 CMES, vol.33, no.3, pp.293-312, 2008

Exact Large Deflection Solutions for Timoshenko Beams with Nonlinear Boundary Conditions

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Abstract: A new analytic solution method is developed to find the exact static deflection of a Timoshenko beam with nonlinear elastic boundary conditions for the first time. The associated mathematic system is shifted and decomposed into six linear differential equations and at most four algebra equations. After finding the roots of the algebra equations, the exact solution of the nonlinear beam system can be reconstructed. It is shown that the proposed method is valid for the problem with strong nonlinearity. Examples, limiting studies and numerical analysis are given to illustrate the analysis. The exact solutions are compared with the perturbation solutions. The influence of the nonlinear spring constant and the slenderness ration on the errors of the perturbation solutions is evaluated.

Keyword: Timoshenko beams, static deflection, nonlinear boundary conditions, shifting function method, perturbation solution.

1 Introduction

Beams are one of the most commonly used structures in the world. It can be widely found in all the engineering fields. Based on the linear theory, including the Bernoulli-Euler and the Timoshenko beam theories, the studies on the static and dynamic response of beam structures are tremendous [Timoshenko (1955); Meirovitch (1967); Lee and Kuo (1992, 1993); Lee and Lin (1992, 1996, 1998); Iura, Suetake, and Atluri (2003); Beda (2003); Zupan and Saje (2003); Andreaus, Batra and Porfiri (2005); Vinod, Gopalakrishnan and Ganguli (2006); Lee and Hsu (2007); Huang and Shih (2007); Lin, Lee and Lin (2008)]. When the physical properties of a beam structure are uniform, the exact solution for the beam structures can be found in many standard text books [Timoshenko (1955); Meirovitch

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(1967)]. When the physic properties of non-uniform beams are in arbitrary polynomial forms, the exact solutions for various kinds of beam can be found in the works done by Lee and Kuo (1992,1993), Lee and Lin (1992, 1996, 1998); Lee and Hsu (2007), Lin, Lee and Lin (2008). In addition, many different kinds of numerical methods were employed to study the problems.

In the non-linear analysis, Emam and Nayfeh (2004), and Saffari, Rahgozar and Tabatabaei (2007) studied the beam problems with geometry nonlinearity. Monasa and Lewis (1983) studied the beam problems with material nonlinearity. Lee and Kuo (1994) and Coºkun (2000) examined the problems for a beam resting on nonlinear elastic foundation. Ma and Silva (2004), Turner (2004), Wolf and Gottlieb (2001), Fung and Huang (2001) and Kuang and Chen (2005), Lee, Lin, Lee, Lu and Liu (2008) investigated the response of a beam with nonlinear elastic boundary conditions.

It is well known that, in general, the exact solutions for the nonlinear beam problems are not available. The problems were mainly solved by approximated methods such as: the perturbation method [Monasa and Lewis (1983); Lee and Kuo (1994); Wolf and Gottlieb (2001)], the iterative method [Ma and Silva (2004)], the Galerkin's method [Emam and Nayfeh (2004); Cao and Zhang (2005); Lee and Soh (1994)], the finite element method [Saffari, Rahgozar and Tabatabaei (2007); Fung and Huang (2001)] and the Adomian decomposition method [Kuang and Chen (2005)]. One exact static deflection solution for a Bernoulli-Euler beam with particularly designed nonlinear boundary conditions was found in the paper by Ma and Silva (2004). Recently, Lee, Lin, Lee, Lu and Liu (2008) developed a new solution method to find the exact large deflection of a Bernoulli-Euler beam with nonlinear boundary conditions.

From the existing literature, it can be found that a systematic analytical method to find the exact solutions for the deflection of a Timoshenko beam with various nonlinear elastic boundary conditions still is not available. In this paper, a systematic analytical method which is an extension of the method developed by Lee and Lin (1998) and Lee, Lin, Lee, Lu and Liu (2008) is developed to find the exact large deflection solutions for Timoshenko beams with nonlinear elastically restrained end supports. The associated nonlinear mathematic system is changed and decomposed into six linear differential equations and at most four algebra equations. After finding the roots of the algebra equations, the exact solution of the nonlinear beam system can be reconstructed. The proposed method is valid for the problem with strong nonlinearity. Examples, limiting studies and numerical analysis are given to illustrate the analysis. Exact solution is compared with perturbation solution which is also a kind of analytic solution and widely used in the existing literature. The influence of the nonlinear spring constant and the slenderness ration on the errors of the perturbation solutions is evaluated.

2 Mathematical Modeling of the Beam System

Consider the static deflection of a uniform Timoshenko beam with nonlinear elastic boundary conditions, as shown in Figure 1. In terms of the following non-dimensional quantities:

$$\xi = \frac{x}{L}, \quad w(\xi) = \frac{W(\xi)}{L}, \quad p(\xi) = \frac{P(\xi)L^3}{EI}, \quad \beta_1 = \frac{K_{\theta L}L}{EI},$$

$$\mu = \frac{EI}{\kappa GAL^2}, \quad \beta_2 = \frac{K_{TL}L^3}{EI}, \quad \beta_3 = \frac{K_{\theta R}L}{EI}, \quad \beta_4 = \frac{K_{TR}L^3}{EI}, \quad K = \frac{kL^4}{EI}, \tag{1}$$

$$\gamma_1 = \frac{K_{N\theta L}L}{EI}, \quad \gamma_2 = \frac{K_{NTL}L^5}{EI}, \quad \gamma_3 = \frac{K_{N\theta R}L}{EI}, \quad \gamma_4 = \frac{K_{NTR}L^5}{EI}, \quad s^2 = \frac{AL^2}{I},$$

the two couple governing differential equations of the system are

$$-\frac{d}{d\xi}\left[\frac{1}{\mu}\left(\frac{dw(\xi)}{d\xi} - \Psi(\xi)\right)\right] + Kw(\xi) = p(\xi), \quad \xi \in (0,1), \tag{2}$$

$$\frac{d^2\Psi(\xi)}{d\xi^2} + \frac{1}{\mu}\left(\frac{dw(\xi)}{d\xi} - \Psi(\xi)\right) = 0, \quad \xi \in (0,1). \tag{3}$$

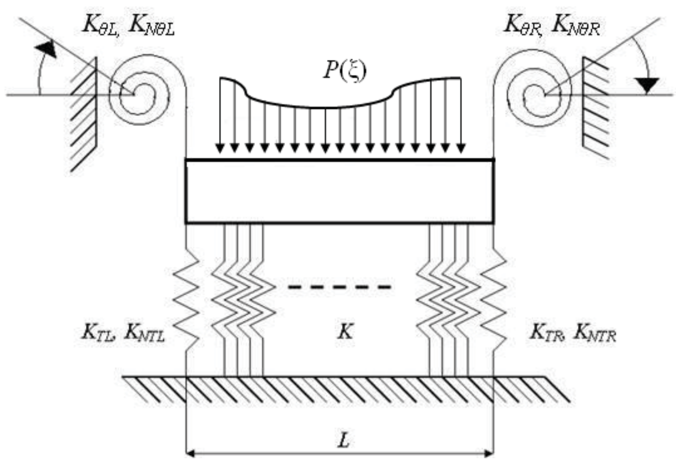

Figure 1: Geometry and coordinate system of a uniform beam with non-linear elastic boundary conditions

The nonlinear elastic boundary conditions are
at ξ = 0:

$$\frac{d\Psi(\xi)}{d\xi} - \beta_1\Psi(\xi) - \gamma_1\Psi^3(\xi) = 0, \tag{4}$$

$$-\frac{1}{\mu}\left(\frac{dw(\xi)}{d\xi} - \Psi(\xi)\right) + \beta_2 w(\xi) + \gamma_2 w^3(\xi) = 0, \tag{5}$$

at ξ = 1:

$$\frac{d\Psi(\xi)}{d\xi} - \beta_3\Psi(\xi) - \gamma_3\Psi^3(\xi) = 0, \tag{6}$$

$$\frac{1}{\mu}\left(\frac{dw(\xi)}{d\xi} - \Psi(\xi)\right) + \beta_4 w(\xi) + \gamma_4 w^3(\xi) = 0. \tag{7}$$

Here, $W(\xi)$ is the flexural displacement, $\Psi(\xi)$ is the angle of rotation due to bending and ξ is the space variable along the beam. E, G, κ, I, A, L and s are the Young's modulus, the shear modulus, the shear correction factor, the area moment of inertia, the cross section area, the length and the slenderness ratio of the beam, respectively. $P(\xi)$ is the applied distributed transverse force per unit length. K_{TL}, $K_{\theta L}$, K_{TR} and $K_{\theta R}$ are the linear translational spring constants and the linear rotational spring constants at the left end and the right end of the beam, respectively. K_{NTL}, $K_{N\theta L}$, K_{NTR} and $K_{N\theta R}$ are the nonlinear translational spring constants and the nonlinear rotational spring constants at the left end and the right end of the beam, respectively. K is the spring constant of elastic foundation.

3 Shifting Function Method

3.1 Change of variable

To find the solution for the two couple differential equation with nonlinear elastic boundary conditions, one extends the method developed by Lee and Lin (1998) and Lee, Lin, Lee, Lu and Liu (2008) by taking

$$w(\xi) = v(\xi) + \sum_{i=1}^{4} f_i g_i(\xi), \quad \Psi(\xi) = \varphi(\xi) + \sum_{i=1}^{4} \bar{f}_i \bar{g}_i(\xi), \tag{8}$$

where

$$f_1 = 0, \quad \bar{f}_1 = -\gamma_1\varphi^3(0) \tag{9}$$

$$f_2 = \mu\gamma_2 w^3(0), \quad \bar{f}_2 = 0 \tag{10}$$

$$f_3 = 0, \quad \bar{f}_3 = -\gamma_3 \varphi^3(1) \tag{11}$$

$$f_4 = \mu \gamma_4 w^3(1), \quad \bar{f}_4 = 0 \tag{12}$$

Here $v(\xi)$ and $\phi(\xi)$ are the transformed functions. $g_i(\xi)$ and $\bar{g}_i(\xi)$, i =1, 2, 3, 4 are the shifting functions to be specified. It should be mentioned that among the eight shifting functions, only four of them, $g_2(\xi)$, $g_4(\xi)$, $\bar{g}_1(\xi)$ and $\bar{g}_3(\xi)$, are required in this analysis.

Substituting equations (8-12) into equations (2-7), one has the differential equations for $v(\xi)$ and $\phi(\xi)$

$$\begin{aligned} &-\frac{1}{\mu}\left[\frac{d^2 v(\xi)}{d\xi^2} - \frac{d\varphi(\xi)}{d\xi}\right] + Kv(\xi) \\ &= p(\xi) + \frac{1}{\mu}\left[\sum_{i=1}^{4} f_i \left(\frac{d^2 g_i(\xi)}{d\xi^2}\right) - \sum_{i=1}^{4} \bar{f}_i \left(\frac{d\bar{g}_i(\xi)}{d\xi}\right)\right] - K\sum_{i=1}^{4} f_i g_i(\xi), \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{d^2\varphi(\xi)}{d\xi^2} + \frac{1}{\mu}\left(\frac{dv(\xi)}{d\xi} - \varphi(\xi)\right) \\ &= -\frac{1}{\mu}\sum_{i=1}^{4} f_i \left(\frac{dg_i(\xi)}{d\xi}\right) + \sum_{i=1}^{4} \bar{f}_i \left(\frac{1}{\mu}\bar{g}_i(\xi) - \frac{d^2\bar{g}_i(\xi)}{d\xi^2}\right), \end{aligned} \tag{14}$$

and the associated boundary conditions
at ξ = 0:

$$\frac{d\varphi(\xi)}{d\xi} - \beta_1 \varphi(\xi) = -\bar{f}_1 - \sum_{i=1}^{4} \bar{f}_i \left[\left(\frac{d\bar{g}_i(\xi)}{d\xi} - \beta_1 \bar{g}_i(\xi)\right)\right], \tag{15}$$

$$\begin{aligned} &-\left(\frac{dv(\xi)}{d\xi} - \varphi(\xi)\right) + \mu\beta_2 v(\xi) \\ &= -f_2 + \sum_{i=1}^{4} f_i \left(\frac{dg_i(\xi)}{d\xi} - \mu\beta_2 g_i(\xi)\right) - \sum_{i=1}^{4} \bar{f}_i \bar{g}_i(\xi), \end{aligned} \tag{16}$$

at ξ = 1:

$$\frac{d\varphi(\xi)}{d\xi} - \beta_3 \varphi(\xi) = -\bar{f}_3 - \sum_{i=1}^{4} \bar{f}_i \left[\left(\frac{d\bar{g}_i(\xi)}{d\xi} - \beta_1 \bar{g}_i(\xi)\right)\right], \tag{17}$$

$$\begin{aligned} &\left(\frac{dv(\xi)}{d\xi} - \varphi(\xi)\right) + \mu\beta_4 v(\xi) \\ &= -f_4 - \sum_{i=1}^{4} f_i \left(\frac{dg_i(\xi)}{d\xi} + \mu\beta_4 g_i(\xi)\right) + \sum_{i=1}^{4} \bar{f}_i \bar{g}_i(\xi). \end{aligned} \tag{18}$$

3.2 Shifting Functions

If the shifting functions $g_i(\xi)$ and $\bar{g}_i(\xi)$, i =1, 2, 3, 4 in equation (8) are chosen to satisfy the differential equations

$$\frac{d^2 g_i(\xi)}{d\xi^2} = 0, \quad \frac{1}{\mu}\bar{g}_i(\xi) - \frac{d^2 \bar{g}_i(\xi)}{d\xi^2} = 0, \tag{19}$$

and the following boundary conditions

$$\frac{d\bar{g}_i(\xi)}{d\xi} - \beta_1 \bar{g}_i(\xi) = \delta_{ij}, \quad j = 1, \tag{20}$$

$$\frac{dg_i(\xi)}{d\xi} - \mu\beta_2 g_i(\xi) = \delta_{ij}, \quad j = 2, \tag{21}$$

$$\frac{d\bar{g}_i(\xi)}{d\xi} - \beta_3 \bar{g}_i(\xi) = \delta_{ij}, \quad j = 3, \tag{22}$$

$$\frac{dg_i(\xi)}{d\xi} + \mu\beta_4 g_i(\xi) = \delta_{ij}, \quad j = 4, \tag{23}$$

where δ_{ij} is a Kronecker symbol, then the differential equations (13-14), the associated boundary conditions (15-18) can be reduced to

$$-\frac{1}{\mu}\left[\frac{d^2 v(\xi)}{d\xi^2} - \frac{\partial\varphi(\xi)}{\partial\xi}\right] = p(\xi) + \frac{1}{\mu}\left[\sum_{i=1}^{4} \bar{f}_i \left(\frac{d\bar{g}_i(\xi)}{d\xi}\right)\right] - K\sum_{i=1}^{4} f_i g_i(\xi), \tag{24}$$

$$\frac{d^2\varphi(\xi)}{d\xi^2} + \frac{1}{\mu}\left(\frac{dv(\xi)}{d\xi} - \varphi(\xi)\right) = -\frac{1}{\mu}\sum_{i=1}^{4} f_i \left(\frac{dg_i(\xi)}{d\xi}\right), \tag{25}$$

at ξ = 0:

$$\frac{\partial\varphi(\xi)}{\partial\xi} - \beta_1\varphi(\xi) = 0, \tag{26}$$

$$-\left(\frac{dv(\xi)}{d\xi} - \varphi(\xi)\right) + \mu\beta_2 v(\xi) = 0, \tag{27}$$

at ξ = 1:

$$\frac{d\varphi(\xi)}{d\xi} - \beta_3\varphi(\xi) = 0, \tag{28}$$

$$\frac{dv(\xi)}{d\xi} - \varphi(\xi) + \mu\beta_4 v(\xi) = 0. \tag{29}$$

Once the transformed functions$v(\xi)$ and $\phi(\xi)$ and the shifting functions $g_2(\xi)$, $g_4(\xi)$, $\bar{g}_1(\xi)$ and $\bar{g}_3(\xi)$ are determined, one substitutes these functions into equation (8). It leads to

$$w(\xi) = v(\xi) + \gamma_2 w^3(0) g_2(\xi) + \gamma_4 w^3(1) g_4(\xi), \tag{30}$$

$$\Psi(\xi) = \varphi(\xi) - \gamma_1 \Psi^3(0) \bar{g}_1(\xi) - \gamma_3 \Psi^3(1) \bar{g}_3(\xi), \tag{31}$$

where, $w(0)$, $w(1)$, $\Psi(0)$, $\Psi(1)$ are four constants to be determined.

Setting ξ = 0 and ξ = 1 into equations (30-31), one has the following algebra equations

$$w(0) = v(0) + \gamma_2 w^3(0) g_2(0) + \gamma_4 w^3(1) g_4(0), \tag{32}$$

$$w(1) = v(1) + \gamma_2 w^3(0) g_2(1) + \gamma_4 w^3(1) g_4(1), \tag{33}$$

$$\Psi(0) = \varphi(0) - \gamma_1 \Psi^3(0) \bar{g}_1(0) - \gamma_3 \Psi^3(1) \bar{g}_3(0), \tag{34}$$

$$\Psi(1) = \varphi(1) - \gamma_1 \Psi^3(0) \bar{g}_1(1) - \gamma_3 \Psi^3(1) \bar{g}_3(1). \tag{35}$$

As a result, the mathematic system of the nonlinear problem is shifted and decomposed into six linear differential equations, in terms of the transformed functions $v(\xi)$ and $\phi(\xi)$ and the shifting functions $g_2(\xi)$, $g_4(\xi)$, $\bar{g}_1(\xi)$and $\bar{g}_3(\xi)$, and at most four algebra equations. After finding the roots of the four algebra equations (32 - 35), the exact solution of the nonlinear beam system can be reconstructed from equations (30-31).

From equations (8-12, 24-29, 30-31), it can be observed that total solution is the superposition of the linear and the nonlinear parts of the solution. The transformed function $v(\xi)$ and $\phi(\xi)$ is corresponding to the solution of the associated linear system. The rest of terms in equation (8) are contributed from the nonlinear parts of the boundary conditions.

4 Verification and Examples

To illustrate the previous analysis, the following examples, limiting cases and numerical analysis are studied.

4.1 Clamped-nonlinear translational spring supported Timoshenko beam subjected to uniform distributed load

4.1.1 Timoshenko beam

Consider the deflection of a beam subjected to uniform distributed load P. The beam is clamped at the left end and is nonlinear translational spring supported at the other end. The governing differential equation and the boundary conditions are:

$$-\frac{1}{\mu}\frac{d^2w(\xi)}{d\xi^2}+\frac{1}{\mu}\frac{d\Psi(\xi)}{d\xi}=p, \tag{36}$$

$$\frac{d^2\Psi(\xi)}{d\xi^2}+\frac{1}{\mu}\left(\frac{dw(\xi)}{d\xi}-\Psi(\xi)\right)=0, \tag{37}$$

at ξ = 0:

$$\Psi=0,\ w=0, \tag{38}$$

at ξ = 1:

$$\frac{d\Psi(\xi)}{d\xi}=0,\ \frac{1}{\mu}\left(\frac{dw(\xi)}{d\xi}-\Psi(\xi)\right)+\beta_4 w(\xi)+\gamma_4 w^3(\xi)=0, \tag{39}$$

One lets

$$w(\xi)=v(\xi)+f_4 g_4(\xi), \tag{40}$$

where

$$f_4=\mu\gamma_4 w^3(1). \tag{41}$$

Here $g_4(\xi)$ is the shifting function to be specified. $v(\xi)$ is the transformed function which satisfies the differential equation

$$-\frac{1}{\mu}\frac{d^2v(\xi)}{d\xi^2}+\frac{1}{\mu}\frac{d\Psi(\xi)}{d\xi}=p, \tag{42}$$

$$\frac{d^2\Psi(\xi)}{d\xi^2}+\frac{1}{\mu}\left[\frac{dv(\xi)}{d\xi}-\Psi(\xi)\right]=\frac{f_4}{\mu\left(1+\mu\beta_4\right)}, \tag{43}$$

and the homogeneous boundary conditions
at ξ = 0:

$$\Psi(\xi)=0,\quad v(\xi)=0,$$
$$\frac{1}{1+\beta_2}\left(\frac{\partial^3 V}{\partial\xi^3}+n\frac{\partial V}{\partial\xi}+\beta_2 V\right)=\bar{f}_2-\bar{f}_i\left[\frac{1}{1+\beta_2}\left(\frac{\partial^3 g}{\partial\xi^3}+n\frac{\partial g}{\partial\xi}+\beta_2 g\right)\right] \tag{44}$$

at ξ = 1:

$$\frac{d\Psi(\xi)}{d\xi}=0,\quad \left(\frac{dv(\xi)}{d\xi}-\Psi(\xi)\right)+\mu\beta_4 v(\xi)=0. \tag{45}$$

It can be easily found that the function $\psi(\xi)$ and the transformed function $v(\xi)$ are

$$\Psi(\xi) = \frac{P\xi^3}{6} + m\xi + n\xi^2, \tag{46}$$

$$v(\xi) = \frac{\xi(24\mu\gamma_4 w^3(1) + (P\xi^3 - 12P\xi\mu + 12\xi m + 8\xi^2 n - 48\mu n)(1+\mu\beta_4))}{24(1+\mu\beta_4)}, \tag{47}$$

where

$$m = -\frac{-12P + 24\gamma_4 w^3(1) - P\beta_4}{8(3+\beta_4+3\mu\beta_4)}, \quad n = -\frac{24P - 24\gamma_4 w^3(1) + 5P\beta_4 + 12P\mu\beta_4}{16(3+\beta_4+3\mu\beta_4)}.$$

The shifting function $g_4(\xi)$ satisfies the following differential equation and the homogeneous boundary conditions:

$$\frac{d^2 g_4(\xi)}{d\xi^2} = 0, \tag{48}$$

at $\xi = 0$:

$$g_4(0) = 0, \tag{49}$$

at $\xi = 1$:

$$\frac{dg_4(1)}{d\xi} + \mu\beta_4 g_4(1) = -1. \tag{50}$$

The shifting function $g_4(\xi)$ is determined as

$$g_4(\xi) = \frac{-\xi}{1+\mu\beta_4}. \tag{51}$$

Substituting the transformed function $v(\xi)$ and the shifting function $g_4(\xi)$ back into equation (40), one has

$$\Psi(\xi) = m\xi + n\xi^2 + \frac{1}{6}p\xi^3, \tag{52}$$

$$\begin{aligned} w(\xi) = & \frac{\xi(24\mu\gamma_4 w^3(1) + (p\xi^3 - 12p\xi\mu + 12\xi m + 8\xi^2 n - 48\mu n)(1+\mu\beta_4))}{24(1+\mu\beta_4)} \\ & + \mu\gamma_4 w^3(1)\frac{-\xi}{1+\mu\beta_4}, \end{aligned} \tag{53}$$

where

$$m=-\frac{-12p+24\gamma_4 w^3(1)-p\beta_4}{8\left(3+\beta_4+3\mu\beta_4\right)},\quad n=-\frac{24p-24\gamma_4 w^3(1)+5p\beta_4+12p\mu\beta_4}{16\left(3+\beta_4+3\mu\beta_4\right)}.$$

Setting ξ = 1 in the equations (52-53), and using the Cardano's formula, one obtains $w(1)$.

$$w(1)=\frac{-2^{1/3}8\left(3+\beta_4+3\mu\beta_4\right)}{A}+\frac{A}{2^{1/3}24(\gamma_4+3\mu\gamma_4)}, \tag{54}$$

where

$$A=\left(\Lambda+\sqrt{28311552\left(3+\beta_4+3\mu\beta_4\right)^3\left(\gamma_4+3\mu\gamma_4\right)^3+\Lambda^2}\right)^{1/3},$$

$$\Lambda=p\gamma_4^2\left(5184+51840\mu+171072\mu^2+186624\mu^2\gamma_4\right).$$

After substituting $w(1)$ back to equation (53), one obtains the exact dimensionless deflection of the problem.

When the nonlinear spring constant is zero, γ_4 = 0, the system turns to be a linear one. The exact solution is reduced to

$$w(\xi)=\frac{\xi\left(p\xi^3-12p\xi\mu+12\xi m+8\xi^2 n-48\mu n\right)(1+\mu\beta_4)}{24(1+\mu\beta_4)}. \tag{55}$$

When both linear and nonlinear spring constants are zeros, γ_4 = 0 and β_4 = 0, it is further reduced to

$$w(\xi)=\frac{\xi^4+4\xi^3-6\left(1-2\mu\right)\xi^2-24\mu\xi}{24}p. \tag{56}$$

It is the exact dimensionless deflection of a cantilevered Timoshenko beam subjected to uniformly dimensionless distributed load p.

4.1.2 Bernoulli-Euler beam

In the Bernoulli-Euler beam theory, the shear deformation is not considered. By setting μ = 0 in equations (53-54), they are reduced to

$$\begin{aligned} w(\xi)=&\left(\frac{\beta_4+12}{16\beta_4+48}\right)p\xi^2-\left(\frac{5\beta_4+24}{48\beta_4+144}\right)p\xi^3+\frac{1}{24}p\xi^4\\ &+\gamma_4 w^3(1)\left(-\frac{3}{2\beta_4+6}\xi^2+\frac{1}{2\beta_4+6}\xi^3\right), \end{aligned} \tag{57}$$

where

$$w(1) = \frac{-2^{1/3}8\,(3+\beta_4)}{H^{1/3}} + \frac{H^{1/3}}{2^{1/3}24(\gamma_4+3\mu\gamma_4)}, \tag{58}$$

$$H = 5184p\gamma_4^2 + \sqrt{28311552\,(3+\beta_4)^3\,\gamma_4^3 + 26873856p^2\gamma_4^4}.$$

The nonlinear deflection of the Bernoulli-Euler beam is the same as that given by Lee, Lin, Lee, Lu and Liu (2008).

4.1.3 Perturbation solutions

To compare the derived exact solution with the perturbation solution which is a kind of approximated analytic solution commonly used in the literature, one lets

$$\begin{aligned} w(\xi) &= w_0(\xi) + \gamma_4 w_1(\xi) + \gamma_4^2 w_2(\xi) + \ldots.. \\ &= w_0(\xi) + \varepsilon w_1(\xi) + \varepsilon^2 w_2(\xi) + \ldots.., \end{aligned} \tag{59}$$

$$\Psi(\xi) = \Psi_0(\xi) + \varepsilon\Psi_1(\xi) + \varepsilon^2\Psi_2(\xi) + \ldots.., \tag{60}$$

Substituting the two equations above into the governing differential equations, equations (2-3), and the associated boundary conditions, equations (4-7), and collecting terms with like power of ε, one obtains the governing differential equations and the associated boundary conditions for $w_0(\xi)$, $\Psi_0(\xi)$, $w_1(\xi)$, $\Psi_1(\xi)$, $w_2(\xi)$ and $\Psi_2(\xi)$.

The governing differential equations for $w_0(\xi)$ and $\Psi_0(\xi)$ are

$$-\frac{1}{\mu}\frac{d^2w_0(\xi)}{d\xi^2} + \frac{1}{\mu}\frac{d\Psi_0(\xi)}{d\xi} = p, \tag{61}$$

$$\frac{d^2\Psi_0(\xi)}{d\xi^2} + \frac{1}{\mu}\left(\frac{dw_0(\xi)}{d\xi} - \Psi_0(\xi)\right) = 0. \tag{62}$$

The associated boundary conditions are

$$\Psi_0(0) = 0, \quad w_0(0) = 0, \tag{63}$$

$$\frac{d\Psi_0}{d\xi}(1) = 0, \quad \frac{1}{\mu}\left(\frac{dw_0(1)}{d\xi} - \Psi_0(1)\right) + \beta_4 w_0(1) = 0. \tag{64}$$

The solution for $w_0(\xi)$ is

$$w_0(\xi) = \frac{\xi\left(p\xi\mu\left(\xi^2-12\mu\right) + 4m_0\left(-\xi^2+6\mu+n_0\xi\right)\right)}{24\mu}, \tag{65}$$

where

$$m_0 = \frac{p\mu\,(24+\beta_4\,(5+12\mu))}{8\,(3+\beta_4+3\beta_4\mu)}, \quad n_0 = \frac{3\,(12+\beta_4)}{24+5\beta_4+12\beta_4\mu}. \tag{66}$$

The governing differential equations for $w_1(\xi)$ and $\Psi_1(\xi)$ are

$$\frac{d^2 w_1(\xi)}{d\xi^2} - \frac{d\Psi_1(\xi)}{d\xi^2} = 0, \tag{67}$$

$$\frac{d^2\Psi_1(\xi)}{d\xi^2} + \frac{1}{\mu}\left(\frac{dw_1(\xi)}{d\xi} - \Psi_1(\xi)\right) = 0, \tag{68}$$

and the associated boundary conditions are

$$\Psi_1(0) = 0, \quad w_1(0) = 0, \tag{69}$$

$$\frac{d\Psi_1}{d\xi}(1) = 0, \quad \frac{1}{\mu}\frac{dw_1(1)}{d\xi} - \frac{1}{\mu}\Psi_1(1) + \beta_4 w_1(1) + w_0^3(1) = 0. \tag{70}$$

The solution for $w_1(\xi)$ is

$$w_1(\xi) = \frac{\xi\left(\xi^2 - 3\xi - 6\mu\right)}{6\mu}\frac{81\mu\,(p+4C\mu)^3}{512\,(3+\beta_4+3\beta_4\mu)^4}. \tag{71}$$

The governing differential equations for $w_2(\xi)$ and $\Psi_2(\xi)$ are

$$\frac{d^2 w_2(\xi)}{d\xi^2} - \frac{d\Psi_2(\xi)}{d\xi^2} = 0, \tag{72}$$

$$\frac{d^2\Psi_2(\xi)}{d\xi^2} + \frac{1}{\mu}\left(\frac{dw_2(\xi)}{d\xi} - \Psi_2(\xi)\right) = 0, \tag{73}$$

The associated boundary conditions are

$$\Psi_1(0) = 0, \quad w_1(0) = 0, \tag{74}$$

$$\frac{d\Psi_1}{d\xi}(1) = 0, \frac{1}{\mu}\frac{dw_2(1)}{d\xi} - \frac{1}{\mu}\Psi_2(1) + \beta_4 w_2(1) + 3w_0^2(1)w_1(1) = 0. \tag{75}$$

The solution for $w_2(\xi)$ is

$$w_2(\xi) = -\frac{\xi\left(\xi^2 - 3\xi - 6\mu\right)}{6\mu}\frac{2187\mu\,(1+3\mu)\,(p+4C\mu)^5}{32768\,(3+\beta_4+3\beta_4\mu)^7}. \tag{76}$$

After substituting $w_0(\xi)$, $w_1(\xi)$, and $w_2(\xi)$ back to equation (59), one has the perturbation solution of the nonlinear system in term of $w(\xi)$.

By setting $\mu = 0$, it yields to the perturbation solution for the Bernoulli-Euler beam

$$\begin{aligned} w(\xi) =& \frac{\beta_4+12}{16\beta_4+48}p\xi^2 - \frac{5\beta_4+24}{48\beta_4+144}p\xi^3 + \frac{1}{24}p\xi^4 \\ &+\varepsilon\left(\frac{3p}{8\beta_4+24}\right)^3\left(-\frac{3}{2\beta_4+6}\xi^2+\frac{1}{2\beta_4+6}\xi^3\right) \\ &+\varepsilon^2\left(\frac{3p}{8\beta_4+24}\right)^5\left(\frac{3}{3+\beta_4}\right)^2\left(\frac{1}{2}\xi^2-\frac{1}{6}\xi^3\right)+\ldots \end{aligned} \tag{77}$$

4.1.4 Numerical analysis

To compare the developed exact solutions with the perturbation solutions, one defines:

$$\begin{aligned} &\mathrm{Error}(1) = (|B_P - B_E|/B_E)\times 100\%, \\ &\mathrm{Error}(2) = (|T_P - T_E|/T_E)\times 100\%, \\ &\mathrm{Error}(3) = (|B_E - T_E|/T_E)\times 100\%, \\ &\mathrm{Error}(4) = (|B_P - T_E|/T_E)\times 100\%, \end{aligned} \tag{78}$$

where T_E is the exact solution based on the Timoshenko beam theory, equations (53-54). T_P is the perturbation solution based on the Timoshenko beam theory, equations (59, 65, 71, and 76). B_E is the exact solution based on the Bernoulli-Euler beam theory, equations (57-58). B_P is the perturbation solution based on the Bernoulli-Euler beam theory, equation (77).

In the following numerical analysis, one considers a beam of square cross section with width d and length $10d$. The beam structure is constructed by the material AISI 1020 (E: 210GPa; G: 80GPa, μ = 0.0025) and subjected to a dimensionless uniformly distributed load $p = 5$.

In Tables 1-2, based on two different beam theories, the deflections of the beam evaluated via two different approaches are presented. The errors are also evaluated. It can be observed that the errors of the perturbation solutions increase as the nonlinear spring constant $K_2$is increased. The perturbation solutions are not accurate enough for a system of high nonlinearity. The conclusions are consistent with those we are familiar with.

In figures 2-3, the influence of the nonlinear spring constant K_2on Errors (1) $\sim$ (4) at two different positions are shown. It can be observed that

a. The influence of the nonlinear spring constant K_2on Error (3) is not significant. Errors (3) at ξ = 1 are less than those at ξ = 0.1.

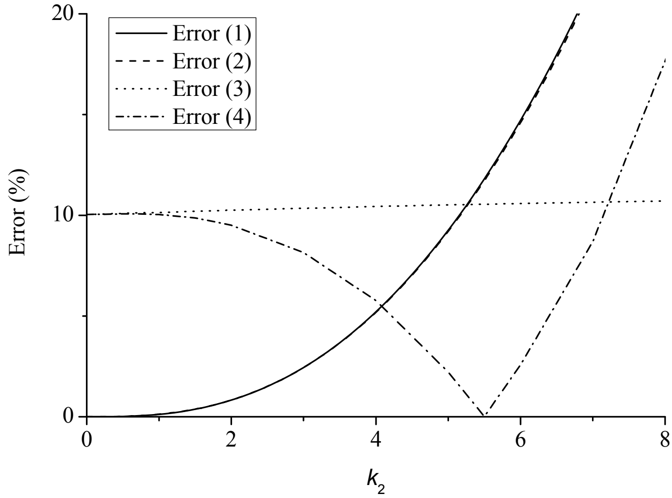

Figure 2: Influence of the nonlinear spring constant k_2 on Errors (1) $\sim$ (4) at ξ = 0.1

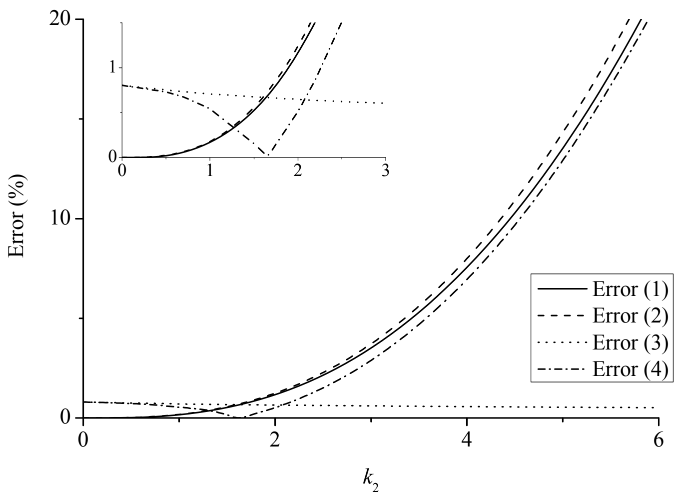

Figure 3: Influence of the nonlinear spring constant k_2 on Errors (1) $\sim$ (4) at ξ = 1

Table 1: Exact and perturbation solutions for the nonlinear deflections of the Bernoulli-Euler beam

Bernoulli-Euler beam (p = 5, k_1= 1)									
	k_2= 1			k_2= 5			k_2= 10		
ξ	B$_E$	B$_P$	Error (1) (%)	B$_E$	B$_P$	Error (1) (%)	B$_E$	B$_P$	Error (1) (%)
0	0	0	0	0	0	0	0	0	0
0.1	0.009	0.009	0.00	0.008	0.009	12.50	0.008	0.012	50.00
0.5	0.166	0.166	0.00	0.149	0.165	10.74	0.137	0.225	64.23
1	0.447	0.447	0.00	0.393	0.446	13.49	0.356	0.636	78.65

Table 2: Exact and perturbation solutions for the nonlinear deflections of the Timoshenko beam

Timoshenko beam (p = 5, k_1= 1, μ= 0.0025)									
	k_2= 1			k_2= 5			k_2= 10		
ξ	T$_E$	T$_P$	Error (2) (%)	T$_E$	T$_P$	Error (2) (%)	T$_E$	T$_P$	Error (2) (%)
0	0	0	0	0	0	0	0	0	0
0.1	0.010	0.010	0.00	0.009	0.010	11.11	0.009	0.013	44.44
0.5	0.169	0.169	0.00	0.152	0.170	11.84	0.140	0.233	66.43
1	0.450	0.450	0.00	0.395	0.452	14.43	0.358	0.654	82.68

b. Independent of the beam theory employed, the errors of the perturbation solutions, Errors (1) and (2), increase as the nonlinear spring constant K_2 is increased. Since the beam considered is a slender beam, the difference between the two Errors is relatively small.

c. Error (4) will decrease to zero first, then, increase rapidly as the nonlinear spring constant K_2 is increased. The reflection of the curve at Error (4) = 0 is due to the definition of the Error which is the absolute value of the difference of two different solutions.

d. Errors (1, 2, 4) at ξ = 1 are greater than those at ξ = 0.1.

Figure 4 illustrates the influence of the slenderness ratio on the Errors (2) $\sim$ (4) at ξ = 1 when k_2 = 5. It can be observed that

a. The difference between the deflections evaluated via two different beam theories, Error (3), decreases and approaches to zero as the slenderness ratio is increased.

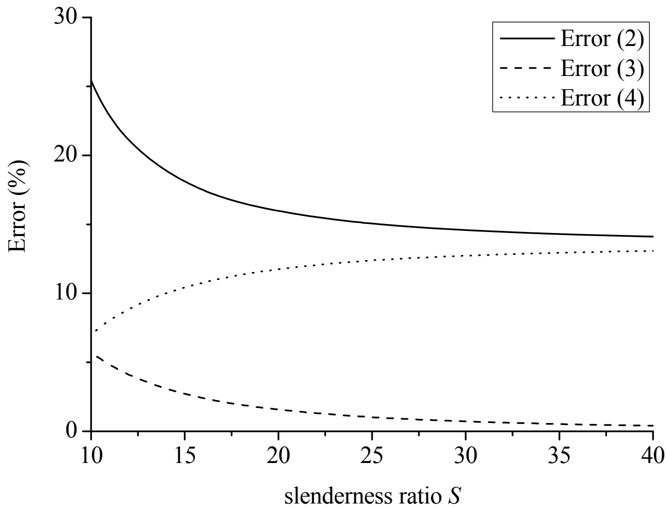

Figure 4: Influence of the slenderness ratio s on Errors (2) $\sim$ (4) at ξ = 1 (k_2 = 5)

b. As the slenderness ratio is increased, Error (2) decreases and Error (4) increases. Finally, they will approach to the same value.

4.2 Clamped-nonlinear rotational spring supported Timoshenko beam subjected to uniform distributed load

Consider the deflection of a beam subjected to uniformly distributed load P with clamped one end and nonlinear rotational spring support at the other end. The governing differential equations are the same as equations (61-62) and the boundary conditions are
at ξ = 0:

$$\Psi = 0, \quad w = 0, \tag{79}$$

at ξ = 1:

$$\frac{d\Psi(\xi)}{d\xi} - \beta_3\Psi(\xi) - \gamma_3\Psi^3(\xi) = 0\,, \quad \frac{1}{\mu}\left(\frac{dw(\xi)}{d\xi} - \Psi(\xi)\right) = 0. \tag{80}$$

One lets

$$\Psi(\xi) = \varphi(\xi) + \bar{f}_3\bar{g}_3(\xi), \tag{81}$$

where

$$\bar{f}_3 = -\gamma_3\Psi^3(1), \tag{82}$$

Following the procedures revealed in the last section, one has

$$\varphi(\xi) = \bar{m}\xi + \bar{n}\xi^2 + \frac{1}{6}p\xi^3, \tag{83}$$

$$w(\xi) = \frac{\xi\left[-12\xi\gamma_3\psi^3(1) + (-1+\beta_3)(p\xi^3 - 12p\xi\mu + 12\xi\bar{m} + 8\xi^2\bar{n} - 48\mu\bar{n})\right]}{24(-1+\beta_3)}, \tag{84}$$

where $\bar{m} = -\frac{3p-2p\beta_3}{6(-1+\beta_3)}$, $\bar{n} = -\frac{p}{2}$ and

$$\overline{g}_3(\xi) = \frac{\xi}{\beta_3 - 1}. \tag{85}$$

After substituting the equations (83) and (85) above back to equation (81), one has

$$\Psi(\xi) = \bar{m}\xi + \bar{n}\xi^2 + \frac{1}{6}p\xi^3 - \gamma_3\Psi^3(1)\frac{\xi}{\beta_3 - 1}, \tag{86}$$

$$w(\xi) = \frac{\xi(-12\xi\gamma_3\Psi^3(1) + (-1+\beta_3)(p\xi^3 - 12p\xi\mu + 12\xi\bar{m} + 8\xi^2\bar{n} - 48\mu\bar{n}))}{24(-1+\beta_3)}, \tag{87}$$

where $\bar{m} = -\frac{p(3-2\beta_3)}{6(-1+\beta_3)}$ and $\bar{n} = -\frac{p}{2}$.
Setting $\xi = 1$ in the equations (86-87), and Using the Cardano's formula, one obtains $\Psi(1)$.

$$\Psi(1) = \frac{Q}{2^{1/3}18\gamma_3} - \frac{2^{1/3}6\,(-1+\beta_3)}{Q}, \tag{88}$$

where

$$Q = \left(-972p\gamma_3^2 + \sqrt{5038848\,(-1+\beta_3)^3\,\gamma_3^3 + 944784p^2\gamma_3^4}\right)^{1/3}.$$

After substituting $\Psi(1)$ back to equations (86-87), one obtains the exact solutions of the problem.
For a Bernoulli-Euler beam, equations (87) is reduced to the following equation by setting μ being zero

$$w(\xi) = p\xi^2\left[\left(\frac{2\beta_3 - 3}{12\beta_3 - 12}\right) - \frac{1}{6}\xi + \frac{1}{24}\xi^2\right] + \gamma_3\left(\frac{dw(1)}{d\xi}\right)^3\left(\frac{1}{2-2\beta_3}\xi^2\right). \tag{89}$$

Setting ξ = 1 in the equation (89), and Using the Cardano's formula, one obtains

$$\frac{dw(1)}{d\xi} = \frac{Q}{2^{1/3}18\gamma_3} - \frac{2^{1/3}6(-1+\beta_3)}{Q}. \tag{90}$$

The solution form, equation (90), is the same as that of equation (88)

When γ_3 = 0, the system turns to be a linear problem. Equation (89) is reduced to

$$w(\xi) = p\xi^2 \left[\left(\frac{2\beta_3 - 3}{12\beta_3 - 12} \right) - \frac{1}{6}\xi + \frac{1}{24}\xi^2 \right]. \tag{91}$$

5 Conclusions

In this paper, an analytic solution method is developed to find the exact static deflection of a Timoshenko beam with nonlinear elastic springs supports at ends for the first time. The associated mathematic system is shifted and decomposed into six linear differential equations and at most four algebra equations. After finding the roots of the algebra equations, the exact solution of the nonlinear beam system can be reconstructed. It is shown that the proposed method is valid for the problem with strong nonlinearity. Examples and limiting studies are given to illustrate the analysis. The exact solutions are compared with the perturbation solutions. The errors of the perturbation solutions are evaluated. In the present study, the loading considered is a distributed force only. However, it can be easily extended to the problems with various kind of loading. It will be of interesting to extend the proposed solution method to study different kinds of nonlinear problems.

Acknowledgement: This research work was supported by the National Science Council of Taiwan, the Republic of China, under grant: NSC95-2221-E-006-033 and the Ministry of Economic Affairs of Taiwan, the Republic of China, under grant: TDPA: 95-EC-17-A-05-S1-0014.

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