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Modeling of Structural Sandwich Plates with 'Through-the-Thickness' Inserts: Five-Layer Theory

Song-Jeng Huang[1,2] **and Lin-Wei Chiu**[2]

Abstract: The composite sandwich plate is one of the most common composite structures. Local stress concentrations can be caused by localized bending effects where a load is introduced. Although a sandwich structure with an insert is one of the classical load bearing structures, little work has been conducted on the adhesive layers or inserts. This study involves a linear elasticity analysis of five-layer sandwich plates with "through-the-thickness" inserts, using sandwich plate theory to analyze deformation behavior. Governing equations are formulated as partial differential equations, which are solved numerically using the multi-segment integration method. Sandwich plates with "through-the-thickness" inserts subjected to axisymmetric external loading are considered as examples. Stress concentrations closest to the intersection between the potting material, core and adhesive layer are likely to fail, as observed. A comparison with degraded three-layer theory and Thomsen's theory [Thomsen and Rits (1998)] confirmed the accuracy of the proposed five-layer theory. The finite element method results in this study were obtained using the ABAQUS software and compared with analytical results. The validity of the analytical solution (five-layer theory) was also demonstrated.

Keyword: sandwich, through-the-thickness insert, adhesive joints, elastic properties, multi-segment integration method.

Nomenclature

A	radius of the whole sandwich structure
b_i	distance from the center of insert to the potting material
b_p	distance from the center of insert to the core
c	thickness of the core

[1] Corresponding author. Tel: +886-5-2720411 ext: 33307, Fax: +886-5-2720589, E-mail: ime_hsj@ccu.edu.tw

[2] Department of Mechanical Engineering, National Chung Cheng University, 168, University Rd., Ming-Hsiung, Chia-Yi, 621, Taiwan, ROC

E_i elastic modulus of the i-th layer ($i = 1-5$)
f_i thickness of the i-th face ($i = 1,2$)
G total potential energy
G_i shear modulus of the i-th layer ($i = 1-5$)
k_i coefficient of the displacement function ($i = 1-32$)
M_i coefficient of the governing equations ($i = 1-78$)
t thickness of the adhesive layer
u_i displacement of the i-th layer in the r-direction ($i = 1-5$)
v_i displacement of the i-th layer in the θ-direction ($i = 1-5$)
V work done by external loads
w_i deflection of the i-th layer in the z-direction ($i = 1-5$)
υ_i Poisson's ratio of the i-th layer ($i = 1-5$)
Γ strain energy
σ_z normal stress in the z-direction
σ_r normal stress in the r-direction
τ_{rz} shear stress

1 Introduction

Sandwich plates have been extensively used in aerospace, shipbuilding, construction and other industries. Fasteners or inserts of the "partially potted," "through-the-thickness" or "fully potted" type [Thomas (1998)], commonly introduce loads into structural elements. This work considers the "through-the-thickness" type, which is frequently adopted in aerospace sandwich plates to transfer severe external loads.

Thomsen (1998) introduced a high-order sandwich plate theory to derive governing equations for sandwiches with both 'through-the-thickness' inserts and 'fully potted' inserts. Governing equations were formulated as a set of coupled first-order differential equations, which are solved numerically using multi-segment integration. Thomsen and Rits (1998) developed high-order sandwich plate theory to derive governing equations for sandwiches with 'through the thickness' inserts. They investigated the local effect of a structural sandwich plate insert. Sandwich plates with "through-the-thickness" inserts under axisymmetric and non-axisymmetric external loadings are considered as examples.

Huang and Lin (2004) studied the application of an electronic speckle pattern interferometer (ESPI) to single-inserted sandwich plates. The proposed ESPI measures microscopic out-of-plane elastic region displacement without specimen waste in advantageous full-field, non-destructive tests. Results of a finite element method (FEM) analysis were compared with ESPI results around the inserts for validation. Thomsen (1995) presented theoretical and experimental results concerning local bending effects in a clamped circular foam-cored sandwich plate subjected to a central point load. The theoretical study utilized a local bending analysis approximation method, while the experimental study employed holographic interferometry. Bozhevolnaya and Lyckegaard (2005) designed a new core insert that significantly reduces the impairment caused by the local face effect. The new core insert design and its design parameters were studied experimentally with the help of finite element modeling.

Frostig and Peled (1995) presented high-order bending of a piecewise uniform sandwich beam with a tapered transition zone and a transversely flexible core. The tapered region inclined skin longitudinal and shear forces yielded concentrated forces at the point where the skin longitudinal layout changes, causing extensive peeling and shear stress concentrations at the interface between the skin and the core, as well as a high local bending moment in the skin. Kim, Kim and Lee (2004) combined the Koiter's asymptotic method with the assumed strain solid shell element formulation to conduct the postbuckling analysis of composite and sandwich structures. Sharnappa, Ganesan and Sethuraman (2007) presented the study on buckling and free vibration behavior of sandwich general shells of revolution under thermal environment using Wilkins theory. The analysis is carried out for different geometry such as truncated conical and hemispherical shells with various facing and core materials under clamped-clamped boundary condition. Huang (2002) developed an analytical sandwich beam model, accounting for the contribution of the adhesive layer to overall beam stiffness. Five constituent layer sandwich beam displacements are computed in this model, considering overall sandwich beam continuity conditions. Analysis results were compared with experimental results. Huang and Wang (2003) added the two-dimensional sandwich beam without an insert to two viscoelastic adhesive layers on both sides of the core and facing sheets. Displacement, stress and strain analytical formulations on structure interfaces were previously derived based on the proposed geometric assumption. A referred numerical case was based on analytical and finite element solutions). Huang and Liu (2003) added three-dimensional sandwich plates without an insert to two viscoelasticity adhesive layers. They determined that the sandwich plate exhibited creep over time. The mechanical behavior of the adhesive layer must be considered in the structural analysis of a sandwich structure. Most researchers did not consider an adhesive layer in sandwich plates with inserts, as the references show. The presents study addresses structural analysis of five circular sandwich plate layers with inserts, with adhesive layers added onto both core sides and facing sheets.

2 Hypothesis of Proposed Analytical Model

Insert-sandwich plate system modeling assumes that the interactions between both adjacent insert and an insert, and the plate boundaries or other sources of local disturbances, are negligible. Figures 1 and 2 define constituent parts and the geometry of a circular sandwich-insert plate with adhesive layers. The analytical model is based on the simplest adhesively bonded sandwich, comprising two thin and stiff facing sheets as well as two thin adhesive sheets, separated by a thicker layer of low-density material (core) with lower stiffness and strength.

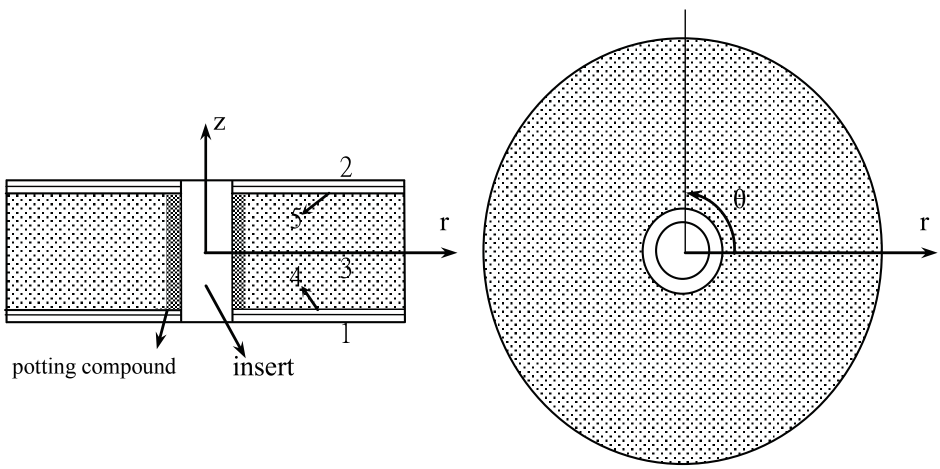

Figure 1: Sandwich plate with insert and adhesive layers; 1: lower face, 2: upper face, 3: core and potting compound, 4: lower adhesive, 5: upper adhesive

The formulation is based on the following restrictive assumptions.

a. The face sheets are treated as homogeneous, isotropic and linear elastic bodies. The thickness of the facing sheet and the adhesive layer is much smaller than that of the core. Facing sheets operate according to the Kirchhoff-Love hypothesis: $\varepsilon_z = \gamma_{rz} = \gamma_{\theta z} = 0$;

b. The core, such as a honeycomb, is treated as an orthotropic body, resisting longitudinal and transversal forces;

c. Adhesive layers behave as isotropic elastic materials, but resist longitudinal and transversal forces, with constant thickness throughout the structure;

d. The insert is treated as a rigid body to which the face sheets and potting material are rigidly connected;

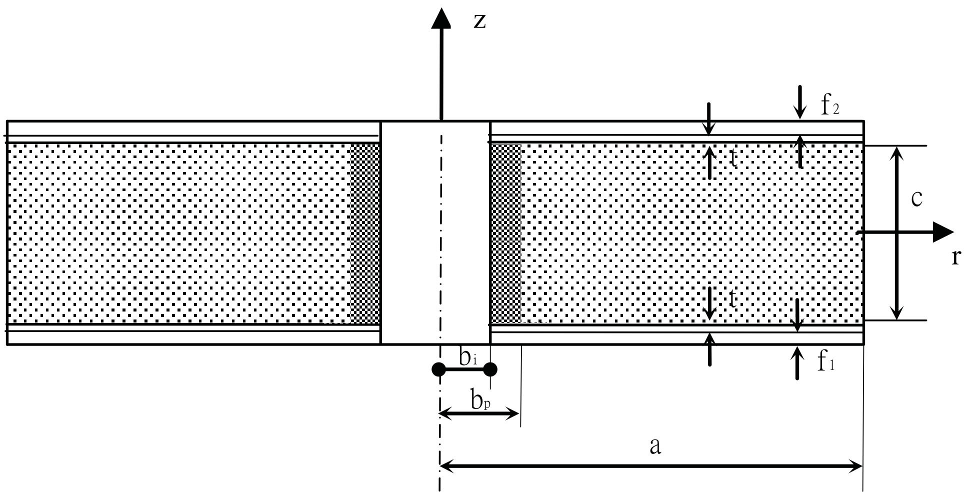

Figure 2: Geometrical definition of sandwich plate structure

e. The potting materials are treated as isotropic elastic materials, which resist both longitudinal and transverse forces.

3 Method of Formulation of Governing Equations

The energy method is utilized to derive equilibrium governing equations [Chiu (2004)] to reduce the complexity of the formulation of the equations for all five layers (two facing sheet layers, one core layer and two adhesive layers).

3.1 Displacements (refer to Figs. 1 and 2)

The displacement distributions of the core and the two adhesive layers are assumed to be functions of r, θ and z. Core and adhesive layer displacements are given by

$$\begin{aligned} u_i &= A_i(r,\theta) + B_i(r,\theta)\cdot z \\ v_i &= C_i(r,\theta) + D_i(r,\theta)\cdot z \\ w_i &= E_i(r,\theta) + F_i(r,\theta)\cdot z \quad i = 3,4,5 \end{aligned} \tag{1}$$

where the subscript $i = 3$ designates core; 4 and 5 designate lower and upper adhesive layers, respectively; $A_i(r,\theta)$, $B_i(r,\theta)$, $C_i(r,\theta)$, $D_i(r,\theta)$, $E_i(r,\theta)$ and $F_i(r,\theta)$ can be obtained by applying the following continuity conditions among the five layers.

3.2 Interface continuity condition

Substituting the displacement field equations (Eq. (1)) into six continuity conditions yields the values of the six preceding unknowns.

For the core, at the core and adhesive junction,

$$u_3 = u_5 \quad v_3 = v_5 \quad w_3 = w_5 \quad \text{as } z = \frac{c}{2} \tag{2a}$$

$$u_3 = u_4 \quad v_3 = v_4 \quad w_3 = w_4 \quad \text{as } z = -\frac{c}{2} \tag{2b}$$

where c is the thickness of the core.

For the fourth layer (lower adhesive layer),

$$u_4 = u_3^0 + \frac{c}{2}\frac{dw_3}{dr} \quad v_4 = v_3^0 + \frac{c}{2}\left(\frac{1}{r}\frac{dw_3}{d\theta}\right) \quad w_4 = w_3 \quad \text{as } z = -\frac{c}{2} \tag{3a}$$

$$u_4 = u_1^0 - \frac{f_1}{2}\frac{dw_1}{dr} \quad v_4 = v_1^0 - \frac{f_1}{2}\left(\frac{1}{r}\frac{dw_1}{d\theta}\right) \quad w_4 = w_1 \quad \text{as } z = -\frac{c}{2} - t \tag{3b}$$

where subscript 1 denotes the lower facing sheet (first layer); superscript 0 denotes the mid-plan of the core or the lower facing sheet; f_1 is the thickness of the first layer, and t is the thickness of the lower adhesive layers.

For the fifth layer (upper adhesive layer),

$$u_5 = u_3^0 - \frac{c}{2}\frac{dw_3}{dr} \quad v_5 = v_3^0 - \frac{c}{2}\left(\frac{1}{r}\frac{dw_3}{d\theta}\right) \quad w_5 = w_3 \quad \text{as } z = \frac{c}{2} \tag{4a}$$

$$u_5 = u_2^0 + \frac{f_2}{2}\frac{dw_2}{dr} \quad v_5 = v_2^0 + \frac{f_2}{2}\left(\frac{1}{r}\frac{dw_2}{d\theta}\right) \quad w_5 = w_2 \quad \text{as } z = \frac{c}{2} + t \tag{4b}$$

where subscript 2 denotes the upper facing sheet (second layer); superscript 0 denotes the mid-plan of core or upper facing sheet; f_2 is the thickness of the second layer, and t is the thickness of the upper adhesive layer.

3.3 Layer displacements

After Eq. (1) is solved by applying continuity conditions, mathematical operations can be performed to yield layer displacements:

$$u_3 = k_1 u_1 + k_2 u_2 + k_3 \frac{\partial}{\partial r} w_1 + k_4 \frac{\partial}{\partial r} w_2 \tag{5}$$

$$v_3 = k_5 v_1 + k_6 v_2 + k_7 \frac{1}{r}\frac{\partial}{\partial \theta} w_1 + k_8 \frac{1}{r}\frac{\partial}{\partial \theta} w_2 \tag{6}$$

$$w_3 = k_9 w_1 + k_{10} w_2 \tag{7}$$

$$u_4 = k_{11}u_1 + k_{12}u_2 + k_{13}\frac{\partial}{\partial r}w_1 + k_{14}\frac{\partial}{\partial r}w_2 \tag{8}$$

$$v_4 = k_{15}v_1 + k_{16}v_2 + k_{17}\frac{1}{r}\frac{\partial}{\partial \theta}w_1 + k_{18}\frac{1}{r}\frac{\partial}{\partial \theta}w_2 \tag{9}$$

$$w_4 = k_{19}w_1 + k_{20}w_2 \tag{10}$$

$$u_5 = k_{21}u_1 + k_{22}u_2 + k_{23}\frac{\partial}{\partial r}w_1 + k_{24}\frac{\partial}{\partial r}w_2 \tag{11}$$

$$v_5 = k_{25}v_1 + k_{26}v_2 + k_{27}\frac{1}{r}\frac{\partial}{\partial \theta}w_1 + k_{28}\frac{1}{r}\frac{\partial}{\partial \theta}w_2 \tag{12}$$

$$w_5 = k_{29}w_1 + k_{30}w_2 \tag{13}$$

where k_i are constants that are composed of the geometric parameters (Appendix A). Core and adhesive layer displacement fields can thus be expressed by various face displacements, so governing equations can be expressed in face variable terms (first and second layers), markedly simplifying computation.

3.4 Variation integral for the circular sandwich plate with insert

The strain can now be obtained using strain-displacement equations, since the five layer displacement expressions have already been derived. Total potential energy (G), comprising strain energy (Γ) and work done by external loads (V) can be expressed in terms of layer strains, as given by Eq. (14):

$$G = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 - V \tag{14}$$

where the subscript $i = 1$, 2 designates lower and upper face strain energy; 3 designates core strain energy, and 4, 5 designates lower and upper adhesive layer strain energy, respectively.

3.5 Governing equations for a circular sandwich plate with insert

The calculated total potential energy is substituted into the following Euler-Lagrange equations to yield equilibrium equations in terms of the displacement of facing sheets ($u_i, v_i, w_i, i = 1,2$, where subscript $i = 1$ denotes the first face and 2 denotes the second face).

$$\begin{aligned}
&\frac{\partial G}{\partial I}-\frac{\partial}{\partial r}\cdot\frac{\partial G}{\partial\left(\frac{\partial I}{\partial r}\right)}-\frac{\partial}{\partial\theta}\cdot\frac{\partial G}{\partial\left(\frac{\partial I}{\partial\theta}\right)}-\frac{\partial}{\partial z}\cdot\frac{\partial G}{\partial\left(\frac{\partial I}{\partial z}\right)}+\frac{\partial^2}{\partial r^2}\cdot\frac{\partial G}{\partial\left(\frac{\partial^2 I}{\partial r^2}\right)}\\
&+\frac{\partial^2}{\partial\theta^2}\cdot\frac{\partial G}{\partial\left(\frac{\partial^2 I}{\partial\theta^2}\right)}+\frac{\partial^2}{\partial z^2}\cdot\frac{\partial G}{\partial\left(\frac{\partial^2 I}{\partial z^2}\right)}+2\cdot\frac{\partial^2}{\partial r\partial z}\cdot\frac{\partial G}{\partial\left(\frac{\partial^2 I}{\partial r\partial z}\right)}+2\cdot\frac{\partial^2}{\partial\theta\partial z}\cdot\frac{\partial G}{\partial\left(\frac{\partial^2 I}{\partial\theta\partial z}\right)}\\
&+2\cdot\frac{\partial^2}{\partial r\partial\theta}\cdot\frac{\partial G}{\partial\left(\frac{\partial^2 I}{\partial r\partial\theta}\right)}=0
\end{aligned} \tag{15}$$

where I denotes $u_1, u_2, v_1, v_2, w_1, w_2$, respectively.

The system of sandwich beam governing equations can finally be expressed:

$$\begin{aligned}
&M_1u_1+M_2u_2+M_3\frac{\partial}{\partial\theta}v_1+M_4\frac{\partial}{\partial r}w_1+M_5\frac{\partial}{\partial r}w_2+M_6\frac{\partial^2}{\partial r^2}u_1\\
&+M_7\frac{\partial^2}{\partial\theta^2}u_1+M_8\frac{\partial^2}{\partial\theta\partial r}v_1+M_9\frac{\partial^2}{\partial\theta^2}w_1+M_{10}\frac{\partial^3}{\partial\theta^2\partial r}w_1=0
\end{aligned} \tag{16}$$

$$\begin{aligned}
&M_{11}u_1+M_{12}u_2+M_{13}\frac{\partial}{\partial\theta}v_2+M_{14}\frac{\partial}{\partial r}w_1+M_{15}\frac{\partial}{\partial r}w_2+M_{16}\frac{\partial^2}{\partial r^2}u_2\\
&+M_{17}\frac{\partial^2}{\partial\theta^2}u_2+M_{18}\frac{\partial^2}{\partial\theta\partial r}v_2+M_{19}\frac{\partial^2}{\partial\theta^2}w_2+M_{20}\frac{\partial^3}{\partial\theta^2\partial r}w_2=0
\end{aligned} \tag{17}$$

$$\begin{aligned}
&M_{21}v_1+M_{22}v_2+M_{23}\frac{\partial}{\partial\theta}u_1+M_{24}\frac{\partial}{\partial\theta}w_1+M_{25}\frac{\partial}{\partial\theta}w_2+M_{26}\frac{\partial^2}{\partial\theta\partial r}u_1\\
&+M_{27}\frac{\partial^2}{\partial r^2}v_1+M_{28}\frac{\partial^2}{\partial\theta^2}v_1+M_{29}\frac{\partial^2}{\partial\theta\partial r}w_1+M_{30}\frac{\partial^3}{\partial\theta^3}w_1=0
\end{aligned} \tag{18}$$

$$\begin{aligned}
&M_{31}v_1+M_{32}v_2+M_{33}\frac{\partial}{\partial\theta}u_2+M_{34}\frac{\partial}{\partial\theta}w_1+M_{35}\frac{\partial}{\partial\theta}w_2+M_{36}\frac{\partial^2}{\partial\theta\partial r}u_2\\
&+M_{37}\frac{\partial^2}{\partial r^2}v_2+M_{38}\frac{\partial^2}{\partial\theta^2}v_2+M_{39}\frac{\partial^2}{\partial\theta\partial r}w_2+M_{40}\frac{\partial^3}{\partial\theta^3}w_2=0
\end{aligned} \tag{19}$$

$$
\begin{aligned}
&M_{41}w_1 + M_{42}w_2 + M_{43}\frac{\partial}{\partial r}u_1 + M_{44}\frac{\partial}{\partial r}u_2 + M_{45}\frac{\partial}{\partial \theta}v_1 + M_{46}\frac{\partial}{\partial \theta}v_2 \\
&+ M_{47}\frac{\partial}{\partial r}w_1 + M_{48}\frac{\partial^2}{\partial \theta^2}u_1 + M_{49}\frac{\partial^2}{\partial \theta \partial r}v_1 + M_{50}\frac{\partial^2}{\partial r^2}w_1 + M_{51}\frac{\partial^2}{\partial r^2}w_2 \\
&+ M_{52}\frac{\partial^2}{\partial \theta^2}w_1 + M_{53}\frac{\partial^2}{\partial \theta^2}w_2 + M_{54}\frac{\partial^3}{\partial \theta^2 \partial r}u_1 + M_{55}\frac{\partial^3}{\partial \theta^3}v_1 \\
&+ M_{56}\frac{\partial^3}{\partial \theta^2 \partial r}w_1 + M_{57}\frac{\partial^4}{\partial r^4}w_1 + M_{58}\frac{\partial^4}{\partial \theta^4}w_1 \\
&+ M_{59}\frac{\partial^4}{\partial \theta^2 \partial r^2}w_1 = Z_1
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&M_{60}w_1 + M_{61}w_2 + M_{62}\frac{\partial}{\partial r}u_1 + M_{63}\frac{\partial}{\partial r}u_2 + M_{64}\frac{\partial}{\partial \theta}v_1 + M_{65}\frac{\partial}{\partial \theta}v_2 \\
&+ M_{66}\frac{\partial}{\partial r}w_2 + M_{67}\frac{\partial^2}{\partial \theta^2}u_2 + M_{68}\frac{\partial^2}{\partial \theta \partial r}v_2 + M_{69}\frac{\partial^2}{\partial r^2}w_1 + M_{70}\frac{\partial^2}{\partial r^2}w_2 \\
&+ M_{71}\frac{\partial^2}{\partial \theta^2}w_1 + M_{72}\frac{\partial^2}{\partial \theta^2}w_2 + M_{73}\frac{\partial^3}{\partial \theta^2 \partial r}u_2 + M_{74}\frac{\partial^3}{\partial \theta^3}v_2 \\
&+ M_{75}\frac{\partial^3}{\partial \theta^2 \partial r}w_2 + M_{76}\frac{\partial^4}{\partial r^4}w_2 + M_{77}\frac{\partial^4}{\partial \theta^4}w_2 \\
&+ M_{78}\frac{\partial^4}{\partial \theta^2 \partial r^2}w_2 = Z_2
\end{aligned} \tag{21}
$$

where Z_1, Z_2 denote the surface forces along the z axes of the first and second layers. M_i are the integral constants that are composed of elastic and geometric parameters (Appendix B and Appendix C).

The dependency of the θ is eliminated by the Fourier series expansion of fundamental variables, which reduces the problem to a set of 16 first-order ordinary differential equations, which can be solved numerically using the multi-segment integration method.

4 Case Study

Circular sandwich plates with “through-the-thickness” inserts under axisymmetric external loading are considered as an example.

4.1 Material properties and boundary conditions

Tables 1 and 2 present the geometry, material properties, external load and boundary conditions. The circular sandwich plate consists of an insert, a potting compound, the core, two adhesive layers and two face layers.

Table 1: Geometry, material properties

geometry	$b_i = 10mm$, $b_p = 30mm$, $a = 150mm$, $c = 10mm$, $f_1 = f_2 = 1mm$, $t = 0.1mm$ (see Fig. 2)
face (FRP-laminate)	$E_1 = E_2 = 40000N/mm^2$, $\upsilon_1 = \upsilon_2 = 0.3$
core (Hexcel honey-comb)	$E_{3r} = 310N/mm^2$, $G_{3rz} = G_{3\theta z} = 138N/mm^2$
adhesive (Film adhe-sive)	$E_4 = E_5 = 854.7N/mm^2$, $\upsilon_4 = \upsilon_5 = 0.4$
insert ($r = b_i \sim b_p$)	Same material properties as the face sheets
potting compound (Bulk epoxy)	$E_{3r} = 2500N/mm^2$, $G_{3rz} = G_{3\theta z} = 930N/mm^2$

Table 2: Boundary conditions

$r = b_i$	$u_1 = u_2 = v_1 = v_2 = 0$, $\frac{d^2u_1}{dr^2} = \frac{d^2u_2}{dr^2} = \frac{d^2v_1}{dr^2} = \frac{d^2v_2}{dr^2} = 0$	The through-the-thickness insert is considered an infinitely rigid body to which face sheets, potting compound, and adhesive are rigidly connected.
$r = a$	$w_1 = w_2 = \frac{d^2w_1}{dr^2} = \frac{d^2w_2}{dr^2} = 0$, $\frac{d^2u_1}{dr^2} = \frac{d^2u_2}{dr^2} = \frac{d^2v_1}{dr^2} = \frac{d^2v_2}{dr^2} = 0$	Face sheet mid-surfaces are assumed as simply supported.
$r = b_i$	$\frac{d^3w_1}{dr^3} = 0$, $\frac{d^3w_2}{dr^3} = Q$	Out-of-plane load Q in the top of insert (see Fig. 3).

4.2 Numerical solution: multi-segment integration method

The set of governing equations along with the boundary condition constitute a boundary value problem. A general closed form solution to this problem is difficult to obtain, and a numerical solution approach is therefore developed [Kalnins (1964)].

The boundary value problem is solved numerically by multi-segment integration. This method has the following features.

a. Easily implemented;

b. Conveniently applied to first-order ordinary differential equation systems;

c. Permits arbitrary radial variations, including discontinuities, of all variables in the problem.

The multi-segment method transforms the boundary value problem into a series of interconnected initial value problems. The insert-sandwich plate configuration is divided into a finite number of segments, and the solution in OR each segment is derived by direct integration. Further details of the multi-segment method of integration can be found elsewhere [Kalnins (1964)]. The plate configuration herein is divided into 2800 segments, to yield the best solution.

4.3 Results

The external out-of-plane loading Q is probably the typical load for industrial usage. In this case study, $Q = 1kN$ is assumed to be loaded at the junction of the insert and the upper facing sheet (Fig. 3).

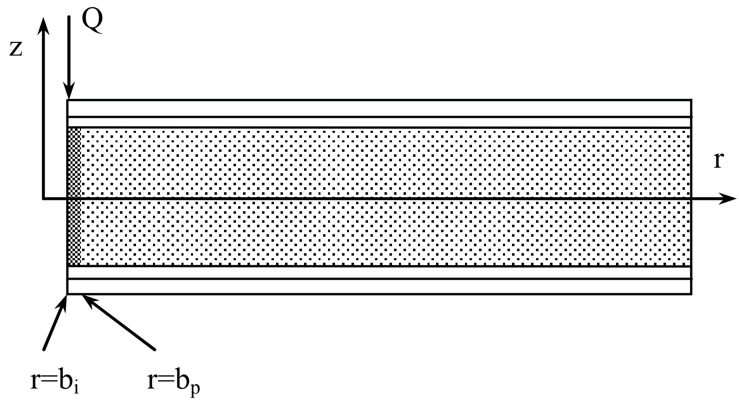

Figure 3: Out-of-plane load Q

Figure 4 depicts the lateral deflections of the facing sheets (w_1, w_2, at z = -5.6, 5.6 mm) and the core mid-surface (w_3, at z = 0 mm). In the third layer $r \leq 30mm$ corresponds to the potting region, while $r > 30mm$ corresponds to the honeycomb region. Lateral deflections of the two facing sheets and the core material mid-surface are almost identical, as the results reveal. As expected, the symmetry of the sandwich plate makes lateral displacements of the two facing sheets w_1 and w_2 identical.

Figure 5 presents lateral deflections of the adhesive layers (w_4, w_5, at z = -5.05, 5.05 mm); $r \leq 30mm$ corresponds to the insert region, whereas $r > 30mm$ corresponds to the honeycomb region. Lateral deflections of the two adhesive layers appear to be identical, because of the considered sandwich plate symmetry.

Figure 6 plots the core material stress distribution. The values of the transverse normal stress σ_z are given at the interface between the upper adhesive layer and

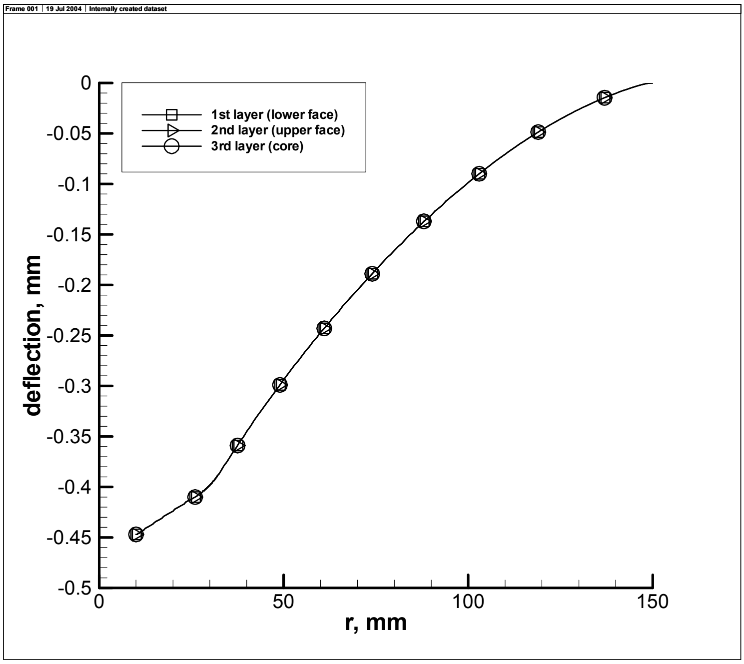

Figure 4: Lateral deflections w_1, w_2 and w_3

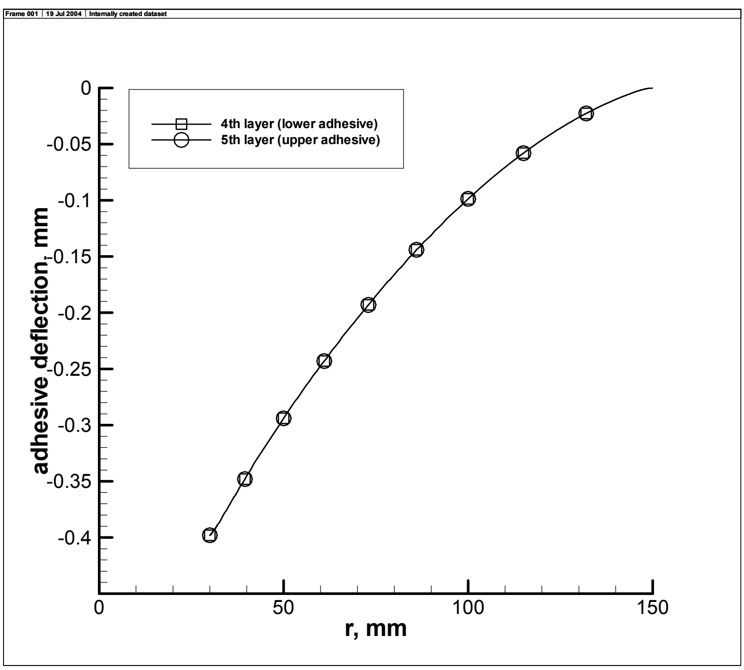

Figure 5: Lateral deflections w_4 and w_5

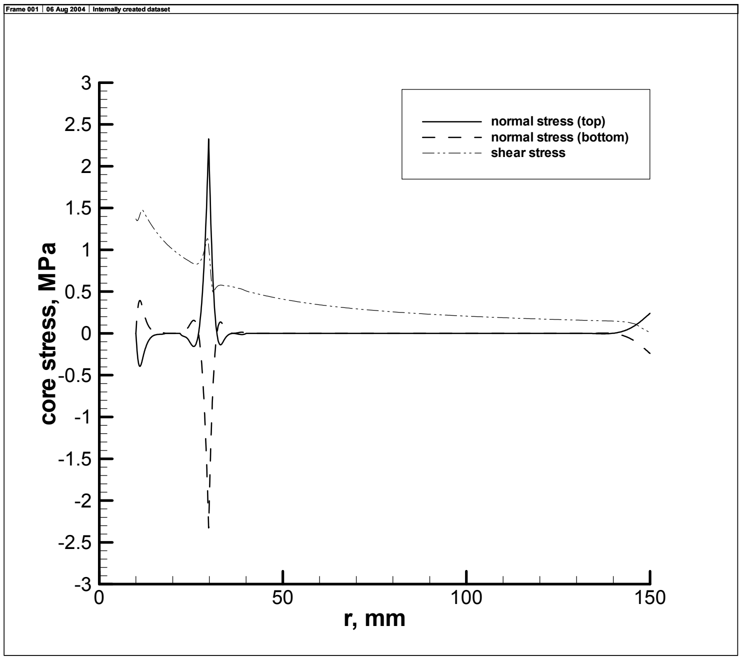

Figure 6: Core stress components σ_z^{top}, σ_z^{bottom} and τ_{rz}

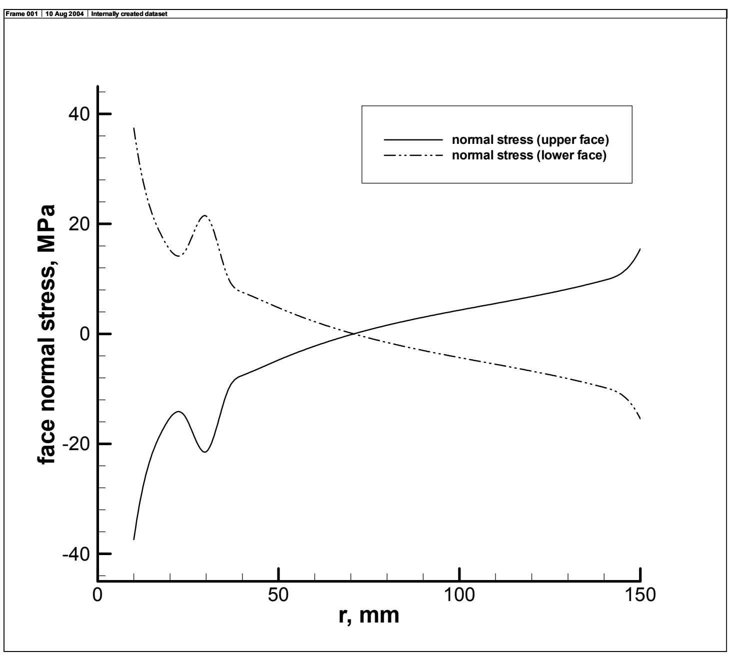

Figure 7: Normal stress of lower facing sheet $\sigma_{r,1}$ and upper facing sheet $\sigma_{r,2}$ in the radius direction

the core (σ_z^{top}) and at the interface between the lower adhesive layer and the core (σ_z^{bottom}). Figure 6 also plots the distribution of the transverse core shear stress componentτ_{rz}, which is assumed to be constant over the height of core material. When the σ_z-distribution is considered, transverse normal stress is observably a very local phenomenon, as significant σ_z-contributions are present only close to $r = 10mm$ (close to the insert) and close to $r = 30mm$ (close to the potting-honeycomb intersection). σ_z^{top} and σ_z^{bottom} have opposite signs. The peak stress magnitudes close to $r = 30mm$ regions exceed those close to$r = 10mm$. Stress concentrations closest to the interface between the potting material and the core are likely to fail. However, stress concentrations at the potting-honeycomb intersection, and immediately adjacent to it, may cause premature failure.

The overall core material shear stress distribution tendency is that τ_{rz} declines as r increases. When the effects of combined normal transverse and shear stress component on potting and honeycomb materials are considered, the mechanical properties of the two materials differ significantly. Therefore, the stiffness and strength of the honeycomb material are usually an order of magnitude lower than those of the potting compound.

Figure 7 plots the normal stress of the lower facing sheet $\sigma_{r,1}$ and the upper facing sheet $\sigma_{r,2}$ in the radial direction. Normal stresses of both facing sheets are maximal at the junction with insert. Figure 8 plots the normal stress of the lower adhesive layer$\sigma_{r,4}$ and the upper adhesive layer $\sigma_{r,5}$ in the radial direction. The normal stresses of both adhesive layers are maximal (in the vicinity of OR near) r=b$_p$. Normal stresses of both adhesive layers have a smaller magnitude than those of the facing sheets. Facing sheets observably resist more normal stress than adhesive layers in the radial direction, as revealed by comparing Figs. 7 and 8.

4.4 Comparison with degraded three-layer theory and Thomsen's theory [Thomsen and Rits (1998)]

For validation, the present five -layer theory is degraded to be a three -layer theory by replacing the mechanical properties of an adhesive layer with those of the face. Then, the results are compared with the results of three-layer theory and Thomsen and Rits (1998). Figure 9 to Figure 12 compare the aforementioned theories. The degraded five-layer theory (three -layer theory) is very consistent with the theory of Thomsen and Rits, which fact verifies the accuracy of the proposed five-layer theory. Figures 10 and 11 reveal that the peak normal stress of the core close to $r = 30mm$ according to five -layer theory exceeds that close to $r = 10mm$, which differs from result of other two theories, because the adhesive effect (reducing the stiffness of the whole structure) is considered in the present five-layer analytical model. However, the peak shear stress occurs near $r = b_i$ ($r = 10mm$) , as displayed

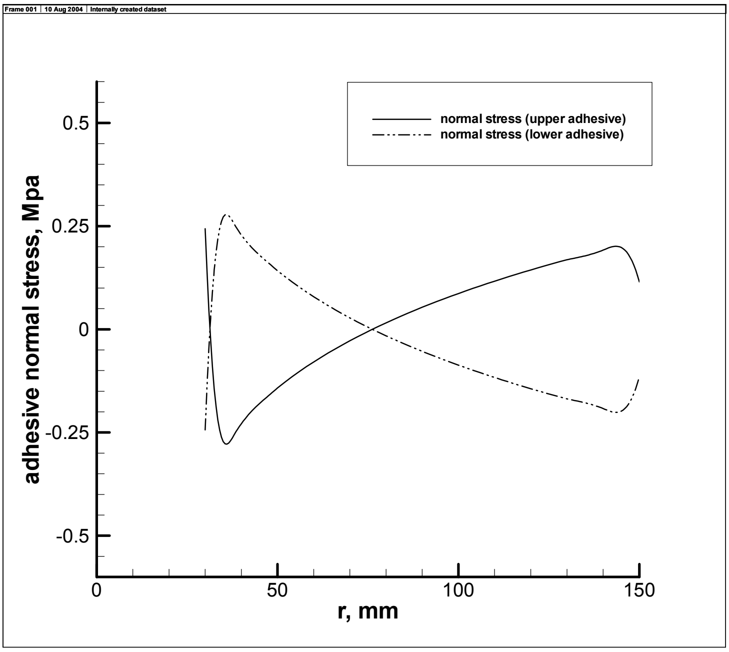

Figure 8: Normal stress of lower adhesive layer $\sigma_{r,4}$ and upper adhesive layer $\sigma_{r,5}$ in the radius direction

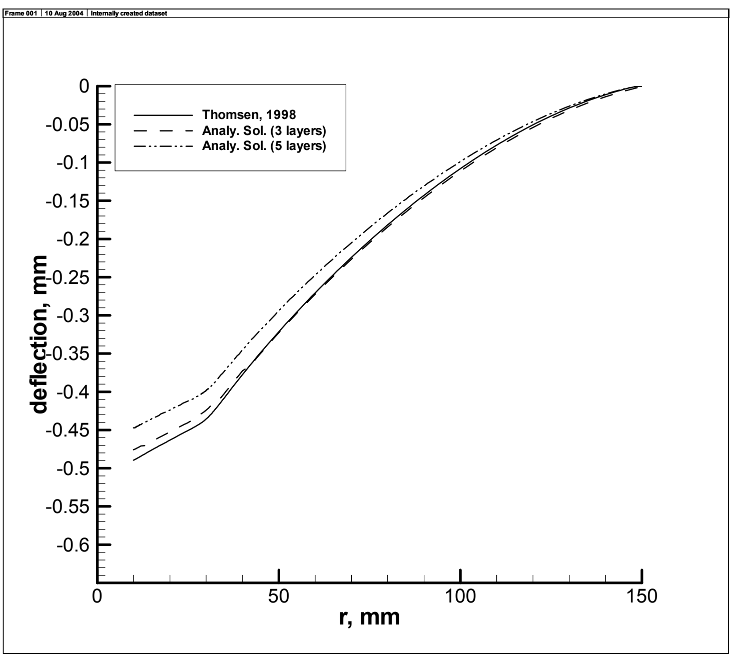

Figure 9: Deflection of the 3rd layer (core) in the z-direction, compared with other theories

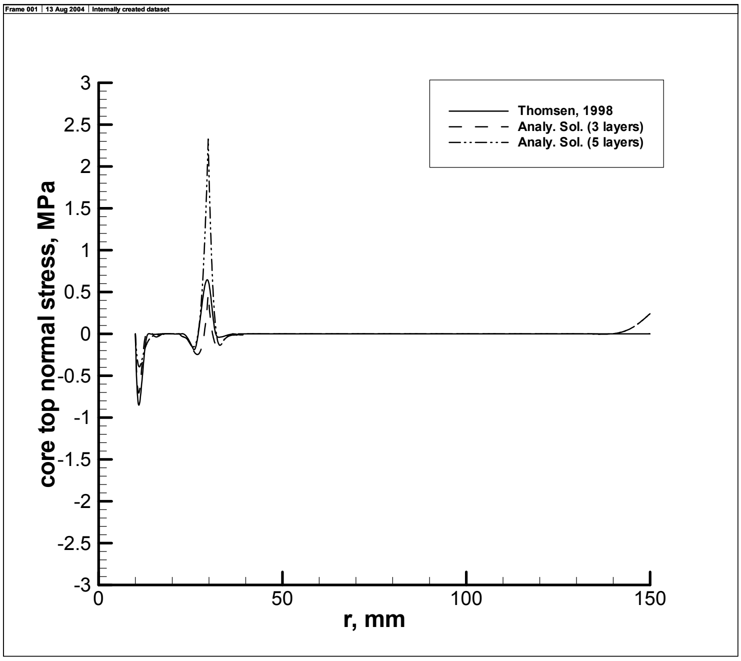

Figure 10: Normal stress σ_z^{top} of the 3rd layer ($z = 5mm$), compared with other theories

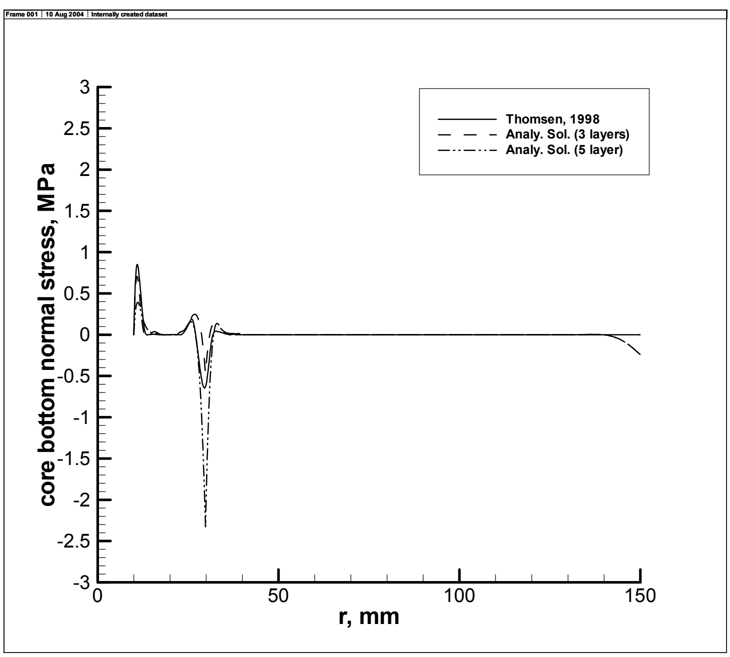

Figure 11: Normal stress σ_z^{bottom} of the 3rd layer ($z = -5mm$), compared with other theories

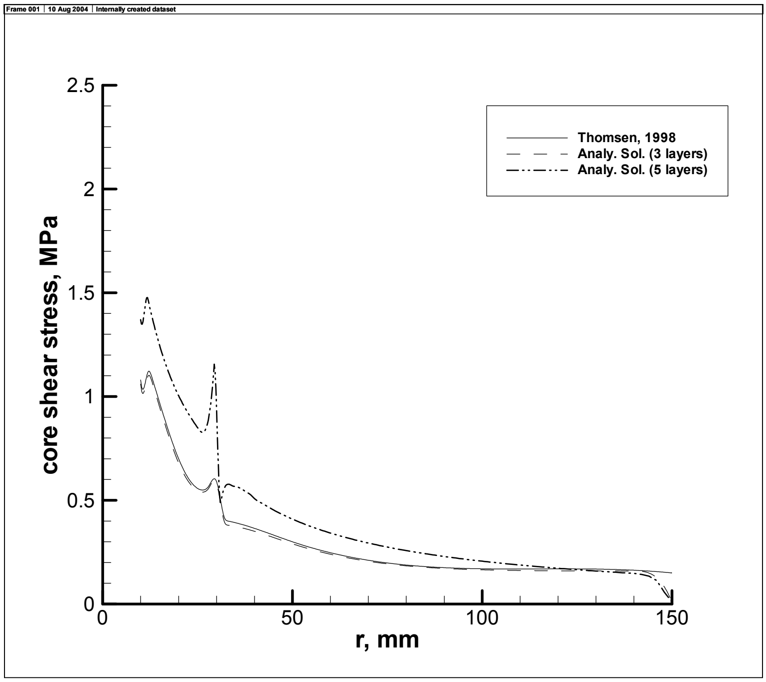

Figure 12: Shear stress $\tau(rz)$ of the 3rd layer, compared with other theories

in Fig. 12.

4.5 Comparison with FEM Solution

A finite element method (FEM) model of the five-layered sandwich plates with a "through-the-thickness" insert is generated using the finite element package ABAQUS. The model consists of numerous 20-node quadratic hexahedral elements with six degrees of freedom in each node for the whole sandwich structure (Fig. 13). A study of mesh density is performed to validate the finite element results, and indicates that the displacement and stress results converge as the total number of elements ranges from 44820 to 70070 with a calculation time of 4 to 9 minutes (on a PC with CPU: 2.13GHz and RAM: 0.99G). Accordingly, the total number of elements used in this FEM model is 66500. The geometry, material properties, boundary conditions and load conditions are the same as in the analytical model (Tables 1 and 2).

Figures 14 and 15 demonstrate the analytic and FEM deflections of difference for the 2nd facing layer and the 5^{th} adhesive layer, respectively. The FEM deflection

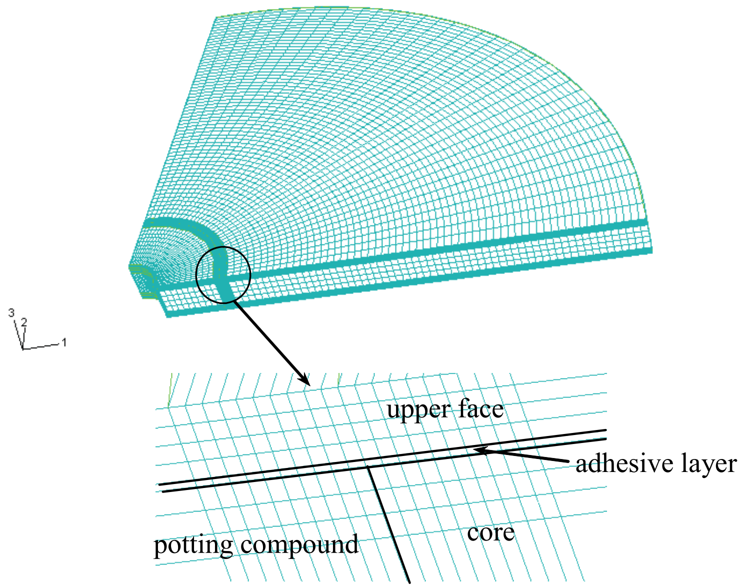

Figure 13: Finite element mesh

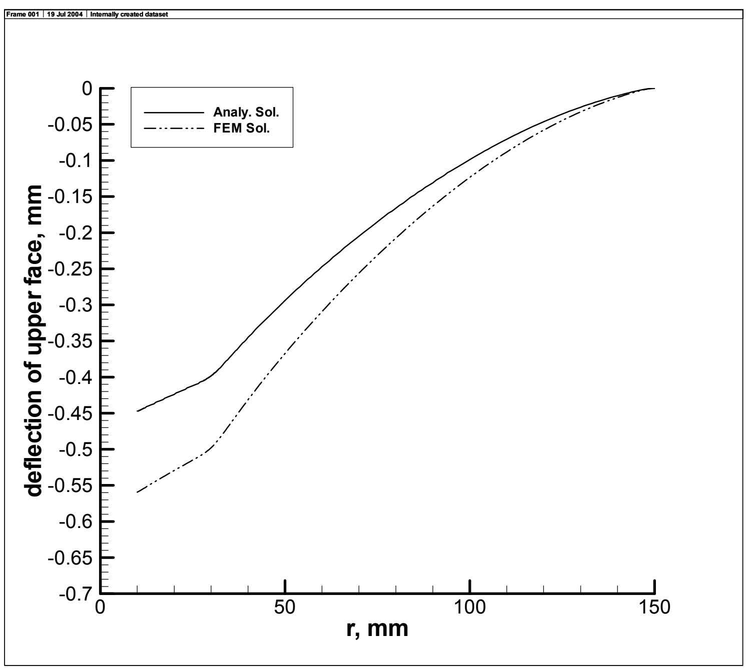

Figure 14: Deflection of the 2nd layer in the z-direction, compared with FEM

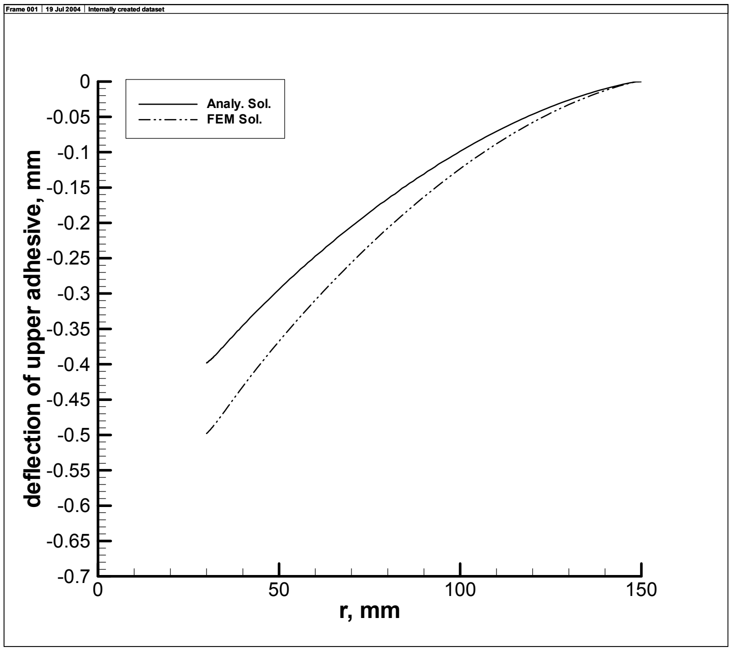

Figure 15: Deflection of the 5th layer in the z-direction, compared with FEM

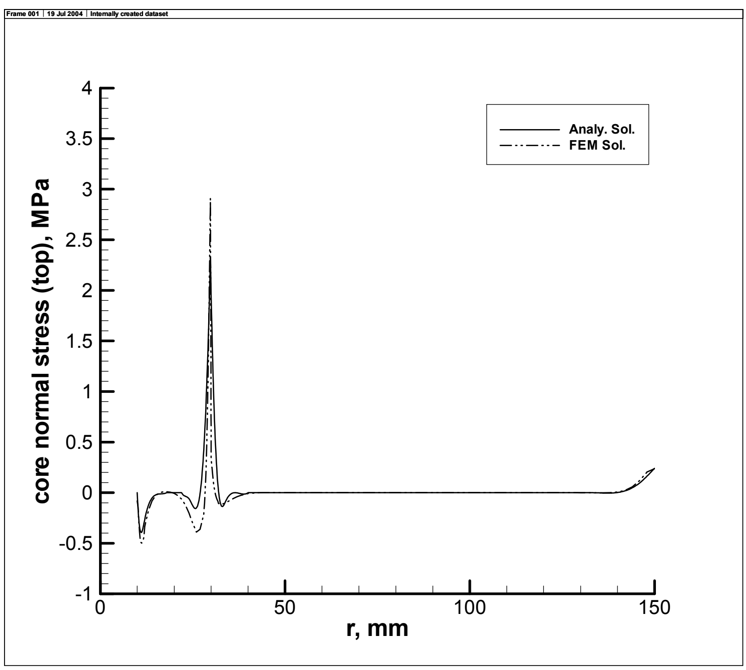

Figure 16: Normal stress σ_z^{top} of the 3rd layer ($z = 5mm$), compared with FEM

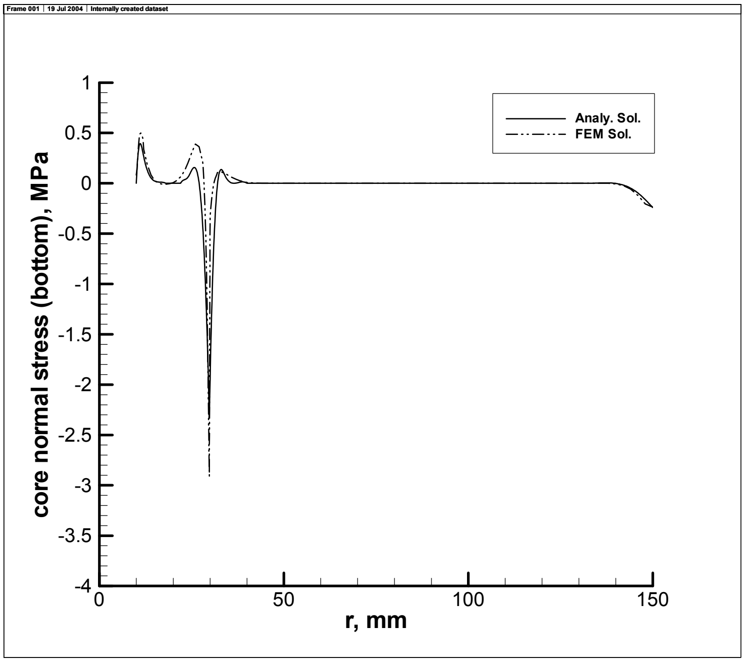

Figure 17: Normal stress σ_z^{bottom} of the 3rd layer ($z = -5mm$), compared with FEM

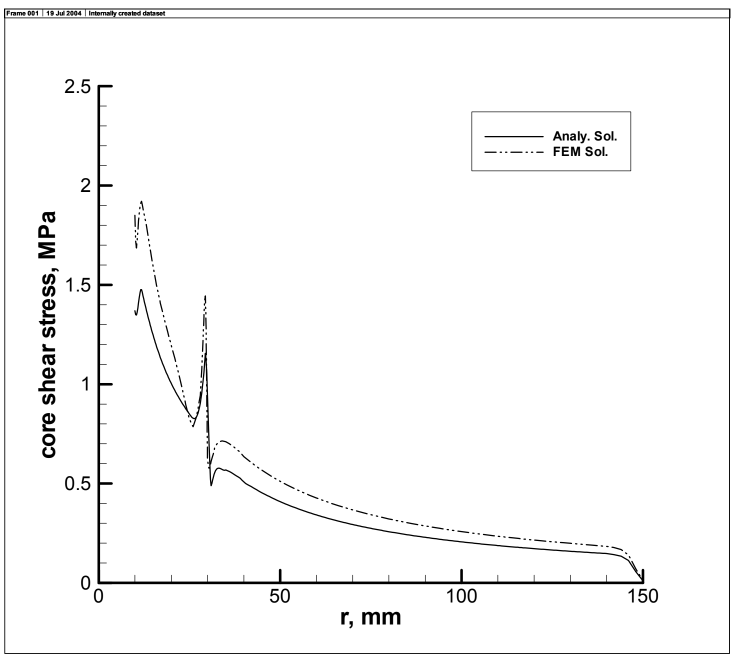

Figure 18: Shear stress $\tau(rz)$ of the 3rd layer, compared with FEM

magnitudes of the second and fifth layers always exceed the analytical values. Figures 16, 17 and 18 show analytical solution of normal and shear stresses of the 3rd layer, compared with the FEM solutions. Core transverse normal stress is observably a very local phenomenon, as significant σ_z-contributions are present only close to $r = 10mm$ (close to the insert) and close to $r = 30mm$ (close to the potting-honeycomb core intersection). Peak normal stress magnitudes close to $r = 30mm$ exceed those close to $r = 10mm$. Strong agreement exists between the analytical solution and the FEM solution, as revealed by Figs. 14 and 18. The comparisons also confirm the accuracy of the analytical results.

5 Conclusions

A new five-layered sandwich plate theory is developed and adapted for sandwich plate inserts. The present formulation is developed for sandwich plates with potted through-the-thickness inserts, but can be extended and adapted to analyze fully and partially potted insert-sandwich plates by considering the effect of the adhesive layer. Insert-sandwich plate problems are specified mathematically as sets of first-order differential equations, which are solved numerically in a convenient and cost-effective manner using multi-segment integration. The sandwich plate normal stress is maximal at the intersection between (the potting material, the core, and the adhesive layer, where failure occurs.

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Appendix A. Constants k_i $(i = 1 \sim 32)$

$$k_1 = \frac{1}{2}; \quad k_2 = \frac{1}{2};$$

$$k_3 = \frac{-1}{4}\left(\frac{-4z^2 f_1 + c^2 f_1 - 4ztc}{c^2 - 4z^2}\right);$$

$$k_4 = \frac{1}{4}\left(\frac{c^2 f_2 - 4z^2 f_2 + 4ztc}{c^2 - 4z^2}\right);$$

$$k_5 = \frac{1}{2}; \quad k_6 = \frac{1}{2};$$

$$k_7 = \frac{-1}{4}\left(\frac{-4z^2 f_1 + c^2 f_1 - 4ztc}{(c^2 - 4z^2)}\right);$$

$$k_8 = \frac{1}{4}\left(\frac{c^2 f_2 - 4z^2 f_2 + 4ztc}{(c^2 - 4z^2)}\right);$$

$$k_9 = \frac{1}{2}; \quad k_{10} = \frac{1}{2};$$

$$k_{11} = \frac{1}{4}(\frac{c+2t+2z}{t}); \quad k_{12} = \frac{1}{4}(\frac{c+2t+2z}{t});$$

$$k_{13} = \frac{1}{8}(-4cf_1z^2 + c^4 + 2zf_1c^2 + 8zt^2c - 2tc^2f_1 + 8z^2tf_1 - 8z^3c - 8f_1z^3 + c^3f_1 + 4ztc^2 + 2zc^3 + 2tc^3 - 4z^2c^2/t(c^2 - 4z^2));$$

$$k_{14} = \frac{1}{8}(-4cf_2z^2 + c^4 + 2zf_2c^2 + 8zt^2c - 2tc^2f_2 + 8z^2tf_2 - 8z^3c - 8f_2z^3 + c^3f_2 + 4ztc^2 + 2zc^3 + 2tc^3 - 4z^2c^2/t(c^2 - 4z^2));$$

$$k_{15} = \frac{1}{4}(\frac{-c+2t-2z}{t}); \quad k_{16} = \frac{1}{4}(\frac{c+2t+2z}{t});$$

$$k_{17} = \frac{1}{8}(-4cf_1z^2 + c^4 + 2zf_1c^2 + 8zt^2c - 2tc^2f_1 + 8z^2tf_1 - 8z^3c - 8f_1z^3 + c^3f_1 + 4ztc^2 + 2zc^3 + 2tc^3 - 4z^2c^2/t(c^2 - 4z^2));$$

$$k_{18} = \frac{1}{8}(-4cf_2z^2 + c^4 + 2zf_2c^2 + 8zt^2c - 2tc^2f_2 + 8z^2tf_2 - 8z^3c - 8f_2z^3 + c^3f_2 + 4ztc^2 + 2zc^3 + 2tc^3 - 4z^2c^2/t(c^2 - 4z^2));$$

$$k_{19} = \frac{1}{4}(\frac{-c+2t-2z}{t}); \quad k_{20} = \frac{1}{4}(\frac{c+2t+2z}{t});$$

$$k_{21} = \frac{1}{4}(\frac{c+2t-2z}{t}); \quad k_{22} = \frac{1}{4}(\frac{-c+2t+2z}{t});$$

$$k_{23} = \frac{-1}{8}(-4cf_1z^2 + c^4 + 2zf_1c^2 + 8zt^2c - 2tc^2f_1 + 8z^2tf_1 - 8z^3c - 8f_1z^3 + c^3f_1 + 4ztc^2 + 2zc^3 + 2tc^3 - 4z^2c^2/t(c^2 - 4z^2));$$

$$k_{24} = \frac{1}{8}(-4cf_2z^2 + c^4 + 2zf_2c^2 + 8zt^2c - 2tc^2f_2 + 8z^2tf_2 - 8z^3c - 8f_2z^3 + c^3f_2 + 4ztc^2 + 2zc^3 + 2tc^3 - 4z^2c^2/t(c^2 - 4z^2));$$

$$k_{25} = \frac{1}{4}(\frac{c+2t-2z}{t}); \quad k_{26} = \frac{1}{4}(\frac{-c+2t+2z}{t});$$

$$k_{27} = \frac{-1}{8}(-4cf_1z^2 + c^4 + 2zf_1c^2 + 8zt^2c - 2tc^2f_1 + 8z^2tf_1 - 8z^3c - 8f_1z^3 + c^3f_1 + 4ztc^2 + 2zc^3 + 2tc^3 - 4z^2c^2/t(c^2 - 4z^2));$$

$$k_{28} = \frac{1}{8}(-4cf_2z^2 + c^4 + 2zf_2c^2 + 8zt^2c - 2tc^2f_2 + 8z^2tf_2 - 8z^3c - 8f_2z^3 + c^3f_2 + 4ztc^2 + 2zc^3 + 2tc^3 - 4z^2c^2/t(c^2 - 4z^2));$$

$$k_{29} = \frac{1}{4}(\frac{c+2t-2z}{t}); \quad k_{30} = \frac{1}{4}(\frac{-c+2t+2z}{t});$$

$$k_{31} = -z + \frac{1}{2}c + t + \frac{1}{2}f_1; \quad k_{32} = -z + \frac{1}{2}c - t - \frac{1}{2}f_2.$$

Appendix B. Constants $M_i (i = 1 \sim 78)$

$$M_1 = G_3J_{11} + G_4J_{34} + G_5J_{57} + \frac{E_1J_{77}}{r^2} \quad M_2 = \frac{1}{2}G_4J_{28} + \frac{1}{2}G_3J_5 + \frac{1}{2}G_5J_{51}$$

$$M_3 = \frac{1}{2}\frac{E_1J_{74}}{r^2} - \frac{1}{2}\frac{G_1J_{85}}{r^2} \quad M_4 = \frac{1}{2}G_3J_6 + \frac{\frac{1}{2}E_1J_{79}}{r^2} + \frac{1}{2}G_5J_{52} + \frac{1}{2}G_4J_{29}$$

$$M_5 = \frac{1}{2}G_3J_7 + \frac{1}{2}G_4J_{30} + \frac{1}{2}G_5J_{53} \quad M_6 = -E_1J_{70}$$

$$M_7 = -\frac{G_1J_{83}}{r^2} \quad M_8 = -\frac{1}{2}\frac{G_1J_{88}}{r} \quad M_9 = \frac{1}{2}\frac{E_1J_{78}}{r^3} - \frac{1}{2}\frac{G_1J_{95}}{r^3}$$

$$M_{10} = -\frac{1}{2}\frac{G_1J_{96}}{r^2} \quad M_{11} = \frac{1}{2}G_4J_{28} + \frac{1}{2}G_3J_5 + \frac{1}{2}G_5J_{51}$$

$$M_{12} = \frac{E_2J_{105}}{r^2} + G_4J_{35} + G_3J_{12} + G_5J_{58} \quad M_{13} = \frac{1}{2}\frac{E_2J_{102}}{r^2} - \frac{1}{2}\frac{G_2J_{113}}{r^2}$$

$$M_{14} = \frac{1}{2}G_3J_8 + \frac{1}{2}G_4J_{31} + \frac{1}{2}G_5J_{54} \quad M_{15} = \frac{1}{2}G_3J_9 + \frac{1}{2}G_4J_{32} + \frac{1}{2}G_5J_{55} + \frac{\frac{1}{2}E_2J_{107}}{r^2}$$

$$M_{16} = -E_2J_{98} \quad M_{17} = -\frac{G_2J_{111}}{r^2} \quad M_{18} = -\frac{1}{2}\frac{G_2J_{116}}{r}$$

$$M_{19} = \frac{1}{2}\frac{E_2J_{106}}{r^3} - \frac{1}{2}\frac{G_2J_{123}}{r^3} \quad M_{20} = -\frac{1}{2}\frac{G_2J_{124}}{r^2}$$

$$M_{21} = \frac{G_1J_{84}}{r^2} + G_4J_{44} + \frac{\frac{1}{2}G_1J_{87}}{r^2} + G_3J_{21} + G_5J_{67}$$

$$M_{22} = \frac{1}{2}G_3J_{23} + \frac{1}{2}G_4J_{46} + \frac{1}{2}G_5J_{69} \quad M_{23} = \frac{1}{2}\frac{G_1J_{85}}{r^2} + \frac{\frac{1}{2}G_1J_{88}}{r^2} - \frac{1}{2}\frac{E_1J_{74}}{r^2}$$

$$M_{24}=\frac{1}{2}\frac{G_1J_{92}}{r^3}+\frac{\frac{1}{2}G_5J_{60}}{r}+\frac{G_1J_{89}}{r^3}+\frac{\frac{1}{2}G_4J_{37}}{r}+\frac{\frac{1}{2}G_3J_{14}}{r}$$

$$M_{25}=\frac{1}{2}\frac{G_4J_{38}}{r}+\frac{\frac{1}{2}G_3J_{15}}{r}+\frac{\frac{1}{2}G_5J_{61}}{r}\quad M_{26}=-\frac{1}{2}\frac{G_1J_{88}}{r}\quad M_{27}=-G_1J_{86}$$

$$M_{28}=-\frac{E_1J_{73}}{r^2}\quad M_{29}=\frac{1}{2}\frac{G_1J_{93}}{r^2}-\frac{1}{2}G_1\left(\frac{J_{89}}{r^2}-\frac{J_{90}}{r^2}\right)-\frac{1}{2}\frac{E_1J_{76}}{r^2}$$

$$M_{30}=-\frac{1}{2}\frac{E_1J_{75}}{r^3}\quad M_{31}=\frac{1}{2}G_3J_{23}+\frac{1}{2}G_4J_{46}+\frac{1}{2}G_5J_{69}$$

$$M_{32}=\frac{G_2J_{112}}{r^2}+G_4J_{45}+G_3J_{22}+G_5J_{68}+\frac{\frac{1}{2}G_2J_{115}}{r^2}$$

$$M_{33}=\frac{1}{2}\frac{G_2J_{113}}{r^2}+\frac{1}{2}\frac{G_2J_{116}}{r^2}-\frac{1}{2}\frac{E_2J_{102}}{r^2}\quad M_{34}=\frac{1}{2}\frac{G_3J_{16}}{r}+\frac{\frac{1}{2}G_4J_{39}}{r}+\frac{\frac{1}{2}G_5J_{62}}{r}$$

$$M_{35}=\frac{1}{2}\frac{G_2J_{120}}{r^3}+\frac{G_2J_{117}}{r^3}+\frac{\frac{1}{2}G_3J_{17}}{r}+\frac{\frac{1}{2}G_4J_{40}}{r}+\frac{\frac{1}{2}G_5J_{63}}{r}$$

$$M_{36}=-\frac{1}{2}\frac{G_2J_{116}}{r}\quad M_{37}=-G_2J_{114}\quad M_{38}=-\frac{E_2J_{101}}{r^2}$$

$$M_{39}=\frac{1}{2}\frac{G_2J_{121}}{r^2}-\frac{1}{2}G_2\left(\frac{J_{117}}{r^2}-\frac{J_{118}}{r^2}\right)-\frac{1}{2}\frac{E_2J_{104}}{r^2}\quad M_{40}=-\frac{1}{2}\frac{E_2J_{103}}{r^3}$$

$$M_{41}=E_3J_1+E_4J_{24}+E_5J_{47}\quad M_{42}=\frac{1}{2}E_3J_2+\frac{1}{2}E_4J_{25}+\frac{1}{2}E_5J_{48}$$

$$M_{43}=-\frac{1}{2}G_5J_{52}-\frac{1}{2}G_4J_{29}-\frac{1}{2}\frac{E_1J_{79}}{r^2}-\frac{1}{2}G_3J_6\quad M_{44}=-\frac{1}{2}G_3J_8-\frac{1}{2}G_5J_{54}-\frac{1}{2}G_4J_{31}$$

$$M_{45}=-\frac{1}{2}\frac{G_1J_{92}}{r^3}-\frac{1}{2}\frac{G_3J_{14}}{r}-\frac{1}{2}\frac{G_4J_{37}}{r}-\frac{1}{2}\frac{G_5J_{60}}{r}+\frac{E_1J_{76}}{r^3}-\frac{2G_1J_{93}}{r^3}$$

$$M_{46}=-\frac{1}{2}\frac{G_3J_{16}}{r}-\frac{1}{2}\frac{G_4J_{39}}{r}-\frac{1}{2}\frac{G_5J_{62}}{r}\quad M_{47}=2\frac{E_1J_{82}}{r^3}$$

$$M_{48}=-2\frac{G_1J_{96}}{r^3}+\frac{\frac{1}{2}E_1J_{78}}{r^3}-\frac{1}{2}\frac{G_1J_{95}}{r^3}\quad M_{49}=-\frac{1}{2}\frac{E_1J_{76}}{r^2}-\frac{1}{2}\frac{G_1J_{89}}{r^2}+G_1\left(-\frac{J_{90}}{r^2}+\frac{J_{93}}{r^2}\right)$$

$$M_{50}=-\frac{E_1J_{82}}{r^2}-G_3J_{13}-G_5J_{59}-G_4J_{36}\quad M_{51}=-\frac{1}{2}G_3J_{10}-\frac{1}{2}G_5J_{56}-\frac{1}{2}G_4J_{33}$$

$$M_{52}=-\frac{G_1J_{94}}{r^4}-\frac{G_3J_{19}}{r^2}+\frac{\frac{3}{2}E_1J_{81}}{r^4}-\frac{G_5J_{65}}{r^2}-\frac{3G_1J_{97}}{r^4}-\frac{G_4J_{42}}{r^2}$$

$$M_{53}=-\frac{1}{2}\frac{G_3J_{18}}{r^2}-\frac{1}{2}\frac{G_4J_{41}}{r^2}-\frac{1}{2}\frac{G_5J_{64}}{r^2}\quad M_{54}=\frac{G_1J_{96}}{r^2}\quad M_{55}=\frac{1}{2}\frac{E_1J_{75}}{r^3}$$

$$M_{56} = -\frac{1}{2}\frac{G_1 J_{97}}{r^3} + G_1\left(-4\frac{J_{91}}{r^3} + \frac{J_{97}}{r^3}\right) \quad M_{57} = E_1 J_{72} \quad M_{58} = \frac{E_1 J_{80}}{r^4}$$

$$M_{59} = 2\frac{G_1 J_{91}}{r^2} \quad M_{60} = \frac{1}{2}E_3 J_2 + \frac{1}{2}E_4 J_{25} + \frac{1}{2}E_5 J_{48}$$

$$M_{61} = E_3 J_3 + E_4 J_{26} + E_5 J_{49} \quad M_{62} = -\frac{1}{2}G_4 J_{30} - \frac{1}{2}G_5 J_{53} - \frac{1}{2}G_3 J_7$$

$$M_{63} = -\frac{1}{2}\frac{E_2 J_{107}}{r^2} - \frac{1}{2}G_5 J_{55} - \frac{1}{2}G_3 J_9 - \frac{1}{2}G_4 J_{32}$$

$$M_{64} = -\frac{1}{2}\frac{G_3 J_{15}}{r} - \frac{1}{2}\frac{G_4 J_{38}}{r} - \frac{1}{2}\frac{G_5 J_{61}}{r}$$

$$M_{65} = -2\frac{G_2 J_{121}}{r^3} + \frac{E_2 J_{104}}{r^3} - \frac{1}{2}\frac{G_3 J_{17}}{r} - \frac{1}{2}\frac{G_4 J_{40}}{r} - \frac{1}{2}\frac{G_5 J_{63}}{r} - \frac{1}{2}\frac{G_2 J_{120}}{r^3}$$

$$M_{66} = 2\frac{E_2 J_{110}}{r^3} \quad M_{67} = -2\frac{G_2 J_{124}}{r^3} - \frac{1}{2}\frac{G_2 J_{123}}{r^3} + \frac{\frac{1}{2}E_2 J_{106}}{r^3}$$

$$M_{68} = -\frac{1}{2}\frac{G_2 J_{117}}{r^2} - \frac{1}{2}\frac{E_2 J_{104}}{r^2} + G_2\left(-\frac{J_{118}}{r^2} + \frac{J_{121}}{r^2}\right)$$

$$M_{69} = -\frac{1}{2}G_3 J_{10} - \frac{1}{2}G_5 J_{56} - \frac{1}{2}G_4 J_{33}$$

$$M_{70} = -\frac{E_2 J_{110}}{r^2} - G_3 J_4 - G_5 J_{50} - G_4 J_{27} \quad M_{71} = -\frac{1}{2}\frac{G_3 J_{18}}{r^2} - \frac{1}{2}\frac{G_4 J_{41}}{r^2} - \frac{1}{2}\frac{G_5 J_{64}}{r^2}$$

$$M_{72} = -\frac{G_2 J_{122}}{r^4} - \frac{G_3 J_{20}}{r^2} - \frac{G_4 J_{43}}{r^2} - \frac{G_5 J_{66}}{r^2} + \frac{\frac{3}{2}E_2 J_{109}}{r^4} - \frac{3G_2 J_{125}}{r^4}$$

$$M_{73} = \frac{G_2 J_{124}}{r^2} \quad M_{74} = \frac{1}{2}\frac{E_2 J_{103}}{r^3}$$

$$M_{75} = G_2\left(\frac{J_{125}}{r^3} - \frac{4J_{119}}{r^3}\right) - \frac{1}{2}\frac{G_2 J_{125}}{r^3} \quad M_{76} = E_2 J_{100} \quad M_{77} = \frac{E_2 J_{108}}{r^4}$$

$$M_{78} = 2\frac{G_2 J_{119}}{r^2}$$

Appendix C. Integral Constants $J_i (i = 1 \sim 125)$

$$J_1 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} \left(\frac{\partial}{\partial z} k_9(z)\right)^2 dz \quad J_2 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_9(z)\right)\left(\frac{\partial}{\partial z} k_{10}(z)\right) dz$$

$$J_3 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} \left(\frac{\partial}{\partial z} k_{10}(z)\right)^2 dz$$

$$J_4 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_4(z)\right) k_{10}(z) + \left(\frac{\partial}{\partial z} k_4(z)\right)^2 + k_{10}(z)^2 dz$$

$$J_5 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_1(z)\right)\left(\frac{\partial}{\partial z} k_2(z)\right) dz$$

$$J_6 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_1(z)\right)\left(\frac{\partial}{\partial z} k_3(z)\right) + 2\left(\frac{\partial}{\partial z} k_1(z)\right) k_9(z) dz$$

$$J_7 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_1(z)\right)\left(\frac{\partial}{\partial z} k_4(z)\right) + 2\left(\frac{\partial}{\partial z} k_1(z)\right) k_{10}(z) dz$$

$$J_8 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_2(z)\right) k_3(z) + 2\left(\frac{\partial}{\partial z} k_2(z)\right) k_9(z) dz$$

$$J_9 = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_2(z)\right)\left(\frac{\partial}{\partial z} k_4(z)\right) + 2\left(\frac{\partial}{\partial z} k_2(z)\right) k_{10}(z) dz$$

$$J_{10} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_3(z)\right)\left(\frac{\partial}{\partial z} k_4(z)\right) + 2\left(\frac{\partial}{\partial z} k_3(z)\right) k_{10}(z) + 2k_9(z)\left(\frac{\partial}{\partial z} k_4(z)\right) + 2\left(\frac{\partial}{\partial z} k_1(z)\right)^2 dz$$

$$J_{11} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2k_3(z) k_{10}(z) dz \quad J_{12} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} \left(\frac{\partial}{\partial z} k_2(z)\right)^2 dz$$

$$J_{13} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} \left(\frac{\partial}{\partial z} k_3(z)\right)^2 + k_9(z)^2 + 2\left(\frac{\partial}{\partial z} k_3(z)\right) k_9(z) dz$$

$$J_{14} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_5(z)\right)\left(\frac{\partial}{\partial z} k_7(z)\right) + 2k_9(z)\left(\frac{\partial}{\partial z} k_5(z)\right) dz$$

$$J_{15} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z} k_5(z)\right)\left(\frac{\partial}{\partial z} k_8(z)\right) + 2k_{10}(z)\left(\frac{\partial}{\partial z} k_5(z)\right) dz$$

$$J_{16} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_6(z)\right)\left(\frac{\partial}{\partial z}k_7(z)\right) + 2k_9(z)\left(\frac{\partial}{\partial z}k_6(z)\right)dz$$

$$J_{17} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_6(z)\right)\left(\frac{\partial}{\partial z}k_8(z)\right) + 2k_{10}(z)\left(\frac{\partial}{\partial z}k_6(z)\right)dz$$

$$J_{18} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2k_9(z)k_{10}(z) + 2\left(\frac{\partial}{\partial z}k_8(z)\right)k_9(z) + 2k_{10}(z)\left(\frac{\partial}{\partial z}k_7(z)\right) + 2\left(\frac{\partial}{\partial z}k_7(z)\right)\left(\frac{\partial}{\partial z}k_8(z)\right)dz$$

$$J_{19} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} k_9(z)^2 + 2k_9(z)\left(\frac{\partial}{\partial z}k_7(z)\right) + \left(\frac{\partial}{\partial z}k_7(z)\right)^2 dz$$

$$J_{20} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} k_{10}(z)^2 + 2k_{10}(z)\left(\frac{\partial}{\partial z}k_8(z)\right) + \left(\frac{\partial}{\partial z}k_8(z)\right)^2 dz$$

$$J_{21} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} \left(\frac{\partial}{\partial z}k_5(z)\right)^2 dz \quad J_{22} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} \left(\frac{\partial}{\partial z}k_6(z)\right)^2 dz$$

$$J_{23} = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_5(z)\right)\left(\frac{\partial}{\partial z}k_6(z)\right)dz \quad J_{24} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} \left(\frac{\partial}{\partial z}k_{19}(z)\right)^2 dz$$

$$J_{25} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{19}(z)\right)\left(\frac{\partial}{\partial z}k_{20}(z)\right)dz \quad J_{26} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} \left(\frac{\partial}{\partial z}k_{20}(z)\right)^2 dz$$

$$J_{27} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{14}(z)\right)k_{20}(z) + \left(\frac{\partial}{\partial z}k_{14}(z)\right)^2 + k_{20}(z)^2 dz$$

$$J_{28} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{11}(z)\right)\left(\frac{\partial}{\partial z}k_{12}(z)\right)dz$$

$$J_{29} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{11}(z)\right)\left(\frac{\partial}{\partial z}k_{13}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{11}(z)\right)k_{19}(z)dz$$

$$J_{30} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{11}(z)\right)\left(\frac{\partial}{\partial z}k_{14}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{11}(z)\right)k_{20}(z)dz$$

$$J_{31} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{12}(z)\right)k_{13}(z) + 2\left(\frac{\partial}{\partial z}k_{12}(z)\right)k_{19}(z)dz$$

$$J_{32} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{12}(z)\right)\left(\frac{\partial}{\partial z}k_{14}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{12}(z)\right)k_{20}(z)dz$$

$$J_{33} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{13}(z)\right)\left(\frac{\partial}{\partial z}k_{14}(z)\right)+2\left(\frac{\partial}{\partial z}k_{13}(z)\right)k_{20}(z)+2k_{19}(z)\left(\frac{\partial}{\partial z}k_{14}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{11}(z)\right)^2 dz$$

$$J_{34} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2k_{13}(z)k_{20}(z)dz \quad J_{35} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} \left(\frac{\partial}{\partial z}k_{12}(z)\right)^2 dz$$

$$J_{36} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} \left(\frac{\partial}{\partial z}k_{13}(z)\right)^2 + k_{19}(z)^2 + 2\left(\frac{\partial}{\partial z}k_{13}(z)\right)k_{19}(z)dz$$

$$J_{37} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{15}(z)\right)\left(\frac{\partial}{\partial z}k_{17}(z)\right)+2k_{19}(z)\left(\frac{\partial}{\partial z}k_{15}(z)\right)dz$$

$$J_{38} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{15}(z)\right)\left(\frac{\partial}{\partial z}k_{18}(z)\right)+2k_{20}(z)\left(\frac{\partial}{\partial z}k_{15}(z)\right)dz$$

$$J_{39} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{16}(z)\right)\left(\frac{\partial}{\partial z}k_{17}(z)\right)+2k_{19}(z)\left(\frac{\partial}{\partial z}k_{16}(z)\right)dz$$

$$J_{40} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{16}(z)\right)\left(\frac{\partial}{\partial z}k_{18}(z)\right)+2k_{20}(z)\left(\frac{\partial}{\partial z}k_{16}(z)\right)dz$$

$$J_{41} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2k_{19}(z)k_{20}(z)+2\left(\frac{\partial}{\partial z}k_{18}(z)\right)k_{19}(z)+2k_{20}(z)\left(\frac{\partial}{\partial z}k_{17}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{17}(z)\right)\left(\frac{\partial}{\partial z}k_{18}(z)\right)dz$$

$$J_{42} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} k_{19}(z)^2+2k_{19}(z)\left(\frac{\partial}{\partial z}k_{17}(z)\right)+\left(\frac{\partial}{\partial z}k_{17}(z)\right)^2 dz$$

$$J_{43} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} k_{20}(z)^2+2k_{20}(z)\left(\frac{\partial}{\partial z}k_{18}(z)\right)+\left(\frac{\partial}{\partial z}k_{18}(z)\right)^2 dz$$

$$J_{44} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} \left(\frac{\partial}{\partial z}k_{15}(z)\right)^2 dz \quad J_{45} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} \left(\frac{\partial}{\partial z}k_{16}(z)\right)^2 dz$$

$$J_{46} = \int_{-\frac{1}{2}c-t}^{-\frac{1}{2}c} 2\left(\frac{\partial}{\partial z}k_{15}(z)\right)\left(\frac{\partial}{\partial z}k_{16}(z)\right)dz \quad J_{47} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} \left(\frac{\partial}{\partial z}k_{29}(z)\right)^2 dz$$

$$J_{48} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{29}(z)\right)\left(\frac{\partial}{\partial z}k_{30}(z)\right)dz \quad J_{49} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} \left(\frac{\partial}{\partial z}k_{30}(z)\right)^2 dz$$

$$J_{50} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{24}(z)\right)k_{30}(z) + \left(\frac{\partial}{\partial z}k_{24}(z)\right)^2 + k_{30}(z)^2 dz$$

$$J_{51} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{21}(z)\right)\left(\frac{\partial}{\partial z}k_{22}(z)\right) dz$$

$$J_{52} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{21}(z)\right)\left(\frac{\partial}{\partial z}k_{23}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{21}(z)\right)k_{29}(z)dz$$

$$J_{53} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{21}(z)\right)\left(\frac{\partial}{\partial z}k_{24}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{21}(z)\right)k_{30}(z)dz$$

$$J_{54} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{22}(z)\right)k_{23}(z) + 2\left(\frac{\partial}{\partial z}k_{22}(z)\right)k_{29}(z)dz$$

$$J_{55} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{22}(z)\right)\left(\frac{\partial}{\partial z}k_{24}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{22}(z)\right)k_{30}(z)dz$$

$$J_{56} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{23}(z)\right)\left(\frac{\partial}{\partial z}k_{24}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{23}(z)\right)k_{30}(z) + 2k_{29}(z)\left(\frac{\partial}{\partial z}k_{24}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{21}(z)\right)^2 dz$$

$$J_{57} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2k_{23}(z)k_{30}(z)dz \quad J_{58} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} \left(\frac{\partial}{\partial z}k_{22}(z)\right)^2 dz$$

$$J_{59} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} \left(\frac{\partial}{\partial z}k_{23}(z)\right)^2 + k_{29}(z)^2 + 2\left(\frac{\partial}{\partial z}k_{23}(z)\right)k_{29}(z)dz$$

$$J_{60} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{25}(z)\right)\left(\frac{\partial}{\partial z}k_{27}(z)\right) + 2k_{29}(z)\left(\frac{\partial}{\partial z}k_{25}(z)\right) dz$$

$$J_{61} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{25}(z)\right)\left(\frac{\partial}{\partial z}k_{28}(z)\right) + 2k_{30}(z)\left(\frac{\partial}{\partial z}k_{25}(z)\right) dz$$

$$J_{62} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{26}(z)\right)\left(\frac{\partial}{\partial z}k_{27}(z)\right) + 2k_{29}(z)\left(\frac{\partial}{\partial z}k_{26}(z)\right) dz$$

$$J_{63} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{26}(z)\right)\left(\frac{\partial}{\partial z}k_{28}(z)\right) + 2k_{30}(z)\left(\frac{\partial}{\partial z}k_{26}(z)\right) dz$$

$$J_{64} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2k_{29}(z)k_{30}(z) + 2\left(\frac{\partial}{\partial z}k_{28}(z)\right)k_{29}(z) + 2k_{30}(z)\left(\frac{\partial}{\partial z}k_{27}(z)\right) + 2\left(\frac{\partial}{\partial z}k_{27}(z)\right)\left(\frac{\partial}{\partial z}k_{28}(z)\right)dz$$

$$J_{65} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} k_{29}(z)^2 + 2k_{29}(z)\left(\frac{\partial}{\partial z}k_{27}(z)\right) + \left(\frac{\partial}{\partial z}k_{27}(z)\right)^2 dz$$

$$J_{66} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} k_{30}(z)^2 + 2k_{30}(z)\left(\frac{\partial}{\partial z}k_{28}(z)\right) + \left(\frac{\partial}{\partial z}k_{28}(z)\right)^2 dz$$

$$J_{67} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} \left(\frac{\partial}{\partial z}k_{25}(z)\right)^2 dz \quad J_{68} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} \left(\frac{\partial}{\partial z}k_{26}(z)\right)^2 dz$$

$$J_{69} = \int_{\frac{1}{2}c}^{\frac{1}{2}c+t} 2\left(\frac{\partial}{\partial z}k_{25}(z)\right)\left(\frac{\partial}{\partial z}k_{26}(z)\right)dz \quad J_{70} = f_1$$

$$J_{71} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 2k_{31}(z)dz \quad J_{72} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} k_{31}(z)^2 dz \quad J_{73} = f_1$$

$$J_{74} = 2f_1 \quad J_{75} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 2k_{31}(z)dz \quad J_{76} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 2k_{31}(z)dz$$

$$J_{77} = f_1 \quad J_{78} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 2k_{31}(z)dz \quad J_{79} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 2k_{31}(z)dz$$

$$J_{80} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} k_{31}(z)^2 dz \quad J_{81} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 2k_{31}(z)^2 dz \quad J_{82} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} k_{31}(z)^2 dz$$

$$J_{83} = f_1 \quad J_{84} = f_1 \quad J_{85} = -2f_1$$

$$J_{86} = f_1 \quad J_{87} = -2f_1 \quad J_{88} = 2f_1$$

$$J_{89} = -4f_1 \quad J_{90} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 4k_{31}(z)dz \quad J_{91} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 4k_{31}(z)^2 dz$$

$$J_{92} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 4k_{31}(z)dz \quad J_{93} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} -4k_{31}(z)^2 dz \quad J_{94} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 4k_{31}(z)^2 dz$$

$$J_{95} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} -4k_{31}(z)dz \quad J_{96} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} 4k_{31}(z)dz \quad J_{97} = \int_{-\frac{1}{2}c-t-f_1}^{-\frac{1}{2}c-t} -8k_{31}(z)^2 dz$$

$$J_{98} = f_2 \quad J_{99} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 2k_{32}(z)dz \quad J_{100} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} k_{32}(z)^2 dz$$

$$J_{101} = f_2 \quad J_{102} = 2f_2 \quad J_{103} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 2k_{32}(z)dz$$

$$J_{104} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 2k_{32}(z)dz \quad J_{105} = f_2 \quad J_{106} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 2k_{32}(z)dz$$

$$J_{107} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 2k_{32}(z)dz \quad J_{108} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} k_{32}(z)^2dz \quad J_{109} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 2k_{32}(z)^2dz$$

$$J_{110} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} k_{32}(z)^2dz \quad J_{111} = f_2 \quad J_{112} = f_2$$

$$J_{113} = -2f_2 \quad J_{114} = f_2 \quad J_{115} = -2f_2$$

$$J_{116} = 2f_2 \quad J_{117} = -4f_2 \quad J_{118} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 4k_{32}(z)dz$$

$$J_{119} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 4k_{32}(z)^2dz \quad J_{120} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 4k_{32}(z)dz \quad J_{121} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} -4k_{32}(z)^2dz$$

$$J_{122} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 4k_{32}(z)^2dz \quad J_{123} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} -4k_{32}(z)dz \quad J_{124} = \int_{\frac{1}{2}c+t}^{\frac{1}{2}c+t+f_2} 4k_{32}(z)dz$$