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A Fictitious Time Integration Method for Two-Dimensional Quasilinear Elliptic Boundary Value Problems

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Abstract: Dirichlet boundary value problem of quasilinear elliptic equation is numerically solved by using a new concept of fictitious time integration method (FTIM). We introduce a fictitious time coordinate t by transforming the dependent variable $u(x,y)$ into a new one by $(1+t)u(x,y) =: v(x,y,t)$, such that the original equation is naturally and mathematically equivalently written as a quasilinear parabolic equation, including a viscous damping coefficient to enhance stability in the numerical integration of spatially semi-discretized equation as an ordinary differential equations set on grid points. Six examples of Laplace, Poisson, reaction diffusion, Helmholtz, the minimal surface, as well as the explosion equations are tested. It is interesting that the FTIM can easily treat the nonlinear boundary value problems without any iteration and has high efficiency and high accuracy. Due to the dissipation nature of the resulting parabolic equation, the FTIM is insensitive to the guess of initial conditions and approaches the true solution very fast.

Keyword: Quasilinear elliptic equation, Laplace equation, Poisson equation, Helmholtz equation, Fictitious Time Integration Method (FTIM)

1 Introduction

Partial differential equations (PDEs) are first divided into two categories: non-evolutionary and evolutionary. Then, the latter is further classified into as parabolic and hyperbolic types according to the number of real characteristic lines. The non-evolutionary PDE is usually named elliptic type PDE because it exists no real characteristic line.

Dirichlet problem of elliptic type PDE is a classical one, and has a broad application in engineering problems. Although for some linear PDEs in simple domains, the analytic solutions could be found, in general, for a given quasilinear PDE or

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when the domain of considered problem having complexity, the finding of analytical solution is not easy. Indeed, the explicit solutions are exceptional. If one encounters nonlinear case in an arbitrary domain, both the geometric complexity and the problem nonlinearity appear, and then typically the numerical solutions should be resorted.

In the past few decades many numerical methods were developed for elliptic type PDEs. This sort of PDEs arises from many fields, like as mechanics, electromagnetism, and biology. For quasilinear elliptic boundary value problems (BVPs), Chen and Zhou (2000) have presented some iteration methods, which include the mountain iteration algorithm, the scaling iterative algorithm, the monotone iterative algorithm, as well as direct iterative algorithm. In general, a sequence of iterations is generated by different methods, but they are not guaranteed to converge to the true solution. Generally speaking, a stronger condition of the differential equation is needed to guarantee the convergence.

On the other hand, there are many papers to concern with the numerical solutions of linear elliptic type BVPs, like as, Liu (2007a,2007b,2007c,2008a). Meshless and meshfree methods are nowadays the main stream in numerical computations as intensively advocated by many researchers, to name a few, Zhu, Zhang and Atluri (1998, 1999), Atluri and Zhu (1998a, 1998b), Atluri, Kim and Cho (1999), Atluri and Shen (2002), Cho, Golberg, Muleshkov and Li (2004), Jin (2004), Li, Lu, Huang and Cheng (2007), Liu (2007d,2007e), Tsai, Lin, Young and Atluri (2006), and Young, Chen, Chen and Kao (2007). However, it cannot be overemphasized the effectiveness of those methods, when nonlinear problems are encountered. Atluri (2004) has extended the meshless local Petro-Galerkin method to solve nonlinear boundary value problems.

Many collocation techniques together with bases expansions were also employed to solve elliptic type BVPs; see, for example, Cheng, Golberg, Kansas and Zammito (2003), Hu, Li and Cheng (2005), Algahtani (2006), and Libre, Emdadi, Kansa, Rahimian and Shekarchi (2008). Recently, Li, Lu, Hu and Cheng (2008) gave a very detailed description of the collocation Trefftz method on engineering problems. Basically, the above bases expansion methods are effective for linear problems. For nonlinear problems some iterations of those methods are unavoidable.

Our task is to develop a non-iterative algorithm having the advantages of easy to numerical implementation, and a great flexibility applying to the most elliptic type BVPs without resorting on special treatments. Let us begin with a discussion of the

following quasilinear elliptic equation:

$$\Delta u(x,y) = F(x,y,u,u_x,u_y,\ldots), \quad (x,y) \in \Omega, \tag{1}$$

$$u(x,y) = H(x,y), \quad (x,y) \in \Gamma, \tag{2}$$

where Δ is a Laplacian operator, Γ is the boundary of the problem domain Ω, and F and H are given functions.

It is known that for the evolutionary type PDEs a semi-discretization of the spatial coordinates together with numerical time integrators for initial value problems (IVPs) can help us to find numerical solutions effectively [Ames (1992)]. However, this technique may be not applicable on the non-evolutionary type PDEs due to an inherent ill-posed property of an IVP version of elliptic type BVP as first pointed out by Hadamard. Indeed, the problem consists of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and the initial conditions

$$u(x,0) = 0,$$

$$\frac{\partial u}{\partial y}(x,0) = k^{-1}\sin(kx)$$

is known to be ill-posed, since the work of Hadamard.

It is easily verified that the exact solution

$$u(x,y) = k^{-2}\sin(kx)\sinh(ky)$$

of the above problem does not become small for any nonzero y, even the initial condition $k^{-1}\sin(kx)$ can be made arbitrary small by increasing k. It is obvious that the solution does not depend continuously on the initial data, and it is not a well-posed problem. Therefore, we cannot view it as an initial value problem and apply the numerical integration method on the above Laplace equation in the y direction by discretizing the coordinate of x.

It may have different strategies by adding a time variable t in Eq. (1), and change it to an initial boundary value problem. The first is viewed the solution u of Eqs. (1) and (2) as a steady state of the following equation:

$$\frac{\partial u}{\partial t} = \Delta u(x,y) - F(x,y,u,u_x,u_y,\ldots). \tag{3}$$

Another is making a perturbation of Eq. (1) by a small parameter:

$$\varepsilon\frac{\partial u}{\partial t} = \Delta u(x,y) - F(x,y,u,u_x,u_y,\ldots). \tag{4}$$

When $\varepsilon = 0$, it recovers to the original equation.

Even there are many better time integration techniques to solve the above two equations by using a semi-discretization technique, it has a big problem of the above two approaches, because Eqs. (3) and (4) are much deviating from the original equation (1). In order to approach the steady state we should calculate Eq. (3) to a long time such that it is very time consumption, and on the other hand we do not know that whether Eq. (3) has a steady state or not. For the second approach, there appears a small quantity ε which will lend a highly singular differential equation when dividing both the sides of Eq. (4) by ε, and this will be very difficult to numerical integration due to instability.

This paper is motivated by using the evolutionary property of parabolic type PDE and the accurcay of numerical time integrators, and proposes a natural and mathematical equivalent approach to transform Eq. (1) into a parabolic PDE without destroying the original structure.

For one-dimensional second-order BVP, Liu (2006a, 2006b, 2006c) has made a breakthrough to extend the method of group preserving scheme (GPS) previously developed by Liu (2001) for ODEs to BVPs, namely the Lie-group shooting method (LGSM), and the numerical results revealed that the LGSM is a rather promising method to effectively solve the nonlinear two-point BVPs. With this in mind, we hope that the present paper could provide an effective numerical solver for the nonlinear BVPs of second-order quasilinear elliptic equations.

2 A fictitious time integration approach

2.1 Transformation into an evolutional PDE

First we propose the following transformation:

$$v(x,y,t) = (1+t)u(x,y), \tag{5}$$

and introduce a viscosity damping coefficient ν in Eq. (1):

$$0 = \nu\Delta u - \nu F(x,y,u,u_x,u_y,\ldots). \tag{6}$$

Multiplying the above equation by $1+t$ and using Eq. (5) we have

$$0 = \nu\Delta v - \nu(1+t)F(x,y,u,u_x,u_y,\ldots). \tag{7}$$

Recalling that $\partial v/\partial t = u(x,y)$ by Eq. (5), and adding it on both the sides of the above equation we obtain

$$\frac{\partial v}{\partial t} = \nu\Delta v - \nu(1+t)F(x,y,u,u_x,u_y,\ldots) + u. \tag{8}$$

Finally by using $u = v/(1+t)$, $u_x = v_x/(1+t)$ and $u_y = v_y/(1+t)$, etc., we can change Eqs. (1) and (2) into a parabolic type PDE:

$$\frac{\partial v}{\partial t} = \nu\Delta v - \nu(1+t)F\left(x, y, \frac{v}{1+t}, \frac{v_x}{1+t}, \frac{v_y}{1+t}, \ldots\right) + \frac{v}{1+t}, \quad (x,y) \in \Omega, \tag{9}$$

$$v(x,y,t) = (1+t)H(x,y), \quad (x,y) \in \Gamma. \tag{10}$$

There is maybe another selection of Eq. (5) by using $v(x,y,t) = q(t)u(x,y)$, where we require that $q(0) = 1$. However, when $q(t)$ is more complex than $1+t$ the resulting PDE is more complex than Eq. (9), and there seems no good reason to select a more complex $q(t)$.

The above idea is first proposed by Liu (2008b) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Then, Liu and his coworkers [Liu (2008c, 2008d); Liu, Chang, Chang and Chen (2008)] extended this idea to develop new methods for estimating parameters in the inverse vibration problems. Recently, Liu and Atluri (2008) have employed the above technique of fictitious time integration method (FTIM) to solve large system of nonlinear algebraic equations, and showed that high performance can be achieved by using the FTIM. Furthermore, Liu (2008e) has used the FTIM technique to solve the nonlinear complementarity problems, whose numerical results are very well.

2.2 Semi-discretization

Let $v_{i,j}(t) := v(x_i, y_j, t)$ be a numerical value of v at the grid point $(x_i, y_j) \in \Omega$ and at the time t. Applying a semi-discrete procedure on the above PDE in Eq. (9) yields a coupled system of ordinary differential equations (ODEs):

$$\dot{v}_{i,j} = \frac{\nu}{(\Delta x)^2}[v_{i+1,j} - 2v_{i,j} + v_{i-1,j}] + \frac{\nu}{(\Delta y)^2}[v_{i,j+1} - 2v_{i,j} + v_{i,j-1}] + \frac{v_{i,j}}{1+t} - \nu(1+t)F\left(x_i, y_j, \frac{v_{i,j}}{1+t}, \frac{v_{i+1,j} - v_{i-1,j}}{2(1+t)\Delta x}, \frac{v_{i,j+1} - v_{i,j-1}}{2(1+t)\Delta y}, \ldots\right), \tag{11}$$

where Δx and Δy are uniform spatial grid lengths in x and y directions, and m is the number of subintervals in each direction, assuming the same.

In this section we have transformed the boundary value problem of the second-order elliptic PDE in Eq. (1) to an evolutionary problem of a parabolic PDE in Eq. (9), and finally arrived to an initial value problem in the n-dimensional ODE system (11) with dimensions $n = m^2$. The initial value of Eq. (11) is given through a guess because the true initial condition of $v(x,y,0) = u(x,y)$ is not known a priori; however, when (x_i, y_j) is located on the boundary, the boundary condition (10) has to be satisfied.

2.3 GPS for differential equations system

Upon letting $\mathbf{v} = (v_{1,1}, v_{1,2}, \ldots, v_{m,m})^{\mathrm{T}}$ and $\mathbf{f}$ denoting a vector with the ij-th component being the right-hand side of Eq. (11) we can write it as a vector form:

$$\dot{\mathbf{v}} = \mathbf{f}(\mathbf{v}, t), \quad \mathbf{v} \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{12}$$

where n is the number of total grid points inside the domain Ω.

Group-preserving scheme (GPS) can preserve the internal symmetry group of the considered ODE system. Although we do not know previously the symmetry group of differential equations system, Liu (2001) has embedded it into an augmented differential system, which concerns with not only the evolution of state variables themselves but also the evolution of the magnitude of the state variables vector. Let us note that

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}} = \sqrt{\mathbf{v} \cdot \mathbf{v}}, \tag{13}$$

where the superscript T signifies the transpose, and the dot between two n-dimensional vectors denotes their inner product. Taking the derivatives of both the sides of Eq. (13) with respect to t, we have

$$\frac{d\|\mathbf{v}\|}{dt} = \frac{(\dot{\mathbf{v}})^{\mathrm{T}}\mathbf{v}}{\sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}}}. \tag{14}$$

Then, by using Eqs. (12) and (13) we can derive

$$\frac{d\|\mathbf{v}\|}{dt} = \frac{\mathbf{f}^{\mathrm{T}}\mathbf{v}}{\|\mathbf{v}\|}. \tag{15}$$

It is interesting that Eqs. (12) and (15) can be combined together into a simple matrix equation:

$$\frac{d}{dt}\begin{bmatrix} \mathbf{v} \\ \|\mathbf{v}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{v},t)}{\|\mathbf{v}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{v},t)}{\|\mathbf{v}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \|\mathbf{v}\| \end{bmatrix}. \tag{16}$$

It is obvious that the first row in Eq. (16) is the same as the original equation (12), but the inclusion of the second row in Eq. (16) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{v}^{\mathrm{T}}, \|\mathbf{v}\|)^{\mathrm{T}}$, which satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \tag{17}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{18}$$

is a Minkowski metric, and $\mathbf{I}_n$ is the identity matrix of order n. In terms of $(\mathbf{v}, \|\mathbf{v}\|)$, Eq. (17) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{v}\cdot\mathbf{v} - \|\mathbf{v}\|^2 = \|\mathbf{v}\|^2 - \|\mathbf{v}\|^2 = 0. \tag{19}$$

It follows from the definition given in Eq. (13), and thus Eq. (17) is a natural result. Consequently, we have an $n+1$-dimensional augmented system:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} \tag{20}$$

with a constraint (17), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{v},t)}{\|\mathbf{v}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{v},t)}{\|\mathbf{v}\|} & 0 \end{bmatrix}, \tag{21}$$

satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{22}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$. This fact prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{23}$$

$$\det \mathbf{G} = 1, \tag{24}$$

$$G_0^0 > 0, \tag{25}$$

where G_0^0 is the 00-th component of $\mathbf{G}$.

Although the dimension of the new system is raised one more, it has been shown that the new system permits a GPS given as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell)\mathbf{X}_\ell, \tag{26}$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at t_ℓ, and $\mathbf{G}(\ell) \in SO_o(n,1)$ is the group value of $\mathbf{G}$ at t_ℓ. If $\mathbf{G}(\ell)$ satisfies the properties in Eqs. (23)-(25), then $\mathbf{X}_\ell$ satisfies the cone condition in Eq. (17).

The Lie group can be generated from $\mathbf{A} \in so(n,1)$ by an exponential mapping,

$$\begin{aligned}\mathbf{G}(\ell) &= \exp[\Delta t \mathbf{A}(\ell)] \\ &= \begin{bmatrix} \mathbf{I}_n + \frac{(a_\ell - 1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell \mathbf{f}_\ell^{\mathrm{T}} & \frac{b_\ell \mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell \mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix},\end{aligned} \tag{27}$$

where

$$a_\ell := \cosh\left(\frac{\Delta t \|\mathbf{f}_\ell\|}{\|\mathbf{v}_\ell\|}\right), \tag{28}$$

$$b_\ell := \sinh\left(\frac{\Delta t \|\mathbf{f}_\ell\|}{\|\mathbf{v}_\ell\|}\right). \tag{29}$$

Substituting Eq. (27) for $\mathbf{G}(\ell)$ into Eq. (26), we obtain

$$\mathbf{v}_{\ell+1} = \mathbf{v}_\ell + \eta_\ell \mathbf{f}_\ell, \tag{30}$$

$$\|\mathbf{v}_{\ell+1}\| = a_\ell \|\mathbf{v}_\ell\| + \frac{b_\ell}{\|\mathbf{f}_\ell\|}\mathbf{f}_\ell \cdot \mathbf{v}_\ell, \tag{31}$$

where

$$\eta_\ell := \frac{b_\ell \|\mathbf{v}_\ell\| \|\mathbf{f}_\ell\| + (a_\ell - 1)\mathbf{f}_\ell \cdot \mathbf{v}_\ell}{\|\mathbf{f}_\ell\|^2} \tag{32}$$

is an adaptive factor. From $\mathbf{f}_\ell \cdot \mathbf{v}_\ell \geq -\|\mathbf{f}_\ell\| \|\mathbf{v}_\ell\|$ we can prove that

$$\eta_\ell \geq \left[1 - \exp\left(-\frac{\Delta t \|\mathbf{f}_\ell\|}{\|\mathbf{v}_\ell\|}\right)\right] \frac{\|\mathbf{v}_\ell\|}{\|\mathbf{f}_\ell\|} > 0, \quad \forall \Delta t > 0. \tag{33}$$

This scheme is group properties preserved for all $\Delta t > 0$, and is called the group-preserving scheme.

2.4 Numerical procedure

Starting from an initial value of $v_{i,j}$ which can be guessed in a rather free way, we employ the above GPS to integrate Eq. (11) from $t = 0$ to a selected final time t_f. In the numerical integration process we can check the convergence of $v_{i,j}$ at the ℓ- and $\ell+1$-steps by

$$\sqrt{\sum_{i,j=1}^{m} [v_{i,j}^{\ell+1} - v_{i,j}^{\ell}]^2} \leq \varepsilon, \tag{34}$$

where ε is a selected convergent criterion. If at a time $t_0 \leq t_f$ the above criterion is satisfied, then the solution of u is given by

$$u_{i,j} = \frac{v_{i,j}(t_0)}{1+t_0}. \tag{35}$$

In practice, if a suitable t_f is selected we find that the numerical solution is also approached very well to the true solution, even the above convergent criterion is not satisfied. The viscosity coefficient ν introduced in Eq. (11) can increase the stability of numerical integration. For example, for the Laplace equation we require that $\nu\Delta t/(\Delta x)^2 < 1/2$ for a reason of numerical stability [Ames (1992)], where Δt is a time stepsize used in the numerical integration; hence, if a small ν is selected, the above inequality can be satisfied.

In particular we would emphasize that the present method is a new fictitious time integration method (FTIM). Because it does not need to face the nonlinearity in the spatial domain, this new FTIM can calculate the boundary value problem of quasilinear elliptic equation very stably and effectively without needing of any iteration. Below we give numerical examples to display some advantages of the present FTIM.

3 Numerical examples

In this section we will apply the new method on both linear, semilinear and quasilinear boundary value problems. In order to focus on the study of the effect of our new method, we suppose that some exact solutions are known and can be compared with the numerical solutions, and the boundary is a rectangle, such that we do not need to consider the interpolation of boundary data when rectangular grids are used in the finite difference.

3.1 Example 1

We first consider an analytical solution of Laplace equation:

$$u(x,y) = e^x \cos y. \tag{36}$$

The domain is given by $\Omega = \{(x,y)|0 \leq x \leq 1,\ 0 \leq y \leq 1\}$. The exact boundary data $H(x,y)$ can be obtained by inserting the exact u on the boundary.

A straightfoward derivation leads to

$$\dot{v}_{i,j} = \frac{\nu}{(\Delta x)^2}[v_{i+1,j} - 2v_{i,j} + v_{i-1,j}] + \frac{\nu}{(\Delta y)^2}[v_{i,j+1} - 2v_{i,j} + v_{i,j-1}] + \frac{v_{i,j}}{1+t}, \tag{37}$$

where

$$v_{0,j}(t) = (1+t)\cos y_j,\ v_{m+1,j}(t) = (1+t)e\cos y_j, \tag{38}$$

$$v_{i,0}(t) = (1+t)e^{x_i},\ \ v_{i,m+1}(t) = (1+t)e^{x_i}\cos 1 \tag{39}$$

are time-varying boundary conditions. We fix $\Delta x = \Delta y = 1/m$ with $m = 20$, and the number of equations in Eq. (37) is $n = 19 \times 19$.

We start by an initial value of $v_{i,j} = 0.3$ and integrate Eq. (37) by using the GPS with a time stepsize $\Delta t = 0.001$. The final time is $t_f = 10$. Under a given $\nu = 0.1$ and $\varepsilon = 0.01$ the convergence is not performed within the range of $t < t_f = 10$.

At the point $y_0 = 0.5$ the error of u was plotted with respect to x in Fig. 1 by the dashed line, of which the maximum error is about 5×10^{-5}. At the point $x_0 = 0.5$ the error of u was plotted with respect to y in Fig. 1 by the solid line, of which the maximum error is about 4.9×10^{-5}. It can be seen that even the numerical solutions are not convergent, they are still rather accurate. If a larger $\varepsilon = 0.03$ is employed, at the 2126 steps the numerical solution is convergent; however, the maximum errors are both enlarged to 2.8×10^{-2}.

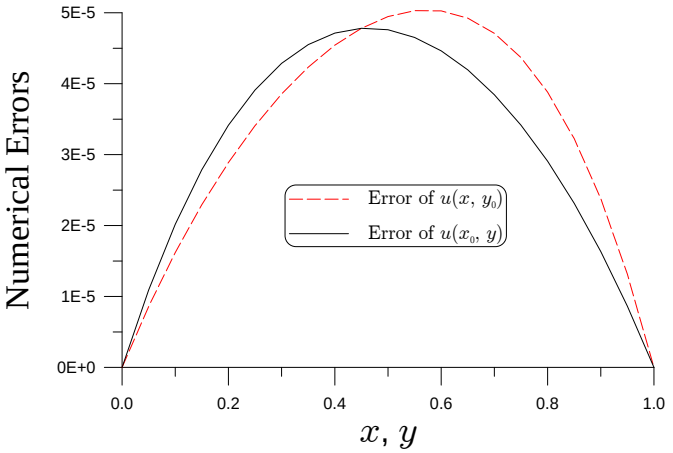

Figure 1: Plotting the numerical errors of Example 1 for a Laplace equation.

3.2 Example 2

Then we consider an analytical solution

$$u(x,y) = x^3 + 2xy \tag{40}$$

of a linear Poisson equation:

$$\Delta u = 6x. \tag{41}$$

The domain is given by $\Omega = \{(x,y)| -1 \le x \le 1,\ -1 \le y \le 1\}$. The exact boundary data $H(x,y)$ can be obtained by inserting the exact u on the boundary. Liu, Yeih, Kuo and Chen (2006) have solved this problem by a Trefftz method with iterations and using SVD regularization technique; however, their results are not good with an error in the order of 10^0.

A derivation leads to

$$\dot{v}_{i,j} = \frac{\nu}{(\Delta x)^2}[v_{i+1,j} - 2v_{i,j} + v_{i-1,j}] + \frac{\nu}{(\Delta y)^2}[v_{i,j+1} - 2v_{i,j} + v_{i,j-1}] + \frac{v_{i,j}}{1+t} - 6\nu(1+t)x_i. \tag{42}$$

Here we fix $\nu = 0.3$ and $m = 30$. Starting from an initial value of $v_{i,j} = 1$ we integrate Eq. (42) by using the GPS with a time stepsize $\Delta t = 0.001$. At the point $y_0 = 1/3$ the error of u was plotted with respect to x in Fig. 2(a) by the dashed line, of which the maximum error is about 3.2×10^{-7}. At the point $x_0 = 1/3$ the error of u was plotted with respect to y in Fig. 2(a) by the solid line, of which the maximum error is about 2.2×10^{-7}. Very accurate numerical results are obtained because we let $t_f = 15$ be larger than that of $t_f = 10$ used in the previous example.

We use the above closed-form solution again, but with the following nonlinear equation:

$$\Delta u = u^2 + 6x - x^6 - 4x^4 y - 4x^2 y^2. \tag{43}$$

By using the same parameters, at the point $y_0 = 1/3$ the error of u was plotted with respect to x in Fig. 2(b) by the dashed line, of which the maximum error is about 3.6×10^{-7}. At the point $x_0 = 1/3$ the error of u was plotted with respect to y in Fig. 2(b) by the solid line, of which the maximum error is about 1.8×10^{-7}. Similarly, for the nonlinear case very accurate numerical results are also obtained.

We have also written a program to calculate this example according to the steady state concept as shown in Eq. (3) by multiplying the right-hand side by a viscous damping constant ν; however, no matter which Δt and ν are used, that method cannot work. For example, under the following parameters of $\nu = 0.01$, $m = 30$, and $\Delta t = 0.0002$, the numerical result after 10000 time steps shows a maximum error of 1.8, which is already over the value of u itself.

3.3 Example 3

The following nonlinear diffusion reaction equation is considered:

$$\Delta u = 4u^3(x^2 + y^2 + a^2). \tag{44}$$

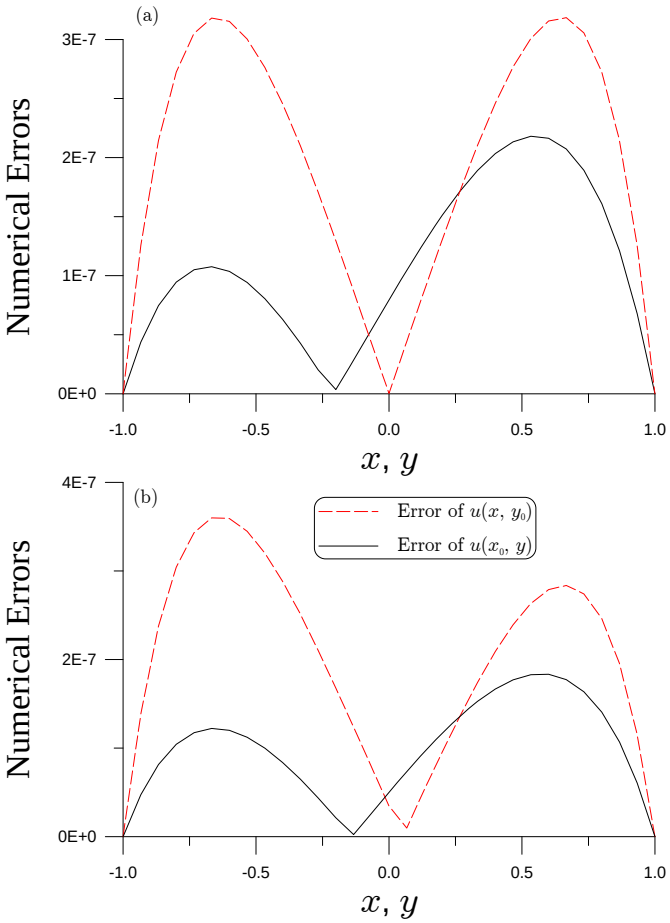

Figure 2: Plotting the numerical errors of Example 2: (a) for a linear Poisson equation, and (b) for a nonlinear Poisson equation.

The domain is same as that given in Example 2. The analytic solution

$$u(x,y) = \frac{-1}{x^2+y^2-a^2} \tag{45}$$

is singular on the circle with a radius a.

Algahtani (2005) has solved this problem by using a radial basis method, whose results as shown there in Fig. 3 are not matched well to the exact solution. Here we consider two cases of $a = 1.5$ and $a = 1.1$. By fixing $\nu = 0.1$, $m = 20$, $\Delta t = 0.002$ and $t_f = 10$ we solve this problem by our method, starting from an initial value of

$v_{i,j} = 1$. At the point $y_0 = 0$ the error of u was plotted with respect to x in Fig. 3(a) by the dashed line, of which the maximum error is about 2.1×10^{-3}. At the point $x_0 = 0.5$ the error of u was plotted with respect to y in Fig. 3(a) by the solid line, of which the maximum error is about 3.1×10^{-3}. When the singular circle is inside the domian with $a = 1.1$ the errors are plotted in Fig. 3(b) with $x_0 = 0.1$ and $y_0 = 0$. It can be seen that even the singularity is appeared in the problem domain, the numerical solutions are also acceptable with errors smaller than 0.25.

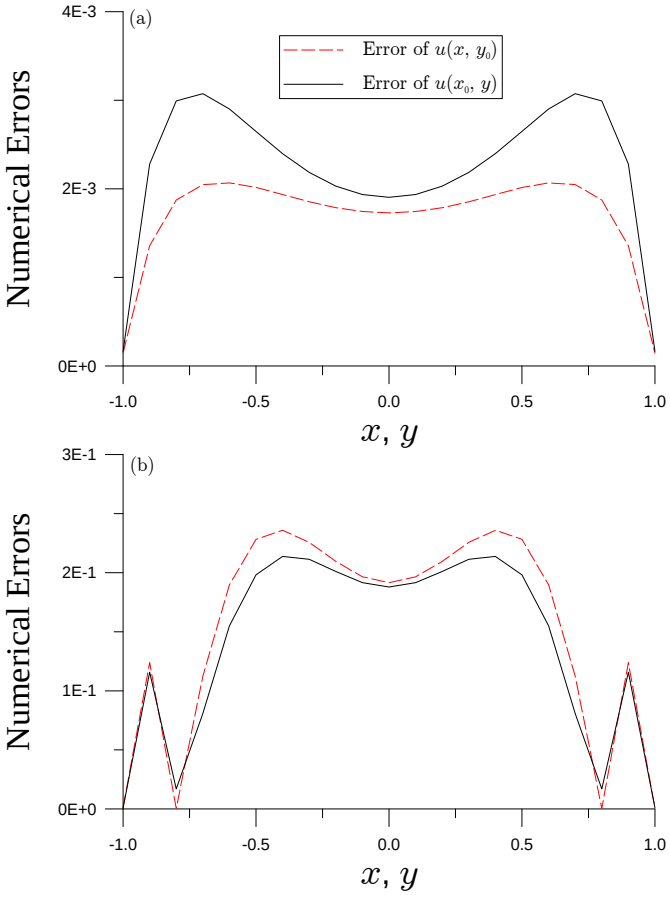

Figure 3: Plotting the numerical errors of Example 3 for (a) $a = 2$ and (b) $a = 1.1$ of a nonlinear reaction-diffusion equation.

3.4 Example 4

The following nonlinear Helmholtz equation is investigated:

$$\Delta u = k^2(u)u, \tag{46}$$

where we fix $k^2 = 4u^2$. The domain is same as that given in Example 1. The analytic solution

$$u(x,y) = \frac{1}{x+y+1} \tag{47}$$

is singular on the straight line $x+y=-1$.

By fixing $\nu = 0.1$, $m = 20$, $\Delta t = 0.002$ and $t_f = 20$ we solve this problem by our method, starting from an initial value of $v_{i,j} = 1$. At the point $y_0 = 0.75$ the error of u was plotted with respect to x in Fig. 4 by the dashed line, of which the maximum error is about 2×10^{-5}. At the point $x_0 = 0.5$ the error of u was plotted with respect to y in Fig. 4 by the solid line, of which the maximum error is about 3.3×10^{-5}. It can be seen that when t_f is increased the accuracy of numerical solutions is increased to the fifth order.

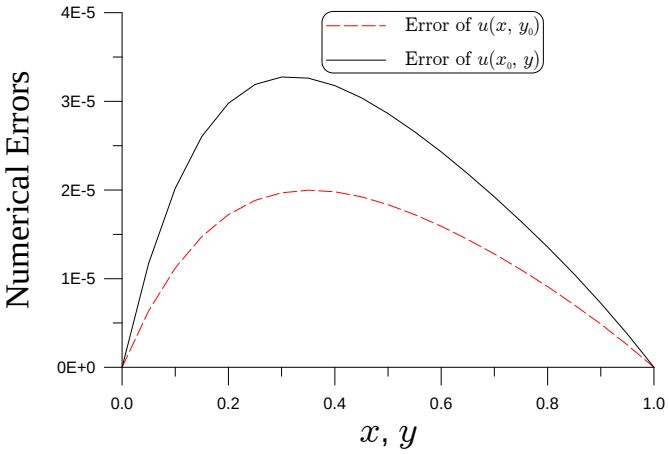

Figure 4: Plotting the numerical errors of Example 4 for a nonlinear Helmholtz equation.

3.5 Example 5

The following quasilinear equation is investigated:

$$\Delta u + u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy} = 0, \tag{48}$$

which is known a minimal surface equation. For this equation we can write

$$\frac{\partial v}{\partial t} = \nu\Delta v + \frac{v}{1+t} + \frac{\nu}{(1+t)^2}(v_x^2 v_{yy} + v_y^2 v_{xx} - 2v_x v_y v_{xy}), \quad (x,y) \in \Omega, \tag{49}$$

$$v(x,y,t) = 0, \quad (x,y) \in \Gamma. \tag{50}$$

The domain Ω is the same as that used in Example 1.

Under the following parameters of $\Delta x = \Delta y = 1/30$, $\Delta t = 0.0001$, $v_{i,j}(0) = 0.3$, $\nu = 0.1$ and $\varepsilon = 10^{-3}$ we find that the numerical solution is convergent at the 1688 time steps. We plot a minimal surface above the unit square in Fig. 5.

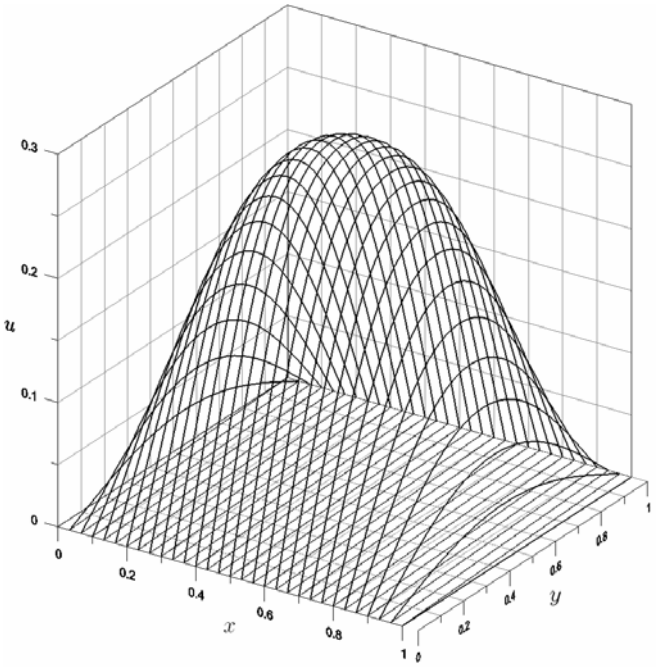

Figure 5: The minimal surface obtained by solving a quasilinear elliptic equation.

3.6 Example 6

A highly nonlinear case to test the effectiveness of FTIM is a thermal explosion problem of the Frank-Kamenettski equation:

$$\Delta u + \delta \exp[u] = 0. \tag{51}$$

This equation is frequently encountered for determining the regimes of safe operation for combustion and other exothemic processes.

For the present calculation we fix the parameter of $\delta = 1.5$. As remarked by Balakrishnan and Ramachandran (1999), this problem is highly nonlinear and can be used a benchmark for testing numerical methods. Under the following parameters of $\Delta x = \Delta y = 1/40$, $\Delta t = 0.0001$, $v_{i,j}(0) = 0.3$, $\nu = 0.1$ and $\varepsilon = 10^{-3}$ we find that the numerical solution is convergent at the 2146 time steps. We plot a surface of the solution above the unit square in Fig. 6. As compared with the method used by Chen (1995), it can be seen that the present FTIM is much simpler to obtain stable solution.

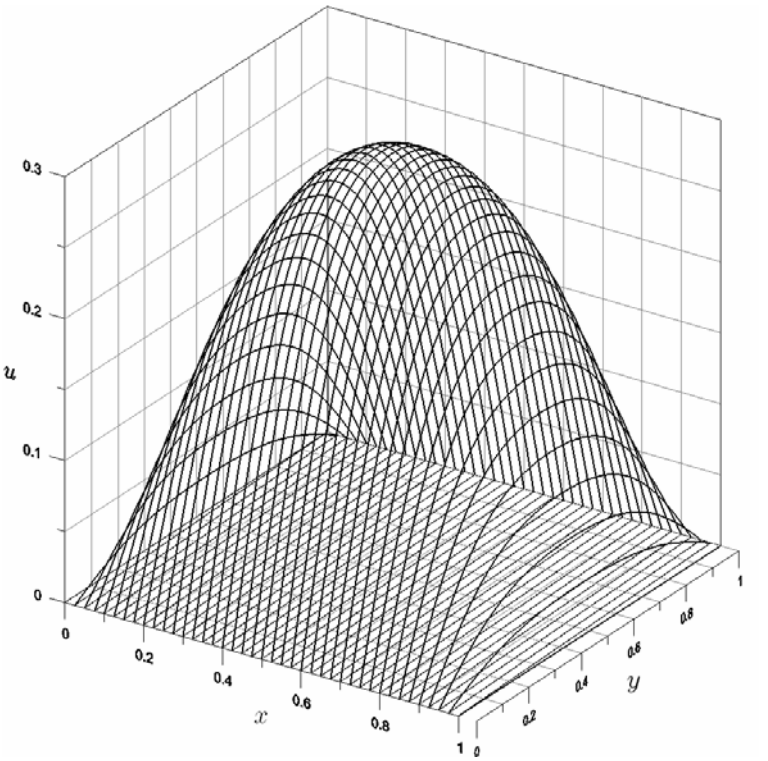

Figure 6: A surface obtained by solving a thermal explosion problem.

4 Conclusions

The present paper was the first time that the original quasilinear elliptic equation is mathematically transformed into a parabolic type evolutionary equation by in-

troducing a fictitious time coordinate, and adding a viscous damping coefficient to enhance the stability of numerical integration of the discretized equations by using a group preserving scheme. In the past several decades the methods developed to numerically solve the elliptic boundary value problems, in addition the linear cases, are frustrated by nonlinearity, and then require some iterations, because the methods were carried out in the spatial domain. In the present paper the nonlinearity of quasilinear elliptic equation is detoured by adding a fictitious time coordinate, and we only required to numerical integration the discretized equations to a certain time to obtain numerical solution. We must stress that the resulting parabolic equation is mathematically equivalent to the original equation, and no approximation is made. Hence, the present FTIM can work very effectively and accurately for the solution of boundary value problem of quasilinear elliptic equation. Because no iteration is required, the present method is very time saving.

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References

Algahtani, H. J. (2006): A meshless method for non-linear Poisson problems with high gradients. *Computer Assisted Mechanics and Engineering Sciences*, vol. 13, pp. 367-377.

Ames, W. F. (1992): Numerical Methods for Partial Differential Equations. 3rd ed., Academic Press, New York.

Atluri, S. N. (2004): The Meshless Method (MLPG) for Domain & BIE Discretizations. Tech Science Press, Forsyth, GA, USA.

Atluri, S. N.; Kim, H. G.; Cho, J. Y. (1999): A critical assessment of the truly meshless local Petrov-Galerkin (MLPG), and local boundary integral equation (LBIE) methods. *Computational Mechanics*, vol. 24, pp. 348-372.

Atluri, S. N.; Shen, S. (2002): The meshless local Petrov-Galerkin (MLPG) method: a simple & less-costly alternative to the finite element and boundary element methods. *CMES: Computer Modeling in Engineering & Sciences*, vol. 3, pp. 11-51.

Atluri, S. N.; Zhu, T. L. (1998a): A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. *Computational Mechanics*, vol. 22, pp. 117-127.

Atluri, S. N.; Zhu, T. L. (1998b): A new meshless local Petrov-Galerkin (MLPG) approach to nonlinear problems in computer modeling and simulation. *Computational Modeling and Simulation in Engineering*, vol. 3, pp. 187-196.

Balakrishnan, K.; Ramachandran, P. A. (1999): A particular solution Trefftz method for non-linear Poisson problems in heat and mass transfer. *Journal of Computational Physics*, vol. 150, pp. 239-267.

Chen, C. S. (1995): The method of fundamental solutions for non-linear thermal explosions. *Communications in Numerical Methods in Engineering*, vol. 11, pp. 675-681.

Chen, G.; Zhou, J. X. (2000): Algorithms and visualization for solutions of nonlinear elliptic equations. *International Journal of Bifurcation and Chaos*, vol. 10, pp. 1565-1612.

Cheng, A. H. D.; Golberg, M. A.; Kansa, E. J.; Zammito, G. (2003): Exponential convergence and H-c multiquadric collocation method for partial differential equations. *Numerical Method for Partial Differential Equations*, vol. 19, pp. 571-594.

Cho, H. A., Golberg, M. A.; Muleshkov, A. S.; Li, X. (2004): Trefftz methods for time-dependent partial differential equations. *CMC: Computers, Materials & Continua*, vol. 1, pp. 1-37.

Hu, H. Y.; Li, Z. C.; Cheng, A. H. D. (2005): Radial basis collocation methods for elliptic boundary value problems. *Computers and Mathematics with Applications*, vol. 50, pp. 289-320.

Jin, B. (2004): A meshless method for the Laplace and biharmonic equations subjected to noisy boundary data. *CMES: Computer Modeling in Engineering & Sciences*, vol. 6, pp. 253-261.

Li, Z. C.; Lu, T. T.; Huang, H. T.; Cheng, A. H. D. (2007): Trefftz, collocation, and other boundary methods–A comparison. *Numerical Method for Partial Differential Equations*, vol. 23, pp. 93-144.

Li, Z. C.; Lu, T. T.; Huang, H. T.; Cheng, A. H. D. (2008): Trefftz and Collocation Methods. WIT Press, Southampton.

Libre, A. N.; Emdadi, A.; Kansa, E. J.; Rahimian, M.; Shekarchi, M. (2008): A stabilized RBF collocation scheme for Neumann type boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 24, pp. 61-80.

Liu, C.-S. (2001): Cone of non-linear dynamical system and group preserving schemes. *International Journal of Non-Linear Mechanics*, vol. 36, pp. 1047-1068.

Liu, C.-S. (2006a): The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, pp. 149-163.

Liu, C.-S. (2006b): Efficient shooting methods for the second order ordinary differential equations. *CMES: Computer Modeling in Engineering & Sciences*, vol.

15, pp. 69-86.

Liu, C.-S. (2006c): The Lie-group shooting method for singularly perturbed two-point boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, pp. 179-196.

Liu, C.-S. (2007a): A modified Trefftz method for two-dimensional Laplace equation considering the domain's characteristic length. *CMES: Computer Modeling in Engineering & Sciences*, vol. 21, pp. 53-66.

Liu, C.-S. (2007b): A highly accurate solver for the mixed-boundary potential problem and singular problem in arbitrary plane domain. *CMES: Computer Modeling in Engineering & Sciences*, vol. 20, pp. 111-122.

Liu, C.-S. (2007c): An effectively modified direct Trefftz method for 2D potential problems considering the domain's characteristic length. *Engineering Analysis of Boundary Element*, vol. 31, pp. 983-993.

Liu, C.-S. (2007d): A meshless regularized integral equation method for Laplace equation in arbitrary interior or exterior plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 99-109.

Liu, C.-S. (2007e): A MRIEM for solving the Laplace equation in the doubly-connected domain. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 145-161.

Liu, C.-S. (2008a): A highly accurate collocation Trefftz method for solving the Laplace equation in the doubly-connected domains. *Numerical Method for Partial Differential Equations*, vol. 24, pp. 179-192.

Liu, C.-S. (2008b): Solving an inverse Sturm-Liouville problem by a Lie-group method. *Boundary Value Problems*, vol. 2008, Article ID 749865.

Liu, C.-S. (2008c): Identifying time-dependent damping and stiffness functions by a simple and yet accurate method. *Journal of Sound and Vibration*, vol. 318, pp. 148-165.

Liu, C.-S. (2008d): A Lie-group shooting method for simultaneously estimating the time-dependent damping and stiffness coefficients. *CMES: Computer Modeling in Engineering & Sciences*, vol. 27, pp. 137-149.

Liu, C.-S. (2008e): A time-marching algorithm for solving non-linear obstacle problems with the aid of an NCP-function. *CMC: Computers, Materials & Continua*, in press.

Liu, C.-S.; Atluri, S. N. (2008): A novel time integration method for solving a large system of non-linear algebraic equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 31, pp. 71-83.

Liu, C.-S.; Chang, J. R.; Chang, K. H.; Chen, Y. W. (2008): Simultaneously

estimating the time-dependent damping and stiffness coefficients with the aid of vibrational data. *CMC: Computers, Materials & Continua*, vol. 7, pp. 97-107.

Liu, R. F.; Yeih, W.; Kuo, S. R.; Chen, W. Y. (2006): Indirect T-Trefftz and F-Trefftz methods for solving boundary value problem of Poisson equation. *Journal of the Chinese Institute of Engineers*, vol. 29, pp. 989-1006.

Tsai, C. C.; Lin, Y. C.; Young, D. L.; Atluri, S. N. (2006): Investigations on the accuracy and condition number for the method of fundamental solutions. *CMES: Computer Modeling in Engineering & Sciences*, vol. 16, pp. 103-114.

Young, D. L.; Chen, K. H.; Chen, J. T.; Kao, J. H. (2007): A modified method of fundamental solutions with source on the boundary for solving Laplace equations with circular and arbitrary domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 197-222.

Zhu, T,; Zhang, J.; Atluri, S. N. (1998): A meshless local boundary integral equation (LBIE) method for solving nonlinear problems. *Computational Mechanics*, vol. 22, pp. 174-186.

Zhu, T,; Zhang, J.; Atluri, S. N. (1999): A meshless numerical method based on the local boundary integral equation (LBIE) to solve linear and non-linear boundary value problems. *Engineering Analysis of Boundary Element*, vol. 23, pp. 375-389.