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A Meshless Method for Nonlinear, Singular and Generalized Sturm-Liouville Problems

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Abstract: A new numerical technique for solving generalized Sturm–Liouville problem $\frac{d^2w}{dx^2}+q(x,\lambda)w=0$, $b_l[\lambda,w(a)]=b_r[\lambda,w(b)]=0$ is presented. In particular, we consider the problems when the coefficient $q(x,\lambda)$ or the boundary conditions depend on the spectral parameter λ in an arbitrary nonlinear manner. The method presented is based on mathematically modelling of physical response of a system to excitation over a range of frequencies. The response amplitudes are then used to determine the eigenvalues. The same technique can be applied to a very wide class of the eigenproblems: the Sturm–Liouville problems, the Schrodinger equation, the non-classical non-linear Sturm–Liouville problems, periodic problems. The results of the numerical experiments justifying the method are presented.

Keyword: Non-linear eigenvalue problems; Singular Sturm–Liouville problems; Numerical solution; Periodic eigenvalue problems; Parameter-dependent boundary conditions

1 Introduction

In this paper we deal with the numerical solution of the following generalized Sturm-Liouville problems:

$$\frac{d^2w}{dx^2}+q(x,\lambda)w=0 \tag{1}$$

$$b_l[\lambda,w(a)]=b_r[\lambda,w(b)]=0 \tag{2}$$

Here $q(x,\lambda)$ is a known function and is assumed to be sufficiently smooth and separated from zero, so that

$$0<q_1\le q(x,\lambda)\le q_2<\infty, \quad a\le x\le b. \tag{3}$$

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The set of admissible values of λ is such that conditions (3) hold. For positive λ we also use the notation $\lambda = k^2$. The operators $b_l[...]$ and $b_r[...]$ specify the boundary conditions. So, we assume that the boundary conditions depend on the eigenparameter λ. When a and b are finite, the Sturm- Liouville eigenvalues problem is regular; otherwise, it is singular.

Considering periodic Sturm-Liouville problems we assume that $q(x,\lambda)$ is a periodic function $q(x+T,\lambda) = q(x,\lambda)$. Such problems arise naturally in the investigation of parametric vibrations and parametric instability, e.g.,[Akulenko and Nesterov (2005); McLachlan (1947)]. In this case the boundary conditions have the form: $w(0) = w(T)$, $w'(0) = w'(T)$.

The problem (1), (2) with $q(x,\lambda) = \lambda q_1(x)$ is a classical problem of mathematical physics [Morse and Feshbach (1953)] and many efforts have been applied to develop its theory (see, e.g. [Courant and Hilbert (1989); Titchmarsh (1962); Boyce and DiPrima (2004)]). However, apart from a few analytically solvable cases, there is no general solution of this problem. Therefore, a large number of numerical methods have been developed for many practical problems. The usual approach to the numerical solution of the Sturm–Liouville problem is to use the Rayleigh minimal principle. Then, using an approximation for w with a finite number of free parameters, one gets the same problem in a finite-dimensional subspace which can be solved by a standard procedure of linear algebra. This results in an effective algorithm which is usually applied for studying lower modes and described in [Gould (1995); Collatz, Albrecht, and Velte (1987); Ghelardoni (1997)]. Pryce (1993) has provided an excellent review of the mathematical background of Sturm-Liouville eigenvalues problems and their numerical solutions, as well as a detailed discussion of applications. A shooting technique for computing eigenvalues was proposed by Ghelardoni and Gheri (2001). Its very effective modification - the Lie-group shooting method has been recently suggested for the computations of second order two-point boundary value problems (BVPs) [Liu (2008a, 2006b)] as well as for singularly perturbated BVPs [Liu (2006c)]. It was suggested for solving time-varying linear systems in [Liu (2007)] and for inverse vibration problems in [Liu (2008b)]. In [Liu (2008a)] the Lie-group shooting method was suggested for computing eigenvalues and eigenfunctions of Sturm-Liouville problems. This technique is applicable for regular as well as for singular Sturm-Liouville eigenvalues problems.

When the coefficient $q(x,\lambda)$ depends on the spectral parameter λ in an arbitrary manner one gets a nonlinear Sturm–Liouville problem. This also concerns the problems with parameter dependent boundary conditions. This class of problems essentially differs from the classical case, and so far no regular method has been proposed for solving nonlinear Sturm–Liouville problems. However, methods for computing the eigenvalues of the problems with parameter dependent boundary conditions have been developed recently by Aliyev and Kerimov (2008); Annaby and Tharwat (2006); Chanane (2008, 2007, 2005).

So, in this paper we focused on the numerical algorithm for solving 1) nonlinear Sturm–Liouville problems; 2) Sturm–Liouville problems with parameter dependent boundary conditions; 3) singular Sturm–Liouville problems; 4) periodic Sturm–Liouville problems. The goal of the paper is to present a new numerical technique which could cope with all these kinds of Sturn-Liouville eigenvalue problems.

Note that without a loss of generality we assume that $0 \leq x \leq 1$ through the paper. If the argument x varies on an arbitrary bounded segment $[a,b]$, we can pass onto another variable defined on the unit segment by letting $x_1 = (x-a)/(b-a)$.

The method presented in the paper is based on the following quite trivial statement. Let $w_e(x)$ be an arbitrary smooth enough function defined in the interval $[0,1]$ named below as the *exciting field*. If the *response field* w_r is a solution of the boundary value problem (BVP)

$$\frac{d^2 w_r}{dx^2} + q(x,\lambda) w_r = -\frac{d^2 w_e}{dx^2} - q(x,\lambda) w_e, \tag{4}$$

$$b_l[\lambda, w_r(0)] = -b_l[\lambda, w_e(0)], \tag{5}$$

$$b_r[\lambda, w_r(1)] = -b_r[\lambda, w_e(1)], \tag{6}$$

then the sum $w(x,\lambda) = w_r + w_e$ satisfies the original problem (1), (2). Let $F(\lambda)$ be some norm of the solution w. This function of λ has maxima at the eigenvalues and, under some conditions described below, can be used for their determining. The growth of the amplitude of response near the eigenvalue is a sequence of the degeneracy of the matrix of the linear algebraic system which approximates the BVP. From this point of view the presented approach is similar to the one presented by Li, Hu, Lu, Tsai, and Cheng (2006); Li (2008), where the degeneracy is measured by the infinitesimal values of the minimal eigenvalue of the stiffness matrix of the problem. Recently this technique has been applied for solving problems of free vibrations of beams, membranes and plates (see [Reutskiy (2005, 2006, 2007a,b,c)]) and for analysis of arbitrarily-shaped waveguides described in [Reutskiy (2008)].

The outline of this paper is as follows: the main algorithm with regularizing procedures is described in Section 2. In Section 3 we present some examples of application of the method presented to the standard and to the non-classical generalized Sturm–Liouville problems. The modes calculation we present in Section 4. Finally, in Section 5, we give the conclusion.

2 Main algorithm

To illustrate the method presented in the simplest case, let us consider the wave equation in homogeneous medium $\partial_{tt}^2 u = \partial_{xx}^2 u$ with the Dirichlet conditions at the endpoints of the interval $[0,1]$, i.e., $u(0,t) = u(1,t) = 0$. Considering the time dependence $u(x,t) = e^{-ikt} w(x)$, we get the following eigenvalue problem on the interval $[0,1]$:

$$\frac{d^2 w}{dx^2} + \lambda w = 0,\ \lambda = k^2, \tag{7}$$

$$w(0) = w(1) = 0, \tag{8}$$

which admits of an analytic solution $\lambda_n = k_n^2 = (n\pi)^2$.

According to the method presented in the paper, we take an arbitrary smooth enough w_e and get the *response field* w_r as a solution of the BVP:

$$\frac{d^2 w_r}{dx^2} + \lambda w_r = -\frac{d^2 w_e}{dx^2} - \lambda w_e, \tag{9}$$

$$w_r(0) = -w_e(0),\ w_r(1) = -w_e(1). \tag{10}$$

Then, the sum $w = w_e + w_r$ satisfies the original BVP (7), (8) for any choice of w_e. Note that from physical point of view the right hand side of (9) can be considered as an external exciting source in the wave equation. And w_r can be treated as a response to this excitation. Let us introduce the norm of the solution as

$$F(\lambda) = \sqrt{\sum_{n=1}^{N_t} |w(x_n)|^2 / N_t}, \tag{11}$$

where the points x_n are randomly distributed in $[0,1]$. We also use the dimensionless form of this function: $F_d(\lambda) = F(\lambda)/F(1)$. The function $F(\lambda)$ characterizes the value of the response of the system to the excitation with the wave number $k = \sqrt{\lambda}$. Varying λ, we get the response curve and calculate the eigenvalues as positions of maxima.

However, this initial form of the method is unfit for our goal. Indeed, a particular solution of (9) is $\widetilde{w}_r = -w_e$. Looking for the response field in the form

$$w_r = A_r \exp(ikx) + B_r \exp(-ikx) - w_e(x),$$

$$k = \lambda^{1/2},$$

we get the linear system for A_r, B_r

$$A_r + B_r - w_e(0) = -w_e(0), \tag{12}$$
$$A_r \exp(ik) + B_r \exp(-ik) - w_e(1) = -w_e(1). \tag{13}$$

For $k \neq n\pi$ the system has the unique solution $A_r = 0,\ B_r = 0$. Thus, $w \equiv 0$ and $F = 0$ with the machine precision. In Fig. 1 we place the response curve corresponding to the exciting field

$$w_e(x) = 1 + x^2. \tag{14}$$

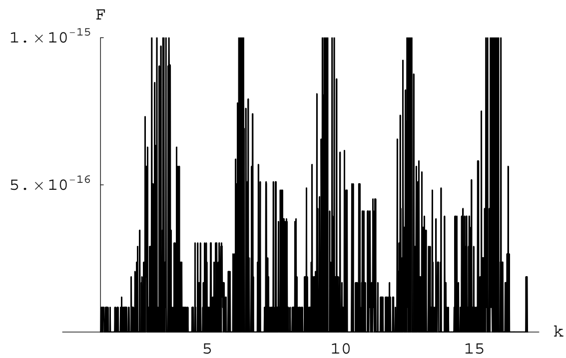

Figure 1: The response curve $F(k)$, $k = \sqrt{\lambda}$ corresponding to the exciting field $w_e(x) = 1 + x^2$. No smoothing.

To get a smooth response curve $F(\lambda)$ we use the following two regularizing procedures. Applying the first one, we substitute BVP (9), (10) as follows:

$$\frac{d^2 w_r}{dx^2} + (\lambda + i\varepsilon) w_r = -\frac{d^2 w_e}{dx^2} - \lambda w_e(x),$$
$$w_r(0) = -w_e(0),\ w_r(1) = -w_e(1),$$

where $\varepsilon > 0$ is a small value. So we shift the spectra of differential operator from the real axis. On the other hand, from the physical point of view, this means that the wave propagates in a weakly absorbing medium and the initial equation is replaced by the equation $\partial_{tt}^2 u = \partial_{xx}^2 u - \varepsilon \partial_t u$. This wave equation also describes vibrations of the string with friction [Morse and Feshbach (1953)]. Resulting BVP has a unique non zero solution for all real λ.

To illustrate this technique we take the same *exciting* field $w_e(x) = 1 + x^2$. The particular solution can also be taken in the same polynomial form

$$\widetilde{w}_r(x,\lambda,\varepsilon) = -\frac{\lambda}{\lambda + i\varepsilon}x^2 - \frac{2+\lambda}{\lambda + i\varepsilon} + \frac{2\lambda}{(\lambda + i\varepsilon)^2}, \tag{15}$$

If $\varepsilon \to 0$, then $\widetilde{w}_r(x,\lambda,\varepsilon) \to -w_e$. But $\widetilde{w}_r \neq -w_e$ for $\varepsilon \neq 0$. As a result, we get the following system instead of (12), (13):

$$A_r + B_r + \widetilde{w}_r(0,\lambda,\varepsilon) = -w_e(0), \tag{16}$$

$$A_r e^{ik_\varepsilon} + B_r e^{-ik_\varepsilon} + \widetilde{w}_r(1,\lambda,\varepsilon) = -w_e(1), \tag{17}$$

where $k_\varepsilon = \sqrt{\lambda + i\varepsilon}$. The response curves $F_d(\lambda)$ depicted in Fig. 2 correspond to $\varepsilon = 10^{-15}$ (left) and $\varepsilon = 10^{-10}$ (right). The value $\varepsilon = 10^{-15}$ is too small to regularize the solution. The value $\varepsilon = 10^{-10}$ provides a smooth curve.

The second regularizing procedure is as follows. We take w_r as a solution of the

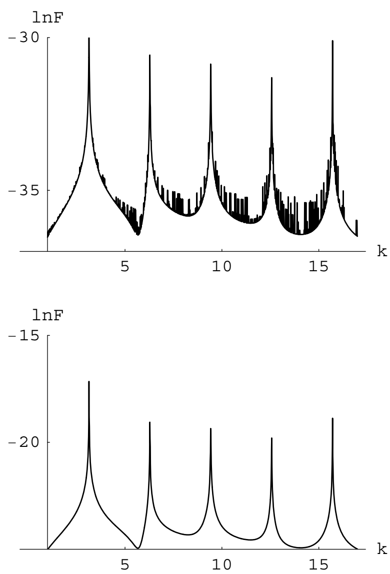

Figure 2: The response curve $\ln(F(k))$. $\varepsilon-$procedure with $\varepsilon = 10^{-15}$ (top) and $\varepsilon = 10^{-10}$ (bottom)

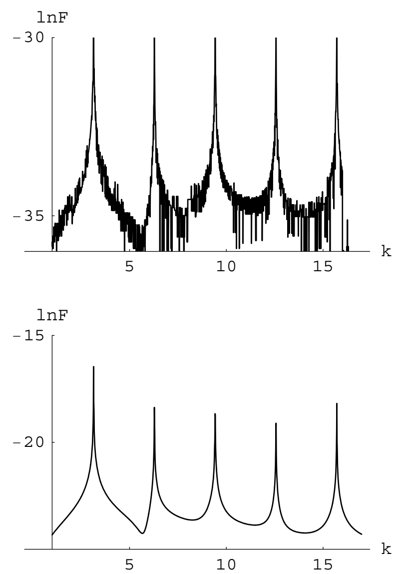

Figure 3: The response curve $\ln(F(k))$. $\lambda-$procedure with $\Delta\lambda = 10^{-15}$ (top) and $\Delta\lambda = 10^{-10}$ (bottom)

BVP

$$\frac{d^2w_r}{dx^2}+\lambda w_r=-\frac{d^2w_e}{dx^2}-(\lambda+\Delta\lambda)\,w_e(x),$$
$$w_r(0)=-w_e(0),\quad w_r(1)=-w_e(1).$$

For example, taking the same $w_e(x)=1+x^2$, we get the particular solution

$$\widetilde{w}_r(x,\lambda,\Delta\lambda)=-\frac{\lambda+\Delta\lambda}{\lambda}x^2-\frac{2+\lambda+\Delta\lambda}{\lambda}+\frac{2\lambda+\Delta\lambda}{\lambda^2} \tag{18}$$

and $\widetilde{w}_r(x,\lambda,\Delta\lambda)\neq -w_e$. So, the linear system for A_r, B_r takes the form

$$A_r+B_r+\widetilde{w}_r(0,\lambda,\Delta\lambda)=-w_e(0),$$
$$A_re^{ik}+B_re^{-ik}+\widetilde{w}_r(1,\lambda,\Delta\lambda)=-w_e(1),$$

with $\widetilde{w}_r$ given in (18). The system has non zero solutions for all λ except the eigenvalues λ_n when the system becomes degenerate. However, due to the iterative procedure of the solution and rounding errors we never solve the system with the *exact* λ_n. We observe degeneration of the system as a considerable growth of the solution in a neighbourhood of the eigenvalues. The data corresponding to $\Delta\lambda=10^{-15}$ and $\Delta\lambda=10^{-10}$ are presented in Fig. 3.

The value $\Delta\lambda=10^{-15}$ is too small to regularize the solution. But the value $\Delta\lambda=10^{-10}$ yields a smooth curve.

These two regularizing procedures are called the $\varepsilon-$procedure and the $\lambda-$procedure. Numerous examples of application of this technique to different eigenvalue problems can be found in [Reutskiy (2005, 2006, 2007b, 2008)].

Having a smooth response curve, we apply the following simple algorithm. First, we localize these maxima of $F(\lambda)$ on the intervals $[a_i,b_i]$. Next, we solve the univariate optimization problem inside each one. In particular, we apply Brent's method based on a combination of parabolic interpolation and bisection of the function near the extremum(see [Press, Teukolsky, Vetterling, and Flannery (2002)]).

3 Numerical implementation of the algorithm

The same approach of the external excitation can be combined with an approximate solution of BVP (4), (5), (6) for the response field w_r. To solve the equation

$$\frac{d^2w}{dx^2}+q(\lambda,x)\,w=f(\lambda,x) \tag{19}$$

through the paper, we apply the asymmetric radial basis functions (RBF) collocation method proposed by Kansa (1990). This method is chosen as a truly meshless technique which can be extended easily onto the 2D case. Meshless methods rely on a group of points. This means that the burdensome work of mesh generation is avoided and more accurate description of irregular complex geometries can be achieved. For more basic details about RBFs interested readers are referred to [Buhmann (2003)].

In application to the BVP considered the Kansa's method is as follows. We look for an approximate solution of the BVP in the form of the linear combination:

$$w = \sum_{j=1}^{N+2} u_j \Psi(x - \xi_j). \tag{20}$$

The points ξ_i are taken as follows: $\xi_i = \Delta x(i-0.5)$, $i = 1,...,N$, $\xi_{N+1} = -0.5/N$, $\xi_{N+2} = 1+0.5/N$. So the endpoints of the interval $[0,1]$ are not included; u_j are the free parameters of the problem. We use only the multiquadrics basis (MQ) RBFs in this paper

$$\Psi(x-\xi) = \left[c^2 + (x-\xi)^2\right]^{1/2}, \tag{21}$$

where c is the shape parameter, see [Fasshauer and Zhang (2007)]. The collocation with the right hand side f at the interior points ξ_i gives N the equations:

$$\sum_{j=1}^{N+2} u_j \left[\frac{d^2\Psi(\xi_i - \xi_j)}{dx^2} + q(\lambda, \xi_i)\Psi(\xi_i - \xi_j)\right] = f(\lambda, \xi_i).$$

Considering the boundary conditions of the type

$$\alpha_0 \frac{dw(0)}{dx} + \beta_0 w(0) = \gamma_0, \quad \alpha_1 \frac{dw(1)}{dx} + \beta_1 w(1) = \gamma_1$$

we get the rest two equations:

$$\sum_{j=1}^{N+2} u_j \left[\alpha_0 \frac{d\Psi(0-\xi_j)}{dx} + \beta_0 \Psi(0-\xi_j)\right] = \gamma_0,$$

$$\sum_{j=1}^{N+2} u_j \left[\alpha_1 \frac{d\Psi(1-\xi_j)}{dx} + \beta_1 \Psi(1-\xi_j)\right] = \gamma_1.$$

Here

$$\frac{d\Psi(x-\xi)}{dx} = (x-\xi)\left[c^2 + (x-\xi)^2\right]^{-1/2}$$

$$\frac{d^2\Psi(x-\xi)}{dx^2} = \frac{1}{\left[c^2+(x-\xi)^2\right]^{1/2}} - \frac{(x-\xi)^2}{\left[c^2+(x-\xi)^2\right]^{5/2}}.$$

The $(N+2)\times(N+2)$ system is solved by a standard procedure of the Gauss elimination.

Applying the method presented to the initial problem (1), (2) we get the sequence of BVPs:

$$\frac{d^2 w_r}{dx^2} + q(\lambda + i\varepsilon, x) w_r = -\frac{d^2 w_e}{dx^2} - q(\lambda, x) w_e \tag{22}$$

$$b_l[\lambda + i\varepsilon, w_r(0)] = -b_l[\lambda, w_e(0)], \tag{23}$$

$$b_r[\lambda + i\varepsilon, w_r(1)] = -b_r[\lambda, w_e(1)], \tag{24}$$

when the $\varepsilon-$procedure is used. Using the $\lambda-$procedure, we replace (22), (23), (24) by the equation

$$\frac{d^2 w_r}{dx^2} + q(\lambda, x) w_r = -\frac{d^2 w_e}{dx^2} - q(\lambda + \Delta\lambda, x) w_e \tag{25}$$

$$b_l[\lambda, w_r(0)] = -b_l[\lambda + \Delta\lambda, w_e(0)], \tag{26}$$

$$b_r[\lambda, w_r(1)] = -b_r[\lambda + \Delta\lambda, w_e(1)], \tag{27}$$

Having $w_r(x,\lambda)$, $w_e(x,\lambda)$ and using the sum $w(x,\lambda) = w_e(x,\lambda) + w_r(x,\lambda)$, we calculate the norm function $F(\lambda)$ like (11) and get the eigenvalues as the maxima of the response curve.

Example 1. Consider the regular Sturm–Liouville problem

$$\frac{d^2 w}{dx^2} + \frac{\lambda}{(1+x)^2} w = 0, \quad w(0) = w(1) = 0 \tag{28}$$

with the exact solution

$$\begin{aligned} w_n &= const \times \sqrt{1+x}\sin\left(n\pi \frac{\ln(1+x)}{\ln 2}\right), \\ \lambda_n &= k_n^2 = \frac{1}{4} + \left(\frac{n\pi}{\ln 2}\right)^2. \end{aligned}$$

Some results are placed in Table 1. corresponding to $w_e = 1 + x^2$, $\Delta\lambda = 10^{-3}$

Table 1: The Sturm-Liouville problem (28). The exciting solution $w_e = 1+x^2$. The value of λ is shown. The top part: $\lambda-$ procedure with $\Delta\lambda = 10^{-3}$; the bottom part: $\varepsilon-$ procedure with $\varepsilon = 10^{-3}$.

i	λ_{ex}	$N = 25$	$N = 50$
$\Delta\lambda = 10^{-3}$			
1	20.79228845	20.792292	20.79228845
2	82.41915382	82.419185	82.41915736
3	185.13059609	185.13042	185.1306146
4	328.92661528	328.92475	328.9266485
5	513.80721138	513.79933	513.8072091
$\varepsilon = 10^{-3}$			
1	20.79228845	20.792293	20.79228847
2	82.41915382	82.419255	82.41915481
3	185.13059609	185.13095	185.1306048
4	328.92661528	328.92699	328.9266313
5	513.80721138	513.80503	513.8072333

The data placed in the three columns of Table 2 are obtained with $w_e(x) = \sin(x)$, $w_e(x) = \exp(x)$ and $w_e(x) = 1+x^2$ correspondingly, $N = 50$ and with the help of the $\varepsilon-$ procedure with $\varepsilon = 10^{-3}$. Thus, we found that the solution is not very sensible to the particular choice of $w_e(x)$. However, the optimal choice of the exciting field needs further investigations.

The shape parameter c of the multiquadrics basis (MQ) RBFs through the paper is taken in such a way that $cN \simeq 5 \div 10$. For example, $c = 0.4$ for $N = 25$ and $c = 0.2$ for $N = 50$.

3.1 Non-linear Sturm–Liouville problems

Example 2. Let us consider the non-linear Sturm–Liouville problem

$$\frac{d^2w}{dx^2} + \frac{1}{(\lambda + x^2)^2} w = 0, \quad w(0) = w(1) = 0, \tag{29}$$

when $q(x,\lambda)$ depends on the spectral parameter λ in a nonlinear manner. With the help of some analytical tricks [Akulenko and Nesterov (2005)] it is possible to construct the exact solution of the form

$$w(x,\lambda) = const \times \left(\lambda + x^2\right)^{1/2} \sin\varphi(x,\lambda) \tag{30}$$

Table 2: The Sturm-Liouville problem (28). Different exciting solutions $w_e(x)$. The $\varepsilon-$ procedure with $\varepsilon = 10^{-3}$.

i	$w_e = \sin(x)$	$w_e = exp(x)$	$w_e = 1+x^2$
1	20.79228844	20.79228843	20.79228845
2	82.41915508	82.41915491	82.41915736
3	185.1306036	185.1306041	185.1306146
4	328.9266316	328.9266286	328.9266485
5	513.8072262	513.8072301	513.8072091

$$\varphi(x,\lambda) = \left(1+\lambda^{-1}\right)^{1/2} \arctan\left(x\lambda^{-1/2}\right) \tag{31}$$

The eigenvalues are roots of the equation:

$$\varphi(1,\lambda_n) = n\pi.$$

The method of solution does not differ from the one described above. Applying the $\varepsilon-$procedure, we consider the sequence of the BVPs

$$\frac{d^2w_r}{dx^2} + \frac{1}{\left(\lambda + i\varepsilon + x^2\right)^2} w_r = -\frac{d^2w_e}{dx^2} - \frac{1}{\left(\lambda + x^2\right)^2} w_e,$$

$$w_r(0) = -w_e(0),\ w_r(1) = -w_e(1).$$

To solve the problems we use the asymmetric RBF collocation method described at the beginning of the section. The rest part of the algorithm is the same. Using Brent's method we find the eigenvalues as maxima of the function $F(\lambda)$.

The data placed in Table 3 correspond to $w_e = 1+x^2$ and the smoothing $\lambda-$ procedure with $\lambda = 10^{-6}$.

Example 3. Let us consider the generalized Sturm–Liouville problem with the eigenparameter in the boundary conditions.

$$\frac{d^2w}{dx^2} + \lambda w = 0,\ 0 \le x \le 1, \tag{32}$$

$$\frac{dw(0)}{dx} + \lambda w(0) = 0, \quad \frac{dw(1)}{dx} - \lambda w(1) = 0. \tag{33}$$

The eigenvalues are roots of the equation [Annaby and Tharwat (2006)]

$$2\lambda \cos\sqrt{\lambda} + \sqrt{\lambda}\,(1-\lambda)\sin\sqrt{\lambda} = 0.$$

Table 3: Non-linear Sturm-Liouville problem (29). Exciting solution $w_e = 1 + x^2$. The $\lambda-$regularizing procedure with $\Delta\lambda = 10^{-6}$ is applied.

i	λ_{ex}	$N = 200$	$N = 400$
1	0.16564262693	0.1656426340	0.1656426307
2	0.04867383829	0.0486738399	0.0486738398
3	0.02321423872	0.0232142359	0.0232142391
4	0.01358401918	0.0135840131	0.0135840186
5	0.00891601673	0.0089160161	0.0089160151
6	0.00630115758	0.0063011890	0.0063011553
7	0.00468943806	0.0046895426	0.0046894369
8	0.00362594868	0.0036260682	0.0036259527
9	0.00288737785	0.0028879770	0.0028873943
10	0.00235360368	0.0023777447	0.0023536416

Using the $\lambda-$procedure, we solve the sequence of BVPs

$$\frac{d^2w_r}{dx^2} + \lambda w_r = -\frac{d^2w_e}{dx^2} - (\lambda + \Delta\lambda) w_e, \quad 0 \le x \le 1,$$

$$\frac{dw_r(0)}{dx} + \lambda w_r(0) = -\frac{dw_e(0)}{dx} - (\lambda + \Delta\lambda) w_e(0),$$
$$\frac{dw_r(1)}{dx} - \lambda w_r(1) = -\frac{dw_e(1)}{dx} + (\lambda + \Delta\lambda) w_e(1).$$

To solve the BVPs we apply Kansa's method described above The data placed in Table 4 correspond to $w_e(x) = 1 + x^2$ and $\Delta\lambda = 10^{-3}$.

Example 4. Let us consider a non-linear Sturm–Liouville problem with the eigenparameter in the boundary conditions as the next example. This example is taken from [Chanane (2005)].

$$\frac{d^2w}{dx^2} + (\lambda - e^x) w = 0, \tag{34}$$

$$w(0) = 0, \quad \cos\sqrt{\lambda}\frac{dw(1)}{dx} - \sqrt{\lambda}\sin\sqrt{\lambda}w(1) = 0. \tag{35}$$

Applying the method presented with the smoothing $\lambda-$ procedure, we get the BVPs depending on the λ:

Table 4: The Sturm-Liouville problem with the eigenparameter in the boundary conditions (32), (33). The exciting solution $w_e = 1 + x^2$. The $\lambda-$ procedure with $\Delta\lambda = 10^{-3}$ is applied.

i	$\sqrt{\lambda_{ex}}$	$N = 100$	$N = 200$
1	1.3065424	1.3070222	1.3065471
2	3.6731944	3.6734714	3.6731971
3	6.5846200	6.5848369	6.5846222
4	9.6316843	9.6317953	9.6316858
5	12.7232409	12.7233156	12.7232416
6	15.8341055	15.8341382	15.8341058
7	18.9549713	18.9549861	18.9549717
8	22.0816593	22.0816602	22.0816597
9	25.2120266	25.2120215	25.2120269
10	28.3448639	28.3448657	28.3448642

$$\frac{d^2 w_r}{dx^2} + (\lambda - e^x)\, w_r = -\frac{d^2 w_e}{dx^2} - (\lambda + \Delta\lambda - e^x)\, w_e,$$

$$w_r(0) = -w_e(0),$$

$$\cos\sqrt{\lambda}\frac{dw_r(1)}{dx} - \sqrt{\lambda}\sin\sqrt{\lambda}w_r(1) = -\cos\sqrt{\lambda+\Delta\lambda}\frac{dw_e(1)}{dx} + \sqrt{\lambda+\Delta\lambda}\sin\sqrt{\lambda+\Delta\lambda}w_e(1)\,.$$

Having $w_r(x,\lambda)$, $w_e(x,\lambda)$ and using the sum $w(x,\lambda) = w_e(x,\lambda) + w_r(x,\lambda)$, we calculate the norm function $F(\lambda)$ and the eigenvalues as maxima of $F(\lambda)$ with the help of Brent's procedure. The exact eigenvalues are taken from the original work by Chanane (2005). The RBF solution shown in Table 5 is obtained using $N = 50$ MQ RBFs (21) with $c = 0.2$.

3.2 Singular Sturm–Liouville problems

Here we consider the important case of the singular Sturm–Liouville problem defined on the infinite or semi-infinite interval. This class of problems includes the

Table 5: Non-linear Sturm-Liouville problem with the eigenparameter in the boundary conditions (34), (35). The exciting solution $w_e = 1 + x^2$. The smoothing $\lambda-$ procedure with $\Delta\lambda = 10^{-6}$. RBF solution: $N = 50$, $c = 0.2$; FD solution: $\Delta x = 2 \times 10^{-4.}$

i	λ_{ex}	RBF	FD
1	0.929062009	0.929049	0.92906204
2	6.747881413	6.747921	6.74788121
3	16.124547958	16.124566	16.12454756
4	31.220275879	31.220419	31.22027703
5	50.733928680	50.734036	50.73392672
6	75.581466675	75.581594	75.58147128

Schrödinger equation

$$\frac{d^2w}{dx^2} + [\lambda - V(x)]\,w = 0 \tag{36}$$

widely used in quantum calculations [Chen and Shizgal (1998, 2001); Shizgal and Chen (1996)]. Here $V(x)$ is the potential.

Example 5. First consider the Sturm–Liouville problem defined in the semi-infinite interval $[0,\infty)$ studied by Titchmarsh (1962):

$$\frac{d^2w}{dx^2} + (\lambda - x)\,w = 0,\; w(0) = w(\infty) = 0. \tag{37}$$

The eigenvalues are known to be zeros of

$$J_{1/3}\left(\frac{2}{3}\lambda^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\lambda^{3/2}\right) = 0.$$

Using the transformation

$$y = \frac{x}{1+x} \tag{38}$$

one gets the problem in $[0,1]$:

$$(1-y)^4\frac{d^2w}{dy^2} - 2(1-y)^3\frac{dw}{dy} + \left(\lambda - \frac{y}{1-y}\right)w = 0,$$

$$w(0) = w(1) = 0.$$

Note that the third term in the equation has the singularity at $y = 1$. Applying the method presented with the $\lambda-$ procedure described in the previous section, we consider the following BVPs

$$(1-y)^4 \frac{d^2 w_r}{dy^2} - 2(1-y)^3 \frac{dw_r}{dy} + \left(\lambda - \frac{y}{1-y}\right) w_r$$
$$= -(1-y)^4 \frac{d^2 w_e}{dy^2} + 2(1-y)^3 \frac{dw_e}{dy} - \left(\lambda + \Delta\lambda - \frac{y}{1-y}\right) w_e \quad (39)$$

$$w_r(0) = -w_e(0), \quad w_r(1) = -w_e(1). \quad (40)$$

Here w_e is any smooth enough function. To solve the sequence of BVPs we apply the asymmetric RBF collocation method described above. Note that the singular point $y = 1$ is not included in the set of the collocation points $\xi_i = \Delta x(i-0.5)$, $i = 1,\ldots,N$, $\xi_{N+1} = -0.5/N$, $\xi_{N+2} = 1+0.5/N$.

Then we calculate the sum $w = w_e + w_r$ and varying k, get the response curve $F(k)$ (11). We get the eigenvalues as the positions of its maxima.

Table 6: The Sturm-Liouville problem in the infinite interval $[0,\infty)$ (37). RBF solution with $w_e = 1+x^2$. Convergence with the growth of the number of free parameters N. The value of λ is shown. The $\lambda-$ regularizing procedure with $\Delta\lambda = 10^{-6}$ is applied.

i	λ_{ex}	$N = 100$	$N = 200$
1	2.3381074105	2.3381074	2.338107403
2	4.0879494441	4.0879494	4.087949437
3	5.5205598281	5.5205598	5.520559821
4	6.7867080901	6.7867081	6.786708083
5	7.9441335871	7.9441356	7.944133580
6	9.0226508533	9.0226563	9.022650845
7	10.0401743416	10.040173	10.04017433
8	11.0085243037	11.008497	11.00852424
9	11.9360155632	11.935301	11.93601528
10	12.8287767529	12.833094	12.82877603

We take $c = 0.15$ and $c = 0.1$ for $N = 50$ and $N = 100$. For $N = 150$ and $N = 200$ the parameter $c = 0.05$. The results of the calculations are placed in Table 6.

Example 6. Many calculations of the Schrödinger equation were performed with the use of the non-polynomial oscillator (NPO) potential of the form [Chen and

Shizgal (1998); Shizgal and Chen (1996); Chen and Shizgal (2001)]

$$V_1(x) = x^2 + \frac{px^2}{1+gx^2},\ p,g \geq 0. \tag{41}$$

We have also considered the potential given by

$$V_2(x) = x^6 - 3x^2 \tag{42}$$

considered in Sinha, Roychoudhury, and Varshni (1996). The potentials $V_1(x)$, $V_2(x)$ are symmetric $V(-x) = V(x)$. Under this condition all the eigenmodes can be divided into two groups: the symmetric modes with the boundary condition at $x = 0$

$$\frac{dw_r(0)}{dx} = 0$$

and the anti-symmetric modes with the boundary condition

$$w_r(0) = 0.$$

Using the same transformation (38) and applying the $\lambda-$ procedure , one gets the BVPs

$$\begin{aligned}(1-y)^4 \frac{d^2 w_r}{dy^2} - 2(1-y)^3 \frac{dw_r}{dy} + \left(\lambda - V\left(\frac{y}{1-y}\right)\right) w_r \\ = -(1-y)^4 \frac{d^2 w_e}{dy^2} + 2(1-y)^3 \frac{dw_e}{dy} - \left(\lambda + \Delta\lambda - V\left(\frac{y}{1-y}\right)\right) w_e,\end{aligned}$$

$$\frac{dw_r(0)}{dx} = -\frac{dw_e(0)}{dx} \text{ (symmetric) or } w_r(0) = -w_e(0) \text{ (anti-symmetric)},$$
$$w_r(1) = -w_e(1),$$

which should be solved with different λ. We use Kansa's method described above. The data placed in Tables 7, 8 are compared with the calculations presented in [Chen and Shizgal (1998), Fack and den Berghe (1985)].

3.3 Periodic problems.

Consider the periodic problem on the interval $[0,T]$

$$\frac{d^2 w}{dx^2} + [\lambda - p(x)]w = 0, \quad w(0) = w(T), \quad \frac{dw(0)}{dx} = \frac{dw(T)}{dx} \tag{43}$$

Table 7: The Schrödinger equation in the infinite interval $(-\infty,\infty)$ (36) with the potential $V_1(x) = x^2 + 100x^2/\left(1+100x^2\right)$. The $\lambda-$ regularizing procedure with $\Delta\lambda = 10^{-6}$ is applied.

	$V_1(x) = x^2 + 100x^2/\left(1+100x^2\right)$		
i	$N = 50$	[CS]	[FB]
1	1.836567	1.836336	1.836334
2	3.983098	3.983099	3.983098
3	5.928499	5.928332	5.928328
4	7.984443	7.984498	7.903154
5	9.949297	9.949607	9.882298
6	11.98536	-	-
7	13.95955	-	-
8	15.98606	-	-
9	17.96661	-	-
10	19.98658	-	-

Here $p(x+T) = p(x)$ is a periodic function. Using the transform $x = Ty$ one gets

$$\frac{d^2w}{dy^2} + T^2\left[\lambda - p(x(y))\right]w = 0, \quad w(0) = w(1), \quad \frac{dw(0)}{dy} = \frac{dw(1)}{dy}$$

Applying the method presented with the smoothing $\lambda-$ procedure, we consider the sequence of the BVPs

$$\frac{d^2w_r}{dy^2} + T^2\left[\lambda - p(x(y))\right]w_r = -\frac{d^2w_e}{dy^2} - T^2\left[\lambda + \Delta\lambda - p(x(y))\right]w_e,$$

$$\begin{aligned} w_r(0) - w_r(1) &= w_e(1) - w_e(0), \\ \frac{dw_r(0)}{dy} - \frac{dw_r(1)}{dy} &= \frac{dw_e(1)}{dy} - \frac{dw_e(0)}{dy} \end{aligned}$$

with some arbitrary smooth function w_e. To solve this BVP we use the Kansa's method described above.

Example 7. Consider the particular case $T = \pi$, $p(x) = 10\cos(2\pi x)$ Yücel (2007). The data placed in Table 9 are obtained with $w_e = 1 + y^2$ and $\Delta\lambda = 10^{-6}$.

Table 8: The Schrödinger equation in the infinite interval $(-\infty,\infty)$ (36) with the potential $V_2(x) = x^6 - 3x^2$. The $\lambda-$ regularizing procedure with $\Delta\lambda = 10^{-6}$ is applied.

	$V_2(x) = x^6 - 3x^2$	
i	$N = 50$	[CS]
1	1.935484	1.935482
2	6.301338	-
3	11.68098	11.68097
4	18.04670	-
5	25.25460	25.25460
6	33.23177	-
7	41.89099	-
8	51.20523	-
9	61.10533	-
10	71.58820	71.57904

3.4 Others BVP solvers

The method presented in the paper leads to a sequence of the BVPs depending on λ as a parameter. To solve the BVPs we apply Kansa's method through the paper. However, this technique can be combined with any appropriate BVP solver. For example, the following four-order accurate finite-difference (FD) scheme:

$$\frac{\delta^2[w_n]}{\Delta x^2} + q_n w_n + \frac{1}{12}\delta^2[q_n w_n] = f_n + \frac{1}{12}\delta^2[f_n], \tag{44}$$

can be used to approximate (19). Here we denote

$$\delta^2[w_n] = w_{n+1} - 2w_n + w_{n-1},$$
$$\delta^2[q_n w_n] = q_{n+1}w_{n+1} - 2q_n w_n + q_{n-1}w_{n-1},$$
$$w_n = w(x_n), \quad x_n = \Delta x(n-1), \quad \Delta x = 1/(N-1)$$

For the boundary condition of the type

$$\alpha\frac{dw(0)}{dx} + \beta w(0) + \gamma = 0$$

we use following FD approximation

$$\frac{\alpha}{\Delta x}(w_2 - w_1) + \beta w_1 + \frac{\alpha\Delta x}{2}[q(\lambda,0) - f(\lambda,0)] + \gamma = 0,$$

Table 9: Periodic problem (43). Kansa's method with the smoothing $\lambda-$ procedure $\Delta\lambda = 10^{-6}$ is applied.

i	λ_{ex}	$N=20$	$N=50$	$N=100$
1	2.09946	2.09938	2.09946	2.09946
2	7.44911	7.44814	7.44903	7.44910
3	16.64822	16.64831	16.64821	16.64822
4	17.09658	17.11382	17.09810	17.09673
5	36.35887	36.36549	36.36362	36.35886

which has the error $\sim \Delta x^2$. The similar approximation is used at the right endpoint $x=1$. As a result one gets the linear system

$$\begin{aligned} a_1 w_2 + b_1 w_1 &= g_1, \\ a_n w_{n+1} + b_n w_n + c_n w_{n-1} &= g_n, \quad n = 2, \ldots, N-1, \\ b_N w_N + c_N w_{N-1} &= g_N, \end{aligned}$$

which can be effectively solved by the double-sweep method.

Example 8 Consider a Sturm–Liouville eigenvalues problem with Liu (2008a); Pryce (1993)

$$\frac{d^2 w}{dy^2} + (\lambda - \exp y)\, w = 0, \quad w(0) = w(\pi) = 0 \tag{45}$$

Using the transform $y = \pi x$, one gets the eigenvalue problem on $[0,1]$

$$\frac{d^2 w}{dx^2} + \pi^2 (\lambda - \exp \pi x)\, w = 0, \quad w(0) = w(1) = 0.$$

Applying the method presented with the smoothing $\lambda-$procedure, we get the sequence of BVPs

$$\frac{d^2 w_r}{dx^2} + \pi^2 (\lambda - \exp \pi x)\, w_r = -\frac{d^2 w_e}{dx^2} - \pi^2 (\lambda + \Delta\lambda - \exp \pi x)\, w_e,$$
$$w_r(0) = -w_e(0), \quad w_r(1) = -w_e(1).$$

The data placed in the first column of Table 10 are obtained using FD solver with $\Delta x = 2 \times 10^{-4}$. The exciting solution $w_e = 1 + x^2$. We compare the eigenvalues with those obtained in Liu (2008a); Pryce (1993). Another example of applying the FD solver is shown in the last column of Table 5. Here we place the data obtained by FD method solving the non-linear eigenvalue problem (34), (35).

Table 10: Sturm-Liouville problem (45). FD solution with $\Delta x = 2 \times 10^{-4}$. The smoothing $\lambda-$ procedure with $\Delta\lambda = 10^{-6}$ is applied.

i	*present*	Liu (2008a)	Pryce (1993)
1	4.8966696497	4.89666937998	4.8966693800
5	32.26370706	32.26370704581	32.263707046
10	107.11667611	107.116676138	107.11667614
15	232.07881195	232.078811987	232.07881198
20	407.06523523	407.065235278	407.06523527
25	632.05890781	632.058907930	632.05890789
30	907.05546070	907.055460697	907.05546058

4 Eigenmodes

The algorithm described above is focused on the problem of finding eigenvalues. Let us dwell in brief on the problem of calculation of the corresponding eigenmodes. The method of finding eigenmodes proposed here is based on the simple physical fact that when a system approaches resonance, then, just the resonance (or eigen) mode is excited in the system. So, when the spectral parameter λ in the BVP (4), (5), (6) is closed to an eigenvalue of the initial eigenvalue problem (1), (2), then the sum $w = w_e + w_r$ is closed to the corresponding eigenmode.

Algorithm is as follows. Using the procedure described above we calculate the approximate value λ_{ap}. Then we calculate the function $w(x,\lambda_{ap}) = w_e(x,\lambda_{ap}) + w_r(x,\lambda_{ap})$ at the representative points $x_i \in [0,1]$, $i = 1,\ldots,M$ and the normed values.

$$\widehat{w}_i(\lambda_{ap}) = w(x_i,\lambda_{ap})/w_{\max}, \quad w_{\max} = \max_{i=1,\ldots,M} |w(x_i,\lambda_{ap})|. \tag{46}$$

In Fig. 4 we present the eigenmodes of *Example 2* (29).

The exact solution (30), (31) was normed in the same way as (46). The data are compared in Table 11. It demonstrates that we obtain the approximate eigenmodes with a high precision.

5 Conclusion

We present a numerical method for solving generalized Sturm-Liouville problems and the problems with parameter dependent boundary conditions. It is convenient for determining some first eigenvalues of the system which are often of the most interest for engineering applications. It leads to the solution of a sequence of bound-

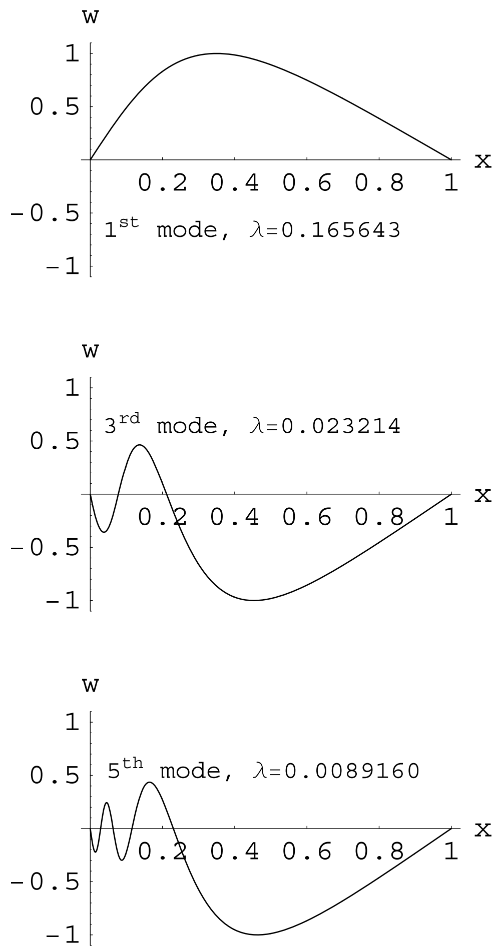

Figure 4: The 1^{st}, 3^{rd} and 5^{th} eigenmodes of Sturm-Liouville problem (29).

ary value problems which depend on the spectral parameter. Varying this parameter, one gets the eigenvalues as positions of maxima of the norm function $F(\lambda)$. The growth of the amplitude of response near the eigenvalue is a sequence of the degeneracy of the matrix approximating the BVP under consideration. From this point of view the presented approach is similar to the one described in Li (2008), where the degeneracy is measured by the infinitesimal values of the minimal eigenvalue of the stiffness matrix of the problem.

Table 11: Eigenmodes from Example 2. Comparison between the exact and computed data.

	1^{th} eigenmode		2^{th} eigenmode	
x	$\widetilde{w}_{ex}$	$\widetilde{w}_{ap}$	$\widetilde{w}_{ex}$	$\widetilde{w}_{ap}$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.831375	0.831374	-0.180994	-0.180999
0.4	0.986162	0.986162	-0.987911	-0.987912
0.6	0.749069	0.749069	-0.835626	-0.835624
0.8	0.388873	0.388872	-0.441099	-0.441095
1.0	0.000000	0.000000	0.000000	0.000000

Approximating the original eigenvalue problem by a linear system

$A(\lambda)\mathbf{x} = \mathbf{b},\ \mathbf{x},\mathbf{b} \in R^N,\ A(\lambda) \in R^{N\times N},$

one gets the eigenvalue λ_i $i = 1,...,N$ as roots of the equation

$$\det A(\lambda) = 0. \tag{47}$$

On the other hand, according to Cramer's rule we can write the solution in the form

$$\mathbf{x}(\lambda) = \frac{1}{\det A(\lambda)}\left[\det A_1(\lambda),...,\det A_N(\lambda)\right]^T,$$

where $A_i(\lambda)$ is the matrix found by replacing the ith column of $A(\lambda)$ with $\mathbf{b}$. So, when λ approaches a root of (47), one observes the growth of the norm $F(\lambda)$ of the solution $\mathbf{x}(\lambda)$. And from the physical point of view this is well-known that the amplitude of oscillations increases when an electrical or mechanical system approaches resonance.

The key moment of the algorithm is the use of the special regularizing procedures which provides a smooth curve $F(\lambda)$ and, as a sequence, provides a high precision in determining eigenvalues. We would like to attract readers attention to the fact that the same technique can be applied to a very wide class of eigenvalue problems: the Sturm–Liouville problems, the Schrödinger equation, the non-classical generalized Sturm–Liouville problems, periodic problems. In the paper we mainly use the asymmetric radial basis functions (RBF) collocation method as the BVPs solver. However, this technique can be combined with any appropriate BVP solver. It seems possible to extend the same approach to eigenvalue problems with other differential equations, e.g. to problems of fourth order and to the case of multi-dimensional systems. This will be the subject of further investigations.

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