\mathrm{B}$. Then

$$\mathbf{M}\boldsymbol{\phi} = \frac{1}{6}\sum_{1}^{2L}{}_h \mathbf{B}_h^T \begin{bmatrix} \mathrm{l}_{1h}\cdot\mathrm{H} \\ \mathrm{l}_{2h}\cdot\mathrm{H} \\ \mathrm{l}_{3h}\cdot\mathrm{H} \end{bmatrix} =$$

$$\begin{bmatrix} \dfrac{1}{6}\dfrac{\tilde{\mathrm{e}}_1}{|v|}\cdot\left(\sum_{1}^{2L}{}_h \mathrm{l}_{1h}\otimes\mathrm{s}_{1h} + \sum_{1}^{2L}{}_h \mathrm{l}_{2h}\otimes\mathrm{s}_{2h} + 2|v|\mathrm{I}\right)\mathrm{H} \\ \vdots \\ \dfrac{1}{6}\dfrac{\tilde{\mathrm{e}}_F}{|v|}\cdot\left(\sum_{1}^{2L}{}_h \mathrm{l}_{1h}\otimes\mathrm{s}_{1h} + \sum_{1}^{2L}{}_h \mathrm{l}_{2h}\otimes\mathrm{s}_{2h} + 2|v|\mathrm{I}\right)\mathrm{H} \end{bmatrix}.$$

Thus from Property 3 , and since from Property 2 it is

$$\sum_{1}^{2L}{}_h \mathrm{l}_{1h}\otimes\mathrm{s}_{1h} = 2\sum_{1}^{L}{}_j \mathrm{e}_j\otimes\tilde{\mathrm{f}}_j = 2|v|\mathrm{I},$$

It results in

$$\mathbf{M}\boldsymbol{\phi} = \begin{bmatrix} \tilde{\mathrm{e}}_1\cdot\mathrm{H} \\ \vdots \\ \tilde{\mathrm{e}}_F\cdot\mathrm{H} \end{bmatrix}$$

and $\mathbf{M}$ satisfies the consistency property i).

Since $\mathbf{M}_h^T = \mathbf{M}_h$, for each $h = 1 \ldots 2L$, it results in

$$\mathbf{M}^T = \sum_{1}^{2L}{}_h \mathbf{B}_h^T\mathbf{M}_h^T\mathbf{B}_h = \sum_{1}^{2L}{}_h \mathbf{B}_h^T\mathbf{M}_h\mathbf{B}_h = \mathbf{M}$$

and $\mathbf{M}$ is *symmetric*.

Since $\boldsymbol{\Phi}_h^T\mathbf{M}_h\boldsymbol{\Phi}_h \geq 0$, for each $h = 1 \ldots 2L$, it results in

$$\begin{aligned}\frac{1}{2}\boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi} &= \frac{1}{12}\sum_{1}^{2L}{}_h \boldsymbol{\Phi}_h^T\mathbf{B}_h^T\mathbf{M}_h\mathbf{B}_h\boldsymbol{\Phi}_h \\ &= \frac{1}{12}\sum_{1}^{2L}{}_h \boldsymbol{\Phi}_h^T\mathbf{M}_h\boldsymbol{\Phi}_h \\ &\geq 0.\end{aligned}$$

Also $\boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi} = 0$ implies $\boldsymbol{\Phi}_h^T\mathbf{M}_h\boldsymbol{\Phi}_h = 0$ and thus $\boldsymbol{\Phi}_h = \mathbf{B}_h\boldsymbol{\phi} = \mathbf{0}$ for all $h = 1 \ldots 2L$. Then $\Phi_{1h} = 0$ for all $h = 1 \ldots 2L$, or equivalently $\phi_i = 0$ for all $i = 1 \ldots F$ that is $\boldsymbol{\phi} = \mathbf{0}$. Thus $\mathbf{M}$ is *positive definite*. Thus $\mathbf{M}$ also satisfies the stability property ii). ∎

8 Numerical results

As a numerical test, we consider a geometry consisting of a circular coil placed above an aluminum plate ($\sigma = 4\cdot 10^7\,S/m$). The domain of interest D of the eddy-current problem consists of a cylinder of diameter of 60 mm and height 44.5 mm. It contains a circular current driven coil of 18 mm of outer diameter, 12 mm of inner diameter and 10 mm height. The coil is placed above an aluminum plate, denoted with D_c, 4 mm thick and with a radius of 30 mm. The coil and the plate are surrounded by an air region. In the coil we force a sinusoidal current $I_s = \sin(\omega t)$ with a frequency of f =5 kHz.

We introduced in D a number of different primal grids made of a variable number of hexahedra up to 42 000 elements.

We assemble the final system of algebraic equations using the conductance and reluctance constitutive matrices $\mathbf{N}$ and $\mathbf{M}$ here introduced.

Figures 4 and 5 show the convergence rate of the magnetic induction and of the current density for four meshes, one finer with respect to the other.

We calculate the error in energy norm defined as

$$\varepsilon_B = \sqrt{\frac{\int_D \nu |\mathrm{B} - \mathrm{B}_{REF}|^2 \, dv}{\int_D \nu |\mathrm{B}_{REF}|^2 \, dv}},$$

where B_{REF} is the reference induction field computed by means of a 2D axisymmetric FE accurate solution with 200 000 triangular elements. As quality factor for the mesh we choose the mean length of the edges. In a similar way, we introduce the quantity

$$\varepsilon_J = \sqrt{\frac{\int_{D_c} \sigma |\mathrm{J} - \mathrm{J}_{REF}|^2 \, dv}{\int_{D_c} \sigma |\mathrm{J}_{REF}|^2 \, dv}},$$

where J_{REF} is the reference current density field computed by means of the 2D axisymmetric FE solution. For comparison, we repeated the computations using tetrahedra primal grids where, as constitutive matrices, those described in [Codecasa, Specogna, and Trevisan (2007)], [Codecasa, Minerva, and Politi (2004)], [Specogna and Trevisan (2005)] for the case of tetrahedra can be equivalently used. We observe that the solution obtained over hexahedra grids is more accurate than the solution computed over tetrahedra grids, for each value of the mean length of the primal edges.

A typical CPU time (on a Pentium IV 2GHz) needed to iteratively solve the linear system with a *stop criterion* on the residual 2-norm less then 10^{-10}, is about 88 sec. The assembly process of the overall linear system requires less then 9 sec.

9 Conclusions

We proposed an approach to construct discrete constitutive matrices for solving eddy-current problems over hexahedral primal grids. The motivation of the paper stems from the fact that the so called “mass matrices" of the FEM for hexahedral primal grids, computed using mixed elements, do *not* satisfy the consistency property of DGA. Instead the novel constitutive matrices we propose, were shown to ensure both the consistency and the stability properties of DGA. Numerical experiments demonstrated that the novel constitutive matrices lead to accurate approximations of the solution of a reference eddy-current problem. Moreover the solution over hexahedra grids seems to be more accurate than the solution over tetrahedra grids for the same value of the mean length of primal edges. Finally the proposed matrices can be obtained with a reduced computational effort and without a numerical volume integration like for the mass matrices in finite elements.

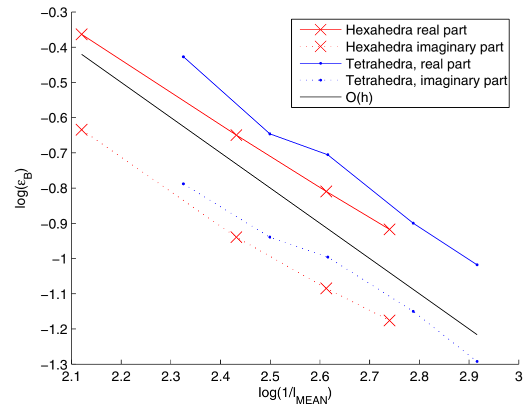

Figure 4: The real and imaginary parts of the relative error ε_B associated with magnetic induction in D is shown, using different hexahedra primal grids and the novel constitutive matrices **M**, **N**. For comparison, the same error is computed using primal grids of tetrahedra.

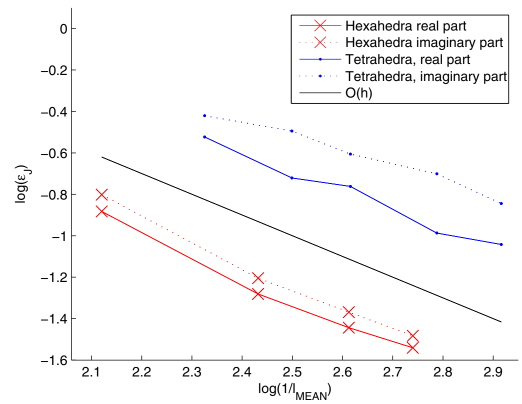

Figure 5: The real and imaginary parts of the relative error ε_J associated with the current density in D_c is shown using with different hexahedra primal grids the novel constitutive matrices **M**, **N**. For comparison, the same error is computed also using tetrahedral primal grids.

Appendix A: Counter-example

In this section we propose a counter-example, in order to show the inconsistency of the mass matrices computed for a simple hexahedron v for *any* choice of the dual grid.

Let the coordinates of the nodes of the hexahedron v in Fig. 6A be $p_1 = (0,0,0)$, $p_2 = (2,0,0)$, $p_3 = (0,1,0)$, $p_4 = (1,1,0)$, $p_5 = (0,0,1)$, $p_6 = (2,0,1)$, $p_7 = (0,1,1)$, $p_8 = (1,1,1)$. We denote with p_{f_i}, with $i = 1,\ldots,F$ the intersection between a primal face f_i and the corresponding dual edge $\tilde{e}_i$. Let p_{e_i} be the intersection between a primal edge e_i with the corresponding dual face $\tilde{f}_i$, with $i = 1,\ldots,L$. Let $\tilde{p}$ be the dual node in v. We stress that the points p_{e_i}, p_{f_i}, and $\tilde{p}$ do not coincide, in general, with the barycenter of edge e_i, face f_i and hexahedron v respectively. In addition, a dual face $\tilde{f}_i$ is not required to be planar, in general. For example $\tilde{f}_1$ in Fig. 6B is not planar. Nevertheless its area vector can always be written as

$$\tilde{\mathrm{f}}_1 = \frac{1}{2}(\mathrm{p}_{f_5} - \mathrm{p}_{f_1}) \times (\mathrm{p}_{e_1} - \mathrm{p}_{\tilde{p}}), \tag{10}$$

where p_{f_i} denotes the position vector corresponding to the point p_{f_i} with $i = 1,\ldots,F$.

We recall that the entries of the mass matrices are

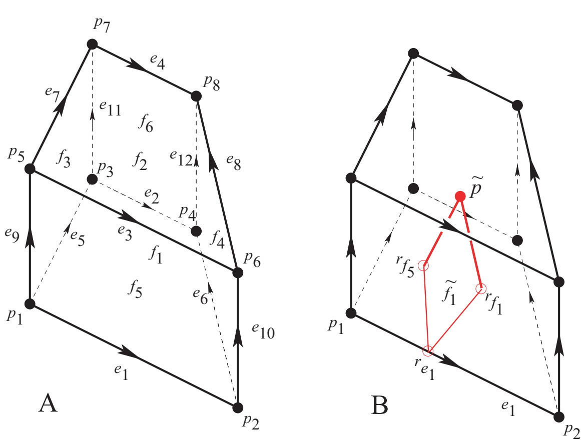

Figure 6: An hexahedron is shown with specified orientations; for simplicity we label the faces f_k, with $k = 1,\ldots,F$ in such a way that opposite faces have successive subscripts.

defined as

$$(\mathbf{M}^f)_{ij} = \int_v \mathrm{w}_i^f \cdot \mathrm{w}_j^f \, dv, \quad (\mathbf{M}^e)_{ij} = \int_v \mathrm{w}_i^e \cdot \mathrm{w}_j^e \, dv,$$

where $(\mathbf{M}^f)_{ij}$ is the generic entry of the $F \times F$ face mass matrix constructed from the w_i^f face vector basis functions described in [Dular, Hody, Nicolet, Genon, and Legros (1994)]. We note that a unitary material parameter has been assumed. Similarly $(\mathbf{M}^e)_{ij}$ is the generic entry of the $L \times L$ edge mass matrix constructed from the w_i^e edge vector basis functions described in [Dular, Hody, Nicolet, Genon, and Legros (1994)]. We note that a unitary material parameter has been assumed.

A necessary and sufficient condition for the consistency of $\mathbf{M}^f$ according to the definition reported in [Bossavit and Kettunen (2000)], [Bossavit (2002)] and [Codecasa, Minerva, and Politi (2004)] is

$$\mathbf{M}^f \mathbf{f} = \tilde{\mathbf{e}}, \tag{11}$$

where $\mathbf{f}$ and $\tilde{\mathbf{e}}$ are F×3 arrays, whose i-th row represent the three components of the face vector f_i and of the dual edge vector $\tilde{\mathrm{e}}_i$ respectively, with $i = 1,\ldots,F$.

Similarly a necessary and sufficient condition for the consistency of $\mathbf{M}^e$, according to the definition reported in [Bossavit and Kettunen (2000)], [Bossavit (2002)] and [Codecasa, Minerva, and Politi (2004)], is

$$\mathbf{M}^e \mathbf{e} = \tilde{\mathbf{f}}, \tag{12}$$

where $\mathbf{e}$, $\tilde{\mathbf{f}}$ are L×3 arrays, whose i-th row represent the three components of the edge vector e_i and of the dual face vector $\tilde{\mathrm{f}}_i$ respectively, with $i = 1,\ldots,L$.

Hereafter we will prove that conditions (12) and (11) are *not* satisfied for any choice of the dual grid.

Inconsistency of $\mathbf{M}^f$

By direct computation, the right hand side of (11) yields

$$\begin{aligned} \mathrm{row}_i(\mathbf{M}^f\mathbf{f}) &= (0,0,3\log 2/4), && \text{with } i = 1,2 \\ \mathrm{row}_i(\mathbf{M}^f\mathbf{f}) &= (0,3/4,0), && \text{with } i = 3,4 \\ \mathrm{row}_i(\mathbf{M}^f\mathbf{f}) &= (-1/4,1/2,0), && \text{with } i = 5,6. \end{aligned}$$

where with row_i we denote the i-th row of an array. Let us consider the edge vectors $\tilde{\mathrm{e}}_1$, $\tilde{\mathrm{e}}_2$ associated with the dual edges $\tilde{e}_1$, $\tilde{e}_2$ respectively; in order to guarantee that $\tilde{\mathrm{e}}_1$, $\tilde{\mathrm{e}}_2$ are parallel to the vectors $\mathrm{row}_i(\mathbf{M}^f\mathbf{f}) = (0,0,3\log 2/4)$, with $i = 1, 2$, it is necessary that $\tilde{p}$, p_{f_1}, p_{f_2} are on a straight line. Thus assuming for $\tilde{p} = (x_2,y_2,x_2)$ it results in $p_{f_1} = (x_2,y_2,0)$, $p_{f_1} = (x_2,y_2,1)$. Then it is

$$\frac{3}{2}\log 2 = (\mathrm{row}_1(\mathbf{M}^f\mathbf{f}) + \mathrm{row}_2(\mathbf{M}^f\mathbf{f}))\cdot \mathrm{a}_z = (y_2-0)+(1-y_2) = 1$$

which is clearly impossible.

Inconsistency of $\mathbf{M}^e$

By direct computation, the right hand side of (12) yields

$$\begin{array}{ll} \mathrm{row}_i(\mathbf{M}^e\mathbf{e}) = (1/4,1/8,0), & \text{with } i = 1,\dots,4 \\ \mathrm{row}_i(\mathbf{M}^e\mathbf{e}) = (0,3/8,0), & \text{with } i = 5,\dots,8 \\ \mathrm{row}_i(\mathbf{M}^e\mathbf{e}) = (0,0,5/12), & \text{with } i = 9,\ 10 \\ \mathrm{row}_i(\mathbf{M}^e\mathbf{e}) = (0,0,1/3), & \text{with } i = 11,\ 12. \end{array}$$

Let $\tilde{\mathrm{f}}_j$ be the face vector of the dual face $\tilde{f}_j$, with $j = 1,\dots,L$ computed as in (10). It is straightfoward to see that in order to guarantee that $\tilde{\mathrm{f}}_j$ are parallel to the vectors $\mathrm{row}_i(\mathbf{M}^e\mathbf{e})$ with $i = 1,\dots,12$, it is necessary that three planes π_1, π_2, π_3 exist, having normals $(2/\sqrt{5},1/\sqrt{5},0)$, $(0,1,0)$ and $(0,0,1)$ respectively, such that p_{f_1}, p_{f_2} lay on the intersection of π_1, π_2, p_{f_3}, p_{f_4} lay on the intersection of π_2, π_3 and p_{f_5}, p_{f_6} lay on the intersection of π_1, π_3. Similarly it is necessary that three planes ρ_1, ρ_2, ρ_3 exist, having normals $(2/\sqrt{5},1/\sqrt{5},0)$, $(0,1,0)$ and $(0,0,1)$ respectively, such that p_{e_1}, p_{e_2}, p_{e_3}, p_{e_4} lay on ρ_1, p_{e_5}, p_{e_6}, p_{e_7}, p_{e_8} lay on ρ_2, p_{e_9}, $p_{e_{10}}$, $p_{e_{11}}$, $p_{e_{12}}$ lay on ρ_3. We note that π_1, π_2, π_3 are parallel respectively to ρ_1, ρ_2, ρ_3, but it is not necessary that they coincide.

Thus, assuming $p_{f_3} = (0,y_1,z_1)$, $p_{f_6} = (x_1,1,z_1)$, it results in $p_{f_1} = ((1-y_1)/2+x_1,y_1,0)$, $p_{f_2} = ((1-y_1)/2+x_1,y_1,1)$, $p_{f_4} = (2-y_1,y_1,z_1)$, $p_{f_5} = (x_1+1/2,0,z_1)$. Besides, assuming $\tilde{p} = (x_2,y_2,z_2)$, it results in $p_{e_1} = (y_2/2+x_2,0,0)$, $p_{e_2} = (y_2/2+x_2,1,0)$, $p_{e_3} = (y_2/2+x_2,0,1)$, $p_{e_4} = (y_2/2+x_2,1,1)$, $p_{e_5} = (0,y_2,0)$, $p_{e_6} = (2-y_2,y_2,0)$, $p_{e_7} = (0,y_2,1)$, $p_{e_8} = (2-y_2,y_2,1)$, $p_{e_9} = (0,0,z_2)$, $p_{e_{10}} = (2,0,z_2)$, $p_{e_{11}} = (0,1,z_2)$, $p_{e_{12}} = (1,1,z_2)$.

Then it results in

$$\frac{1}{2} = (\mathrm{row}_1(\mathbf{M}^e\mathbf{e}) + \mathrm{row}_3(\mathbf{M}^e\mathbf{e}))\cdot \mathrm{a}_x = \frac{1}{2}(y_1+y_2) \tag{13}$$

and in

$$\frac{5}{6} = (\mathrm{row}_9(\mathbf{M}^e\mathbf{e}) + \mathrm{row}_{10}(\mathbf{M}^e\mathbf{e}))\cdot \mathrm{a}_z = \frac{1}{2}(2-y_1)(y_1+y_2)+\frac{1}{2}y_1^2, \tag{14}$$

being $\mathrm{a}_x = (1,0,0)$, $\mathrm{a}_y = (0,1,0)$, $\mathrm{a}_z = (0,0,1)$. By using Eq. (13) in Eq. (14) it follows

$$y_1^2 - y_1 + \frac{1}{3} = 0$$

which clearly has no real solution.

Appendix B: Reciprocal basis

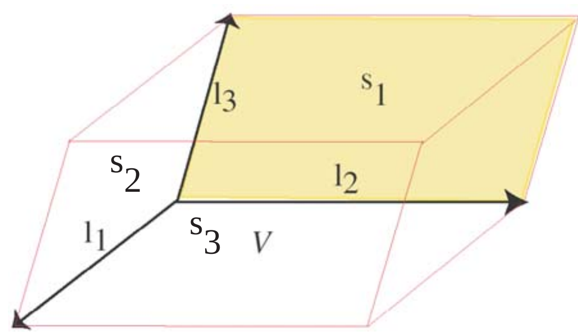

Figure 7: Parallelepiped V.

Let l_1, l_2, l_3 be a triplet of vectors which are not coplanar. They can be interpreted as the edge vectors of a triplet of edges l_1, l_2, l_3 of a parallelepiped V as in Fig. 7. We recall that he *reciprocal* basis l_1^r, l_2^r, l_3^r associated with the basis l_1, l_2, l_3 is uniquely defined by

$$\sum_{1}^{3}{}_i\, \mathrm{l}_i \otimes \mathrm{l}_i^r = \mathrm{I}$$

and it is such that

$$\mathrm{l}_i^r = \frac{\mathrm{l}_{i-1}\times \mathrm{l}_{i+1}}{\mathrm{l}_{i-1}\times \mathrm{l}_{i+1}\cdot \mathrm{l}_i}, \tag{15}$$

in which $i = 1,\ldots,3$ and index operations are modulo 3. From (15) an arbitrary vector a can be expressed as

$$a = \sum_{i=1}^{3} (a \cdot l_i^r)\, l_i,$$

and an arbitrary vector b can be expressed as

$$b = \sum_{i=1}^{3} (b \cdot l_i)\, l_i^r.$$

Now let s_i be the face of the parallelepiped V in the plane of l_{i-1} and l_{i+1}, and oriented in such a way that $s_i \cdot l_i = |V|$, s_i being the face vector of s_i, $|V|$ being the volume of V and index operations being modulo 3. Then it is

$$s_i = l_i^r |V|, \qquad i = 1,\ldots,3, \tag{16}$$

and thus it results in

$$a = \frac{1}{|V|} \sum_{i=1}^{3} (a \cdot s_i)\, l_i, \tag{17}$$

$$b = \frac{1}{|V|} \sum_{j=1}^{3} (b \cdot l_j)\, s_j, \tag{18}$$

Let now T be a tensor relating vectors a, b by $a = Tb$. Then from (18) it results in

$$a \cdot s_i = \sum_{j=1}^{3} \frac{s_i \cdot T s_j}{|V|}, (b \cdot l_j) \qquad i = 1,\ldots,3. \tag{19}$$

Thus the fluxes of vector a through the faces s_i, with $i = 1,\ldots,3$, are expressed by a linear combination of the circulations of vector b along the edges l_i, with $i = 1,\ldots,3$. This mapping is represented by a 3×3 matrix whose entries are $s_i \cdot T s_j / |V|$ with $i, j = 1,\ldots,3$. We note that if the tensor T is symmetric positive definite, also such matrix is symmetric, positive definite.

Similarly from (17) it results in

$$a \cdot l_i = \sum_{j=1}^{3} \frac{l_i \cdot T l_j}{|V|}, (b \cdot s_j) \qquad i = 1,\ldots,3. \tag{20}$$

Thus the circulations of vector a along the edges l_i, with $i = 1,\ldots,3$, are expressed by a linear combination of the fluxes of vector b through the faces s_i, with $i = 1,\ldots,3$. This mapping is represented by a 3×3 matrix whose entries are $l_i \cdot T l_j / |V|$ with $i, j = 1,\ldots,3$. We note that if the tensor T is symmetric positive definite, also such matrix is symmetric, positive definite.

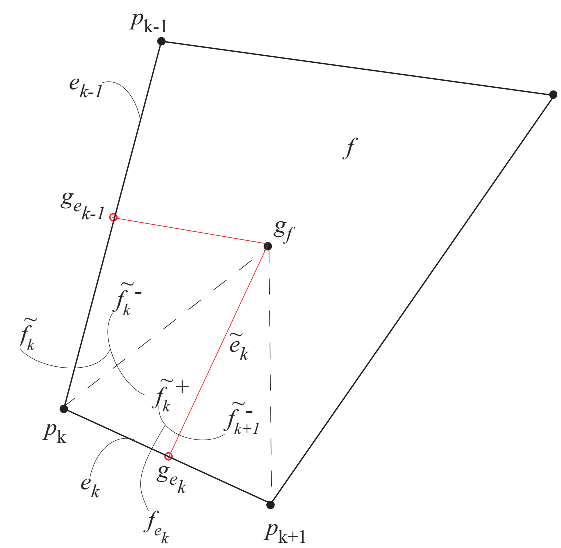

Figure 8: Geometric elements of the f quadrangle.

Appendix C: Geometric relations for quadrangles

Let f be a generic quadrangle. Let p_k be the nodes of f, having position vectors p_k, with $k = 1,\ldots,4$. Let e_k be the edges of f, with $k = 1,\ldots,4$. Nodes are assumed to be numbered counterclockwise. Edges e_k are assumed to be oriented from node p_k to node p_{k+1}; operations on indexes are modulo 4.

The dual grid of f has faces $\tilde{f}_k$ and edges $\tilde{e}_k$ with $k = 1,\ldots,4$. Let the dual node of f be the barycenter of f denoted as g_f and let the dual edge $\tilde{e}_k$ be a segment drawn from g_f to the barycenter g_{e_k} of e_k, with $k = 1,\ldots,4$.

The dual face $\tilde{f}_k$ is the union of triangle $\tilde{f}_k^-$ (having vertices g_f, p_k, $g_{e_{k-1}}$) and triangle $\tilde{f}_k^+$ (having vertices g_f, p_k, g_{e_k}). The union of faces $\tilde{f}_k^+$ and $\tilde{f}_{k+1}^-$ is referred to as f_{e_k}. The following relations are proven

Lemma 2 *It results in*

$$\sum_{1}^{4}{}_k |\tilde{f}_k|(\mathrm{p}_k - \mathrm{g}_f) = 0 \tag{21}$$

$$\sum_{1}^{4}{}_k |f_{e_k}|(\mathrm{g}_{e_k} - \mathrm{g}_f) = 0 \tag{22}$$

$$\sum_{1}^{4}{}_k \int_{\tilde{f}_k} (\mathrm{r} - \mathrm{p}_k)\, ds = 0 \tag{23}$$

Proof. It is

$$\sum_{1}^{4}{}_k |\tilde{f}_k|(\mathrm{p}_k - \mathrm{g}_f) =$$

$$\sum_{1}^{4}{}_k |\tilde{f}_k^+|(\mathrm{p}_k - \mathrm{g}_f) + |\tilde{f}_{k+1}^-|(\mathrm{p}_{k+1} - \mathrm{g}_f) =$$

$$\sum_{1}^{4}{}_k \frac{1}{2}(|\tilde{f}_k^+| + |\tilde{f}_{k+1}^-|)((\mathrm{p}_k - \mathrm{g}_f) + (\mathrm{p}_{k+1} - \mathrm{g}_f)) +$$

$$+ \sum_{1}^{4}{}_k \frac{1}{2}(|\tilde{f}_k^+| - |\tilde{f}_{k+1}^-|)(\mathrm{p}_k - \mathrm{p}_{k+1}).$$

Thus since $|f_k^+| = |f_{k+1}^-|$ holds and since

$$\frac{1}{2}(|\tilde{f}_k^+| + |\tilde{f}_{k+1}^-|)((\mathrm{p}_k - \mathrm{g}_f) + (\mathrm{p}_{k+1} - \mathrm{g}_f)) =$$

$$\frac{3}{2} \int_{f_k^+ \cup f_{k+1}^-} (\mathrm{r} - \mathrm{g}_f)\, ds,$$

it results in

$$\sum_{1}^{4}{}_k |\tilde{f}_k|(\mathrm{p}_k - \mathrm{g}_f) = \frac{3}{2} \int_f (\mathrm{r} - \mathrm{g}_f)\, ds = 0.$$

and (21) follows. Besides, since

$$\sum_{1}^{4}{}_k |\tilde{f}_k|(\mathrm{p}_k - \mathrm{g}_f) =$$

$$\sum_{1}^{4}{}_k |f_k^+|(\mathrm{p}_k - \mathrm{g}_f) + \sum_{1}^{4}{}_k |f_{k+1}^-|(\mathrm{p}_{k+1} - \mathrm{g}_f) =$$

$$\sum_{1}^{4}{}_k \frac{|f_{e_k}|}{2}(\mathrm{p}_k - \mathrm{g}_f) + \frac{|f_{e_k}|}{2}(\mathrm{p}_{k+1} - \mathrm{g}_f) =$$

$$\sum_{1}^{4}{}_k |f_{e_k}|(\mathrm{g}_{e_k} - \mathrm{g}_f),$$

from (21) also (22) follows. Lastly, since it is

$$\sum_{1}^{4}{}_k \int_{\tilde{f}_k} (\mathrm{r} - \mathrm{p}_k)\, ds =$$

$$\int_f (\mathrm{r} - \mathrm{g}_f)\, ds - \sum_{1}^{4}{}_k \int_{\tilde{f}_k} (\mathrm{p}_k - \mathrm{g}_f)\, ds =$$

$$- \sum_{1}^{4}{}_k |\tilde{f}_k|(\mathrm{p}_k - \mathrm{g}_f),$$

from (21) also (23) follows. ■

We note that clearly Lemma 2 holds also for arbitrary numerations and orientations of the edges and nodes of f.

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