

A New Shooting Method for Solving Boundary Layer Equations in Fluid Mechanics

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Abstract: In this paper, we propose a new method to tackle of two famous boundary layer equations in fluid mechanics, namely, the Falkner-Skan and the Blasius equations. We can employ this method to find unknown initial conditions. The pivotal point is based on the erection of a one-step Lie group element $\mathbf{G}(T)$ and the formation of a generalized mid-point Lie group element $\mathbf{G}(r)$. Then, by imposing $\mathbf{G}(T) = \mathbf{G}(r)$ we can seek the missing initial conditions through a minimum discrepancy from the target in terms of a weighting factor $r \in (0, 1)$. Numerical examples are worked out to persuade that this novel approach has good efficiency and accuracy with a fast convergence speed by searching r with the minimum norm to fit two targets.

Keyword: One-step group preserving scheme, Falkner-Skan equation, Blasius equation, Boundary value problem, Lie-group shooting method, Estimation of missing initial condition.

1 Introduction

The Falkner-Skan equation as coined by Falkner and Skan (1931) arose in the study of two-dimensional incompressible laminar

boundary layers of fluids exhibiting similarity. It is a third-order nonlinear two-point boundary value problem for which no closed-form solutions are available. The first numerical treatment of this problem was presented by Hartree (1937). The mathematical treatments of this problem by Weyl (1942), Coppel (1960), Rosenhead (1963) and Padé (2003) were principally concentrated on attaining results of existence and uniqueness. Smith (1954), Cebeci and Keller (1971), Na (1979), Veldman and Van der Vooren (1980), Sher and Yakhot (2001), Zaturka and Banks (2001), Kuo (2003), Salama (2004), and Salama and Mansour (2005) have addressed other numerical methods for solving this problem. All those approaches have mainly employed shooting or invariant imbedding. Later, the methods presented by Asaithambi (1998, 2004a, 2004b, 2005), and Asaithambi (1997) improved the performance of the previous methods by reducing the amount of the computational effort.

When a two-dimensional (2D) steady flow of an incompressible constant property fluid with very low viscosity and high Reynolds number moves promptly over a semi-infinite flat plate, the friction between the fluid and the flat plate will obstruct the fluid within a thin region immediately adjacent to the boundary layer. The governing equation describing the boundary layer with such fluid characteristics and boundary conditions is called the Blasius equation; see, e.g., Schlichting (1979), and Özisik (1979). Bla-

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sius (1908) gave a solution in the form of a power series and since then, it has led to much attention on solving this equation by developing different techniques. Töpfer (1912) began to adopt the Runge-Kutta algorithms to solve this equation, and until the age of Howarth (1938), the numerical solution with the Runge-Kutta method as presently shown in tabulated result is still not as accurate and reliable as today is; see, e.g., Schlichting (1979), and Özisik (1979). Apart from this, Lock (1951, 1954) investigated two cases, where the lower stream may be at rest or in motion. Later, Potter (1957) extended the research to two fluids of different viscosities and densities, where both fluids were moving co-current with different velocities. Moreover, Abussita (1994) took a differential equation of mixing layer into account that arises in Blasius solutions for flow passing over a flat plate, and manifested the existence of a solution for this model by using the Weyl techniques. Thereafter, Liao (1997, 1999) proposed a systematic depiction of a new kind of analytic technique for nonlinear problems, namely, the homotopy analysis method (HAM), and applied it to give an explicit and analytic solution of the 2D laminar viscous flow over a semi-infinite flat plate. This method may have higher accuracy but it is very complex in expression. In addition, Yu and Chen (1998) converted the Blasius equation into a pair of initial value problems, and then solved them by a differential transformation method. To speed up the convergent rate and the accuracy of calculation, the entire domain needs to be divided into sub-domains. Besides, Khabibrakhmanov and Summers (1998) employed the generalized Laguerre polynomials to compute a spectral solution of the Blasius equation on a semi-infinite interval; however, this method involves many calculations for nonlinear algebraic equations. For approximating the solution of the Blasius equation,

He (1999) proposed the variational iteration method and Lin (1999) employed the parameter iteration method to cope with. Recently, He (2003) coupled the iteration method with the perturbation method to solve the Blasius equation, and Wang (2004) even proposed the Adomian decomposition method to the transformation of the Blasius equation. Both of their results demonstrate reliability and efficiency of their own proposed algorithms. As for solutions and error estimates of the Blasius equation, Lee and Hung (2002) proposed the modified group preserving (MGP) scheme together with the shooting method; however, their method shows complicated in algorithms and seems indirect to solve the Blasius equation.

The Lie-group shooting method (LGSM) is primarily employed for the boundary value problems as proposed by Liu (2006a, 2006b, 2006c) for direct problems. However, these approaches are limited only for the 2D ordinary differential equations (ODEs), and here we will extend them to the multi-dimensional problems. Authors have used the LGSM to treat the various problems. The backward heat conduction problems (BHCPs) [Chang, Liu and Chang (2007a, 2007b), and Chang, Liu, Chang (2008)] are formulated with a semi-discretization version. In order to evaluate the missing initial conditions for the quasi-boundary value problems of the BHCP, they have employed the LGSM towards the time direction to derive algebraic equations. Hence, they can solve them through a minimum solution in a compact space of $r \in (0, 1)$. The approach is good enough against the noise disturbance. Liu (2008a) used the LGSM to identify nonhomogeneous heat conductivity functions, and it has twofold advantages in that no a priori information of heat conductivity is required and no iterations in the calculation process are needed. After that, the LGSM is examined through numer-

ical examples of estimating an unknown heat conductivity parameter [Liu (2008b)] to convince that it is highly accurate and efficient; the maximum estimation error is smaller than 10^{-5} for smooth parameter and for discontinuous and oscillatory parameter the accuracy is still in the order of 10^{-2} . For the Sturm-Liouville eigenvalues problem, Liu (2008c) constructed a very effective LGSM to search the eigenvalues, and when eigenvalue is determined the author can also search a missing left-boundary condition of the slope through a weighting factor $r \in (0, 1)$. Hence, the eigenvalues and eigenfunctions can be calculated with a better accuracy. For the inverse vibration problem [Liu (2008d)], an LGSM is proposed to simultaneously estimate the time-dependent damping and stiffness functions by using two sets of displacement as inputs. The LGSM approach is very interesting, which resulting to closed-form estimating equations without needing of any iteration and initial guess of coefficient functions, and more importantly, it does not require to assume a priori the functional forms of unknown coefficients. To estimate the missing initial conditions for three-point boundary value problems (BVPs) of second-order ODEs, Liu (2008e) have employed the equation $\mathbf{G}(\mathbf{u}_{t_0}, \mathbf{u}_{t_1}) = \mathbf{G}(r)$ in two-stage of two consecutive intervals to derive four extra algebraic equations, which together with the two given nonlocal boundary conditions lead to totally six equations to solve the six unknowns. The method is also workable to find multiple solutions if the considered equation has. Liu (2008f) proposed a numerical integration method of second-order BVPs resulting from the elastica of slender rods under different loading conditions and boundary conditions. The LGSM is very effective for large deflection problems of elastica even exhibiting multiple solutions. Liu (2008g) studied numerical computations of inverse thermal stress problems. The unknown bound-

ary conditions of an elastically deformable heat conducting rod are not given a priori and are not allowed to measure directly, because the boundary may be not accessible to measure. Although the measured temperature is disturbed by large noise, the LGSM is stable to recover the boundary conditions very well. To evaluate the missing initial conditions for the BVPs of the Blasius equation, [Chang, Chang and Liu (2008)] have employed the equation $\mathbf{G}(T) = \mathbf{G}(r)$ to derive algebraic equations. The numerical implementation of the LGSM is very simple and the computation speed is very fast. For the Falkner–Skan equation, including the Blasius equation as a special case, Liu and Chang (2008) developed a new numerical technique, transforming the governing equation into a non-linear second-order boundary value problem by a new transformation technique, and then solve it by the LGSM. The approach is very effective for searching the multiple-solutions under very complex boundary conditions of suction or injection, and also allowing the motion of plate.

In this paper, we propose an LGSM to tackle these two famous boundary layer equations in fluid mechanics. Our approach is based on the group preserving scheme (GPS) developed by Liu (2001) for the integration of initial value problems. It will be clear that our method can be applied to these two famous boundary layer equations, since we are able to search the missing initial condition through a minimum solution of r in a compact space of $r \in (0, 1)$, where the factor r is used in a generalized mid-point rule for the Lie group of one-step GPS. Especially, the proposed scheme is easy to implement and time saving. Through this study, we may have an easy-implementation and accurate LGSM used in the calculations of these two famous boundary layer equations.

2 One-step GPS

2.1 The GPS

Although we do not know previously the symmetry group of nonlinear differential equations system, Liu (2001) has embedded it into an augmented system and found an internal symmetry of the new system. That is, for an ODEs system with dimensions n :

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u} \in R^n, \quad t \in R, \quad (1)$$

we can embed it into the following $n+1$ -dimensional augmented system:

$$\frac{d}{dt} \mathbf{X} := \frac{d}{dt} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{\mathbf{f}(\mathbf{u}, t)}{\|\mathbf{u}\|} \\ \frac{f^T(\mathbf{u}, t)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix}. \quad (2)$$

It is obvious that the first row in Eq. (2) is the same as the original Eq. (1), but the inclusion of the second row in Eq. (2) gives us a Minkowskian structure of the augmented system for \mathbf{X} satisfying the cone condition:

$$\mathbf{X}^T \mathbf{g} \mathbf{X} = \mathbf{u} \cdot \mathbf{u} - \|\mathbf{u}\|^2 = 0, \quad (3)$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & -1 \end{bmatrix} \quad (4)$$

is a Minkowski metric. \mathbf{I}_n is the identity matrix of order n , and the superscript T denotes the transpose. The cone condition (3) is a natural constraint imposed on the system (2).

Consequently, we have an $n+1$ -dimensional augmented system:

$$\dot{\mathbf{X}} = \mathbf{A} \mathbf{X} \quad (5)$$

with a constraint (3), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{\mathbf{f}(\mathbf{u}, t)}{\|\mathbf{u}\|} \\ \frac{f^T(\mathbf{u}, t)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \quad (6)$$

is an element of the Lie algebra $\mathfrak{so}(n,1)$ satisfying

$$\mathbf{A}^T \mathbf{g} + \mathbf{g} \mathbf{A} = 0. \quad (7)$$

Therefore, Liu (2001) has developed a group-preserving numerical scheme as follows:

$$\mathbf{X}_{l+1} = \mathbf{G}(1) \mathbf{X}_l, \quad (8)$$

where \mathbf{X}_l denotes the numerical value of \mathbf{X} at the discrete time t_l , and $\mathbf{G}(1) \in \text{SO}_o(n,1)$ satisfies

$$\mathbf{G}^T \mathbf{g} \mathbf{G} = \mathbf{g}, \quad (9)$$

$$\det \mathbf{G} = 1, \quad (10)$$

$$\mathbf{G}_0^0 > 0, \quad (11)$$

where G_0^0 is the 00th component of \mathbf{G} .

2.2 Generalized mid-point rule

Applying scheme (8) on Eq. (5) with a specified initial condition $\mathbf{u}(0) = \mathbf{u}_0$, we can compute the solution $\mathbf{u}(t)$ by GPS. Assuming that the total time T is divided by K steps, that is, the time stepsize we use in the GPS is $\Delta t = T/K$, and starting from an initial augmented condition $\mathbf{X}_0 = \mathbf{X}(0) = (\mathbf{u}_0^T, \|\mathbf{u}_0\|)^T$ we may calculate the value $\mathbf{X}(T) = (\mathbf{u}^T(T), \|\mathbf{u}(T)\|)^T$ at a desired time $t = T$.

By applying Eq. (8) step-by-step we can obtain

$$\mathbf{X}_T = \mathbf{G}_K(\Delta t) \dots \mathbf{G}_1(\Delta t) \mathbf{X}_0, \quad (12)$$

where \mathbf{X}_T approximates the exact $\mathbf{X}(T)$ with a certain accuracy depending on $\Delta \tau$. However, let us recall that each \mathbf{G}_i , $i = 1, \dots, K$, is an element of the Lie group $\text{SO}_o(n,1)$, and by the closure property of Lie group $\mathbf{G}_K(\Delta t) \dots \mathbf{G}_1(\Delta t)$ is also a Lie group denoted by \mathbf{G} . Hence, we have

$$\mathbf{X}_T = \mathbf{G} \mathbf{X}_0. \quad (13)$$

This is a one-step transformation from \mathbf{X}_0 to \mathbf{X}_T ; see, e.g., Liu, Chang and Chang (2006). We can calculate \mathbf{G} by a generalized mid-point rule, which is obtained from an exponential mapping of \mathbf{A} by taking the values of the argument variables of \mathbf{A} at a generalized mid-point and viewing it as a constant matrix. The Lie group generated from $\mathbf{A} \in \text{so}(v,1)$ is known as a proper orthochronous Lorentz group, which admits a closed-form representation as follows:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)\hat{\mathbf{f}}\hat{\mathbf{f}}^T}{\|\hat{\mathbf{f}}\|^2} & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^T}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \quad (14)$$

where

$$\hat{\mathbf{u}} = r\mathbf{u}_0 + (1-r)\mathbf{u}_T, \quad (15)$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{t}, \hat{\mathbf{u}}), \quad (16)$$

$$a = \cosh\left(\frac{T\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right), \quad (17)$$

$$b = \sinh\left(\frac{T\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right). \quad (18)$$

Here, we employ the initial \mathbf{u}_0 and the final \mathbf{u}_T through a suitable weighting factor r to calculate \mathbf{G} , where $0 < r < 1$ is a parameter. The above method applied a generalized mid-point rule on the calculation of \mathbf{G} , and the result is a single-parameter Lie group element denoted by $\mathbf{G}(r)$.

2.3 A Lie group mapping between two points

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{u}}\|}, \quad (19)$$

such that Eqs. (14), (17) and (18) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)\mathbf{F}\mathbf{F}^T}{\|\mathbf{F}\|^2} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^T}{\|\mathbf{F}\|} & a \end{bmatrix}, \quad (20)$$

$$a = \cosh(T\|\mathbf{F}\|), \quad (21)$$

$$b = \sinh(T\|\mathbf{F}\|). \quad (22)$$

From Eqs. (13) and (20) it follows that

$$\mathbf{u}_T = \mathbf{u}_0 + \eta\mathbf{F}, \quad (23)$$

$$\|\mathbf{u}_T\| = a\|\mathbf{u}_0\| + b\frac{\mathbf{F} \cdot \mathbf{u}_0}{\|\mathbf{F}\|}, \quad (24)$$

where

$$\eta := \frac{(a-1)\mathbf{F} \cdot \mathbf{u}_0 + b\|\mathbf{u}_0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \quad (25)$$

Substituting

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}_T - \mathbf{u}_0) \quad (26)$$

into Eq. (24) we obtain

$$\frac{\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|} = a + b\frac{(\mathbf{u}_T - \mathbf{u}_0) \cdot \mathbf{u}_0}{\|\mathbf{u}_T - \mathbf{u}_0\|\|\mathbf{u}_0\|}, \quad (27)$$

where

$$a = \cosh\left(\frac{T\|\mathbf{u}_T - \mathbf{u}_0\|}{\eta}\right), \quad (28)$$

$$b = \sinh\left(\frac{T\|\mathbf{u}_T - \mathbf{u}_0\|}{\eta}\right) \quad (29)$$

are obtained by inserting Eq. (26) for \mathbf{F} into Eqs. (21) and (22).

Let

$$\cos\theta := \frac{(\mathbf{u}_T - \mathbf{u}_0) \cdot \mathbf{u}_0}{\|\mathbf{u}_T - \mathbf{u}_0\|\|\mathbf{u}_0\|}, \quad (30)$$

$$S := T\|\mathbf{u}_T - \mathbf{u}_0\|, \quad (31)$$

and from Eqs. (27)-(29) it follows that

$$\frac{\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos\theta \sinh\left(\frac{S}{\eta}\right). \quad (32)$$

By defining

$$Z := \exp\left(\frac{S}{\eta}\right), \quad (33)$$

we obtain a quadratic equation for Z from Eq. (32):

$$(1 + \cos \theta)Z^2 - \frac{2\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|}Z + 1 - \cos \theta = 0. \quad (34)$$

The solution is found to be

$$Z = \frac{\frac{\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|} + \sqrt{\left(\frac{\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|}\right)^2 - 1 + \cos^2 \theta}}{1 + \cos \theta}, \quad (35)$$

and then from Eqs. (33) and (31) we obtain

$$\eta = \frac{T\|\mathbf{u}_T - \mathbf{u}_0\|}{\ln Z}. \quad (36)$$

Thus, between any two points $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ and $(\mathbf{u}_T, \|\mathbf{u}_T\|)$ on the cone, there exists a single-parameter Lie group element $G \in \text{SO}_o(n, 1)$ mapping $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ onto $(\mathbf{u}_T, \|\mathbf{u}_T\|)$, which is given by

$$\begin{bmatrix} \mathbf{u}_T \\ \|\mathbf{u}_T\| \end{bmatrix} = \mathbf{G} \begin{bmatrix} \mathbf{u}_0 \\ \|\mathbf{u}_0\| \end{bmatrix}, \quad (37)$$

where \mathbf{G} is uniquely determined by \mathbf{u}_0 and \mathbf{u}_T through Eqs. (20)-(22), (26) and (36).

3 The LGSM for Falkner-Skan and Blasius equations

Let us consider the Falkner-Skan equation:

$$f''' + ff'' + \nu(1 - f'^2) = 0, \quad (38)$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \quad (39)$$

where the prime stands for the differential with respect to ξ . Note that problem (38) and (39) is depicted on a semi-infinite physical domain.

3.1 The case of $\nu > 0$

Because Eqs. (38) and (39) are defined in an infinite range, they are not easy to compute in practice. To remedy this inefficiency we replace the condition $f'(\infty) = 1$ by $f'(T) = 1$, where T is an unknown variable to be determined.

Let $y_1 = f$, $y_2 = f'$ and $y_3 = f''$. We can rewrite Eqs. (38) and (39)

$$y_1' = y_2, \quad (40)$$

$$y_2' = y_3, \quad (41)$$

$$y_3' = -y_1y_3 - \nu(1 - y_2^2) =: Y(y_1, y_2, y_3), \quad (42)$$

$$y_1(0) = \alpha = 0, \quad y_1(T) = A, \quad (43)$$

$$y_2(0) = \beta = 0, \quad y_2(T) = B = 1, \quad (44)$$

$$y_3(0) = \delta, \quad y_3(T) = C = 0, \quad (45)$$

where A , T and δ are three unknown constants. Here we adopt a physical by plausible assumption $f''(T) = 0$, such that there are still three unknowns to be solved by the LGSM.

Let

$$u := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \quad (46)$$

From Eqs. (23), (43),(44) and (45) it follows that

$$\mathbf{F} := \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} A - \alpha \\ B - \beta \\ C - \delta \end{bmatrix}. \quad (47)$$

Starting from an initial guess of (A, T, δ) , we use the following equation to calculate η :

$$\eta = \frac{T\sqrt{(\alpha - A)^2 + (\beta - B)^2 + (\delta - C)^2}}{\ln Z}, \quad (48)$$

in which Z is calculated by

$$Z = \frac{\frac{\sqrt{A^2+B^2+C^2}}{\sqrt{\alpha^2+\beta^2+\delta^2}} + \sqrt{\frac{A^2+B^2+C^2}{\alpha^2+\beta^2+\delta^2}} - (1 - \cos^2 \theta)}{1 + \cos \theta}, \quad (49)$$

where

$$\cos \theta = \frac{\alpha(A - \alpha) + \beta(B - \beta) + \delta(C - \delta)}{\sqrt{(\alpha - A)^2 + (\beta - B)^2 + (\delta - C)^2}} \frac{1}{\sqrt{\alpha^2 + \beta^2 + \delta^2}}. \quad (50)$$

The above three equations were obtained from Eqs. (36), (35) and (30) by inserting Eq. (46) for \mathbf{u} .

When comparing Eq. (47) with Eq. (19), and with the aid of Eqs. (15), (16) and (40)-(45) we obtain

$$A = \alpha + \frac{\eta[r\beta + (1-r)B]}{\rho}, \quad (51)$$

$$B = \beta + \frac{\eta[r\delta + (1-r)C]}{\rho}, \quad (52)$$

$$C = \delta + \frac{-\eta \left\{ \begin{array}{l} [r\alpha + (1-r)A][r\delta + (1-r)C] \\ + v\{1 - [r\beta + (1-r)B]^2\} \end{array} \right\}}{\rho}. \quad (53)$$

Through some calculations, we obtain

$$T = \frac{-E + \sqrt{E^2 - 4DF}}{2D\eta_0}, \quad (54)$$

$$A = \alpha + \frac{T\eta_0[r\beta + (1-r)B]}{\rho}, \quad (55)$$

$$\delta = \frac{\rho(B - \beta)}{r\eta}, \quad (56)$$

where

$$D := \frac{v\{1 - [r\beta + (1-r)B]^2\}}{\rho}, \quad (57)$$

$$E := (B - \beta)[r\alpha + (1-r)A], \quad (58)$$

$$F := \frac{-\rho(B - \beta)}{r}, \quad (59)$$

$$\hat{Y} := Y(r\alpha + (1-r)A, r\beta + (1-r)B, r\delta + (1-r)C), \quad (60)$$

$$\rho := \sqrt{\begin{array}{l} [r\alpha + (1-r)A]^2 \\ + [r\beta + (1-r)B]^2 \\ + [r\delta + (1-r)C]^2 \end{array}}, \quad (61)$$

$$\eta_0 := \frac{\sqrt{(\alpha - A)^2 + (\beta - B)^2 + (\delta - C)^2}}{\ln Z}. \quad (62)$$

When v of Eq. (42) is equal to 0, Eq. (42) is the Blasius equation. Therefore, we can rewrite Eq. (54) as

$$T = \frac{2\rho}{r[r\alpha + (1-r)A]\eta_0}. \quad (63)$$

The above derivation of the governing equations (48)-(63) is stemmed from by letting the two \mathbf{F} in Eqs. (19) and (26) be equal, which is essentially identical to the specification of $\mathbf{G}(T) = \mathbf{G}(r)$ in terms of the Lie group elements $\mathbf{G}(T)$ and $\mathbf{G}(r)$.

For a specified r , Eqs. (54), (55), (56) and (63) can be used to generate the new (A, T, δ) by repeating the above process in Eqs. (48)-(63) until (A, T, δ) converges according to a given stopping criterion with $\varepsilon_1 = 10^{-10}$:

$$\sqrt{(A_{i+1} - A_i)^2 + (T_{i+1} - T_i)^2 + (\delta_{i+1} - \delta_i)^2} \leq \varepsilon_1. \quad (64)$$

If δ is available, we can return to Eqs. (40)-(44) but with merely integrating the following equations by a forward integration scheme as the one given in Section 2:

$$y'_1 = y_2, \quad (65)$$

$$y_2' = y_3, \quad (66)$$

$$y_3' = -y_1 y_3 - v(1 - y_2^2), \quad (67)$$

$$y_1(0) = \alpha, \quad (68)$$

$$y_2(0) = \beta, \quad (69)$$

$$y_3(0) = \delta. \quad (70)$$

So far, we have not yet said that how to determine r . Let $y_n^r(T)$ denote the above solution of y_n at T . We start from $r = 1/2$ to determine δ by Eqs. (48)-(64) and then numerically integrate Eqs. (65)-(70) from $t = 0$ to $t = T$, and compare the end values of $y_2(T)$ and $y_3(T)$ with the exact B and C . Then, we apply the minimum norm to fit the two targets of $y_2(T) = 1$ and $y_3(T) = 0$, which requires us to calculate Eqs. (65)-(70) at each of the calculation of $\sqrt{(y_2^r - B)^2 + (y_3^r - C)^2}$, until it is small enough to satisfy the criterion of $\sqrt{(y_2^r - B)^2 + (y_3^r - C)^2} < \epsilon_{\min}$, where $\epsilon_{\min} = 0.36$ is a given error tolerance. Because the numerical method is very stable, we can fast carry off the correct range of r through some trials and modifications.

3.2 The case of $v \leq 0$

Because Eqs. (38) and (39) are not very convenient for computations, it is used to replace $\xi = \infty$ in condition (39) with a sufficiently large ξ_{∞} ; see, e.g., Asaithambi (1998, 2004a, 2004b, 2005), and Asaithambi (1997). In terms of the new independent variable $t = x/\xi_{\infty}$ and dependent variables $y_1 = f$, $y_2 = f'$ and $y_3 = f''$, it is straightforward to replace Eqs. (40)-(45) by

$$\dot{y}_1 = \xi_{\infty} y_2, \quad (71)$$

$$\dot{y}_2 = \xi_{\infty} y_3, \quad (72)$$

$$\dot{y}_3 = -\xi_{\infty} y_1 y_3 - \xi_{\infty} v(1 - y_2^2), \quad (73)$$

$$y_1(0) = \alpha = 0, \quad y_1(1) = A, \quad (74)$$

$$y_2(0) = \beta = 0, \quad y_2(1) = B = 1, \quad (75)$$

$$y_3(0) = \delta, \quad y_3(1) = C, \quad (76)$$

where A , C and δ are three unknown constants. Starting from an initial guess of (A, C, δ) , we use the following equation to calculate η :

$$\eta = \frac{\sqrt{(\alpha - A)^2 + (\beta - B)^2 + (\delta - C)^2}}{\ln Z}. \quad (77)$$

Comparing Eq. (47) with Eq. (19) and noting that

$$\hat{\mathbf{u}} := \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{bmatrix} = \begin{bmatrix} r\alpha + (1-r)A \\ r\beta + (1-r)B \\ r\delta + (1-r)C \end{bmatrix} \quad (78)$$

by Eqs. (15), (16) and (71)-(76), we obtain

$$A = \alpha + \eta \frac{\xi_{\infty} \hat{y}_2}{\rho}, \quad (79)$$

$$B = \beta + \eta \frac{\xi_{\infty} \hat{y}_3}{\rho}, \quad (80)$$

$$C = \delta - \eta \frac{\xi_{\infty}}{\rho} [\hat{y}_1 \hat{y}_3 + v(1 - \hat{y}_2^2)], \quad (81)$$

where ρ is still defined by Eq. (61).

Because $\beta = 0$ and $B = 1$ are given constants, we can solve Eqs. (80) and (81) by using $\hat{y}_3 = r\delta + (1-r)C$, such that

$$C = \frac{\rho}{\eta \xi_{\infty}} - \xi_{\infty} \frac{r\eta [\hat{y}_1 \hat{y}_3 + v(1 - \hat{y}_2^2)]}{\rho}, \quad (82)$$

$$\delta = \frac{\rho}{\eta \xi_{\infty}} + \xi_{\infty} \frac{\eta(1-r) [\hat{y}_1 \hat{y}_3 + v(1 - \hat{y}_2^2)]}{\rho}. \quad (83)$$

For a specified r and given constants $\alpha = 0$, $\beta = 0$ and $B = 1$, starting from an initial guess of (A, δ, C) , Eqs. (79), (82) and (83) can be used to generate the new (A, δ, C) by repeating the above iterative process until (A, δ, C)

converges according to a given stopping criterion:

$$\sqrt{(A_{i+1} - A_i)^2 + (\delta_{i+1} - \delta_i)^2 + (C_{i+1} - C_i)^2} \leq \varepsilon_2, \quad (84)$$

which means that the norm of the difference between the $i + 1$ -th and the i -th iterations of (A, δ, X) is smaller than a given stopping criterion $\varepsilon_2 = 10^{-10}$. If δ is available, we can return to Eqs. (71)-(76) and integrate them to obtain $y_2(1)$. The above process can be done for all r in the interval of $r \in (0, 1)$. Among these solutions we pick up r , which leads to the smallest error of $|y_2(1) - 1|$ in Eq. (75), since $y_2(1) = 1$ is our target at the end boundary. That is,

$$\min_{r \in (0,1)} |y_2(1) - 1|. \quad (85)$$

4 Numerical results and discussions

Solutions of the Falkner-Skan equation have been investigated in the literature by varying the values of ν . The solutions corresponding to $\nu > 0$ stand for accelerating flows; those corresponding to $\nu = 0$ are known as constant flows; and those corresponding to $\nu < 0$ are called decelerating flows. It is known that physically relevant solutions exist only for $-0.1988 \leq \nu \leq 2$. Following Section 3, when the factor ν is equal to 0.5 (Homann's solution), 1 (Hiemenz solution) and 2 (exponentially-varying outer flow), we apply the Lie-group shooting method to the Falkner-Skan equation with an initial $(A, T, \delta) = (3, 5, 2)$ and through some trials we take $\rho = 0.6515421$, $\rho = 0.658208$, and $\rho = 0.6625697$, respectively. By using a stepsize $\Delta\xi = 0.0001$ the numerical results are shown in Table 1. Moreover, when the factor ν is equal to -0.1, -0.12, -0.15, -0.18, and -0.1988, we apply the Lie-group shooting method to the Falkner-Skan equation with an initial $(A, \delta, C) = (6, 1, 0)$ and through some trials, we

take $r = 0.6153$, $r = 0.6099$, $r = 0.6065$, $r = 0.6165$, and $r = 0.6225$, respectively. By using a stepsize $\Delta\xi = 0.001$ the numerical results are shown in Table 1.

For the case of $\nu < 0$, there also exists the second solution, which corresponds to negative value of δ as shown in Table 2. When the factor ν is equal to -0.1, -0.12, -0.15, and -0.18, we take $\rho = 0.8543$, $r = 0.8196$, $r = 0.7696$, and $r = 0.7237$. As an example for the case of $\nu < 0$, we are considered $\nu = -0.1$ in Fig. 1, where $\xi_\infty = 10$ was fixed. In Fig. 1(a), we plot the error of target with respect to r . It can be seen that there are two minimum points as marked by a and b . When the ranges for minima are identified, we can pick up more correct value of ρ by searching the minima in more refined ranges. When the missing initial conditions are available, with all given initial conditions we can use the RK4 method to integrate Eqs. (71)-(73). There appear two solutions which are marked by α and β for f' in Fig. 1(b) and f'' in Fig. 1(c). The situation of multiple solutions occurs only for $-0.1988 < \nu < 0$, and these solutions are said to stand for reverse flows. Our numerical method is able to attain the solutions of accelerating, constant, decelerating, and reverse flows. From Tables 1 and 2, it is apparent that our results are in good agreement with those reported previously in the literature.

When the factor ν is equal to 0, we apply the Lie-group shooting method to the Blasius equation with an initial $(A, \delta) = (3, 2)$ and through some trials, we take $r = 0.520748$. By using a stepsize $\Delta\xi = 0.0001$ the numerical results are shown in Fig. 2 and Table 3. From Table 3, it is obvious that our results are in excellent agreement with those given by Cortell (2005).

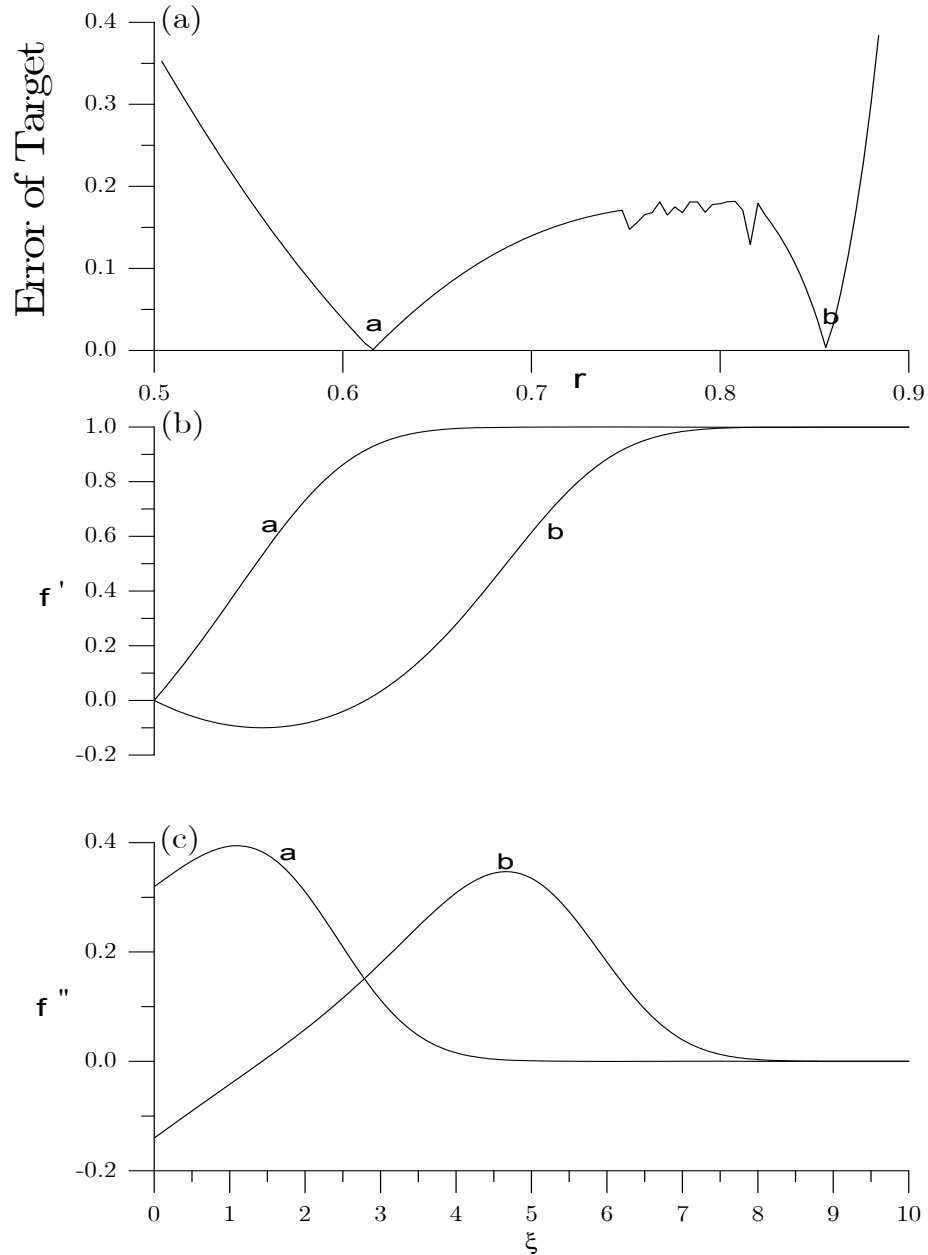


Figure 1: For the Falkner-Skan equation with $\nu = -0.1$ we plot the error of target with respect to r in (a), (b) displaying two different solutions of f' , and (c) displaying two different solutions of f'' .

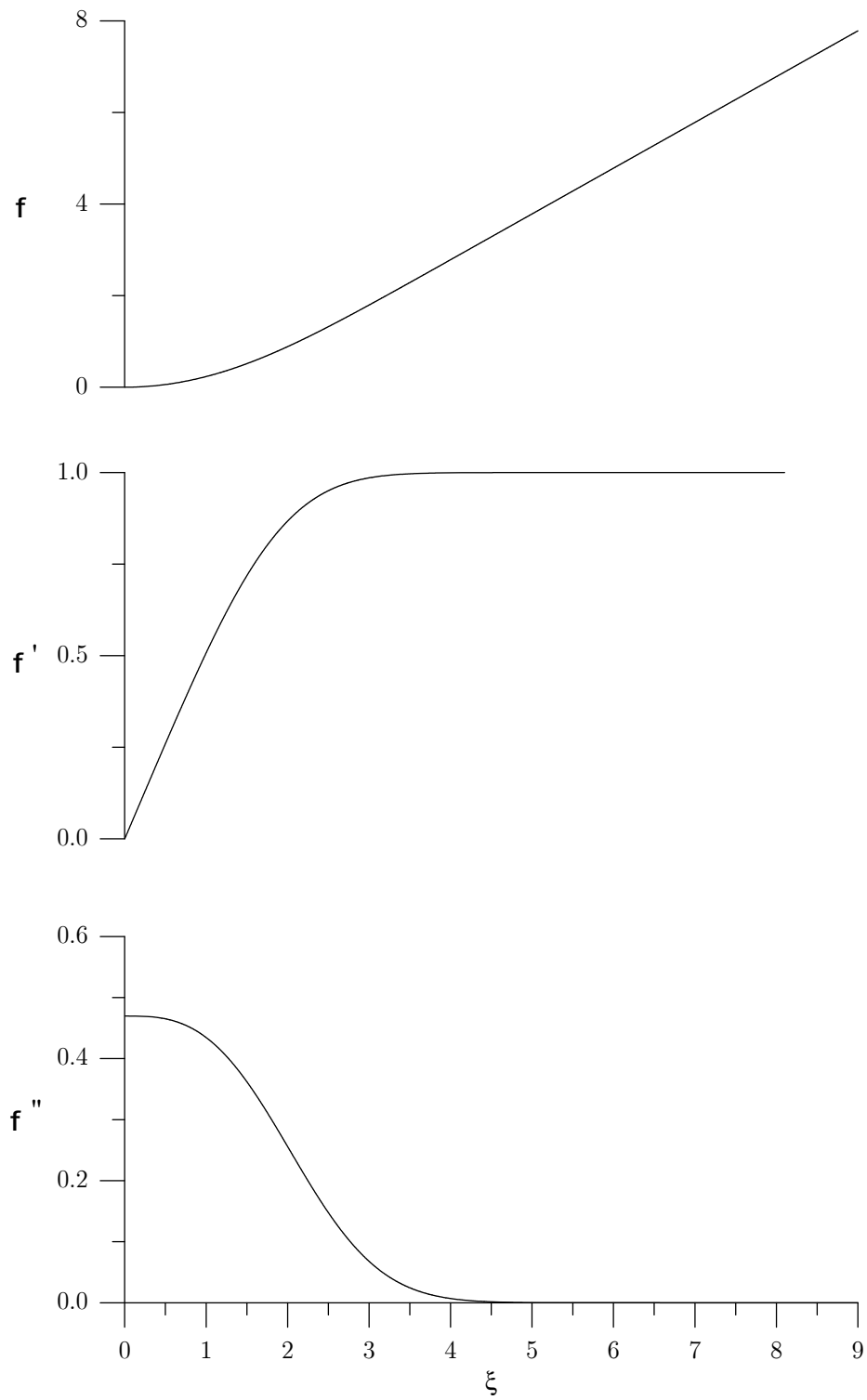


Figure 2: Variations with respect to ξ of functions f , f' and f'' for the Blasius equation.

Table 1: Comparison of computed δ for $-0.1988 \leq \nu \leq 2$.

ν	LGSM	Asaithambi (1998)	Asaithambi (2004)	Asaithambi (2004)	Asaithambi (2005)	Salama (2004)
2	1.687218	1.687222	1.687218	1.687222	1.687222	1.687218
1	1.232588	1.232589	1.232588	1.232588	1.232588	1.232588
0.5	0.927680	0.927682	0.927680	0.927680	0.927680	0.927680
0.0	0.469600	0.469601	0.469600	0.469600	0.469600	0.469600
-0.1	0.319269	0.319270	0.319269	0.319270	0.319270	0.319270
-0.12	0.281761			0.281761		
-0.15	0.216361	0.216360	0.216358	0.216361	0.216361	0.216362
-0.18	0.128637	0.128636	0.128624	0.128636	0.128637	0.128638
-0.1988	0.005190	0.005218	0.005239	0.005220	0.005225	0.005226

Table 2: Comparison of computed δ for $-0.1988 < \nu < 0$.

ν	LGSM	Asaithambi (1997)
-0.1	-0.140546	-0.140546
-0.12	-0.142935	-0.142935
-0.15	-0.133419	-0.133421
-0.18	-0.097690	-0.097692

Table 3: Values of functions f , f' and f'' . LGSM Lie-group shooting method; Cortell's solution.

ξ	f of LGSM	f of C	f' of LGSM	f' of C	f'' of LGSM	f'' of C
0	0.00000	0.00000	0.00000	0.00000	0.46960	0.46960
1	0.23298	0.23299	0.46065	0.46063	0.43439	0.43438
2	0.88682	0.88681	0.81675	0.81670	0.25568	0.25567
3	1.79567	1.79558	0.96912	0.96906	0.06770	0.06771
4	2.78407	2.78390	0.99783	0.99777	0.00687	0.00687
5	3.78349	3.78325	0.99999	0.99994	0.00026	0.00026
6	4.78354	4.78324	1.0	1.0	0.0	0.0

5 Conclusions

In order to evaluate the missing initial conditions for the boundary value problems of boundary layer equations, we have employed the equation $\mathbf{G}(T) = \mathbf{G}(r)$ to derive algebraic equations. Hence, we can solve them through a minimum solution in a compact space of $r \in (0, 1)$. Numerical examples of the Falkner-Skan and the Blasius equations were examined to ensure that the new algorithm has a fast convergence speed on the solution of r

in a pre-selected range smaller than $(0, 1)$ by using the minimum norm to fit two targets, which usually required only a small number of iterations. Through this paper, it can be concluded that the new Lie-group shooting method is accurate, effective and stable. Its numerical implementation is very simple and the computation speed is very fast. Thus, it is highly advocated to be used in the numerical computations of these two famous boundary layer equations in fluid mechanics.

References

- Abussita, A. M. M.** (1994): A note on a certain boundary-layer equation. *Appl. Math. Comput.*, vol. 64, pp. 73–77.
- Asaithambi, A.** (1998): A finite-difference method for the Falkner Skan equation. *Appl. Math. Comput.*, vol. 92, pp. 135–141.
- Asaithambi, A.** (2004a): A second-order finite-difference method for the Falkner Skan equation. *Appl. Math. Comput.*, vol. 156, pp. 779–786.
- Asaithambi, A.** (2004b): Numerical solution of the Falkner Skan equation using piecewise linear functions. *Appl. Math. Comput.*, vol. 159, pp. 267–273.
- Asaithambi, A.** (2005): Solution of the Falkner Skan equation by recursive evaluation of Taylor coefficients. *J. Comput. Appl. Math.*, vol. 176, pp. 203–214.
- Asaithambi, N. S.** (1997): A numerical method for the solution of the Falkner Skan equation. *Appl. Math. Comput.*, vol. 81, pp. 259–264.
- Blasius, H.** (1908): Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Zeitschrift für Mathematik und Physik*, vol. 56, pp. 1–37.
- Cebeci, T.; Keller, H. B.** (1971): Shooting and parallel shooting methods for solving the Falkner-Skan boundary-layer equation. *J. Comput. Phys.*, vol. 7, pp. 289–300.
- Chang, J.-R.; Liu, C.-S.; Chang, C.-W.** (2007a): A new shooting method for quasi-boundary regularization of backward heat conduction problems. *Int. J. Heat Mass Transfer*, vol. 50, pp. 2325–2332.
- Chang, C.-W.; Liu, C.-S.; Chang, J.-R.** (2007b): The Lie-group shooting method for quasi-boundary regularization of backward heat conduction problems. *ICCES on line Journal*, vol. 3, pp. 69–79.
- Chang, J.-R.; Chang, C.-W.; Liu, C.-S.** (2008): The Lie-group shooting method for solving classical Blasius flat-plate problem. *CMC: Computers, Materials & Continua*, vol. 7, pp. 139–153.
- Chang, C.-W.; Liu, C.-S.; Chang, J.-R.** (2008): A new shooting method for quasi-boundary regularization of multi-dimensional backward heat conduction problems. *J. Chin. Inst. Engineers*, accepted.
- Coppel, W. A.** (1960): On a differential equation of boundary layer theory. *Philos. Trans. R. Soc. of London. Series A*, vol. 253, pp. 101–136.
- Cortell, R.** (2005): Numerical solutions of the classical Blasius flat-plate problem. *Appl. Math. Comput.*, vol. 170, pp. 706–710.
- Falkner, V. M.; Skan, S. W.** (1931): Some approximations of the boundary layer equations. *Philos. Magazine*, vol. 12, pp. 865–896.
- Hartree, D. R.** (1937): On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer. *Proc. Cambridge Philos. Soc.*, vol. 33, pp. 223–239.
- He, J. H.** (1999): Approximate analytical solution of Blasius' equation. *Commun. Non-linear Sci. Numer. Simulation*, vol. 4, pp. 75–78.
- He, J. H.** (2003): A simple perturbation approach to Blasius equation. *Appl. Math. Comp.*, vol. 140, pp. 217–222.
- Howarth, L.** (1938): On the solution of the laminar boundary layer equations. *Proc. R. Soc. London A*, vol. 164, pp. 547–579.
- Khabibrakhmanov, I. K.; Summers, D.** (1998): The use of generalized Laguerre polynomials in spectral methods for nonlinear differential equations. *Comp. Math. Appl.*, vol. 36, pp. 65–70.
- Kuo, B.-L.** (2003): Application of the differential transformation method to the solutions of the Falkner Skan wedge flow. *Acta Mech.*,

vol. 164, pp. 161–174.

Lee, H.-C.; Hung, C.-I. (2002): Solutions and error estimates of the Blasius equation by using the modified group preserving scheme, *J. Chin. Soc. Mech. Eng.*, vol. 23, pp. 111-119.

Liao, S.-J. (1997): A kind of approximate solution technique which does not depend upon small parameters II an application in fluid mechanics. *Int. J. Non-Linear Mech.*, vol. 32, pp. 815-822.

Liao, S.-J. (1999): An explicit, totally analytic approximate solution for Blasius' viscous flow problems. *Int. J. Non-Linear Mech.*, vol. 34, pp. 759-778.

Liu, C.-S. (2001): Cone of non-linear dynamical system and group preserving schemes. *Int. J. Non-Linear Mech.*, vol. 36, pp. 1047-1068.

Liu, C.-S. (2006a): The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, pp. 149-163.

Liu, C.-S. (2006b): Efficient shooting methods for the second order ordinary differential equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, pp. 69-86.

Liu, C.-S. (2006c): The Lie-group shooting method for singularly perturbed two-point boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, pp. 179-196.

Liu, C.-S. (2008a): An LGSM to identify nonhomogeneous heat conductivity functions by an extra measurement of temperature. *CMES: Computer Modeling in Engineering & Sciences*, vol. 51, pp. 2603-2613.

Liu, C.-S. (2008b): An LGEM to identify time-dependent heat conductivity function by an extra measurement of temperature gradient. *CMC: Computers, Materials & Continua*, vol. 7, pp. 81-96.

Liu, C.-S. (2008c): A Lie-group shooting method for computing eigenvalues and eigenfunctions of Sturm-Liouville problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 26, pp. 157-168.

Liu, C.-S. (2008d): A Lie-group shooting method for simultaneously estimating the time-dependent damping and stiffness coefficients. *CMES: Computer Modeling in Engineering & Sciences*, vol. 27, pp. 137-149.

Liu, C.-S. (2008e): A two-stage LGSM for three-point BVPs of second-order ODEs. *Boundary Value Problems*, vol. 2008, Article ID 963753.

Liu, C.-S. (2008f): A Lie-group shooting method for post buckling calculations of elastica. *CMES: Computer Modeling in Engineering & Sciences*, vol. 30, pp. 1-16.

Liu, C.-S. (2008g): The Lie-group shooting method for thermal stress evaluation through an internal temperature measurement. *CMC: Computers, Materials & Continua*, vol. 8, pp. 1-16.

Liu, C.-S.; Chang, J.-R. (2008): The Lie-group shooting method for multiple-solutions of Falkner-Skan equation under suction-injection conditions. *Int. J. Non-Linear Mech.*, in press.

Liu, C.-S.; Chang, C.-W.; Chang, J.-R. (2006): Past cone dynamics and backward group preserving scheme for backward heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 12, pp. 67-81.

Lin, J. G. (1999): A new approximate iteration solution of Blasius' equation, *Commun. Nonlinear Sci. Numer. Simulation*, vol. 4, pp. 91-94.

Lock, R. C. (1951): The velocity distribution in the laminar boundary layer between parallel streams. *Q. J. Mech. Appl. Math.*, vol. 4, pp. 42–63.

Lock, R. C. (1954): Hydrodynamic stabil-

ity of the flow in the laminar boundary layer between parallel streams. *Proc. Cambridge Philos. Soc.*, vol. 50, pp. 105–124.

Na, T. Y. (1979): *Computational methods in engineering boundary value problems*. Academic Press, New York.

Özisik, M. N. (1979): *Basic Heat Transfer*. McGraw-Hill, New York.

Padé, O. (2003): On the solution of Falkner-Skan equations, *J. Math. Anal. Appl.*, vol. 285, pp. 264–274.

Potter, O. E. (1957): Mass transfer between co-current fluid streams and boundary layer solutions, *Chem. Eng. Sci.*, vol. 6, pp. 170–182.

Rosenhead, L. (1963): *Laminar boundary layer*. Clarendon Press, Oxford.

Salama, A. A. (2004): Higher-order method for solving free boundary-value problems, *Numer. Heat Transfer, Part B: Fundamentals* vol. 45, pp. 385–394.

Salama, A. A.; Mansour, A. A. (2005): Fourth-order finite-difference method for third-order boundary-value problems, *Numer. Heat Transfer, Part B: Fundamentals* vol. 47, pp. 383–401.

Schlichting, H. (1979): *Boundary layer theory*. McGraw-Hill, New York.

Sher, I.; Yakhot, A. (2001): New approach to the solution of the Falkner-Skan equation, *AIAA J.*, vol. 39, pp. 965–967.

Smith, A. M. O. (1954): Improved solutions of the Falkner and Skan boundary-layer equation. *Institute of the Aeronautical Sciences, SMF Fund Paper*, No. FF-IO.

Töpfer, C. (1912): Bemerkungen zu dem Aufsatz von H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Zeitschrift für Mathematik und Physik*, vol. 61, pp. 397–398.

Veldman, A. E. P.; Van der Vooren, A. I. (1980): On generalized Falkner-Skan equa-

tions, *J. Math. Anal. Appl.*, vol. 75, pp. 101–111.

Wang, L. (2004): A new algorithm for solving classical Blasius equation, *Appl. Math. Comput.*, vol. 157, pp. 1–9.

Weyl, H. (1942): On the differential equations of the simplest boundary-layer problem. *Annals of Math.*, vol. 43, pp. 381–407.

Yu, L.-T.; Chen, C.-K. (1998): The solution of the Blasius equation by the differential transformation method. *Math. Comput. Model.*, vol. 28, pp. 101–111.

Zaturka, M. B.; Banks, W. H. H. (2001): A new solution branch of the Falkner-Skan equation, *Acta Mech.*, vol. 152, pp. 197–201.

